TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 10, October 1995

# $L_2(q)$ AND THE RANK TWO LIE GROUPS: THEIR CONSTRUCTION IN LIGHT OF KOSTANT'S CONJECTURE

MARK R. SEPANSKI

ABSTRACT. This paper deals with certain aspects of a conjecture made by B. Kostant in 1983 relating the Coxeter number to the occurrence of the simple finite groups $L(2, q)$ in simple complex Lie groups. A unified approach to Kostant's conjecture that yields very general results for the rank two case is presented.

## 1. INTRODUCTION

This work centers around a conjecture made by B. Kostant in 1983 in *A Tale of Two Conjugacy Classes* [12]. He proposed a link between a certain intrinsic number of a simple Lie group, called the Coxeter number, and the occurrence of certain finite simple groups. The conjecture has fueled quite a bit of research and turns out to have many connections to other areas of mathematics. Although a complete statement of Kostant's conjecture may be found in Theorem 2.1.1 in [16], let us outline the conjecture broadly in the following paragraphs.

The finite simple groups in question are the families alternately known as $PSL(2, q)$ or $L_2(q)$, where $q$ is a prime power. To recall the definition of these groups, first write $\mathbb{F}_q$ for the field of $q$ elements and write $SL(2, q)$ for the set of two-by-two matrices of determinant one with entries in $\mathbb{F}_q$. Then $L_q(q)$ is defined as $SL(2, q)$ modulo its center (which consists of $\pm I$). These groups are of fundamental importance in group and representation theory since they often play the role of building blocks.

Now if we have some complex simple Lie group $G$, let us write $h$ for the Coxeter number [11]. Then, roughly, Kostant's conjecture states that when $2h + 1$ is an odd prime power, then $L_2(2h + 1)$ sits inside the Lie group. Moreover, it sits inside the Lie group in a special way. The conjecture states that the Lie algebra breaks up into certain "principal series" representations and subrepresentations of $L_2(2h+1)$ depending on the exponents of $G$ [11]. These principal series representations will be dealt with explicitly in Section 2.

Considerable work has been done on finite subgroups of Lie groups (e.g. Cohen and Wales in [3] and [2]) and, in particular, on Kostant's conjecture ([2], [3], [5], [10], [13], and [10] of Cohen, Griess, Kleidman, Lisser, Meurman,

Received by the editors July 1, 1994 and, in revised form, October 1, 1994.
1991 *Mathematics Subject Classification.* Primary 22E40, 17B60, 20F29.
*Key words and phrases.* $L_2(q)$, $PSL(2, q)$, Kostant's conjecture.

Ryba, and Wales). The conjecture is verified easily in the nonexceptional cases by using a character table and Schur indicators (see [10] or [4]). However, the exceptional cases are much more difficult. The following table indicates the papers responsible for checking the conjecture in each case (note: a computer is relied upon in many of the papers below).

| | |
|---|---|
| $G_2$ | [13] and [2] |
| $F_4$ | [3] |
| $E_6$ | [3] |
| $E_7$ | [10] |
| $E_8$ | [5] |

This paper will examine a unified approach to the conjecture that yields very general results in the rank two case. That the conjecture has been checked in all cases has been noted above. However, most proofs have relied on a computer and this was the case for $E_8$ in particular [5]. In fact, the result of $L_2(61)$ sitting inside $E_8$ had been a major stumbling block in the classification of all finite simple subgroups occurring in complex simple Lie groups (only $L_2(2, 29)$ for $E_7$ and $L_2(32)$ and $Sz(8)$ for $E_8$ are still in doubt—see [4, Table 1]). Of course, it is desirable to have a proof that does not need a computer.

One of the aims of my research is to provide such a proof. In fact, the hope is to do something even stronger: begin with $L_2(q)$ and construct the corresponding Lie group out of knowledge of this finite group and its representations.

As an overview, start with three pieces of data: a principal series representation of $L_2(q)$ on a vector space $V$, a $L_2(q)$ invariant symmetric nondegenerate two-form $( \, , \, )$ on $V$, and a $L_2(q)$ invariant alternating three-form $( \, , \, , \, )$ on $V$. With these, define a $L_2(q)$ invariant algebra structure $[ \, , \, ]$ on $V$ according to the rule:

$$(v_1, v_2, v_3) = ([v_1, v_2], v_3) \tag{1}$$

for $v_1, v_2, v_3 \in V$. The idea is to see when $V$ can be made into a Lie algebra by this method, i.e., when can $[ \, , \, ]$ satisfy the Jacobi identity. If this can be done, then automatically $L_2(q)$ injects into the automorphism group of the Lie algebra. For instance, if this were done in the case of $E_8$, it would prove Kostant's conjecture. The central result of this paper is the following theorem (Theorem 12.2, Corollary 13.1, and Theorem 10.1):

**Theorem 1.1.** *For $V$ an irreducible principal series representation of $L_2(q)$ with $q$ an odd prime power subject to Restriction* 2.1, *the above construction can make $V$ into a nontrivial Lie algebra if and only if $q = 7, 9$, or* 13. *Moreover, in these cases, the resulting Lie algebra is $A_2$, $B_2$, and $G_2$, respectively.*

The above theorem proves Kostant's conjecture in the rank two case in a uniform manner. In the course of the proof, certain interesting facts appear. Chief among them is the connection between the Jacobi identity in the rank two case and the problem of tiling the plane. It turns out that, in most cases, the Jacobi identity forces certain integrality conditions (see Theorem 12.2) that are equivalent to the condition of being able to tile the plane with triangles, squares, or hexagons.

*Note*: There is a glossary of notation at the end of this paper that will help in locating definitions.

## 2. The $e$-basis

Throughout this paper, let $q = p^f$ be an odd prime power. The main group under consideration will be $L_2(q) = PSL(2, q) =$ the group of $2 \times 2$ matrices of determinant 1 over the field of $q$ elements, $\mathbb{F}_q$, all modulo its center. Since $q$ is odd, we may write

$$q = 2h + 1 \tag{2}$$

with $h$ an integer. This number, $h$, will end up playing the role of the Coxeter number in Lie theory.

It is well known that

$$|L_2(q)| = \frac{q(q^2-1)}{2} = 2(h+1)h(2h+1) \tag{3}$$

where $|L_2(q)|$ is the order of the group.

The product decomposition exhibited in equation (3) corresponds to three special subgroups of $L_2(q)$. The first, denoted by $\mathscr{A}$, consists of diagonal matrices. It is cyclic of order $h$. The second, denoted by $\mathscr{N}$, consists of the upper triangular matrices with ones on the diagonals. Its order is $q = 2h+1$ and it is cyclic only if $p = q$. Together, these two subgroups generate a *Borel* subgroup, $\mathscr{B}$, of $L_2(q)$ consisting of upper triangular matrices. The third special subgroup, denoted $\mathscr{K}$, is cyclic of order $h+1$. It is more complicated than $\mathscr{A}$ and $\mathscr{N}$ and is discussed in detail in Section 2.5 of [16].

For the present, the study of $\mathscr{B}$ will be the most important task. Of course one has

$$|L_2(q)/\mathscr{B}| = q + 1.$$

$L_2(q)/\mathscr{B}$ may be viewed as the projective line, $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$. Thus if we take a complex character $\pi$ of $\mathscr{B}$ and induce the representation up to $L_2(q)$, we get a $(q+1)$-dimensional representation. It is precisely these *principal series* representations that play a central role in Kostant's conjecture. Even though they are well understood, it will be useful for us to write then out explicitly.

*Notation.* To begin with, we *fix a generator* $\lambda$ for the multiplicative group $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$. This generator will be fixed throughout the paper. Next fix $\pi$ to be a complex *multiplicative character* of $\mathbb{F}_q^*$ such that $\pi(-1) = 1$. Thus, for each integer $m$ where $1 \le m \le h$, there exists such a character uniquely determined by $\pi_m(\lambda) = e^{2\pi i m/h}$. The reason for choosing $\pi$ to be trivial on $-1$ is that, by using the obvious homomorphism from $\mathbb{F}_q^*$ onto $\mathscr{A}$ (with kernel $\{\pm 1\}$), we may view $\pi$ as a character of $\mathscr{A}$. By extending $\pi$ to be trivial on $\mathscr{N}$, we may view $\pi$ as a character of $\mathscr{B}$. We will therefore view $\pi$ interchangeably as a character either of $\mathbb{F}_q^*$ or of $\mathscr{B}$ as context dictates.

Now let $V_\pi$ be $\mathrm{Ind}_{\mathscr{B}}^{L_2(q)}(\pi)$, the induced representation of $\pi$ from $\mathscr{B}$ to $L_2(q)$. The notation will be simplified to just $V$ whenever $\pi$ is understood. As in [14], we may consider $V$ to be the set of all complex-valued functions $f$ on $L_2(q)$ satisfying

$$f(bg) = \pi(b)f(g)$$

for all $b \in \mathscr{B}$ and $g \in L_2(q)$. With this, we have the action $gf(x) = f(xg)$. The appropriate theorem regarding the nature of $V$ is standard. For instance, it may be found in [14, §5.4]:

**Theorem 2.1.** *$V_{\pi_m}$ is an irreducible representation of $L_2(q)$ if and only if $\pi_m^2 \neq 1$. Moreover, $V_{\pi_m}$ and $V_{\pi_n}$ are equivalent if and only if $\pi_m = \pi_n$ or $\pi_m = \pi_n^{-1}$.*

It will be useful to write out a "delta" basis for $V$, i.e. a basis for which each basal element is supported on one right coset of $\mathscr{B}\backslash L_2(q)$. To this end, we choose the following representatives for $\mathscr{B}\backslash L_2(q)$:

$$g_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad g_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for $u \in \mathbb{F}_q$. The *e-basis* is defined by requiring that $e_v \in V$ and

$$e_v(g_w) = \delta_{v,w}$$

for all $v, w \in \mathbb{P}^1(\mathbb{F}_q)$ where $\delta_{v,w}$ is 1 if $v = w$ and 0 otherwise. These functions form a basis for $V$. It is also well known and easily checked that $L_2(q)$ acts on this basis as an (inverse transpose) linear fractional transformation on $\mathbb{P}^1(\mathbb{F}_q)$ with certain nonzero coefficients. This is detailed in the next theorem.

**Theorem 2.2.** *Let $u \in \mathbb{P}^1(\mathbb{F}_q)$ and*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L_2(q).$$

*Then $L_2(q)$ acts on $V$ in the e-basis by $ge_u = ke_v$ where $v \in \mathbb{P}^1(\mathbb{F}_q)$ is determined by*

$$v = \frac{du - c}{-bu + a}$$

*and $k \in \mathbb{C}^*$ is determined by*

$$k = \begin{cases} \pi(-bu + a) & \text{if } u, v \neq \infty, \\ \pi(1/b) & \text{if } u \neq \infty \text{ but } v = \infty, \\ \pi(1/a) & \text{if } u, v = \infty, \\ \pi(-b) & \text{if } u = \infty \text{ but } v \neq \infty. \end{cases}$$

*Proof.* Since this result is well known and just a matter of checking definitions, we omit the details. Part of it may be found in [14]. Note only that everything is well defined in the definition of $k$ since $ad - bc = 1$. □

It will also be useful for us to write out this action for a few elements in $L_2(q)$ that will be particularly important to us. Namely, define

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \tag{4}$$

$$N_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{5}$$

$$M_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \tag{6}$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

where $x \in \mathbb{F}_q^*$ and we recall that $\lambda$ was the fixed generator for $\mathbb{F}_q^*$. For simplicity, we will refer to $N_1$ as $N$ and to $M_1$ as $M$. The action on these elements is given by:

**Corollary 2.1.** *Let $u \in \mathbb{F}_q$. Then with the preceding notation, $L_2(q)$ acts on $V$ by*

(1) $Ae_u = \pi(\lambda)e_{\frac{u}{\lambda^2}}$ *and* $Ae_\infty = \pi(\lambda^{-1})e_\infty$,

(2) $N_x e_u = \pi(-xu+1)e_{\frac{u}{-xu+1}}$ *if* $u \neq 1/x$, $N_x e_{\frac{1}{x}} = \pi(x^{-1})e_\infty$, *and* $N_x e_\infty = \pi(-x)e_{\frac{-1}{x}}$,

(3) $M_x e_u = e_{u-x}$ *and* $M_x e_\infty = e_\infty$,

(4) $Se_u = \pi(u)e_{\frac{-1}{u}}$ *and* $Se_\infty = e_0$.

Our goal in this paper will be to determine when a principal series representation can be made into a nontrivial Lie algebra by the recipe given in equation (1). To get our desired results, we will need to place a restriction on the type of principal series we consider or require more of Kostant's conjecture to hold (see [16]). Specifically, we will do the first in this paper and always assume that:

**Restriction 2.1.** *In this paper, consider only those principal series representations $V_{\pi_m}$ where it is assume that if $h$ is even and $m$ is odd, then $\pi_m(\lambda)$ is a primitive $h$th root of unity, i.e., that $m$ and $h$ are relatively prime.*

## 3. The $f$-basis

For reasons that will become apparent later, it is convenient to introduce a "fourier transform" of our earlier $e$-basis. For now, we can view it as a way of diagonalizing the operators $M_x$. To this end, fix a nontrivial *additive character* $\chi$ of $\mathbb{F}_q$. This character will also be fixed throughout the remainder of the paper. The next definition gives the *$f$-basis.*

**Definition 3.1.** For $u \in \mathbb{F}_q$, define

$$f_u = \sum_{a \in \mathbb{F}_q} \chi(au)e_a$$

and let $f_\infty = e_\infty$.

We will also need the following "Bessel" and "Gamma" functions since they will come up often:

**Definition 3.2.** For $i, j \in \mathbb{F}_q$, let

$$\Gamma_{i,j} = \frac{1}{q} \sum_{a \in \mathbb{F}_q^*} \chi\left(\frac{i}{a} + ja\right)\pi(a).$$

Note that $\Gamma_{0,0} = 0$.

Though we will not need any properties of the $\Gamma_{i,j}$ at this moment, we will eventually need a few of their elementary properties. Thus we note:

**Lemma 3.1.** *The $\Gamma_{i,j}$ satisfy the following relations:*

(1) $\Gamma_{i,j} = \Gamma_{-i,-j}$,

(2) $\overline{\Gamma_{i,j}} = \Gamma_{j,i}$,

(3) $\Gamma_{i,j} = \pi(1/j)\Gamma_{ij,1}$ *for* $i \in \mathbb{F}_q$ *and* $j \in \mathbb{F}_q^*$,

(4) $\Gamma_{i,j} = \pi(i)\Gamma_{1,ij}$ *for* $i \in \mathbb{F}_q^*$ *and* $j \in \mathbb{F}_q$,
(5) $\Gamma_{1,0}\Gamma_{0,1} = 1/q$. *In particular,* $\Gamma_{0,1}$ *is nonzero.*

*Proof.* Part (1) follows by the substitution of $a \to -a$ and the fact that $\pi(-1) = 1$. Part (2) uses the behavior of the characters under conjugation and the substitution $a \to 1/a$. Parts (3) and (4) simply use the substitutions $a \to a/j$ and $a \to ia$, respectively. Part (5) is merely the fact that, in this case, our definitions reduce to Gauss sums. It is a trivial calculation—see [9] or [16]. □

It is now easy to check how the $f$-basis behaves under the operators $M$, $A$, and $S$ from equations (6), (4), and (7). In the following, recall that $\lambda$ is the fixed generator of $\mathbb{F}_q^*$, $\pi$ is the fixed multiplicative character of $\mathbb{F}_q^*$, $\chi$ is the fixed additive character of $\mathbb{F}_q$, and the $\Gamma$'s are as defined above.

**Theorem 3.1.** *For* $u \in \mathbb{F}_q$,
(1) $M_x f_u = \chi(xu) f_u$ *and* $M_x f_\infty = f_\infty$,
(2) $A f_u = \pi(\lambda) f_{\lambda^2 u}$ *and* $A f_\infty = \pi(\lambda^{-1}) f_\infty$,
(3) $S f_u = \sum_{i\in\mathbb{F}_q}(\Gamma_{i,u} f_i) + f_\infty$ *and* $S f_\infty = \frac{1}{q}\sum_{i\in\mathbb{F}_q^*} f_i$.

