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CYCLIC SULLIVAN–DE RHAM FORMS

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ABSTRACT. For a simplicial set X the Sullivan–de Rham forms are defined to be the simplicial morphisms from X to a simplicial rational commutative graded differential algebra (cgda)∇. However ∇ is a cyclic cgda in a standard way. And so, when X is a cyclic set, one has a cgda of cyclic morphisms from X to ∇. It is shown here that the homology of this cgda is naturally isomorphic to the rational cohomology of the orbit space of the geometric realization $|X|$ with its standard circle action. In addition, a cyclic cgda ∇C is introduced; and it is shown that the homology of the cgda of cyclic morphisms from X to ∇C is naturally isomorphic to the rational equivariant (Borel construction) cohomology of $|X|$.

1. INTRODUCTION

Recall that a simplicial set, X, is a graded set, graded over that natural numbers, X_0, X_1, ..., such that, for each $n \geq 1$, there are boundary maps $d_i : X_n \longrightarrow X_{n-1}$, $0 \leq i \leq n$, and, for each $n \geq 0$, there are degeneracy maps $s_j : X_n \longrightarrow X_{n+1}$, $0 \leq j \leq n$; and there are various relations amongst the d_i's and s_j's. (See [M].) More generally, a simplicial object in a category $\mathscr{C}$, for example, a simplicial group, a simplicial algebra or a simplicial topological space, is defined just as for a simplicial set, except that each X_n is required to be an object of $\mathscr{C}$, and all d_i's and s_j's are required to be morphisms of $\mathscr{C}$. A cyclic set is a simplicial set with some additional structure: for each $n \geq 0$, X_n is acted on by the cyclic group of order $n+1$; and, if t_n denotes the generator of the cyclic group acting on X_n, then there are additional relations amongst the d_i's and the t_n's and amongst the s_j's and the t_n's. (See [L], §6.1.) A cyclic object in a category $\mathscr{C}$ (a cyclic group, a cyclic algebra or a cyclic topological space, for example) is a simplicial object in $\mathscr{C}$ with the additional group actions and relations, with each t_n required to be a morphism of $\mathscr{C}$.

Alternatively, there is a simplicial category Δ, and a simplicial object in a category $\mathscr{C}$ can be viewed as a contravariant functor $\Delta \longrightarrow \mathscr{C}$ (or a covariant functor from the opposite category Δ^{op} to $\mathscr{C}$). Similarly there is a cyclic category ΔC, which has the same objects as Δ and some additional morphisms; and a cyclic object in $\mathscr{C}$ is a contravariant functor $\Delta C \longrightarrow \mathscr{C}$ (or a covariant functor $\Delta C^{op} \longrightarrow \mathscr{C}$). (Again see [L], §6.1.)

If X is a simplicial set, then the commutative graded differential algebra (cgda) of Sullivan–de Rham forms on X, $A^*(X)$, is defined to be the cgda of

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all simplicial maps of X into a simplicial rational cgda ∇^*. (See, e.g., [S], [B, G], [H]. The precise definition will be reviewed below.) The homology of $A^*(X)$ is naturally isomorphic to the rational cohomology of X or its geometric realization $|X|$. Now ∇^* has an obvious structure as a cyclic cgda. Thus, if X is a cyclic set, then one can consider the cgda $A^*_{cy}(X)$ of all cyclic maps of X into ∇^*. One purpose of this paper is to show that the homology of $A^*_{cy}(X)$ is naturally isomorphic to the rational cohomology of $|X|/G$, where $G = S^1$, the circle group, acting on $|X|$ in the usual way (to be reviewed in §2 below). In addition we define another cyclic rational cgda, which we denote ∇C^*, and we define $A^*_G(X)$ to be the cgda of all cyclic maps of X into ∇C^*. The other purpose of this paper is to show that the homology of $A^*_G(X)$ is naturally isomorphic to $H^*_G(|X|; \mathbb{Q})$, the rational equivariant (Borel construction) cohomology of $|X|$.