*Proof.* Using Theorem 2.2 and the definitions, these are all simple calculations. For instance, to get part (3), we use the trivial observation that $e_u = \frac{1}{q}\sum_{a\in\mathbb{F}_q}\chi(-au) f_a$, i.e., the "inverse fourier transform," to get:

$$\begin{aligned} S f_u &= \sum_{a\in\mathbb{F}_q^*} \chi(au)\pi(a) e_{-1/a} + e_\infty \\ &= \frac{1}{q}\sum_{b\in\mathbb{F}_q}\sum_{a\in\mathbb{F}_q^*} \chi(au)\pi(a)\chi(b/a) f_b + e_\infty \\ &= \frac{1}{q}\sum_{b\in\mathbb{F}_q} \Gamma_{b,u} f_b + e_\infty. \end{aligned}$$

The remaining computations are similar. □

## 4. $PGL(2, q)$

Since we have noted earlier that the action of $L_2(q)$ on the $e$-basis is basically a linear fractional transformation action on $\mathbb{P}^1(\mathbb{F}_q)$, it will be useful to bring the group $PGL(2, q)$ into the picture. The definition of $PGL(2, q)$ is the set of all $2 \times 2$ nonsingular matrices with entries in $\mathbb{F}_q$ modulo its center. The order of this group is

$$|PGL(2, q)| = q(q^2 - 1).$$

One observes that the $PGL(2, q)$ has twice the order of $L_2(q)$. This is because $L_2(q)$ sits inside of $PGL(2, q)$ as a normal subgroup of index two. This can be seen using the determinant. Now the determinant function on $PGL(2, q)$ is only well defined up to multiples by squares in $\mathbb{F}_q^*$, but this is enough to pick out $L_2(q)$ inside of $PGL(2, q)$. $L_2(q)$ can be viewed as precisely those elements of $PGL(2, q)$ whose determinant is of the form $u^2$, $u \in \mathbb{F}_q^*$.

The usefulness of $PGL(2, q)$ will arise from the fact that it acts on $\mathbb{P}^1(\mathbb{F}_q)$ by linear fractional transformations in a very nice way. Specifically, the Fundamental Theorem of Projective Geometry says that any three distinct points of the projective line may always be sent to any other three distinct points by a unique element of $PGL(2, q)$, i.e., it is sharply 3-transitive. For future use, we give the determinant of the following specific maps (determined up to a square in $\mathbb{F}_q^*$):

**Theorem 4.1.** *Let $s$, $t$, $v$ be distinct elements in $\mathbb{F}_q$. The determinant of the unique element in $PGL(2, q)$ that maps the triple $(1, 0, \infty)$ to the triple $(s, t, v)$, $(\infty, t, v)$, $(s, \infty, t)$, and $(s, t, \infty)$ is, respectively, the following: $(s-t)(s-v)(t-v)$, $(t-v)$, $-(s-v)$, and $(s-t)$.*

*Proof.* One has only to examine the following matrices, bearing in mind that the determinant is only defined up to a square:

$$\begin{pmatrix} -v(s-t) & -(s-t) \\ t(s-v) & s-v \end{pmatrix}, \quad \begin{pmatrix} -v & -1 \\ t & 1 \end{pmatrix},$$
$$\begin{pmatrix} -v & -1 \\ -(s-v) & 0 \end{pmatrix}, \quad \begin{pmatrix} s-t & 0 \\ t & 1 \end{pmatrix}. \quad \square$$

## 5. The invariant two-form

In this section we wish to examine the nature and existence of $L_2(q)$ invariant two-forms on the induced representation $V$. Of course, if $V$ is irreducible, then there is at most one (depending on whether it is self-dual or not). One way to see abstractly there is only one invariant symmetric two-form is by using the Fundamental Theorem of Projective Geometry and the "linear fractional" action of the $e$-basis. While this is easy, we will need an explicit description. The $f$-basis provides a very nice formulation of our invariant two-form.

**Theorem 5.1.** *For $\pi^2$ nontrivial, up to a constant multiple, there exists a unique $L_2(q)$ invariant symmetric nondegenerate two-form $( \, , \, )$ on $V$ characterized uniquely by*

(1) $(f_u, f_{-u}) = 1/\pi(u)$ *for* $u \in \mathbb{F}_q^*$.
(2) $(f_0, f_\infty) = (f_\infty, f_0) = \Gamma_{1,0}$.
(3) *All other pairings between the $f$-basis are zero.*

*Proof.* We first comment on the requirement that $\pi^2 \neq 1$. This will actually be useful in the proof. However, the real reason for it lies in Theorem 2.1 which makes it the requirement for $V$ to be irreducible. If $\pi^2$ were trivial, one could easily check that there would be *two* different invariant two-forms on $V$, namely, the one above and a second one defined only on the diagonal parts $(v, v)$.

As already noted, there are many ways to check the existence of a nonzero $L_2(q)$ invariant two-form. Since this is easy, we merely record that in any character table for $L_2(q)$ (say in [14]) one may check that the characters for $V$ are all real valued so that $V$ is self-dual. Let us write $( \, , \, )$ for a nonzero choice of an invariant two-form.

First, by $A$ invariance (see Corollary 2.1) and the fact that $\pi^2 \neq 1$, it is easy to see that $(e_x, e_x) = 0$ for $x \in \mathbb{P}^1(\mathbb{F}_q)$. Next, since the Fundamental Theorem of Projective Geometry says that $PGL(2, q)$ is strictly three transitive on $\mathbb{P}^1(\mathbb{F}_q)$, it is easy to see that $L_2(q)$ is two transitive. In particular, if $(e_0, e_1)$ were zero, then by invariance we would have $(e_x, e_y)$ zero for all $x, y$ distinct in $\mathbb{P}^1(\mathbb{F}_q)$. But by definition, $(\ ,\ )$ is nonzero which forces $(e_0, e_1) \neq 0$ so that we will be able to renormalize it below.

Now let $(\ ,\ )$ be the unique nonzero invariant two-form on $V$ that we have from the preceding paragraph. For $x, y \in \mathbb{F}_q$, let us calculate $(f_x, f_y)$ using $M_a$ invariance (see Corollary 2.1):

$$\begin{aligned}(f_x, f_y) &= \frac{1}{q^2} \sum_{a, b \in \mathbb{F}_q} \chi(ax + by)(e_a, e_b) \\ &= \frac{1}{q^2} \sum_{a, b \in \mathbb{F}_q} \chi(ax + by)(e_0, e_{b-a}).\end{aligned}$$

Setting $c = b - a$ yields

$$\begin{aligned}(f_x, f_y) &= \sum_{a, c \in \mathbb{F}_q} \chi(ax + zy + cy)(e_0, e_c) \\ &= \sum_{c \in \mathbb{F}_q} \chi(cy)(e_0, e_c) \sum_{a \in \mathbb{F}_q} \chi(a(x + y)) \\ &= q\delta_{x+y=0} \sum_{c \in \mathbb{F}_q} \chi(cy)(e_0, e_c) \\ &= q\delta_{x+y=0} \sum_{c \in \mathbb{F}_q^*} \chi(cy)(e_0, e_c)\end{aligned} \tag{8}$$

where $\delta_{\text{condition}}$ is 1 or 0 depending on whether the condition is satisfied or not.

We will make use of the invariance again. For $b \in \mathbb{F}_q^*$, we know that $N_b e_0 = e_0$ and if $b \neq 1$, then $N_b e_1 = \pi(1 - b) e_{\frac{1}{-b+1}}$ by Corollary 2.1. Observe that the values $\frac{1}{-b+1}$ in $\mathbb{P}^1(\mathbb{F}_q)$ as $b$ varies over $\{b \in \mathbb{F}_q \mid b \neq 1\}$ is precisely $\mathbb{F}_q^*$. Therefore $\{c \in \mathbb{F}_q^*\} = \{\frac{1}{-b+1} \mid b \neq 1\}$. Thus we may continue equation (8) to write:

$$\begin{aligned}(f_x, f_y) &= q\delta_{-x=y} \sum_{b \in \mathbb{F}_q, b \neq 1} \chi\left(\frac{y}{-b+1}\right)\left(e_0, e_{\frac{1}{-b+1}}\right) \\ &= q\delta_{-x=y} \sum_{b \neq 1} \chi\left(\frac{y}{-b+1}\right)(N_b e_0, N_b e_1)\pi(-b+1)^{-1} \\ &= q\delta_{-x=y}(e_0, e_1) \sum_{b \neq 1} \chi\left(\frac{y}{1-b}\right)\pi\left(\frac{1}{1-b}\right).\end{aligned}$$

By setting $a = \frac{1}{1-b}$ and using Definition 3.2 and Lemma 3.1 (3), we get

$$\begin{aligned}(f_x, f_y) &= q\delta_{-x=y}(e_0, e_1) \sum_{a \in \mathbb{F}_q^*} \chi(ya)\pi(a) \\ &= q^2\delta_{-x=y}(e_0, e_1)\Gamma_{0,y} \\ &= q^2\delta_{-x=y}\pi(1/y)\Gamma_{0,1}(e_0, e_1).\end{aligned}$$

Renormalizing $(e_0, e_1) = \frac{1}{q}\Gamma_{0,1}$ gives us the desired formula for $(f_x, f_y)$ when $x, y \in \mathbb{F}_q$.

Let us compute the formulas for the remaining cases, namely when $x$ or $y$ are $\infty$. Again by $A$ invariance, we know that $(f_\infty, f_\infty) = 0$. Thus it only remains to evaluate $(f_\infty, f_x)$ (the calculation for $(f_x, f_\infty)$ is similar). We shall use techniques similar to the ones above; however, let us now use

$$g_d = \begin{pmatrix} 0 & -1 \\ 1 & -d \end{pmatrix}$$

instead of $N_b$. This element (Theorem 2.2) satisfies $g_d e_0 = e_\infty$ and $g_d e_1 = e_{d+1}$. This will allow us to write:

$$\begin{aligned}(f_\infty, f_x) &= \sum_{a\in\mathbb{F}_q} \chi(ax)(e_\infty, e_a)\\ &= \sum_{d\in\mathbb{F}_q} \chi((d+1)x)(g_d e_0, g_d e_1)\\ &= (e_0, e_1)\chi(x) \sum_{d\in\mathbb{F}_q} \chi(dx)\\ &= q\delta_{x=0}(e_0, e_1).\end{aligned}$$

By renormalizing $(e_0, e_1)$ as before, we have finished the proof. Merely recall that $\Gamma_{1,0}\Gamma_{0,1} = 1/q$ by Lemma 3.1 (5). □

As a result of this theorem, we get a formula for the $e$-basis as well.

**Theorem 5.2.** *Suppose $\pi^2 \neq 1$. Extend $\pi$ to $\mathbb{F}_q$ by setting $\pi(0) = 0$. Then for $u, v \in \mathbb{F}_q$, the invariant symmetric two-form on $V$ satisfies*
(1) $(e_u, e_v) = \Gamma_{1,0}/q\pi(u-v)$,
(2) $(e_u, e_\infty) = \Gamma_{1,0}/q$,
(3) $(e_\infty, e_\infty) = 0$.

*Proof*. By $A$ invariance, we have already noted that $(e_u, e_u) = 0$ for all $u \in \mathbb{P}^1(\mathbb{F}_q)$. For $x, y \in \mathbb{F}_q$, take the "inverse fourier transform" of the $e$-basis to get the $f$-basis and use the above theorem:

$$\begin{aligned}(e_x, e_y) &= \frac{1}{q^2} \sum_{a,b\in\mathbb{F}_q} \chi(-ax-by)(f_a, f_b)\\ &= \frac{1}{q^2} \sum_{a,b\in\mathbb{F}_q^*} \chi(-ax-by)\delta_{a+b=0}\pi(a)^{-1}\\ &= \frac{1}{q^2} \sum_{a\in\mathbb{F}_q^*} \chi(a(y-x))\pi(1/a)\\ &= \frac{1}{q}\Gamma_{y-x,0} = \pi(x-y)\Gamma_{1,0}/q.\end{aligned}$$

The calculations for $e_\infty$ are similar. □

## 6. The invariant three-forms

We would like to examine the nature of $L_2(q)$ invariant alternating three-forms on $V$ and get an explicit description of them. In other words, using our $L_2(q)$ invariant symmetric non-degenerate two-form to identify $V$ with $V^*$, we are interested in $\bigwedge^3 V$, the third exterior power of $V$, and its orbit structure under $L_2(q)$. As a first step, let us look at the action on $\otimes^3 V$, the third tensor power of $V$, which may be regarded as the space of *all* 3-forms.

If we work in the $e$-basis, it will be sufficient to look at the action of $L_2(q)$ on elements of the form $e_x \otimes e_y \otimes e_z$ where $x, y, z \in \mathbb{P}^1(\mathbb{F}_q)$ since this basis is preserved by $L_2(q)$ up to nonzero scalars (Theorem 2.2). As a second refinement, it is enough to look at the action of $L_2(q)$ on the $L_2(q)$ invariant subspace of $\otimes^3 V$ spanned by $\{e_x \otimes e_y \otimes e_z \mid x, y, z \text{ are distinct in } \mathbb{P}^1(\mathbb{F}_q)\}$. Let us call this subspace $D \otimes^3 V$. The reason we may restrict our attention to $D \otimes^3 V$ is because we will eventually want to apply anti-symmetrization to get non-zero alternating three-forms.

As a next step, let us "projectivize" the action. That is, for the moment let us ignore the (nonzero) constants of Theorem 2.2 and concentrate on the "linear fractional" aspect of the action. Thus we look at the action of $L_2(q)$ on $D \otimes^3 \mathbb{P}^1(\mathbb{F}_q) = \{x \otimes y \otimes z \mid x, y, z \text{ are distinct in } \mathbb{P}^1(\mathbb{F}_q)\}$. The previous discussion in Section 4 on $PGL(2, q)$ amounts to the fact that whereas $PGL(2, q)$ breaks this set into a single orbit, the $L_2(q)$ action yields two orbits depending on whether a certain determinant is a square or not in $\mathbb{F}_q^*$ (Theorem 4.1). Thus there cannot possibly be more than two independent $L_2(q)$ invariant forms that have a hope of being alternating. Using Theorem 4.1, we see that the determinant of the element in $PGL(2, q)$ mapping $1 \otimes 0 \otimes -1$ to $\lambda \otimes 0 \otimes -\lambda$ (via $1 \otimes 0 \otimes \infty$) is $\lambda^3$, a non-square. Thus, these are representatives of the two $L_2(q)$ orbits in $D \otimes^3 \mathbb{P}^1(\mathbb{F}_q)$.

Let us put the constants back in and look again at $D \otimes^3 V$. First we show that the stabilizer of $1 \otimes 0 \otimes -1$ and the stabilizer of $\lambda \otimes 0 \otimes -\lambda$ also fixes $e_1 \otimes e_0 \otimes e_{-1}$ and $e_\lambda \otimes e_0 \otimes e_{-\lambda}$, respectively. However, both stabilizers are trivial by the Fundamental Theorem of Projective Geometry. Thus the orbit of each defines an invariant 3-form. Hence we are allowed to make the following definition.

**Definition 6.1.** Let $( \ , \ , \ )_+$ and $( \ , \ , \ )_-$ be the two independent $L_2(q)$ invariant three-forms on $V$ defined by

(1) $(e_1, e_0, e_{-1})_+ = 1$ and $(e_\lambda, e_0, e_{-\lambda})_+ = 0$,
(2) $(e_1, e_0, e_{-1})_- = 0$ and $(e_\lambda, e_0, e_{-\lambda})_- = 1$.

As a result of the above discussion, we see that any non-zero invariant 3-form on $V$, for $V$ irreducible, that has a possibility of being an *alternating* form must be a linear combination of $( \ , \ , \ )_+$ and $( \ , \ , \ )_-$. Since each of these forms will be so important to us, we will give explicit descriptions of their structure. First we make the following notational definition:

**Definition 6.2.** Let $u \in \mathbb{F}_q$. Define the symbol $\sqrt{u}^{\delta} \in \mathbb{F}_q$ to be

$$\sqrt{u}^{\delta} = \begin{cases} v & \text{if } u = v^2 \text{ for some } v \in \mathbb{F}_q, \\ 0 & \text{if } u \text{ is not a square in } \mathbb{F}_q. \end{cases}$$

Note that if $u \neq 0$, then $\sqrt{u}^{\delta}$ is only well defined up to $\pm 1$. However, this will be sufficient for our purposes.

**Theorem 6.1.** *Let $x, y, z \in \mathbb{F}_q$. Extending $\pi$ to all of $\mathbb{F}_q$ by $\pi(0) = 0$, one has*:

(1)

$$(e_x, e_y, e_z)_+ = \pi\left(\sqrt{2(x-y)(x-z)(y-z)}^{\delta}\right),$$

*and*

$$(e_x, e_y, e_z)_- = \pi\left(\sqrt{2\lambda^3(x-y)(x-z)(y-z)}^{\delta}\right).$$

(2) $(e_\infty, e_y, e_z)_+ = \pi(\sqrt{2(y-z)}^{\delta})$ *and* $(e_\infty, e_y, e_z)_- = \pi(\sqrt{2\lambda(y-z)}^{\delta})$,

(3) $(e_x, e_\infty, e_z)_+ = \pi(\sqrt{-2(x-z)}^{\delta})$ *and* $(e_x, e_\infty, e_z)_- = \pi(\sqrt{-2\lambda(x-z)}^{\delta})$,

(4) $(e_x, e_y, e_\infty)_+ = \pi(\sqrt{2(x-y)}^{\delta})$ *and* $(e_x, e_y, e_\infty)_- = \pi(\sqrt{2\lambda(x-y)}^{\delta})$.

*In particular, if* $u, v, w$ *are not distinct, then* $(e_u, e_v, e_w)_+ = (e_u, e_v, e_w)_- = 0$.