To define ∇^* precisely one begins with the free rational cgda, E_n, say, generated by indeterminates $t_{n0}, \ldots, t_{nn}$ of degree zero and their differentials $dt_{n0}, \ldots, dt_{nn}$ of degree one. Then $\nabla^*_n := E_n/J_n$ where J_n is the ideal generated by $1 - \sum_{j=0}^{n} t_{nj}$ and $\sum_{j=0}^{n} dt_{nj}$. The vector space of q-forms of simplicial dimension n is denoted ∇^q_n; and the simplicial vector space of q-forms is denoted ∇^q. (See, e.g., [B, G] for details of the simplicial structure. In [B, G] ∇^q_n is denoted $\nabla(n, q)$.)

1.1. **Definitions.** The cyclic operator $t_n : \nabla^*_n \longrightarrow \nabla^*_n$ is induced by the cyclic permutation $(t_{n0}, \ldots, t_{nn}) \mapsto (t_{n1}, \ldots, t_{nn}, t_{n0})$. (Cf. [L], 7.1.3.)

If X is a cyclic set, then let $A^*_{cy}(X) = \mathrm{Mor}_{\Delta C^{op}}(X, \nabla^*)$ and $A^q_{cy}(X) = \mathrm{Mor}_{\Delta C^{op}}(X, \nabla^q)$. Call $A^*_{cy}(X)$, resp. $A^q_{cy}(X)$, the cgda, resp. vector space, of rational cyclic Sullivan–de Rham forms, resp. q–forms, on X.

Now, if X is a cyclic set, and $|X|$ is its geometric realization, then $G = S^1$ acts on $|X|$ in a standard way (see, e.g., [L], 7.1, to be reviewed in §2 below). One purpose of this paper is to prove the following theorem.

1.2. **Theorem.** *Given a cyclic set X there is a natural isomorphism of rational commutative graded algebras*

$$H\left(A^*_{cy}(X)\right) \cong H^*(|X|/G;\ \mathbb{Q}).$$

In §5 below we define the cyclic rational cgda ∇C^*. Then, for a cyclic set X, we define $A^*_G(X) = \mathrm{Mor}_{\Delta C^{op}}(X, \nabla C^*)$. The second result of this paper is the following.

1.3. **Theorem.** *Given a cyclic set X there is a natural isomorphism of rational commutative graded algebras*

$$H(A^*_G(X)) \cong H^*_G(|X|; \mathbb{Q}).$$

Both proofs are basically cyclic versions of the proof in the simplicial case to be found in [B, G], §§14 and 3. They make essential use of some constructions to be found in [B, H, M] and [Sp]: and I would like to thank Jan Spaliński for his very timely visit to Hawaii and for his very helpful paper.

In §2 below we review some basic facts concerning cyclic sets. In §3 we prove the additive part of Theorem 1.2. And in §4 we deal with the multiplicative part. Theorem 1.3 is proven in §5.

2. Review of cyclic sets.

As far as possible we shall follow the notation used in [L]. However we shall frequently write $\Lambda[n]$ instead of $F\Delta[n]$, where F is the left adjoint of the forgetful functor $\Delta C^{op} \longrightarrow \Delta^{op}$ ([L], 7.1.5). And t_n will denote the cyclic operator without sign ([L], 6.1.2).

Now let X be a cyclic set. And let $|X| = \coprod_{n\geq 0} X_n \times \Delta_n/\sim$ be its geometric realization defined, just as in [M], §14, using only the simplicial structure. Let $[x, u] \in |X|$ be the equivalence class of $(x, u) \in X_n \times \Delta_n$, where x is non–degenerate and $u = (u_0, \dots, u_n) \in \Delta_n$ is interior. The canonical circle action on $|X|$ is given by

$$e^{2\pi i v}[x, u] = \left[t_{n+1}^{n+1-j} s_j x, \tau_{n+1}^{j+1}(w_0, \dots, w_{n+1})\right]$$

where $0 \leq v < 1$, τ_{n+1} is the cocyclic operator, *i.e.* $\tau_{n+1}(w_0, \dots, w_{n+1}) = (w_1, \dots, w_{n+1}, w_0)$, and $(w_0, \dots, w_{n+1}) = (u_0, \dots, u_{j-1}, 1-v-u^{j-1}, u^j - (1-v), u_{j+1}, \dots, u_n)$, where $u^j = u_0 + \dots + u_j$, $u^{-1} = 0$ and j is such that $u^{j-1} < 1 - v \leq u^j$. (See [L], 7.1, and [M], proof of Theorem 14.3.)

2.1. **Definition.** For a cyclic set X let

$$X_0^f = \{x \in X_0 ; t_1 s_0 x = s_0 x\}.$$

And let X^f be the cyclic subset of X generated by X_0^f. (For any $y \in X_n^f$, $y = s_0^n x$ for some $x \in X_0^f$, and $t_n y = y$.)

Clearly $|X^f| = |X|^G$. (For a cyclic set X, the fixed point set is always discrete.)

2.2. *Remark.* It is well–known that $|X| \approx \coprod_{n\geq 0} X_n \times \Lambda_n/\sim$ where $\Lambda_n = |\Lambda[n]|$ and, now, the equivalence relation uses all cyclic operators (*i.e.* all operators from ΔC) ([D, H, K], Proposition 2.8). However, Δ_* is also a cocyclic space with τ_n as above. So one may form $|X|_{orb} := \coprod_{n\geq 0} X_n \times \Delta_n/\sim$ using all operators from ΔC. It is easy to see that there is a canonical homeomorphism $|X|_{orb} \approx |X|/G$.

Given a cyclic set X, the group $\mathbb{Z}/(n+1)$ generated by t_n acts on X_n; and so each $x \in X_n$ has an isotropy subgroup equal to a cyclic group K_r of order r for some r dividing $n+1$. The proofs of the following technical lemma and Corollary 2.4 will be given in the appendix.

2.3. **Lemma.** *Let Y, Z be cyclic sets. Let $x \in Y_n$. Suppose that $t_n^q x$ is non–degenerate for all q $(0 \leq q \leq n)$, and that $x \notin Y_0^f$. Suppose that x has isotropy subgroup K_r. Finally suppose that $t_{n+k}^{m_1} s_{i_1} \dots s_{i_k} x = t_{n+k}^{m_2} s_{j_1} \dots s_{j_k} x$ in Y_{n+k} for some $k \geq 0$.*

Then $t_{n+k}^{m_1} s_{i_1} \dots s_{i_k} z = t_{n+k}^{m_2} s_{j_1} \dots s_{j_k} z$ for any $z \in Z_n$ if the isotropy subgroup of z contains K_r.

Recall that if $r|n+1$, then there is a cyclic action of K_r on $\Lambda[n]$. (See [Sp], 3.5. In the notation of [L], 7.1, the action of the generator of K_{n+1} on $\Lambda[n] = F\Delta[n]$ is the map $\Lambda[n] \longrightarrow \Lambda[n]$ corresponding to the point (t_n, ι_n): *i.e.* $(1, \iota_n) \mapsto (t_n, \iota_n)$.) Let $\dot{\Lambda}[n] = F\dot{\Delta}[n]$ be the usual cyclic subset of boundaries. The following corollaries follow from Lemma 2.3.

2.4. **Corollary.** *Let Y be a cyclic set and $X \subseteq Y$ a cyclic subset. Let $x \in Y_n - X_n$. Suppose that $t_n^k x$ is non-degenerate for $0 \le k \le n$, that $x \notin Y_0^f$ and that $d_i x \in X_{n-1}$ for $0 \le i \le n$. Suppose that x has isotropy subgroup K_r.*

Then the following diagram is a push-out.

$$\begin{tikzcd} \dot{\Lambda}[n]/K_r \arrow[r, "\bar{g}"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ \Lambda[n]/K_r \arrow[r, "g"'] & X \cup \langle x \rangle \end{tikzcd}$$

where $X \cup \langle x \rangle$ is the cyclic subset of Y generated by X and x, the vertical maps are the inclusions, and g is induced by $(1, \iota_n) \mapsto x$.