*Proof*. First observe that since $\pi(\pm 1) = 1$, everything above is well defined with respect to the symbol $\sqrt{\cdot}^{\delta}$. Now all we need to use is Theorem 4.1. Consider (1a) first. The element $g' \in PGL(2, q)$ that takes the triple $(x, y, z)$ to $(1, 0, -1)$ is

$$g' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z(x-y) & x-y \\ -y(x-z) & -(x-z) \end{pmatrix}$$

whose determinant is $d = 2(x-y)(x-z)(y-z)$. If $d$ is not a square in $\mathbb{F}_q^*$, then $g'$ is not in $L_2(q)$ and thus the triple $(x, y, z)$ is not in the $L_2(q)$ orbit of $(1, 0, -1)$ but necessarily in the $(\lambda, 0, -\lambda)$ orbit. Hence by definition, if $d$ is not a square, $(e_x, e_y, e_z)_+ = 0$. On the other hand, if $d$ is a square, then we may consider $g = \frac{1}{\sqrt{d}^{\delta}} g'$ as element of $L_2(q)$. Using Theorem 2.2 and invariance of $g$, we write:

$$\begin{aligned}(e_x, e_y, e_z) &= (ge_x, ge_y, ge_z)\\ &= \pi\left(\frac{(x-y)(-x+z)}{\sqrt{d}^{\delta}}\right)\pi(1-2)\pi\left(\frac{(x-y)(-y+z)}{\sqrt{d}^{\delta}}\right)\\ &\quad\cdot \pi(0-2)\pi\left(\frac{\sqrt{d}^{\delta}}{x-y}\right)\pi(-1)(e_1, e_0, e_\infty)_+\\ &= \pi\left(\frac{2(x-y)(x-z)(y-z)}{\sqrt{d}^{\delta}}\right)(e_1, e_0, e_\infty)_+\\ &= \pi\left(\sqrt{2(x-y)(x-z)(y-z)}^{\delta}\right)(e_1, e_0, e_\infty)_+.\end{aligned}$$

Finally, we use that $(e_1, e_0, e_\infty)_+ = 1$ to finish the proof.

For (1b), make use of

$$g'' = \begin{pmatrix} 2\lambda & \lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} z(x-y) & x-y \\ -\lambda y(x-z) & -\lambda(x-z) \end{pmatrix}$$

which has determinant $d'' = 2\lambda^3(x-y)(x-z)(y-z)$. Since this and the remaining calculations are similar, we omit them. □

We note that had we used $(1, 0, \infty)$ and $(\lambda, 0, \infty)$ as our starting points in Definition 6.1 instead of $(1, 0, -1)$ and $(\lambda, 0, -\lambda)$, the "2's" and "3" would have not appeared in the formulas in Theorem 6.1 above. However, we have chosen $(1, 0, -1)$ and $(\lambda, 0, -\lambda)$ since it will make the formulas a bit more symmetrical for the $f$-basis (below) which will be much more important to us. A more fundamental problem with the above formulas for the $e$-basis is the presence of the $\sqrt{x}^{\delta}$. It is very difficult to proceed when one is constantly being concerned with whether or not an object is a square in the field or not. The formulas we present next for the $f$-basis, while not as pretty as those for the $e$-basis, nevertheless avoid talking about things such as $\sqrt{x}^{\delta}$.

**Theorem 6.2.** *Let* $x, y \in \mathbb{F}_q$. *Then*

(1)

$$(f_x, f_{-x-y}, f_y)_+ = \frac{q}{2} \sum_{a, b \in \mathbb{F}_q, a \neq 0, \pm b} \chi\left(\frac{x}{a(a-b)} - \frac{y}{a(a+b)}\right) \pi(a(a-b)(a+b))^{-1},$$

(2)

$$(f_x, f_{-x-y}, f_y)_- = \frac{q}{2} \sum_{a, b \in \mathbb{F}_q, a \neq 0, \pm \lambda b} \chi\left(\frac{\lambda x}{a(a-\lambda b)} - \frac{\lambda y}{a(a+\lambda b)}\right) \pi(a(a-\lambda b)(a+\lambda b))^{-1},$$

(3) $(f_\infty, f_{-x}, f_x)_+ = \frac{q}{2} \sum_{a \in \mathbb{F}_q^*} \chi(\frac{-x}{2a^2}) \pi(2a)^{-1}$,

(4) $(f_\infty, f_{-x}, f_x)_- = \frac{q}{2} \sum_{a \in \mathbb{F}_q^*} \chi(\frac{-x}{2\lambda a^2}) \pi(2\lambda^2 a)^{-1}$,

(5) $(f_\infty, f_{-x}, f_x)_\pm = (f_x, f_\infty, f_{-x})_\pm = (f_{-x}, f_x, f_\infty)_\pm$.

(6) *All other pairings in the* $f$*-basis are zero.*

*Proof.* We shall make use of the matrices $M_u$ (equation (6)) and the matrices

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Using $M_u$ invariance in the second line, we calculate:

$$\begin{aligned}(f_x, f_z, f_y)_+ &= \sum_{a, b, c \in \mathbb{F}_q} \chi(ax + cz + by)(e_a, e_c, e_b)_+ \\ &= \sum_{a, b, c \in \mathbb{F}_q} \chi(ax + cz + by)(e_{a-c}, e_0, e_{b-c})_+ \\ &= \sum_{a', b', c \in \mathbb{F}_q} \chi(a'x + b'y + c(x + y + z))(e_{a'}, e_0, e_{b'})_+ \\ &= q\delta_{x+y+z=0} \sum_{a, b \in \mathbb{F}_q} \chi(ax + by)(e_a, e_0, e_b)_+. \end{aligned} \tag{9}$$

It is clear that $(1, 0, -1)$ and $(a, 0, b)$ are in the same $L_2(q)$ orbit if and only if $(a, 0, b)$ is of the form $g(1, 0, -1)$ for some $g \in L_2(q)$. Since $(e_a, e_0, e_b)_+$ vanishes unless $(a, 0, b)$ is in the $(1, 0, -1)$ orbit, we may omit the terms not of the form $g(e_1, 0, e_{-1})\mathbb{C}^*$ in equation (9). Note that by Theorem 2.2, the only $g \in L_2(q)$ that preserve the element $0 \in \mathbb{P}^1(\mathbb{F}_q)$ are of the form $g_{a,b}$ for $a, b \in \mathbb{F}_q$, $a \neq 0$. Explicitly, the action is given by

$$g_{a,b}(e_1, e_0, e_{-1}) = \pi(a-b)\pi(a)\pi(a+b)\left(e_{\frac{a^{-1}}{a-b}}, e_0, e_{\frac{-a^{-1}}{a+b}}\right).$$

For this to be of the form $(e_c, 0, e_d)\mathbb{C}^*$ for $c, d \in \mathbb{F}_q$, we only need that $a(a^2-b^2) \neq 0$. Using this formula and the FT of Projective Geometry, it is clear that $\{g_{a,b}(1, 0, -1) \mid a, b \in \mathbb{F}_q, a(a^2-b^2) \neq 0\}$ is equal to $\{(a, 0, b) \mid a, b \in \mathbb{F}_q, (a, 0, b) \in L_2(q)(1, 0, -1)\}$. It is also clear that the map $(a, b) \to g_{a,b}(1, 0, -1)$ is 2-to-1 since only $g_{a,b} = g_{-a,-b}$ in $L_2(q)$. Putting this together in equation (9), dropping terms that are zero, and using invariance, we continue:

$$(f_x, f_z, f_y)_+ = \frac{q}{2}\delta_{x+y+z=0} \sum_{a, b\in\mathbb{F}_q, a(a^2-b^2)\neq 0} \chi\left(\frac{xa^{-1}}{a-b} - \frac{ya^{-1}}{a+b}\right) \cdot g_{a,b}(e_1, e_0, e_{-1})_+\pi(a(a^2-b^2))^{-1}.$$

Definition 6.1 finishes (1).

For (2), there are similar arguments; however, terms will be nonzero if and only if they are in the $(\lambda, 0, -\lambda)$ orbit. Since these and the remaining calculations contain nothing new, they are omitted. □

To make a simple observation that will be useful later, let us introduce the following notation.

**Definition 6.3.** Let $x \in \mathbb{P}^1(\mathbb{F}_q)$. Define the symbol $|x|$ by

$$|x| = \begin{cases} x & \text{if } x \neq \infty, \\ 0 & \text{if } x = \infty. \end{cases}$$

Then we may note that Theorem 6.2 tells us that

$$(f_x, f_y, f_z)_\pm = 0 \quad \text{if } |x| + |y| + |z| \neq 0. \tag{10}$$

The following theorem gives some useful elementary properties of our invariant 3-forms.

**Theorem 6.3.** *Let* $x, y, c \in \mathbb{F}_q$, $c \neq 0$. *Then*

(1) $(f_x, f_{-x-y}, f_y)_- = (f_{\lambda x}, f_{\lambda(-x-y)}, f_{\lambda y})_+$,
(2) $(f_\infty, f_{-x}, f_x)_- = (f_\infty, f_{-\lambda^3 x}, f_{\lambda^3 x})_+$,
(3) $(f_x, f_{-x-y}, f_y)_\pm = (f_{-y}, f_{x+y}, f_{-x})_\pm$,
(4) $(f_{c^2x}, f_{c^2(-x-y)}, f_{c^2y})_\pm = (f_x, f_{-x-y}, f_y)_\pm\pi(c)^{-3}$,
(5) $(f_\infty, f_{-c^2x}, f_{c^2x})_\pm = (f_\infty, f_{-x}, f_x)_\pm\pi(c)$.

*Proof.* To get (1), use the substitution $b \to b'/\lambda$ in Theorem 6.2 (2) above. (2) follows from (4) of the above theorem by $b \to b'/\lambda^2$. (3) follows from (1) of the above theorem and (2) by $b \to -b'$. (4) and (5) follow by $A$ invariance and Theorem 3.1. □

Next we wish to see if these forms can be fitted together to make alternating forms. To do this, first consider $a, b, c \in \mathbb{P}^1(\mathbb{F}_q)$, distinct, and $\sigma$ a permutation of $a, b, c$. Observe that, by Theorem 4.1, the element in $PGL(2, q)$ mapping $(a, b, c)$ (via $(1, 0, \infty)$) to $(\sigma a, \sigma b, \sigma c)$ has a $-1$ entering into the determinant for each transposition. This implies that the map $g_{\sigma,a,b,c} \in PGL(2, q)$ taking the triple $(a, b, c)$ to $\sigma(a, b, c) = (\sigma a, \sigma b, \sigma c)$ has determinant $\operatorname{sgn}(\sigma)$.

Let us now work in the $e$-basis for $V$. By definition and by the discussion so far, we know that if $(e_a, e_b, e_c)_\pm$ is nonzero, then $\sigma(e_a, e_b, e_c)_\pm =$

$(e_{\sigma a}, e_{\sigma b}, e_{\sigma c})_\pm$ is nonzero if and only if $(a, b, c)$ and $\sigma(a, b, c)$ are in the same $L_2(q)$ orbit, that is, if and only if $\operatorname{sgn}(\sigma)$ is a square in $\mathbb{F}_q^*$.

Say that $\operatorname{sgn}(\sigma)$ is a square. Then we claim that, actually, $\sigma(e_a, e_b, e_c)_\pm = \pi(\sqrt{\operatorname{sgn}(\sigma)})(e_a, e_b, e_c)_\pm$. First observe that it is sufficient to check this statement for any particular $(a_0, b_0, c_0)$ in the $L_2(q)$ orbit. It will be easiest if we choose $(a_0, b_0, c_0) = (1, 0, \infty)$ and use Theorem 6.3 to extend the results from $( \, , \, , \, )_+$ to $( \, , \, , \, )_-$ (or vice versa depending what orbit $(1, 0, \infty)$ is in). However, this case is easily checked using Theorem 4.1 and Theorem 2.2. This allows us to prove (recalling $h$ from equation (2) and that $\lambda$ is the fixed generator for $\mathbb{F}_q^*$):

**Theorem 6.4.** *Using the symbol* $\mathbb{F}_q^{*2}$ *to denote the set of squares in* $\mathbb{F}_q^*$ *and recalling the notation from Section 2 so that* $\pi(\lambda) = e^{2\pi i m/h}$, *then the alternating invariant* 3*-forms* $( \, , \, , \, )$ *on* $V$ *are described explicitly as follows*:

(1) *If* $-1 \in \mathbb{F}_q^{*2}$ (*i.e.*, $h$ *is even*) *and* $m$ *is odd, then there are precisely two linearly independent invariant alternating* 3*-forms. They are of the form*

$$c_+( \, , \, , \, )_+ + c_-( \, , \, , \, )_-$$

*for any* $c_+, c_- \in \mathbb{C}$. *If* $m$ *is even, there are no invariant alternating* 3*-forms.*

(2) *If* $-1 \notin \mathbb{F}_q^{*2}$ (*i.e.*, $h$ *is odd*), *then, there is only one invariant alternating* 3*-form up to scalar multiplication. It is of the form*

$$c_+( \, , \, , \, )_+ + c_-( \, , \, , \, )_-$$

*for any* $c_+ \in \mathbb{C}$ *with* $c_- = -c_+\pi(\sqrt{-1/\lambda})^{-3}$.

*Proof.* Let us recall the result of our above discussion. With the same notation as above so that a permutation $\sigma$ acts by $\sigma(e_a, e_b, e_c) = (e_{\sigma a}, e_{\sigma b}, e_{\sigma c})$, we derived:

$$\sigma(e_a, e_b, e_c)_\pm = \pi(\sqrt{\operatorname{sgn}(\sigma)}^{\delta})(e_a, e_b, e_c)_\pm. \tag{11}$$

In case (1) of the theorem, we see that equation (11) implies that both the "+" form and the "−" form are already alternating if $m$ is odd since then $\pi(\sqrt{-1}) = -1$. If $m$ is even, no linear combination is ever alternating since $\pi(\sqrt{-1}) = 1$. By our discussion at the beginning of this section, we are done.

Consider case (2) above. By equation (11), we see that if $\operatorname{sgn}(\sigma) = 1$, we already have $\sigma(e_a, e_b, e_c)_\pm = \operatorname{sgn}(\sigma)(e_a, e_b, e_c)_\pm$. Thus to see how to combine things in order to get an alternating form, it is sufficient to consider the case where $\operatorname{sgn}(\sigma) = -1$. Therefore it is enough to assume that $\sigma$ is a transposition. Since we will see that all the calculations are similar, let us just do the calculations for the fixed permutation $\sigma$ where $\sigma(a, b, c) = (b, a, c)$.

For $a, b, c \in \mathbb{F}_q$ and $c_+, c_- \in \mathbb{C}$, consider the form $( \, , \, , \, ) = c_+( \, , \, , \, )_+ + c_-( \, , \, , \, )_-$. It will be convenient to work in the $f$-basis to find out what restrictions on $c_+$ and $c_-$ are needed to make the form alternating. Also let $x \in \mathbb{F}_q^*$ so that $x^2\lambda = -1$, that is, $x^2 = -1/\lambda$ (since $-1 \notin \mathbb{F}_q^{*2}$). Making much use of Theorem 6.3 (1), (3), and (4) and equation (11), we calculate:

$$\begin{aligned}
\sigma(f_a, f_b, f_c) &= (f_{\sigma a}, f_{\sigma b}, f_{\sigma c}) \\
&= (f_b, f_a, f_c) \\
&= c_+(f_b, f_a, f_c)_+ + c_-(f_b, f_a, f_c)_- \\
&= c_+(f_{-c}, f_{-a}, f_{-b})_+ + c_-(f_{-c}, f_{-a}, f_{-b})_- \\
&= c_+(f_{x^2\lambda c}, f_{x^2\lambda a}, f_{x^2\lambda b})_+ + c_-(f_{x^2\lambda c}, f_{x^2\lambda a}, f_{x^2\lambda b})_- \\
&= c_+(f_{x^2 c}, f_{x^2 a}, f_{x^2 b})_- + c_-(f_{x^2\lambda^2 c}, f_{x^2\lambda^2 a}, f_{x^2\lambda^2 b})_+ \\
&= c_+(f_c, f_a, f_b)_- \pi(x)^{-3} + c_-(f_c, f_a, f_b)_+ \pi(\lambda x)^{-3} \\
&= c_+(f_a, f_b, f_c)_- \pi(x)^{-3} + c_-(f_a, f_b, f_c)_+ \pi(\lambda x)^{-3}.
\end{aligned}$$

For $(\ ,\ ,\ )$ to be alternating, we need $\sigma(f_a, f_b, f_c) = -(f_a, f_b, f_c)$. By the above calculations, this is true if and only if $c_+/\pi(x)^3 = -c_-$ and $c_-/\pi(\lambda x)^3 = -c_+$. In fact, these two conditions are equivalent. To see this, observe that the second equation gives $c_- = -c_+\pi(\lambda x)^3$. But since $x^2 = -1/\lambda$, we have $(\lambda x)^3 = \lambda^3 x^6/x^3 = x^{-3}\lambda^3(x^2)^3 = x^{-3}$. Thus the two requirements are equivalent. If one of the $a, b, c$ are $\infty$, it is easy to check that the same result appears. □

We record for future use an immediate consequence of this theorem in three cases:

**Corollary 6.1.** *Up to scalar multiplication, there is one $L_2(q)$ invariant alternating 3-form on $V$ for $q = 7$ and two independent $L_2(q)$ invariant alternating 3-forms on $V$ for $q = 9$ or $13$.*

## 7. The algebra structure

Now that we have detailed information on $L_2(q)$ invariant alternating 3-forms and nondegenerate symmetric 2-forms, we are in a position to define an invariant skew-symmetric algebra structure, $[\ ,\ ]$, on $V$. To do this, first fix $(\ ,\ )$, the unique 2-form from Theorem 5.1, and $(\ ,\ ,\ )$, any nonzero invariant alternating 3-form.