2.5. **Corollary.** *Let X be a cyclic set. Let $X(n)$ be the n-skeleton of X, i.e. the cyclic subset generated by $\bigcup_{j=0}^n X_j$. Then X is the direct limit of the sequence $X(-1) := X^f \subseteq X(0) \subseteq X(1) \subseteq \cdots \subseteq X(n-1) \subseteq X(n) \ldots$, and each $X(n-1) \subseteq X(n)$, for $n \ge 0$, is a push-out*

$$\begin{tikzcd} \coprod_{\alpha \in A_n} \dot{\Lambda}[n]/K_\alpha \arrow[r] \arrow[d] & X(n-1) \arrow[d] \\ \coprod_{\alpha \in A_n} \Lambda[n]/K_\alpha \arrow[r] & X(n) \end{tikzcd}$$

where A_n is the set of orbits of simplicies $x \in X_n$ such that $t_n^k x$ is non-degenerate for $0 \le k \le n$, and K_α is the isotropy subgroup of the orbit α. $\left(A_0 = X_0 - X_0^f.\right)$

Since geometric realization is a left adjoint, and so commutes with colimits, one also gets the following.

2.6. **Corollary.** *If X is a cyclic set, then $|X|$ is a G–CW–complex (where $G = S^1$).*

The next lemma is also useful.

2.7. **Lemma.** *Let Z be an acyclic cyclic rational vector space. (I.e., Z is a cyclic rational vector space, and $Z \longrightarrow 0$ is a homotopy equivalence of simplicial abelian groups.) Then the dotted arrow exists in any commutative diagram of the form*

$$\begin{tikzcd} \dot{\Lambda}[n]/K_r \arrow[r] \arrow[d] & Z \arrow[d] \\ \Lambda[n]/K_r \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd} \qquad (r \text{ divides } n+1).$$

Proof. By [D, H, K], λ exists in the diagram

$$\begin{tikzcd} \dot{\Lambda}[n] \arrow[r] \arrow[d] & \dot{\Lambda}[n]/K_r \arrow[r] \arrow[d] & Z \arrow[d] \\ \Lambda[n] \arrow[r] \arrow[urr, dashed, "\lambda"] & \Lambda[n]/K_r \arrow[r] & 0 \end{tikzcd}$$

since the vertical map on the right is an acyclic fibration and the vertical map on the left is a cofibration.

Suppose that $\lambda(1, \iota_n) = \omega$. (Here, as above, $\iota_n \in \Delta[n]$ is the generator, and we are thinking of $\Lambda[n]$ as $F\Delta[n]$ as in [L], 7.1.) Now let $\theta = \frac{1}{r}\sum_{j=0}^{r-1} t_n^{js}\omega$, where $rs = n+1$. Define $\mu : \Lambda[n] \longrightarrow Z$ by $\mu(1, \iota_n) = \theta$. A straightforward check shows that $d_i\theta = d_i\omega$ for $0 \leq i \leq n$. So μ also makes the above diagram commute. And μ factors through $\Lambda[n]/K_r$. □

3. The additive part of Theorem 1.2.

Here we verify that a cyclic version of [B, G], §14 is valid. First, however, recall the Connes cochain complex $S_\lambda^*(X)$ of a cyclic set X with rational coefficients. A cyclic cochain $\varphi \in S_\lambda^n(X)$ is an ordinary cochain $\varphi : X_n \longrightarrow \mathbb{Q}$ such that $\varphi(t_n x) = (-1)^n\varphi(x)$ for all $x \in X_n$ ([L], 2.5.9). Then (see [J] or, e.g., [L], 7.2.3)

$$H(S_\lambda^*(X)) \cong H_G^*(|X|; \mathbb{Q}),$$

the equivariant (Borel construction) cohomology.

Recall, too, the map $\rho : A^*(X) \longrightarrow S^*(X)$ from the Sullivan–de Rham cgda of a simplicial set X to the rational cochain complex of X ([B, G], p. 7). For $\varphi \in A^n(X)$ and $x \in X_n$, $\rho(\varphi)(x) = \int \varphi(x)$, where the integration is over $\Delta_n' := \{(v_1, \dots, v_n) \in \mathbb{R}^n ; \sum_{i=1}^n v_i \leq 1 \text{ and } v_i \geq 0 \text{ for } 1 \leq i \leq n\}$.

3.1. **Lemma.** *For any cyclic set X, the restriction of ρ to $A_{cy}^*(X)$ maps into $S_\lambda^*(X)$: i.e. one has*

$$\rho : A_{cy}^*(X) \longrightarrow S_\lambda^*(X).$$

Proof. We must show that $(-1)^n\rho(\varphi)(x) = \rho(\varphi)(t_n x)$ for $\varphi \in A_{cy}^n(X)$ and $x \in X_n$. Let $\varphi(x) = f(t_{n1}, \dots, t_{nn})dt_{n1}\dots dt_{nn}$. Then

$$\varphi(t_n x) = t_n\varphi(x) = f(t_{n2}, \dots, t_{nn}, 1-t)dt_{n2}\dots dt_{nn}d(1-t),$$

where $t = \sum_{i=1}^n t_{ni}$. So $\varphi(t_n x) = (-1)^n f(t_{n2}, \dots, t_{nn}, 1-t)dt_{n1}\dots dt_{nn}$. Now the change of variable $v_1 = t_{n2}, \dots, v_{n-1} = t_{nn}$, $v_n = 1-t$ shows that $\int\varphi(t_n x) = (-1)^n\int\varphi(x)$. □

3.2. *Notation.* For a cyclic set X let

$$\widetilde{A}_{cy}^*(X) = A_{cy}^*(X, X^f) = \ker[A_{cy}^*(X) \longrightarrow A_{cy}^*(X^f)]$$

and

$$\widetilde{S}_\lambda^*(X) = S_\lambda^*(X, X^f) = \ker[S_\lambda^*(X) \longrightarrow S_\lambda^*(X^f)].$$

Note that ρ induces $\widetilde{\rho} : \widetilde{A}^*_{cy}(X) \longrightarrow \widetilde{S}^*_\lambda(X)$. And, by [J], $H\widetilde{S}^*_\lambda(X)$ is naturally isomorphic to $H^*_G(|X|, |X|^G; \mathbb{Q})$.

If Y is a cyclic set and $X \subseteq Y$ is a cyclic subset, then Y is the direct limit of the sequence

$$X \subseteq X \cup Y^f \subseteq X \cup Y(0) \subseteq \cdots \subseteq X \cup Y(n-1) \subseteq X \cup Y(n) \subseteq \dots ,$$

where $Y(n)$ is the cyclic n–skeleton of Y (as in Corollary 2.5). Thus the next lemma follows easily from Corollary 2.5 and Lemma 2.7.

3.3. **Lemma.** *Let Y be a cyclic set and $X \subseteq Y$ a cyclic subset. Then the restriction homomorphism $A^*_{cy}(Y) \longrightarrow A^*_{cy}(X)$ is surjective.*

Indeed, since the extension to Y^f can be arbitrary, the restriction homomorphism $\widetilde{A}^*_{cy}(Y) \longrightarrow \widetilde{A}^*_{cy}(X)$ is surjective also.

The next lemma is an easy variant of the corresponding simplicial result (see, e.g., [B, G], p. 82).