**Definition 7.1.** Given the fixed 2-form and 3-form above, let $[\ ,\ ] : V \times V \to V$ be the nonzero $L_2(q)$ invariant skew-symmetric algebra structure on $V$ defined by

$$(v_1, v_2, v_3) = ([v_1, v_2], v_3)$$

for all $v_1, v_2, v_3 \in V$. Note that $[\ ,\ ]$ depends on our choice of the 3-form $(\ ,\ ,\ )$.

It is very useful to compute explicitly what this algebra structure looks like. It is particularly nice in the $f$-basis as the next theorem demonstrates.

**Theorem 7.1.** *Let $(\ ,\ ,\ )$ be a nonzero $L_2(q)$ invariant alternating 3-form and write $[\ ,\ ]$ for the corresponding algebra structure on $V$. For $p, q \in \mathbb{F}_q^*$,*
(1) $[f_p, f_q] = (f_q, f_{-q-p}, f_p)\pi(p+q) f_{p+q}$ *if* $p + q \neq 0$,
(2) $[f_0, f_q] = (f_q, f_{-q}, f_0)\pi(q) f_q$,
(3) $[f_\infty, f_q] = (f_q, f_{-q}, f_\infty)\pi(q) f_q$,
(4) $[f_0, f_\infty] = 0$,

(5) $[f_p, f_{-p}] = (f_{-p}, f_\infty, f_p)/\Gamma_{1,0}f_0 + (f_{-p}, f_0, f_p)/\Gamma_{1,0}f_\infty$.

*Proof.* This follows easily by Definition 7.1, Theorem 5.1, and equation (10). For instance, let us check (1). First of all, we may write $[f_p, f_q] = \sum_{r\in\mathbb{P}^1(\mathbb{F}_q)} c_r f_r$ for some constants $c_r \in \mathbb{C}$. Then applying $(\cdot, f_s)$, $s \in \mathbb{F}_q$, to both sides gives

$$(f_p, f_q, f_s) = c_{-s}\frac{1}{\pi(s)}.$$

Thus $c_{-s} = \pi(s)(f_q, f_s, f_p)$ (by equation (11)). In particular, we have $c_{-s} = 0$ unless $s = -p-q$ which gives the stated result. The other cases are similar. □

Of course this algebra structure on $V$ will be a Lie algebra if and only if it satisfies the Jacobi identity

$$[[v_1, v_2], v_3] + [[v_3, v_1], v_2] + [[v_2, v_3], v_1] = 0 \tag{12}$$

for all $v_1, v_2, v_3 \in V$. As a corollary of the above work, we get an expression for the Jacobi identity in terms of the 3-form. Part (3) below will be very useful later in the paper.

**Corollary 7.1.** *Let $p, q, r \in \mathbb{F}_q^*$ and $s = -(p+q+r)$. Then the 3-form $(\ ,\ ,\ )$ makes $V$ into a Lie algebra if and only only if the following three conditions hold*:

(1) *When none of the subscripts are zero, one must have*

$$\frac{(f_p, f_{-p-q}, f_q)(f_r, f_{p+q}, f_s)}{(f_{p+q}, f_{-p-q})} + \frac{(f_r, f_{-r-p}, f_p)(f_q, f_{r+p}, f_s)}{(f_{r+p}, f_{-r-p})} + \frac{(f_q, f_{-q-r}, f_r)(f_p, f_{q+r}, f_s)}{(f_{q+r}, f_{-q-r})} = 0.$$

(2) *When none of the subscripts are zero, one must have*

$$\frac{(f_p, f_{-p}, f_0)(f_q, f_{-q}, f_\infty) + (f_p, f_{-p}, f_\infty)(f_q, f_{-q}, f_0)}{(f_0, f_\infty)} + \frac{(f_q, f_{-p-q}, f_p)^2}{(f_{p+q}, f_{-p-q})} - \frac{(f_q, f_{p-q}, f_{-p})^2}{(f_{p-q}, f_{-p+q})} = 0.$$

(3) *Let $\square \in \{0, \infty\}$. When none of the subscripts are zero, one must have*

$$(f_p, f_{-p-q}, f_q)\left(-\frac{(f_{p+q}, f_{-p-q}, f_\square)}{(f_{p+q}, f_{-p-q})} + \frac{(f_q, f_{-q}, f_\square)}{(f_q, f_{-q})} + \frac{(f_p, f_{-p}, f_\square)}{(f_p, f_{-p})}\right) = 0.$$

*Proof.* The proof of this is just a straightforward application of the various cases of Theorem 7.1 applied to equation (12). We omit the details as they are trivial and not very enlightening. □

## 8. The four-form

We have seen that any $L_2(q)$ invariant alternating 3-form gives rise to an algebra structure. Since we will be concerned with the veracity of the Jacobi identity, let us make the following definition.

**Definition 8.1.** Let $x, y, z, w \in V$. Given a nonzero $L_2(q)$ invariant alternating 3-form and the corresponding algebra structure $[\ ,\ ]$, define a 4-form $(\ ,\ ,\ ,\ )$ on $V$ by

$$(x, y, z, w) = ([[x, y], z] + [[y, z], x] + [[z, x], y], w)$$

where $(\ ,\ )$ is the fixed 2-form. Note that $(\ ,\ ,\ ,\ )$ depends on $[\ ,\ ]$ which in turn depends on the 3-form.

**Theorem 8.1.** *The above 4-form is a $L_2(q)$ invariant alternating form on $V$. Moreover, it is identically zero if and only if $V$ is a Lie algebra under $[\ ,\ ]$.*

*Proof.* That the 4-form is invariant is obvious from the invariance of $(\ ,\ )$ and $[\ ,\ ]$. The fact that it is alternating follows from

$$\begin{aligned}(x, y, z, w) &= ([[x, y], z] + [[y, z], x] + [[z, x], y], w)\\ &= ([x, y], z, w) + ([y, z], x, w) + ([z, x], y, w)\\ &= (z, w, [x, y]) + (x, w, [y, z]) + (y, w, [z, x])\\ &= ([z, w], [x, y]) + ([x, w], [y, z]) + ([y, w], [z, x]).\end{aligned}$$

The statement about being a Lie algebra is clear since $(\ ,\ )$ is nondegenerate and it is precisely the Jacobi identity that appears in the definition of the 4-form. □

We record a simple calculation for future use.

**Theorem 8.2.** *Up to scalar multiplication, there are no $L_2(q)$ invariant alternating 4-forms on $V$ for $q = 7$ and one $L_2(q)$ invariant alternating 4-form on $V$ for $q = 9$ or $13$.*

*Proof.* It is possible to give explicit expression for these 4-forms just as we did for the 3-forms earlier. However, since we will only need results for $q = 7, 9$, and 13 and then only of a quantitative nature, we simply calculate the number of times the trivial representation of $L_2(q)$ occurs in $\bigwedge^4 V$. Here we recall that if $\mu$ is a character of a finite group, then the character of the fourth exterior power of $\mu$ evaluated on some $g \in L_2(q)$ is given by

$$\bigwedge^4 \mu(g) = \frac{\mu(g)^4 - 6\mu(g)^2\mu(g^2) + 8\mu(g^3)\mu(g) + 3\mu(g^2)^2 - 6\mu(g^4)}{24}.$$

Using a character table for $L_2(q)$ [6], the calculations needed to apply the Schur orthogonality relations to the theorem are easy and omitted. □

## 9. The Clifford algebra

It turns out that the 4-form has a nice connection to Clifford algebras which we develop in this section. First recall some notation from exterior algebras. Let $W$ be a finite dimensional vector space over $\mathbb{C}$ equipped with a nondegenerate symmetric two-form $(\ ,\ )$. Then one may extend $(\ ,\ )$ to all of $\bigwedge W$ by requiring $(\bigwedge^k W, \bigwedge^l W) = 0$ if $k \neq l$ and letting

$$(w_1 \wedge \cdots \wedge w_k, w_1' \wedge \cdots \wedge w_k') = \det[(w_i, w_j')]$$

for all $w_i, w_j' \in W$.

There are also two standard maps of $\bigwedge W$ that will be useful. Fix $w \in W$. The first map is $\varepsilon(w) : \bigwedge^i W \to \bigwedge^{i+1} W$ defined by

$$\varepsilon(w)u = w \wedge u$$

for all $u \in \bigwedge W$. Clearly $\varepsilon(w)^2 = 0$. The second map is $\iota(w) : \bigwedge^i W \to \bigwedge^{i-1} W$ by setting $\iota(w) = \varepsilon(w)^t$, the transpose. Thus it is clear that $\iota(w)^2 = 0$ also. It is well known that there is an explicit formula for $\iota(w)$ given by

$$\iota(w)(w_1 \wedge \cdots \wedge w_k) = \sum_{i=1}^{k} (-1)^{i+1}(w, w_i) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k$$

where the $\widehat{w_i}$ means to omit the $w_i$ term.

To prepare our coming connection with Clifford algebras, we define the operator $L_w : \bigwedge W \to \bigwedge W$ by

$$L_w = \varepsilon(w) + \iota(w).$$

It is classical that

$$L_w^2 = (w, w)1. \tag{13}$$

(This follows from the easily checked equation $\iota(w)\varepsilon(w) + \varepsilon(w)\iota(w) = (w, w)1$.)

Let $\bigotimes W$ be the tensor algebra of $W$. Then recall that the Clifford algebra, $C(W)$, is just

$$C(W) = \bigotimes W / (\langle w \otimes w - (w, w) \rangle).$$

Now there is an interesting bijective map of $C(W)$ onto $\bigwedge W$. To see this, first observe that there is a map $\Phi : W \to \operatorname{End}(\bigwedge W)$ by $\Phi : w \to L_w$. This naturally extends to a map of the same name $\Phi : \bigotimes W \to \operatorname{End}(\bigwedge W)$. Equation (13) then tells us that the map $\Phi$ descends to the quotient $\Phi : C(W) \to \operatorname{End}(\bigwedge W)$. At last, define $\Psi : C(W) \to \bigwedge W$ by

$$\Psi(x) = \Phi(x)(1).$$

Explicitly, let $w_i \in W$ and consider elements of the form $w_1, \ldots, w_k \in W \subset C(W)$. One easily checks that

$$\Psi(w_1) = (\varepsilon(w_1) + \iota(w_1))1 = w_1$$

and

$$\Psi(w_1 w_2) = (\varepsilon(w_1) + \iota(w_1))w_2 = w_1 \wedge w_2 + (w_1, w_1)1 \tag{14}$$

and, in general,

$$\Psi(w_1 \cdots w_k) = w_1 \wedge \cdots \wedge w_k + \sum_{i>0} \text{ terms in } \bigwedge^{k-2i} W. \tag{15}$$

With this, we may now state a well-known bijection of $C(W)$ and $\bigwedge W$. This material may be found in many places, e.g., [1, Chapter 1, §6].

**Theorem 9.1.** *With the above definitions,*

$$\Psi : C(W) \to \bigwedge W$$

*is a $\mathbb{C}$-linear one-to-one onto map. Thus, Clifford multiplication induces a second algebra structure on $\bigwedge W$. This new multiplication will be denoted by placing two elements of $\bigwedge W$ next to each other (i.e. with no $\wedge$ in between).*

We observe by equation (14) that, for $w_i \in \bigwedge^1 W$, the new "Clifford" multiplication in $\bigwedge W$ is

$$w_1 w_2 = w_1 \wedge w_2 + (w_1, w_2)1.$$

We also observe by the same source that if $w_1, \dots, w_k$ are mutually orthogonal with respect to the two-form $(\ ,\ )$, then

$$w_1 \cdots w_k = w_1 \wedge \cdots \wedge w_k.$$

It will be useful to have a more general formula for this new multiplication. For our purposes, this will be provided by

**Theorem 9.2.** *Let $x_i$ be a basis for $\bigwedge W$ and let $y_i$ be its dual basis, that is, $(x_i, y_j) = \delta_{i,j}$. Then for any $u, v \in \bigwedge W$, Clifford multiplication is given by*

$$uv = \sum_{i=1}^{2^{\dim(W)}} \imath(x_i)u \wedge \imath(y_i)v.$$

*Proof.* This is a simple matter of checking the result in one particularly nice basis and then using the trick of universality of the tensor product space to show independence of basis. The proof, due to Kostant, may be found in [15]. □

**Corollary 9.1.** *With the above notation,*

$$u^2 = \sum_i \imath(x_i)u \wedge \imath(y_i)u.$$

Let us return to our original concern where $V$ is the induced $L_2(q)$ module, $(\ ,\ )$ is our invariant symmetric nondegenerate 2-form, $(\ ,\ ,\ )$ is a fixed nonzero invariant alternating 3-form, and $(\ ,\ ,\ ,\ )$ is the corresponding invariant alternating 4-form measuring the failure of the Jacobi identity. We may view $(\ ,\ ,\ )$ and $(\ ,\ ,\ ,\ )$ to be elements of $\bigwedge^3 V$ and $\bigwedge^4 V$, respectively, by our 2-form. The remarkable observation of Kostant is that the relation of the 4-form to the 3-form is encapsulated by Clifford multiplication in $\bigwedge V$.

**Theorem 9.3** (Kostant). *Viewing $(\ ,\ ,\ )$ and $(\ ,\ ,\ ,\ )$ as elements in $\bigwedge^3 V$ and $\bigwedge^4 V$, respectively, and using Clifford multiplication,*

$$(\ ,\ ,\ )^2 = 2(\ ,\ ,\ ,\ ) + \text{ a degree zero term.}$$

*Proof.* A priori, the Clifford product of two elements in $\bigwedge^3 V$ would have components in degrees 6, 4, 2, and 0 by equation (15). Let us first check that Clifford squaring of an element $x \in \bigwedge^3 V$ results in only degree 4 and 0 terms. To do this, recall the algebra anti-automorphism of $\bigotimes V$ defined by $(v_1 \otimes \cdots \otimes v_k)^* = v_k \otimes \cdots \otimes v_1$. This anti-automorphism descends compatibly with $\Psi$ to both $C(V)$ and $\bigwedge V$. We observe that "$*$" reduces to $+1$ in degrees 0 and

4 of $\bigwedge V$ while it reduces to $-1$ in degrees 2, 3, and 6. However, this implies that, on the Clifford square of a degree 3 object, "*" acts by $(-1)(-1) = +1$. Hence, "*" must act by $+1$ on each of the components. Thus there are no degree 2 or 6 terms.

We can now make use of Theorem 9.2. First recall that our 2-form $( \, , \, )$ on $V$ extends to all of $\bigwedge V$ as described above. By viewing $( \, , \, , \, )$ and $( \, , \, , \, , \, )$ to be in $\bigwedge V$, we mean that we identify them with elements $\varphi_3$ and $\varphi_4$ in $\bigwedge^3 V$ and $\bigwedge^4 V$, respectively, such that for $v_i \in V$, $(v_1, v_2, v_3) = (\varphi_3, v_1 \wedge v_2 \wedge v_3)$ and $(v_1, v_2, v_3, v_4) = (\varphi_4, v_1 \wedge v_2 \wedge v_3 \wedge v_4)$. To show that the degree 4 component of $\varphi_3^2$ is $2\varphi_4$, it will be sufficient to show that $(\varphi_3^2, v_1 \wedge v_2 \wedge v_3 \wedge v_4) = 2(v_1, v_2, v_3, v_4)$.

Choose $x_i$ to be a basis of homogeneous elements in $\bigwedge V$ and $y_i$ to be the corresponding (homogeneous) dual basis so that $\deg(x_i) = \deg(y_i)$. We know by Corollary 9.1 that

$$\varphi_3^2 = \sum_i \iota(x_i)\varphi_3 \wedge \iota(y_i)\varphi_3.$$

By the fact that $\varphi_3$ is degree three and by the degree lowering nature of $\iota$, $x_i$ and $y_i$ can contribute nontrivially to the fourth degree component of $\varphi_3^2$ only for $x_i$ of degrees 0, 1, or 2 and $y_i$ of degrees 2, 1, or 0, respectively. But since $x_i$ and $y_i$ have the same degree, we only need to consider the above sum for $x_i, y_i \in \bigwedge^2 V$. Hence, we have

$$(\varphi_3^2, v_1 \wedge \cdots \wedge v_4) = \sum_{i, \deg(x_i)=2} (\iota(x_i)\varphi_3 \wedge \iota(y_i)\varphi_3, v_1 \wedge \cdots \wedge v_4). \tag{16}$$

Of course, $\iota(x_i)\varphi_3$ and $\iota(y_i)\varphi_3$ are in $\bigwedge^2 V$. We wish to "rewrite" the above determinant. Let $a_i \in V$. By definition, one has

$$(a_1 \wedge \cdots \wedge a_4, v_1 \wedge \cdots \wedge v_4) = \sum_{\sigma \in S^4} \operatorname{sgn}(\sigma)(a_1, v_{\sigma(1)}) \cdots (a_4, v_{\sigma(4)})$$

and

$$(a_1 \wedge a_2, v_1 \wedge v_2) = \sum_{\sigma \in S^2} \operatorname{sgn}(\sigma)(a_1, v_{\sigma(1)})(a_2, v_{\sigma(2)}).$$

Combining these, one can check that the following sum over $\binom{4}{2} = 6$ permutations holds:

$$(a_1 \wedge \cdots \wedge a_4, v_1 \wedge \cdots \wedge v_4) = \sum_{\{i,j,k,l\}=\{1,2,3,4\}, i<j, k<l} (a_1 \wedge a_2, v_i \wedge v_j)(a_3 \wedge a_4, v_k \wedge v_l).$$

Applying this to equation (16), we may now write

$$(\varphi_3^2, v_1 \wedge \cdots v_4) = \sum_{\substack{\{p,q,r,s\}=\{1,2,3,4\} \\ p<q, r>s}} \sum_{i, \deg(x_i)=2} (\iota(x_i)\varphi_3, v_p \wedge v_q)(\iota(y_i)\varphi_3, v_r \wedge v_s). \tag{17}$$

Using $i(v)^t$ and $(\varphi_3, v_1 \wedge v_2 \wedge v_3) = (v_1, v_2, v_3) = ([v_1, v_2], v_3)$, we have

$$\begin{aligned}(\iota(v)\varphi_3, v_1 \wedge v_2) &= (\varphi_3, v \wedge v_1 \wedge v_2) \\ &= ([v, v_1], v_2).\end{aligned}$$

Putting this in equation (17), we get (using the nature of the dual basis)

$$\begin{aligned}(\varphi_3^2, v_1 \wedge \cdots \wedge v_4) &= \sum_{p,q,r,s} \sum_i ([x_i, v_p], v_q)([y_i, v_r], v_s) \\ &= \sum_{p,q,r,s} \sum_i ([v_p, v_q], x_i)([v_r, v_s], y_i) \\ &= \sum_{p,q,r,s} \sum_i ([v_p, v_q], ([v_r, v_s], y_i)x_i) \\ &= \sum_{p,q,r,s} ([v_p, v_q], [v_r, v_s]) \\ &= \sum_{p,q,r,s} ([[v_r, v_s], v_p], v_q).\end{aligned}$$

Writing out explicitly the 6 terms of $\{\{p, q, r, s\} = \{1, 2, 3, 4\}, p < q, r > s\}$ and rearranging, one sees that we get precisely the desired result. □

**Corollary 9.2.** *There exists a second degree homogeneous polynomial map from $L_2(q)$ invariant alternating 3-forms on $V$ to invariant alternating 4-forms that takes any such 3-form to its corresponding 4-form.*

*Proof.* This is an immediate consequence of the above theorem and the nature of Clifford multiplication. □

## 10. Existence for rank 2

Using the results from our Clifford structure, we can now show that, for $q = 7, 9$, and 13, one may choose an appropriate 3-form on $V$ so that the induced algebra structure yields a Lie algebra. Of course in these cases it is clear that $\dim(V) = 8, 10$, and 14, respectively. We will later see that the Lie algebras obtained in this way are the simple rank two Lie algebras $A_2$, $B_2$, and $G_2$, respectively. Thus in these cases, knowledge of $L_2(q)$ determines the entire Lie algebra which in turn determines the adjoint Lie groups in which $L_2(q)$ lies.

**Theorem 10.1.** *For $q = 7, 9$, and 13, there exist nonzero $L_2(q)$ invariant alternating 3-forms on $V$ making $V$ into a Lie algebra under the corresponding bracket structure.*

*Proof.* Since vanishing of the corresponding 4-form is equivalent to the Jacobi identity, it suffices to show that there always exist nonzero 3-forms whose 4-forms vanish. By Corollary 6.1 and Theorem 8.2, we already know that for $q = 7, 9$, and 13 there is always one more $L_2(q)$ invariant alternating 3-form on $V$ than there are invariant alternating 4-forms. In particular, for $q = 7$ we are done since there are no 4-forms. However, for $q = 9, 13$ there is a 4-form. But by Corollary 9.2, we have a homogeneous polynomial map taking 3-forms to 4-forms. Thus by choosing a basis, we have a map $\mathbb{C}^2 \to \mathbb{C}$ of the form $ax^2 + bxy + cy^2$. But by the quadratic formula this always has a nontrivial zero so we are done. □

Using a character table ([6], [7]), let us work backwards and make some general remarks about the groups in the cases of $q = 7, 9, 13$, and 5:

$L_2(7)$ has exactly two 3 dimensional irreducible representations. This gives two nonisomorphic injections of $L_2(7)$ into $SL(3, \mathbb{C})$. One may arrange things so that the outer automorphism of $L_2(7)$ corresponds to an outer automorphism of $SL(3, \mathbb{C})$ which interchanges the two nonisomorphic representations of $L_2(7)$. Of course, this outer automorphism is a Lie algebra automorphism of $SL(3, \mathbb{C})$ even though it is not an intertwining operator for $L_2(7)$. Next note that $L_2(7)$ has only one irreducible 8 dimensional representation. In fact, one may readily see that it is obtained by composing either of the two 3-dimensional representations with Ad and letting $L_2(7)$ act on $\mathfrak{sl}(3, \mathbb{C})$. The intertwining operator for these two equivalent representations may be obtained by conjugating by the outer automorphism of $SL(3, \mathbb{C})$. This is now simultaneously a Lie algebra automorphism and intertwining operator for $L_2(7)$. This must be the case since there is only one invariant alternating three-form up to scalar multiplication (Corollary 6.1) so that there is only one invariant Lie algebra structure up to automorphisms. This will be reflected in Figures 1 and 2 in later sections which will show that the possible Lie algebras differ by an outer automorphism.

$L_2(9)$ has exactly two 5 dimensional irreducible representations. The Schur indicator of each is $+1$ so that $L_2(9)$ embeds into $SO(5, \mathbb{C})$. We note, however, that $L_2(9)$ does not have any four dimensional representations so that it does not sit in the simply connected covering of $SO(5, \mathbb{C})$, $\mathrm{Spin}_5(\mathbb{C}) \cong Sp_4(\mathbb{C})$ (even though $SL(2, 9)$ does have four dimensional representations, these do not descend to $PSL(2, 9) = L_2(9)$). Regardless, we still have two embeddings of $L_2(9)$ into $SO(5, \mathbb{C})$. Unlike the above case for $q = 7$, these two embeddings are not related by an outer automorphism. Next note that $L_2(9)$ has only one irreducible 10 dimensional representation. It is easily checked that either of the 5 dimensional representations composed with Ad yields the 10 dimensional one. Thus there will be a $L_2(9)$ intertwining operator. However, there is no reason to suppose that this map will be a Lie algebra automorphism as was the case for $q = 7$. This makes perfect sense since we have seen in Corollary 6.1 that, up to scalar multiplication, there are two invariant 3-forms. By the quadratic nature of Corollary 9.2, one would expect there to be two different ways of making $V$ into a Lie algebra under $L_2(9)$. We will actually see that this is the case later on. It will be reflected in Figures 3 and 4.

$L_2(13)$ also has exactly two 7 dimensional irreducible representations. The Schur indicator of each is also $+1$ so that we have an embedding into $SO(7, \mathbb{C})$. In fact, we will later see that $L_2(13)$ actually lies inside of $G_2$ inside of $SO(7, \mathbb{C})$. These two inequivalent 7 dimensional representations also cannot be related by an outer automorphism of $G_2$ as was the case for $q = 7$. Composition of either with Ad will yield a 14 dimensional representation. One may check that both 7 dimensional representations give the same 14 dimensional representation (in fact, in the notation of [6], $\chi_9$ by either Theorem 6.4 (1) or [16]). Just as with $q = 9$ above, one expects that there are two ways of making $V$ into a Lie algebra under $G_2$. Even though there is a $L_2(13)$ intertwining operator, it need not be a Lie algebra automorphism. This will be reflected in Figures 5 and 6 in later sections.

Let us also make a few comments about $q = 5$. For this value of $q$, Theorem 2.1 tells us that the only principal series representation of $L_2(5)$ is reducible. This 6 dimensional representation breaks up into two (nonisomorphic) 3 dimensional representations. In fact, it is easy to see that each of these 3 dimensional spaces becomes (by analogous techniques) the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Thus one can check that we have $L_2(5)$ injecting to the adjoint group $PSL(2, \mathbb{C})$. However, $L_2(5)$ has no two dimensional representations so it does not sit in $SL(2, \mathbb{C})$.

## 11. A family of subalgebras

A fundamental step in understanding the nature of semisimple Lie algebras is examining the various $\mathfrak{sl}(2, \mathbb{C})$'s that naturally embed in the semisimple Lie algebra. This will be important for our study. As usual, fix an alternating, $L_2(q)$ invariant, nonzero 3-form on $V$ so that we have $[\ ,\ ]$ as the corresponding algebra structure. We have already seen in Theorem 7.1 that the subalgebra spanned by the vectors $f_0$ and $f_\infty$ "wants" to be a rank two torus of $V$ with root vectors $f_p$, $p \in \mathbb{F}_q^*$. Because of this, it is natural to consider the following analogues of $\mathfrak{sl}(2, \mathbb{C})$.

**Definition 11.1.** Given an alternating, $L_2(q)$ invariant, nonzero 3-form on $V$, let $\mathfrak{g}_p$ be the subspace of $V$ defined by

$$\mathfrak{g}_p = \operatorname{span}\{f_p, f_{-p}, [f_p, f_{-p}]\}$$

for each $p \in \mathbb{F}_q^*$. Note that $\mathfrak{g}_p = \mathfrak{g}_q$ if and only if $p = \pm q$.

Define the number $d_p \in \mathbb{C}$ by

$$d_p = \frac{(f_p, f_{-p}, f_0)(f_p, f_{-p}, f_\infty)}{(f_p, f_{-p})(f_0, f_\infty)}. \tag{18}$$

Observe that $d_p$ is nonzero if and only if both terms in the numerator are nonzero.

Within $\mathfrak{g}_p$, single out the following elements:

$$x_p = f_p, \quad y_p = f_{-p},$$
$$h_p = [f_p, f_{-p}] = \frac{(f_p, f_{-p}, f_\infty)}{(f_0, f_\infty)} f_0 + \frac{(f_p, f_{-p}, f_0)}{(f_0, f_\infty)} f_\infty.$$

**Theorem 11.1.** *Given the above notation, one has*

$$[x_p, y_p] = h_p, \quad [h_p, x_p] = 2d_p x_p, \quad [h_p, y_p] = -2d_p y_p.$$

*In particular, $\mathfrak{g}_p$ is a Lie algebra.*

*Moreover, if $d_p \neq 0$, then $h_p \neq 0$. Thus by replacing $y_p$ and $h_p$ by $y_p/d_p$ and $h_p/d_p$, respectively, we see that $\mathfrak{g}_p$ is isomorphic to the three dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.*

*Consider what happens when $d_p = 0$ (see equation (18)). In the case where only one term in the numerator of $d_p$ is zero, then $h_p \neq 0$ and $\mathfrak{g}_p$ is still a three dimensional algebra; however, $h_p$ is in the center of $\mathfrak{g}_p$. Thus $\mathfrak{g}_p$ is isomorphic to the three dimensional Heisenberg Lie algebra. In the case where both terms in the numerator of $d_p$ are zero, we see that $h_p = 0$ and so $\mathfrak{g}_p$ is the two-dimensional Abelian Lie algebra.*

*Note that even without $V$ necessarily being a Lie algebra with respect to $[\ ,\ ]$, $\mathfrak{g}_p$ is always a Lie algebra.*

*Proof.* This follows simply from the definitions and Theorem 7.1. □

The next goal is to show that if $V$ is both irreducible and a Lie algebra under $[\ ,\ ]$, then each $\mathfrak{g}_p$ is forced to be isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We will need the following results.

**Lemma 11.1.** *Suppose that* $\pi^2 \neq 1$ *and let* $(\ ,\ ,\ )$ *be as above. If* $(v, v', v'') = 0$ *for all* $v', v'' \in V$, *then* $v = 0$.

*Proof*. Since $(\ ,\ ,\ )$ is $L_2(q)$ invariant and since $V$ is an irreducible representation of $L_2(q)$ (Theorem 2.1), $v \neq 0$ implies that $(V, v', v'') = 0$ which implies that $(\ ,\ ,\ ) = 0$. However, this contradicts the choice of the nonzero 3-form. □

For the next theorem, recall Definition 6.3 for the symbol $|x|$.

**Theorem 11.2.** *For* $\pi^2 \neq 1$ *and the above notation, the center of* $V$ *is trivial. In particular, for* $r \in \mathbb{P}^1(\mathbb{F}_q)$, $(f_r, f_s, f_q)$ *cannot be zero for all* $q, s \in \mathbb{P}^1(\mathbb{F}_q)$ *where* $|r| + |s| + |q| = 0$.

*Proof*. The first part comes from Lemma 11.1, the definition of $(\ ,\ ,\ ) = ([\ ,\ ],\ )$, and the nondegeneracy of $(\ ,\ )$. The second part follows by Theorem 7.1. □

Recall Theorem 11.1 and suppose that for some $p$ one has $(f_p, f_{-p}, f_0) = 0$ or $(f_p, f_{-p}, f_\infty) = 0$. In other words, suppose that $\mathfrak{g}_p$ is not isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Our goal is to show that this would imply that $f_0$ or $f_\infty$, respectively, would be in the center of $V$ and then to use Theorem 11.2 to get a contradiction.

In the case where $(f_p, f_{-p}, f_0) = 0$ or $(f_p, f_{-p}, f_\infty) = 0$, it will follow that all $f_{\pm a^2 p}$, $a \in \mathbb{F}_q^*$, commute with $f_0$ or $f_\infty$, respectively. This follows by the invariance of the three-form under the powers of the element $A$ (equation (4)), Theorem 7.1, and Theorem 6.3 (3).

In the case where $-1$ is not a square in $\mathbb{F}_q$ (i.e., when $\frac{q-1}{2} = h$ is odd or equivalently when there is only one invariant, alternating three-form), then we are already done (without reference to Jacobi!) since the set $\{\pm a^2\}$ exhausts all of $\mathbb{F}_q^*$. However, we will need to do more work (and definitely require Jacobi) for the case where $-1$ is a square in $\mathbb{F}_q$. Nevertheless, we record what we have found.

**Theorem 11.3.** *For* $\pi^2 \neq 1$ *and the above notation, if* $h$ *is odd* (*i.e.,* $-1$ *is not a square in* $\mathbb{F}_q$), *then each* $\mathfrak{g}_p$ *is isomorphic to* $\mathfrak{sl}(2, \mathbb{C})$.

For the general case, let us first proceed towards showing that $\mathfrak{g}_p$ cannot be the Abelian two-dimensional algebra if $V$ satisfies Jacobi (noting that, by the above theorem, we may assume that $-1$ is a square in the field). Suppose the Abelian case were possible (Theorem 11.1). Then fix some $p_0 \in \mathbb{F}_q^*$ so that $(f_{p_0}, f_{-p_0}, f_0) = (f_{p_0}, f_{-p_0}, f_\infty) = 0$. We define the subspaces:

**Definition 11.2.** For $p_0$ fixed in $\mathbb{F}_q^*$, let $F_1$, $F_\lambda$, and $\mathfrak{h}$ be the following subspaces of $V$:

$$F_1 = \operatorname{span}\{f_{a^2 p_0} \mid a \in \mathbb{F}_q^*\},$$
$$F_\lambda = \operatorname{span}\{f_{\lambda a^2 p_0} \mid a \in \mathbb{F}_q^*\},$$
$$\mathfrak{h} = \operatorname{span}\{f_0, f_\infty\},$$

where $\lambda$ was our fixed generator for $\mathbb{F}_q^*$. Observe that $V = F_1 \oplus \mathfrak{h} \oplus F_\lambda$ and that each of the subspaces is invariant under the group elements $A$ and $M_x$ (see equation (4) and equation (6)).

**Definition 11.3.** For $r \in \mathbb{F}_q^*$, define the "root" $\alpha_r \in \mathfrak{h}^*$ to be the linear function defined by the two relations

$$\alpha_r(f_\Box) = \frac{(f_r, f_{-r}, f_\Box)}{(f_r, f_{-r})}$$

where $\Box \in \{0, \infty\}$. Note that by Theorem 7.1 if $h \in \mathfrak{h}$, then

$$[h, f_r] = \alpha_r(h) f_r.$$

**Lemma 11.2.** *For $\pi^2 \neq 1$, if $V$ is a Lie algebra and $\mathfrak{g}_{p_0}$ is Abelian, then $F_1$ and $\mathfrak{h} \oplus F_\lambda$ are both ideals in $V$ with $[\mathfrak{h} \oplus F_\lambda, F_1] = 0$ and $[\mathfrak{h}, F_\lambda] = F_\lambda$.*

*Proof*. Throughout we may take $-1 \in \mathbb{F}_q$ to be a square so that if $f_q \in F_x$, then $f_{-q} \in F_x$ where $x \in \{1, \lambda\}$. We will make much use of the Jacobi relation given in Corollary 7.1(3). Combined with Theorem 7.1 and Definition 11.3, it says that, for $r, s \in \mathbb{F}_q^*$ with $r + s \neq 0$, either

$$[f_r, f_s] = 0 \tag{19}$$

or

$$\alpha_r + \alpha_s - \alpha_{r+s} = 0. \tag{20}$$

For later convenience, let us note that, by using $A$ invariance, one easily checks

$$\begin{aligned} \alpha_{a^2 r}(f_0) &= \pi(a)^{-1} \alpha_r(f_0), \\ \alpha_{a^2 r}(f_\infty) &= \pi(a) \alpha_r(f_\infty). \end{aligned} \tag{21}$$

By checking the definitions, the fact that we are in the Abelian case just comes down to meaning that $\alpha_{a^2 p_0} = 0$. In particular, $\alpha_{a^2 p_0}(f_\Box) = 0$ where $\Box \in \{0, \infty\}$. But since there is no center (Theorem 11.2), neither $f_0$ nor $f_\infty$ can commute with everything. This in turn gives us that $\alpha_{\lambda a^2 p_0}(f_\Box) \neq 0$ by $A$ invariance and Definition 11.3.

Using Theorem 7.1 and these facts, let us check the theorem. We already know that $\mathrm{ad}(\mathfrak{h})$ will commute with itself, kill $F_1$, and preserve $F_\lambda$ (in fact $f_0$ and $f_\infty$ will not kill anything in $F_\lambda$ since they preserve the $f$-basis and must be non-trivial on each $f_{\lambda a^2 p_0}$ to avoid being in the center). Thus we need only consider the action of $F_1$ and $F_\lambda$.

We show first that $[F_1, F_1] \subseteq F_1$. By the Abelian assumption, it is enough to show that $[f_{a^2 p_0}, f_{b^2 p_0}] = 0$ if $a^2 + b^2 = \lambda c^2$ for $a, b, a+b \in \mathbb{F}_q^*$. Thus suppose that $a^2 + b^2 = \lambda c^2$. The left-hand side of equation (20) reduces to $\alpha_{\lambda c^2 p_0}$. But by our previous remarks, this is not zero. Hence, equation (19) must hold which gives us that $[f_{a^2 p_0}, f_{b^2 p_0}] = 0$ as desired.

Next we show that $[F_\lambda, F_1] \subseteq F_\lambda$. Suppose that $\lambda a^2 + b^2 = c^2$. Then equation (20) reduces to $\alpha_{\lambda c^2 p_0}$ which, we have seen, is nonzero. Thus, as before, we get $[F_\lambda, F_1] \subseteq F_\lambda$.

At this point, let us make use of the assumption that $\pi(\lambda)$ is a primitive $h$th root of one (see Restriction 2.1). This assumption will be needed for the following three paragraphs and is sufficient for our needs by the previous discussion and Theorem 6.4. Let us only record that so far (without this assumption) we have already proved:

$$[F_1, F_1] \subseteq F_1, \tag{22}$$

$$[F_\lambda, F_1] \subseteq F_\lambda. \tag{23}$$

Next we show that $[F_\lambda, F_\lambda] \subseteq F_\lambda \oplus \mathfrak{h}$. Suppose that $\lambda a^2 + \lambda b^2 = c^2$. Then the left-hand side of equation (20) evaluated at $f_\infty$ ($f_0$ would work equally well) yields $\alpha_{\lambda p_0}(f_\infty)(\pi(a) + \pi(b))$. The first part of this product is nonzero. The second part will be zero if and only if $\pi(a) = -\pi(b)$. If we denote by $i$ some $\sqrt{-1}$ in $\mathbb{F}_q^*$, then this situation will occur if and only if $a = \pm ib$ (note that $\pi(\lambda)$ is a primitive $h$th root of unity) so that this will occur if and only if $a^2 = -b^2$. For $c \neq 0$, we must therefore have equation (19) which tells us that the corresponding bracket is zero. For $c = 0$, the bracket will lie in $\mathfrak{h}$. Hence $[F_\lambda, F_\lambda]$ will never have a $F_1$ component and we have shown that $[F_\lambda, F_\lambda] \subseteq F_\lambda \oplus \mathfrak{h}$.

Similarly, we show $[F_\lambda, F_1] \subseteq F_1$. Suppose that $\lambda a^2 + b^2 = \lambda c^2$. In this case, equation (20) evaluated at (say) $f_\infty$ yields $\alpha_{\lambda p_0}(f_\infty)(\pi(a) - \pi(c))$. As before, this can only be zero if $a^2 = c^2$. But this would tell us that $b = 0$ which is not possible. Hence we get $[F_\lambda, F_1] \subseteq F_1$.

We complete the picture in the case of $\pi(\lambda)$ primitive by noting that $F_1 \cap F_\lambda = \{0\}$ so that $[F_\lambda, F_1] = 0$ at last. □

It will be useful for us to introduce the following notation for a particular basis for $V$ consisting of eigenvectors for $A$.

**Definition 11.4.** Write $\zeta = e^{2\pi i/h}$. Then for $k = 0, 1, \dots, h-1$, let $w_{1,k}$ and $w_{\lambda,k}$ be as follows.

$$w_{1,k} = \sum_{l=1}^{h} \zeta^{-lk} A^k f_{p_0} = \sum_{l=1}^{h} (\zeta^{-k}\pi(\lambda))^l f_{\lambda^{2l} p_0},$$

$$w_{\lambda,k} = \sum_{l=1}^{h} \zeta^{-lk} A^k f_{\lambda p_0} = \sum_{l=1}^{h} (\zeta^{-k}\pi(\lambda))^l f_{\lambda^{2l+1} p_0}.$$

From this, it is clear that $w_{1,k}$ and $w_{\lambda,k}$ are distinct eigenvectors for $A$ corresponding to the eigenvalue $\zeta^k$. If we also note that $f_0$ and $f_\infty$ are eigenvectors for $A$ of eigenvalue $\pi(\lambda)$ and $\pi(\lambda)^{-1}$, respectively, then it is clear that the set $\{w_{1,k}, w_{\lambda,k}, f_0, f_\infty \mid k = 0, \dots, h-1\}$ is a basis for $V$ consisting of eigenvectors of $A$.

**Lemma 11.3.** *Let $m \in 1, \dots, h-1$ be such that $\zeta^m = \pi(\lambda)$. Then for $x \in \{1, \lambda\}$, we have:*

(1) *the $f_0$ component of $Sw_{x,k}$ in the $f$-basis is $\pi(xp_0)^{-1}\Gamma_{0,1}h\delta^{(h)}_{m+k}$,*

(2) *the $f_\infty$ component of $Sw_{x,k}$ in the $f$-basis is $h\delta^{(h)}_{m-k}$,*

(3) $Sf_0 = \pi(p_0)\Gamma_{1,0}[w_{1,-m} + \pi(\lambda)w_{\lambda,-m}] + f_\infty$,

(4) $Sf_\infty = 1/q[w_{1,m} + w_{\lambda,m} + f_0]$,

*where $\delta_r^{(h)}$ denotes 1 if $r = 0 \bmod (h)$ and 0 otherwise.*

*Proof.* These are all simple calculations that follow from Theorem 3.1, Definition 11.4, and Lemma 3.1. We will only work out part (1) since the rest are similar or obvious. The $f_0$ component of $Sw_{x,k}$ is simply

$$\sum_{l=1}^{h}(\zeta^{-k}\pi(\lambda))^l\Gamma_{0,x\lambda^{2l}p_0} = \sum_{l=1}^{h}\zeta^{l(m-k)}\Gamma_{0,1}\pi(x\lambda^{2l}p_0)^{-1}$$
$$= \pi(xp_0)^{-1}\Gamma_{0,1}\sum_{l=1}^{h}\zeta^{-l(m+k)}$$

which gives us the desired result. □

**Lemma 11.4.** *With the assumptions of Lemma* 11.2, *there exists* $k_0$ *such that* $Sw_{1,k_0} = c_{k_0}w_{\lambda,-k_0}$ *for some* $c_{k_0} \neq 0$.

*Proof*. Recall the group element $S$ from equation (7). First observe that $S$, $A$, and $M_u$ generate $L_2(q)$. Since $V$ is an irreducible $L_2(q)$ module, then any proper subspace of $V$ invariant under $A$ and $M_u$ cannot be $S$ invariant. Thus, $SF_1$ must have vectors with nonzero components in $\mathfrak{h} \oplus F_\lambda$.

In fact, we claim that $SF_1$ must have vectors with nontrivial components in $F_\lambda$. If this were not so, then we would have $SF_1 \subseteq F_1 \oplus \mathfrak{h}$. But we will see that this is not possible. First recall the notation $m$ from Lemma 11.3 and consider the vectors $w_{1,\pm m}$ in $F_1$. We will apply $S$ to them.

Since $SAS = A^{-1}$, $S$ will carry a $\zeta^k$ eigenvector of $A$ into a $\zeta^{-k}$ eigenvector. Thus, a priori, we may always write:

$$\begin{aligned} Sw_{1,-m} &= aw_{1,m} + bw_{\lambda,m} + cf_0, \\ Sw_{1,m} &= ew_{q,-m} + fw_{\lambda,-m} + gf_\infty, \\ Sw_{\lambda,m} &= e'w_{1,-m} + f'w_{\lambda,-m} + g'f_\infty, \end{aligned} \tag{24}$$

where $a, b, c, d, e, f, g, e', f', g'$ are certain numbers in $\mathbb{C}$. By Lemma 11.3, we also know $c$, $g$, $g'$ explicitly. In particular, none of these are zero. Using the fact that $S^2 = \mathrm{Id}$, we get the following equations by applying equations (24) twice and using Lemma 11.3 (3) and (4):

$$1 = ae + be' + c\pi(p_0)\Gamma_{1,0}, \tag{25}$$
$$0 = af + bf' + c\pi(\lambda p_0)\Gamma_{1,0}, \tag{26}$$
$$0 = ag + bg' + c. \tag{27}$$

If we were to assume that $SF_1 \subseteq F_1 \oplus \mathfrak{h}$, this implies that $b = f = 0$ in equations (24). However, equation (26) then would imply that $c = 0$, but we have already computed that this is not so.

Hence, there exists some nonzero $v_0 \in V$ such that $Sv_0 = v_1 + h + v_\lambda$, written with respect to the decomposition $V = F_1 \oplus \mathfrak{h} \oplus F_\lambda$, such that $v_\lambda \neq 0$. We have already seen (say in Lemma 11.2) that $\mathrm{ad}(f_\square)$ does not kill anything in $F_\lambda$. On the other hand, it kills everything in $F_1 \oplus \mathfrak{h}$. Thus, if we let $v_0' = [f_\square, Sv_0]$, then we see that $v_0' \in F_\lambda$ and is nonzero. However, since $S$ preserves the bracket structure and since $F_1$ is an ideal, $SF_1$ is also an ideal. Hence $v_0'$ is also in $SF_1$. Thus we have shown that $SF_1 \cap F_\lambda$ is nontrivial.

To finish the proof, it suffices to note that $F_1$ and $F_\lambda$ are $A$ invariant. Since $A^{-1} = A^{h-1}$ and $SAS = A^{-1}$, $SF_1$ is also $A$ invariant. Thus $SF_1 \cap F_\lambda$ is a nonzero $A$ invariant space. Therefore it consists of eigenvectors of $A$ and we are done. □

**Lemma 11.5.** *With the notation from Lemma* 11.4, *either* $Sw_{1,m}$ *or* $Sw_{1,-m}$ *is contained in* $F_\lambda \oplus \mathfrak{h}$.

*Proof.* Let $k_0$ be from Lemma 11.4 so that $Sw_{1,k_0} = c_{k_0} w_{\lambda,-k_0}$, $c_0 \neq 0$. We know that $[f_0, w_{\lambda,-k_0}] \neq 0$ so that $S$ applied to it is nonzero. Using Lemma 11.3 (3) and the fact that we are in the Abelian case (so that Lemma 11.2 applies), this gives us that $[\pi(p_0)\Gamma_{1,0} w_{1,-m}, c_{k_0}^{-1} w_{1,k_0}] \neq 0$. Thus we have

$$[w_{1,-m}, w_{1,k_0}] \neq 0. \tag{28}$$

With this done, let us use the notation from equation (24) again and make heavy use of the Abelian case while we consider the following.

$$\begin{aligned}
[w_{1,-m}, w_{1,k_0}] &= S^2[w_{1,-m}, w_{1,k_0}] \\
&= S[bw_{\lambda,w} + cf_0, c_{k_0} w_{\lambda,-k_0}] \\
&= [be' w_{1,-m} + c\pi(p_0)\Gamma_{1,0} w_{1,-m}, w_{1,k_0}] \\
&= (be' + c\pi(p_0)\Gamma_{1,0})[w_{1,-m}, w_{1,k_0}].
\end{aligned}$$

Hence equation (28) implies that $be' + c\pi(p_0)\Gamma_{1,0} = 1$. Thus, equation (25) tells us that $ae = 0$. Therefore, we have $a = 0$ or $e = 0$ which is exactly the desired result. □

**Theorem 11.4.** *For* $\pi^2 \neq 1$ *and* $p \in \mathbb{F}_q^*$, *if* $V$ *is a nontrivial Lie algebra, then* $\mathfrak{g}_p$ *is not Abelian.*

*Proof.* Suppose not. Then we would be in the position of Lemma 11.5. Let us carry over all of its notation so that we have either $Sw_{1,m}$ or $Sw_{1,-m}$ contained in $F_\lambda \oplus \mathfrak{h}$. In other words, $ae = 0$. Suppose that $e = 0$ so $Sw_{1,m} \subseteq F_\lambda \oplus \mathfrak{h}$. Then we would have:

$$[Sw_{1,m}, f_\infty] = [fw_{\lambda,-m}, f_\infty] = f[w_{\lambda,-m}, f_\infty].$$

Thus $[Sw_{1,m}, f_\infty]$ is zero if and only if $f = 0$ since $f_\infty$ does not kill anything in $F_\lambda$ and $w_{\lambda,-m} \not\subseteq \mathfrak{h}$. However, by applying $S$ to $[Sw_{1,m}, f_\infty]$, we get $[w_{1,m}, (1/q)w_{1,m}]$ which is equal to $0$. Since $S$ is invertible, this gives us that $f = 0$. In other words, $e = 0$ implies $f = 0$. But looking at equation (24), this translates to saying that $Sw_{1,-m} = cf_0$. This would imply, though, that $w_{1,-m} = cSf_0$. But this is a contradiction by Lemma 11.3.

To see that $a = 0$ also gives a contradiction, repeat the same argument as above, but this time start out with $[Sw_{1,-m}, f_0]$. It is easy to see that everything is similar. Thus we are done. □

Since we have shown that the Abelian possibility of Theorem 11.1 cannot occur if $V$ is actually a Lie algebra, this leaves only the possibilities of $\mathfrak{sl}(2, \mathbb{C})$ or the Heisenberg. Next, we will show that Jacobi also excludes the Heisenberg case.

First we need a "nilpotent" argument.

**Lemma 11.6.** *If* $\pi^2 \neq 1$ *and* $V$ *is a Lie algebra, then for* $p, q \in \mathbb{F}_q^*$ *there exists* $n$ *in* $\mathbb{Z}^+$ *such that* $\operatorname{ad}(f_p)^n f_q = 0$.

*Proof.* First, by Theorem 7.1, we observe that $\operatorname{ad}(f_p)^n f_q \subseteq \mathbb{C} f_{np+q}$. Suppose $\operatorname{ad}(f_p) f_q \neq 0$. Then since Jacobi is satisfied, equation (19) tells us that $\alpha_{p+q} = \alpha_p + \alpha_q$. If $\operatorname{ad}(f_p)^2 f_q$ is also nonzero, then $\operatorname{ad}(f_p) f_{p+q} \neq 0$. Hence we get

$\alpha_{2p+q} = \alpha_p + \alpha_{p+q} = 2\alpha_p + \alpha_q$. In general, suppose $\operatorname{ad}(f_p)^n f_q \neq 0$. Using induction, we get

$$\alpha_{np+q} = n\alpha_p + \alpha_q. \tag{29}$$

Assume that the lemma is false. Since $V$ is a Lie algebra, we have already seen that each $\mathfrak{g}_p$ must be three-dimensional. In particular, this will give us (see Definition 11.3 for $\alpha_p$, Theorem 11.1 for the three-dimensional properties, and Definition 11.1 for $d_p$) that each $\alpha_p$ is a nonzero element of $\mathfrak{h}^*$. However, by definition of the finite field $\mathbb{F}_q$, there are only a finite number of "roots" $\alpha_r$, $r \in \mathbb{F}_q^*$. But equation (29) would imply (by taking arbitrary $n \in \mathbb{Z}^+$) that there were an infinite number of distinct "roots". Hence we have a contradiction. □

Next, we present the standard "bracket" relations for $\mathfrak{g}_p$.

**Theorem 11.5.** *In the case where $\mathfrak{g}_p$ is three dimensional, using the notation of Definition* 11.1, *suppose that $[x, f_q] = 0$ for some $q \in \mathbb{F}_q^*$. Then, if we define $v_0 = f_q$ and $v_i = \frac{1}{i!}\operatorname{ad}(y)^i v_0$ for $i \in \mathbb{Z}^+$, we have*

$$\begin{aligned} \operatorname{ad}(y)v_i &= (i+1)v_{i+1}, \\ \operatorname{ad}(h)v_i &= (\alpha_q(h) - 2id_p)v_i, \\ \operatorname{ad}(x)v_i &= (\alpha_q(h) + (1-i)d_p)v_{i-1}, \end{aligned}$$

*where $\alpha_q$ is given in Definition* 11.3. *In this case,*

$$\alpha_q(h) = \frac{(f_p, f_{-p}, f_0)(f_q, f_{-q}, f_\infty) + (f_p, f_{-p}, f_\infty)(f_q, f_{-q}, f_0)}{(f_q, f_{-q})(f_0, f_\infty)}.$$

*Note that, by Lemma* 11.6, *if $[f_p, f_q]$ were not equal to zero, we could always "push" $f_q$ up (in a nonzero way) with $\operatorname{ad}(f_p)$ to some $f_{q'} = f_{np+q}$ so that $[f_p, f_{q'}] = 0$.*

*Proof*. This is just the standard $\mathfrak{sl}(2, \mathbb{C})$ type proof. It follows by induction on the bracket relations given in Theorem 11.1 and Definitions 11.1 and 11.3. We omit the details. □

We are now in a position to exclude the Heisenberg case.

**Lemma 11.7.** *Let $\pi^2 \neq 1$ and let $V$ be a Lie algebra. Then $\mathfrak{g}_p$ is not a Heisenberg algebra.*

*Proof*. Assume $\mathfrak{g}_p$ is a Heisenberg algebra. Then pick any $q' \in \mathbb{F}_q^*$. By Lemma 11.6, let $q = np + q'$ be such that $\operatorname{ad}(f_p)^n f_{q'} \neq 0$ but $[f_p, f_q] = 0$. We are now in a position to use Theorem 11.5 so we will adopt the theorem's notation. Since $\mathfrak{g}_p$ is a Heisenberg, we know that either $(f_p, f_{-p}, f_0)$ (call this case I) or $(f_p, f_{-p}, f_\infty)$ (call this case II) is equal to zero, but not both. In either case, we have $d_p = 0$. Hence, we have the relations

$$\begin{aligned} [f_{-p}, v_i] &= (i+1)v_{i+1}, \\ [h, v_i] &= \alpha_q(h)v_i, \\ [f_p, v_i] &= \alpha_q(h)v_{i-1}, \end{aligned} \tag{30}$$

where $v_0 = f_q$ and $v_i = (1/i!)\operatorname{ad}(f_{-p})^i v_0$. However, Lemma 11.6 tells us that, for large $i$, $v_i$ is zero. Using an $i$ such that $v_i = 0$ but $v_{i-1} \neq 0$, equation (30) tells us that we must have $\alpha_q(h) = 0$. Hence $f_p$ kills each $v_i$. In particular, looking at the beginning of the proof, we must have $q' = q$. Thus we have $\alpha_{q'}(h) = 0$. However, looking at Theorem 11.5 for an explicit form of $\alpha_{q'}(h)$, we must have $(f_{q'}, f_{-q'}, f_0) = 0$ in case I or $(f_{q'}, f_{-q'}, f_\infty) = 0$ in case II. However, $q'$ was arbitrary. Thus, case I implies that $f_0$ is in the center while case II implies that $f_\infty$ is in the center. Either possibility contradicts Theorem 11.2. This finishes the proof. □

Concluding this section, we state the main result:

**Theorem 11.6.** *Assume that* $\pi^2 \neq 1$. *Suppose there exists a nonzero* $L_2(q)$ *invariant alternating three-form that makes* $V$ *into a Lie algebra. Then each* $\mathfrak{g}_p$, $p \in \mathbb{F}_q^*$, *is isomorphic to* $\mathfrak{sl}(2, \mathbb{C})$.

*Proof.* This is an immediate corollary of the previous theorem and earlier discussion. □

## 12. The roots

In this section and the next, we will be able to show that the only time $V$ can be made into a nontrivial Lie algebra is in the cases of $A_2$, $B_2$, and $G_2$. To do this, we will exploit certain "integrality" conditions that will follow from Jacobi. To some degree, the basic reason that only $q = 5, 7, 9, 13$ are allowable stems from the fact that

$$2\cos(2\pi i/h)$$

is only an integer for $h = 2, 3, 4, 6$ (Theorem 12.2, equation (36)).

It will fall out of previous work that the set of $\{\alpha_p\}$ forms a root system. To start, let us make use of Theorem 11.6 and Theorem 11.1 to redefine the basal elements in $\mathfrak{g}_p$ of Definition 11.1.

**Definition 12.1.** For $V$ a nontrivial Lie algebra, $\pi^2 \neq 1$, and $p \in \mathbb{F}_q^*$, normalize a basis for $\mathfrak{g}_p$ as follows:

$$x_p = f_p, \quad y_p = \frac{1}{d_p} f_{-p},$$
$$h_p = \frac{1}{d_p}[f_p, f_{-p}] = \frac{(f_p, f_{-p})}{(f_p, f_{-p}, f_0)} f_0 + \frac{(f_p, f_{-p})}{(f_p, f_{-p}, f_\infty)} f_\infty.$$

Note that this is isomorphic to the standard $e$, $f$, $h$ basis of $\mathfrak{sl}(2, \mathbb{C})$ given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us note for future reference the value of a "root" on our new $h_p$. From the above definition and Definition 11.3, we see

$$\alpha_q(h_p) = \frac{(f_p, f_{-p})}{(f_q, f_{-q})} \left( \frac{(f_q, f_{-q}, f_0)}{(f_p, f_{-p}, f_0)} + \frac{(f_q, f_{-q}, f_\infty)}{(f_p, f_{-p}, f_\infty)} \right) \tag{31}$$

for $p, q \in \mathbb{F}_q^*$. Now we prove the first integrality condition.

**Theorem 12.1.** *Let $V$ be a nontrivial Lie algebra and let $p, q \in \mathbb{F}_q^*$. Write $r, s \in \mathbb{Z}$ as the largest integers satisfying $\mathrm{ad}(f_p)^s f_q \neq 0$ and $\mathrm{ad}(f_{-p})^r f_q \neq 0$. Then*

$$\alpha_q(h_p) = -(s-r).$$

*In particular, $\alpha_q(h_p)$ is always an integer.*

*Proof.* Since $\mathfrak{g}_p$ is just $\mathfrak{sl}(2, \mathbb{C})$ and we have renormalized $h_p$ so that it is the standard " $h$ ", this is simply a well-known fact that follows easily by the bracket relations and finiteness of $r$ and $s$ (see [8]). One way to see this is the following. Using $v_0 = f_{q+sp}$ and the fact that $v_{s+r+1} = 0$ but $v_{s+r} \neq 0$ in Theorem 11.5 (with $d_p = 1$), we get $0 = \mathrm{ad}(f_{-p})v_{s+r+1} = [\alpha_{q+sp}(h_p) + (1-(s+r+1))]v_{s+r}$ so that we must have $0 = \alpha_{q+sp}(h_p) - s - r = \alpha_q(h_p) - s\alpha_p(h_p) - s - r = \alpha_q(h_p) + s - r$ which gives us our result. □

Since our $L_2(q)$ invariant two-form ( , ) is nondegenerate when restricted to $\mathfrak{h}$, it is a simple matter to transfer all the structure we have on $\mathfrak{h}$ to $\mathfrak{h}^*$. Using the results on the "roots" $\{\alpha_p\}$ and the above integrality condition, it is not too hard to check (Theorem 13.1) that the set of $\alpha_p$ is indeed an honest (reduced) root system [8] when $V$ is a Lie algebra. Hence by the dimension of $\mathfrak{h}$, we would get as a corollary that the root system so obtained must be isomorphic to one of the following root systems: $A_1 + A_1$, $A_2$, $B_2$, $G_2$, or $BC_2$. Then as a corollary of this, one gets limits on the values of $q$.

While this line of attack is possible and will be followed up in the next section, most of the time we do not really need to rely on the classification of root systems. Instead, the Jacobi identity forces us into the situation where each $\mathfrak{g}_p$ is a $\mathfrak{sl}(2, \mathbb{C})$. As it turns out, this alone will usually give us that $q$ must be equal to 5, 7, 9, or 13. However, when $q = 5$, then $\pi^2 = 1$ so $V$ is never irreducible. Thus, we will get that only $q = 7, 9, 13$ is possible. (It is easily seen that, for $q = 5$, $V$ will split into two copies of $\mathfrak{sl}(2, \mathbb{C})$ as we have already remarked at the end of Section 10.)

**Theorem 12.2.** *Assume that $\pi_m^2 \neq 1$ and that $V_{\pi_m}$ is a nontrivial Lie algebra with $\pi_m(\lambda)$ a primitive hth root of unity, i.e., $(m, h) = 1$. Then $q = 7, 9$, or $13$.*

*Proof.* Let $p, q \in \mathbb{F}_q^*$. Using equation (31), the fact that $d_r \neq 0$, and the invariance of the two- and three-forms with respect to $A$ (see Theorem 5.1 and Theorem 6.3), let us calculate (by factoring out the $(f_1, f_{-1}, f_\square)$ and $(f_\lambda, f_{-\lambda}, f_\square)$ part in each and cancelling where appropriate):

(32) $\alpha_{q^2}(h_{p^2}) = \pi(q/p)^2(\pi(p/q) + \pi(p/q)^3) = \pi(q/p) + \pi(p/q)$,

(33) $\alpha_{\lambda q^2}(h_{\lambda p^2}) = \pi(q/p) + \pi(p/q)$,

(34) $$\alpha_{q^2}(h_{\lambda p^2}) = \pi(1/\lambda)\left(\pi(q/p)\frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} + \pi(p/q)\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)}\right),$$

(35) $$\alpha_{\lambda q^2}(h_{p^2}) = \pi(\lambda)\left(\pi(q/p)\frac{(f_\lambda, f_{-\lambda}, f_\infty)}{(f_1, f_{-1}, f_\infty)} + \pi(p/q)\frac{(f_\lambda, f_{-\lambda}, f_0)}{(f_1, f_{-1}, f_0)}\right).$$

We will only need the first equation for this proof, but we listed the others as they will be useful later. Let $r = q/p$. Then as $q$ and $p$ vary, $r$ will vary over all of $\mathbb{F}_q^*$. Thus Theorem 12.1 and equation (32) imply that the expression

(36) $$\pi(r) + \pi(r)^{-1}$$

is always an integer for all $r \in \mathbb{F}_q^*$. However, since $\pi(\lambda)$ is a primitive $h$th root of unity, this is the same as saying that

$$2\cos(2\pi i/h)$$

is an integer. It is trivial to check that this implies that $h = 2, 3, 4$, or $6$. In turn, this gives us that $q = 5, 7, 9$, or $13$. However, as we have already noted, if $q = 5$, then $\pi^2 = 1$. Hence we must have $q = 7, 9$, or $13$ as desired. □

## 13. Structure of the roots

We note that the condition of equation (36) being an integer is precisely the relation needed when one considers the problem of tiling the plane. And in fact it is clear that the $h$ values 3, 4, 6 are precisely the only values for which the tiling may be done: for the triangle, the square, and the hexagon. Another way of saying this is that the only dihedral groups that preserve a lattice in the plane are $D_3$, $D_4$, $D_6$.

Next let us examine the "roots". It will show that only $q = 7, 9, 13$ are allowable under Restriction 2.1. To do this we will first extract roots and so will need to transfer the nondegenerate 2-form $(\ ,\ )|_{\mathfrak{h}}$ to $\mathfrak{h}^*$. For any $\mu \in \mathfrak{h}^*$, let $u_\mu \in \mathfrak{h}$ be the unique element satisfying $\mu(h) = (u_\mu, h)$ for all $h \in \mathfrak{h}$. With this, we define a nondegenerate symmetric $L_2(q)$ invariant 2-form on $\mathfrak{h}^*$ by letting

$$(\mu, \nu) = (u_\mu, u_\nu)$$

for all $\mu, \nu \in \mathfrak{h}^*$. In particular, for each $p \in \mathbb{F}_q^*$, $u_{\alpha_p}$ is the unique element of $\mathfrak{h}$ such that $\alpha_p(h) = (u_{\alpha_p}, h)$ for all $h \in \mathfrak{h}$. Using the definition of the bracket structure and Definition 11.3, we note that $\alpha_p(f_\square) = (f_p, f_{-p}, f_\square)/(f_p, f_{-p}) = ([f_p, f_{-p}], f_\square)/(f_p, f_{-p})$. This gives us

$$u_{\alpha_p} = \frac{[f_p, f_{-p}]}{(f_p, f_{-p})}.$$

Moreover, if we define

$$h_{\alpha_p} = 2u_{\alpha_p}/(\alpha_p, \alpha_p),$$

it is also easy to check that these "coroots" satisfy the relation $h_{\alpha_p} = h_p$ from Definition 12.1.

Let us use the standard notation

$$\langle u \mid \nu \rangle = 2(\mu, \nu)/(\nu, \nu).$$

Thus if we write $\sigma_\mu$ for the reflection across the hyperplane perpendicular to $\mu$, we have the usual formula:

$$\sigma_\mu(\nu) = \nu - \langle \mu \mid \nu \rangle \mu.$$

We will need one more relation. Namely,

$$\langle \alpha_p \mid \alpha_q \rangle = \langle h_q \mid h_p \rangle = \alpha_p(h_q). \tag{37}$$

Since this is the standard phenomenon and since the above is simply a matter of checking the definitions, we leave it to the reader. With this notation, we are now ready to consider root systems.

**Theorem 13.1.** *For $\pi^2 \neq 1$ and $V$ a nontrivial Lie algebra, the set of $\alpha_p$, $p \in \mathbb{F}_q^*$, forms a rank two (reduced) root system of order $q-1$ within their real span.*

*Proof.* First, let us check that the real span of the $\alpha_p$ really is only two dimensional (over $\mathbb{R}$). To do this, we claim that, for $q \in \mathbb{F}_q^*$, the following identity holds:

$$\alpha_q = [2\alpha_q(h_1) - \alpha_q(h_\lambda)\alpha_\lambda(h_1)]\alpha_1 + [2\alpha_q(h_\lambda) - \alpha_q(h_1)\alpha_1(h_\lambda)]\alpha_\lambda. \tag{38}$$

To check this equality, merely evaluate both sides at $f_\square$, $\square \in \{0, \infty\}$, using Definition 11.3 and equation (31). Since it is an easy calculation, we omit the details. However, Theorem 12.1 tells us that we have actually expressed any $\alpha_q$ as an integral linear combination of $\alpha_1$ and $\alpha_\lambda$. In particular, since $\mathbb{Z} \subseteq \mathbb{R}$, the real span of the roots is no more than two dimensional. We will show below that $\alpha_1$ is not a multiple of $\alpha_\lambda$ which will give us that the real span is exactly two dimensional as desired.

Let us show that all the roots are distinct and that the only multiples of roots that are still roots are $\pm 1$. This is just the standard argument. First note that by the alternating nature of the three-form, we have:

$$-\alpha_p = \alpha_{-p}. \tag{39}$$

Fix $\alpha = \alpha_q$. By Theorem 11.6, we know that $\mathfrak{g}_q$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. For each $c \in \mathbb{R}$, let $L_c$ be the real span of all vectors $f_p$, $p \in \mathbb{F}_q^*$, such that $\alpha_p = c\alpha$. Then put $L = \bigoplus_c L_c$. Thus by the Jacobi identity (see equation (20)), $V' = L \oplus \mathfrak{h} \subseteq V$ is a finite dimensional representation of $\mathfrak{g}_q$, i.e., of $\mathfrak{sl}(2, \mathbb{C})$, under ad. Since we have $\mathrm{ad}(h_q)|_{L_c} = (2c)\mathrm{Id}$, we must have $c \in \frac{1}{2}\mathbb{Z}$ by elementary $\mathfrak{sl}(2, \mathbb{C})$ theory. Also since $\mathfrak{g}_q + \mathfrak{h} \subseteq V'$ is an invariant subspace of the representation that contains all occurrences of the 0-weight for $\mathrm{ad}(h_q)$, we see that 0 and $\pm 2$ exhaust the even roots in $V'$. In particular, twice a root is not a root. Then we also have half a root is not allowable either since $\alpha$ is a root. In particular, 1 is not a weight. Thus $\mathfrak{sl}(2, \mathbb{C})$ tells us that only $c = \pm 1$ yield nonzero $L_c$ and that $V' = L_1 \oplus \mathfrak{h} \oplus L_{-1} = \mathfrak{g}_q + \mathfrak{h}$. Since each $f_p$ is distinct, we have shown that each $\alpha_q$ is distinct and that the only multiples of $\alpha_q$ that are roots are $\pm\alpha_q$.

Next, let $\alpha$ and $\beta$ be two roots. Then if $\beta - r\alpha, \dots, \beta + s\alpha$ is the maximal $\alpha$ root string through $\beta$, then Theorem 12.1 has already told us that $\alpha(h_\beta) = r - s$. With this, one can easily check that $\sigma_\alpha(\beta)$ is still a root.

Finally, we have already shown that $\langle \alpha_p \mid \alpha_q \rangle \in \mathbb{Z}$ by equation (37) and Theorem 12.1. This finishes the proof. □

We also prove:

**Theorem 13.2.** *For $\pi^2 \neq 1$ and $V$ a nontrivial Lie algebra, $V$ is semisimple of rank 2.*

*Proof.* Since $\pi^2 \neq 1$, $V$ is irreducible. Since $[\ ,\ ]$ is $L_2(q)$ invariant, the first derived algebra of $V$, $[V, V]$, is invariant and thus either 0 or $V$. But since there is no center, $[V, V] = V$. In particular, $V$ is not solvable. But since the Killing form (being constructed out of invariant things) is $L_2(q)$ invariant, it must be a multiple of our two-form. In fact, it must be a nonzero multiple by Cartan's criterion. Hence, since the two-form is nondegenerate, the Killing form is nondegenerate. Thus $V$ must be semisimple. The rest follows basically from Theorem 13.1. Another way to see it follows.

We claim that $\mathfrak{h}$ is a maximal Abelian semisimple subalgebra of $V$. If not, then by $A$ invariance and the fact that the $f$-basis diagonalizes $\mathrm{ad}(f_0)$ and $\mathrm{ad}(f_\infty)$, $\mathfrak{h}$ would commute with all of $F_1$ or $F_\lambda$ (see Definition 11.2). But we have already seen that this is not possible. Hence we are done since $\mathfrak{h}$ is two dimensional. □

**Corollary 13.1.** *Assume that* $\pi_m^2 \neq 1$ *and that* $V_{\pi_m}$ *is a nontrivial Lie algebra with* $m$ *subject to Restriction* 2.1. *Then* $q = 7, 9$, *or* 13 *and* $m$ *is an exponent of* $A_2$, $B_2$, *or* $G_2$, *respectively.*

*Proof.* This follows easily from Theorem 13.2. Since $V$ has dimension $q+1$ and must be rank two, $q+1$ must be either 6, 8, 10, or 14. Since we have already seen that 5 is not allowable owing to irreducibility, we have $q$ equal to 7, 9, or 13. The exponent statement is obvious for $q = 7, 9$ since the exponents are the only possibilities anyway. For $q = 13$, simply apply Theorem 2.1 and Theorem 6.4 (1). □

**Corollary 13.2.** *For* $\pi^2 \neq 1$ *and* $V$ *a Lie algebra, the root system obtained from the set of* $\alpha_p$ *must be isomorphic to* $A_2$, $B_2$, *and* $G_2$, *for* $q = 7, 9$, *and* 13, *respectively.*

*Proof.* Corollary 13.1 has already told us that the only allowable $q$ are 7, 9, and 13. Now by using Theorem 13.2 and Theorem 13.1 and noting that in each case we must have 6, 8, and 12 roots (since $|\mathbb{F}_q^*| = q-1$), respectively, it is trivial to check that the above listed root systems are the only rank two possibilities with the correct number of roots. Note: had we not insisted on $\pi^2 \neq 1$, then $D_2 = A_1 + A_1$ would have been the corresponding root system for $q = 5$. □

Let us look at each of these cases, $q = 7, 9, 13$, to see how the roots are situated.

$\mathbb{F}_7$: As we have seen, the root system must be $A_2$. Fix $\lambda = -2$ as a generator for $\mathbb{F}_7^*$. Using the fact that $-1 = \lambda^3$ and equations (32), we calculate that $\alpha_1(h_\lambda) = -\alpha_{-1}(h_\lambda) = -\pi(\lambda) - \pi(\lambda)^{-1}$. Since $\pi(\lambda)$ is a primitive third root of unity, this tells us that $\alpha_1(h_\lambda) = 1$. Similarly, we calculate that $\alpha_1(h_{\lambda^5}) = -\alpha_{-1}(h_{\lambda^5}) = 1$. In other words, between $\alpha_1$ and $\alpha_\lambda$ and between $\alpha_1$ and $\alpha_{\lambda^5}$ there is a $60°$ angle. Thus, up to isomorphism, the root system in this case must be as in Figure 1 or Figure 2 (this also can be seen using the additive structure of the roots). Note that the roots go around in order of powers of $\lambda$ and that the element $A$ (multiplication by $\lambda^2$) acts as the Coxeter element, rotation by $120°$.

Also, using equation (34), Theorem 12.1, and our explicit determination of the roots above, it is now easy to calculate that in either case

$$\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)} = -1, \quad \frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} = -\pi(\lambda)^2$$

by solving two linear equations in two unknowns, e.g., $\alpha_{\lambda^2}(h_\lambda) = 1$ and $\alpha_{\lambda^4}(h_{\lambda^5}) = 1$. In principle, these numbers and $L_2(7)$ invariance (up to normalization) determine the three-form which in turn determines the bracket structure.

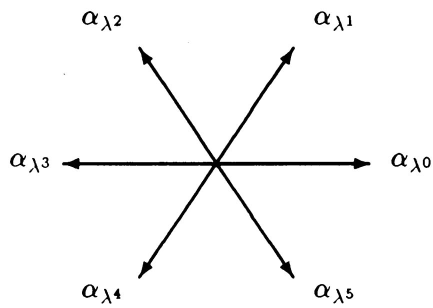


FIGURE 1. $q = 7$, possibility 1

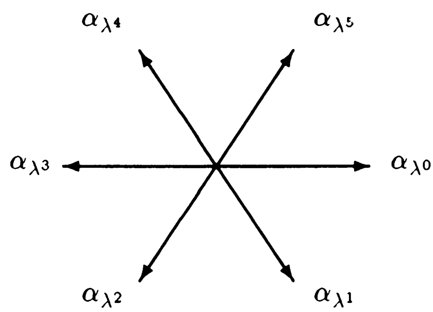


FIGURE 2. $q = 7$, possibility 2

Moreover, using the notation of [6] and looking at a few character values [16], one may say that the representation $\chi_2$ is associated to Figure 1 and $\chi_3$ to Figure 2 in the standard representations.

$\mathbb{F}_9$: Here we know that the root system must be $B_2$. First of all, one may check using the definitions and invariance that

$$(\alpha_{c^2q}, \alpha_{c^2q}) = \frac{2d_{c^2q}}{\pi(c^2q)^2(f_0, f_\infty)} = \frac{2d_q}{\pi(q)^2(f_0, f_\infty)}. \tag{40}$$

In other words, all roots of the form $\alpha_{c^2q}$ have the same length (this can also be seen by $A$ invariance). Fix a generator $\lambda$ of $\mathbb{F}_9^*$ with the property that $\lambda^2 = 1 + \lambda$ and $1 + \lambda^2 = \lambda^7$ (such a generator exists). Then using that the roots must add according to the field (if their sum is another root, then $\alpha_p + \alpha_q = \alpha_{p+q}$, $p, q, p+q \in \mathbb{F}_9$) and that all $\alpha_{q^2}$ have the same length, we see that only two possibilities can happen. If $\alpha_1$ is a short root, then we must have the root system isomorphic to the one in Figure 3. If $\alpha_1$ is a long root, then we must have the root system isomorphic to Figure 4. In either case we may calculate (as above for $q = 7$) that in the second case we have

$$\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)} = -1 - \pi(\lambda),$$
$$\frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} = 1 - \pi(\lambda)$$

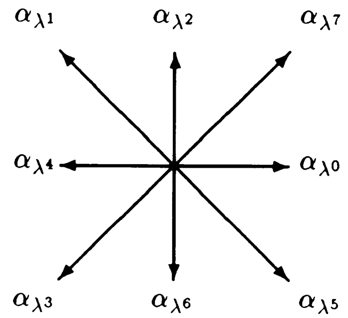


FIGURE 3. $q = 9$ with $\alpha_1$ short

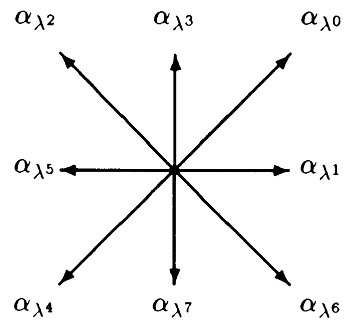


FIGURE 4. $q = 9$ with $\alpha_1$ long

and in the first case

$$\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)} = (1 - \pi(\lambda))^{-1},$$
$$\frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} = (-1 - \pi(\lambda))^{-1}.$$

Moreover, using the notation of [6] and looking at a few character values [16], one may say that the representation $\chi_3$ is associated to Figure 3 and $\chi_2$ to Figure 4.

$\mathbb{F}_{13}$: Here we know that the root system must be $G_2$ and $m$ an exponent of $G_2$. Again by equation (40) or $A$ invariance, we know that all roots $\alpha_{c^2q}$ have the same length. Fix a generator $\lambda = 2$. On the roots of the same length, use a similar argument as with the $A_2$ above. Combining this with the addition being indexed by the field, it is trivial to check that again only two possibilities occur. If $\alpha_1$ happens to be a short root, the roots must be as in Figure 5. If $\alpha_1$ is a long root, then it must be as in Figure 6.

As before, we can calculate that in the first case

$$\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)} = -3\pi(\lambda)/(1 + \pi(\lambda)),$$
$$\frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} = -3\pi(\lambda)^2/(1 + \pi(\lambda)),$$

and in the second case we get

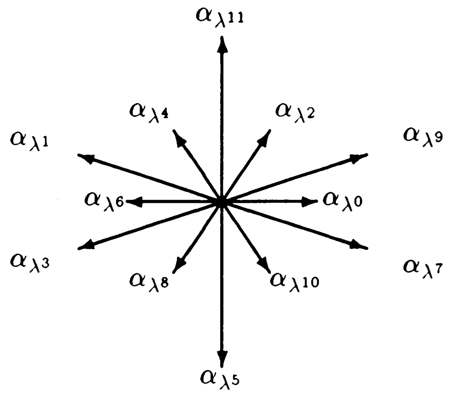


FIGURE 5. $q = 13$ with $\alpha_1$ short

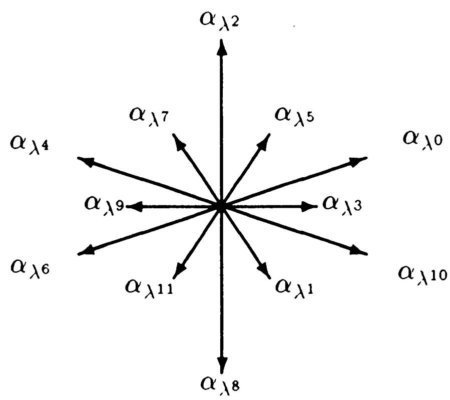


FIGURE 6. $q = 13$ with $\alpha_1$ long

$$\frac{(f_1, f_{-1}, f_0)}{(f_\lambda, f_{-\lambda}, f_0)} = -1/(-1 + \pi(\lambda)^{-2}),$$
$$\frac{(f_1, f_{-1}, f_\infty)}{(f_\lambda, f_{-\lambda}, f_\infty)} = 1/(-1 + \pi(\lambda)^{-2}).$$

Moreover, using the notation of [6] and looking at a few character values [16], one may say that the representation $\chi_2$ is associated to Figure 5 and $\chi_3$ to Figure 6.

*Note* 13.1. As a final observation, we note that the appearance of two possibilities for the root configuration in the above examples is due to our choice of labeling $M_1$ and $M_\lambda$ in $L_2(q)$ or in our choice of $V_{\pi_m}$ and $V_{\pi_m}$. In a sense, they can be interchanged. For more details, see the discussion following Theorem 10.1 and Theorems 3.3.1, 3.3.2, and 3.3.3 in [16].

## Glossary of Notation

| | | | |
|---|---|---|---|
| $d_p$ | 4005 | $\alpha_p$ | 4007, 4012 |
| $e_u$ | 3986 | $\delta_{i,j}$ | 3990 |
| $e_\infty$ | 3986 | $\varepsilon$ | 4000 |
| $f_u$ | 3987 | $\iota$ | 4000 |
| $f_\infty$ | 3987 | $\lambda$ | 3985 |
| $h$ | 3985 | $\pi$ | 3985 |
| $h_p$ | 4005, 4012 | $\pi_m$ | 3985 |
| $h_{\alpha_p}$ | 4014 | $\sigma_\mu$ | 4014 |
| $m$ | 3985 | $\chi$ | 3987 |
| $q$ | 3985 | $\Gamma_{i,j}$ | 3987 |
| $u_{\alpha_p}$ | 4014 | $\Psi$ | 4000 |
| $w_{1,k}$ | 4008 | $\mathscr{A}$ | 3985 |
| $w_{\lambda,k}$ | 4008 | $\mathscr{B}$ | 3985 |
| $x_p$ | 4005, 4012 | $\mathscr{N}$ | 3985 |
| $y_p$ | 4005, 4012 | $\mathfrak{g}_p$ | 4005 |
| $A$ | 3986 | $\mathfrak{h}$ | 4006 |
| $C(W)$ | 4000 | $\mathbb{F}_q$ | 3985 |
| $F_1$ | 4006 | $\mathbb{F}_q^*$ | 3985 |
| $F_\lambda$ | 4006 | $(\mathbb{F}_q^*)^2$ | 3996 |
| $L_2(q)$ | 3985 | $\mathbb{P}^1(\mathbb{F}_q)$ | 3985 |
| $M$ | 3986 | $( \, , \, )$ | 3989 |
| $M_x$ | 3986 | $( \, , \, , \, )_+$ | 3992 |
| $N$ | 3986 | $( \, , \, , \, )_-$ | 3992 |
| $N_x$ | 3986 | $( \, , \, , \, )$ | 3996 |
| $PSL(2,q)$ | 3985 | $( \, , \, , \, , \, )$ | 3999 |
| $PGL(2,q)$ | 3988 | $[ \, , \, ]$ | 3997 |
| $S$ | 3986 | $\langle \, \mid \, \rangle$ | 4014 |
| $V$ | 3985 | $\lvert x \rvert$ | 3995 |
| $V_\pi$ | 3985 | $\sqrt{x}^\delta$ | 3992 |

## References


1. T. Brocker and T. tom Dieck, *Representations of compact Lie groups*, Springer-Verlag, 1985.
2. A. M. Cohen and D. B. Wales, *Finite subgroups of* $G_2(\mathbb{C})$, Comm. Algebra **11** (1983), 441–459.
3. ——, *Finite subgroups of* $E_6(\mathbb{C})$ *and* $F_4(\mathbb{C})$, preprint, 1992.
4. ——, *Finite simple subgroups of semisimple complex Lie groups—a survey*, preprint, 1994.
5. Arjeh M. Cohen, Jr., Robert L. Griess, and Bert Lisser, *The groups* $L(2, 61)$ *embeds in the Lie group of type* $E_8$, Comm. Algebra **21** (1993), 1889–1907.
6. J. H. Conway, R. T. Curtis, et al., *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
7. Larry Dornhoff, *Group representation theory: Part* A, Marcel Dekker, 1971.
8. James E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.
9. Kenneth Ireland and Michael Rosen, *A classical introduction to modern number theory*, Springer-Verlag, 1990.
10. Peter B. Kleidman and A. J. E. Ryba, *Kostant's conjecture holds for* $E_7 : L_2(37) < E_7(\mathbb{C})$, J. Algebra **161** (1993), 535–540.

11. Bertram Kostant, *The principal 3-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.
12. ———, *A tale of two conjugacy classes*, Colloquium Lecture of the Amer. Math. Soc., 1983.
13. A. Meurman, *An embedding of $PSL(2, 13)$ in $G_2(\mathbb{C})$*, Lie Algebras and Related Topics, Lecture Notes in Math., vol. 933, Springer, 1982.
14. M. A. Naimark and A. I. Štern, *Theory of group representations*, Springer-Verlag, 1982.
15. Mohammad Ali Najafi, *Clifford algebra structure on the cohomology algebra of compact symmetric spaces*, Master's thesis, MIT, February, 1979.
16. Mark R. Sepanski, *$L_2(q)$ and the rank two Lie groups: their construction, geometry, and invariants in light of Kostant's Conjecture*, Ph.D. thesis, MIT, May, 1994.

Department of Mathematics, Cornell University, Ithaca, New York 14853
*E-mail address*: `sepanski@math.cornell.edu`