3.4. **Lemma.** *Let $X : J \longrightarrow$ cyclic sets be a functor from a small category $J = \{j\}$ to the category of cyclic sets. Suppose that the map $\varinjlim X(j)^f \longrightarrow (\varinjlim X(j))^f$ is surjective. Then $\widetilde{A}^*_{cy}(\varinjlim X(j)) \cong \varprojlim \widetilde{A}^*_{cy}(X(j))$. (Here $\varinjlim$ is colimit and $\varprojlim$ is limit.) Similarly $\widetilde{S}^*_\lambda(\varinjlim X(j)) \cong \varprojlim \widetilde{S}^*_\lambda(X(j))$.*

It is also easy to verify the following.

3.5. **Lemma.** *Let*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

be a push–out of cyclic sets.

Let $\overline{W}$ be the push–out obtained by replacing X, Y and Z by X^f, Y^f and Z^f respectively. Let $h : \overline{W} \longrightarrow W^f$ be the standard map.

If g is injective, then h is surjective. If, in addition, $Z^f = Y^f = \varnothing$, then h is bijective.

3.6. **Lemma.** *For any $n \geq 0$ and r dividing $n+1$, the map*

$$\widetilde{\rho} : \widetilde{A}^*_{cy}(\Lambda[n]/K_r) \longrightarrow \widetilde{S}^*_\lambda(\Lambda[n]/K_r)$$

induces an isomorphism in homology.

Proof. It is easy to see that $(\Lambda[n]/K_r)^f = \varnothing$. So we are concerned with $\rho : A^*_{cy}(\Lambda[n]/K_r) \longrightarrow S^*_\lambda(\Lambda[n]/K_r)$.

Now with the notation and results of [Sp] we have the following (where $rs = n+1$).

$$\begin{aligned} A^*_{cy}(\Lambda[n]/K_r) &= \mathrm{Mor}_{\Delta C^{op}}(\Psi_r\Delta[s-1], \nabla^*) \\ &= \mathrm{Mor}_{\Delta^{op}}(\Delta[s-1], \Phi_r(\nabla^*)) \\ &\cong \Phi_r(\nabla^*)_{s-1} = (\nabla^*_n)^{K_r}. \end{aligned}$$

Since $H(\nabla_n^*) = \mathbb{Q}$, $H\left((\nabla_n^*)^{K_r}\right) = \mathbb{Q}$ by averaging.

On the other hand,

$$\begin{aligned} H\left(S_\lambda^*\left(\Lambda[n]/K_r\right)\right) &\cong H_G^*\left(\left|\Lambda[n]/K_r\right|; \mathbb{Q}\right) \\ &\cong H_G^*\left(\left|\Lambda[n]\right|/K_r; \mathbb{Q}\right) \\ &\cong H_G^*\left(\left|\Lambda[n]\right|; \mathbb{Q}\right) \text{ since } K_r \text{ is finite} \\ &\cong \mathbb{Q}. \quad \square \end{aligned}$$

We are now in a position to prove the following, which essentially gives the additive part of Theorem 1.2. (See proof of 4.8.)

3.7. **Proposition.** *Let X be a cyclic set. Then $\widetilde{\rho} : \widetilde{A}_{cy}^*(X) \longrightarrow \widetilde{S}_\lambda^*(X)$ induces an isomorphism $\widetilde{\rho}^* : H\left(A_{cy}^*\left(X, X^f\right)\right) \longrightarrow H\left(S_\lambda^*\left(X, X^f\right)\right)$.*

Proof. We can now mimic the proof in [B, G], 14.5.

Step 1 holds for the standard cyclic sets $\Lambda[n]/K_r$ by Lemma 3.6. Step 2 holds by Step 1 and Lemma 3.4. Step 3 follows from Step 2, induction, Corollary 2.5, Lemma 3.4 and, in order to get the required version of [B, G], Lemma 14.1, Lemma 3.3. Finally Step 4 follows from Step 3, Lemma 3.4 and Lemma 3.3, which permits the required version of [B, G], Lemma 14.4. $\square$

4. The multiplicative part of Theorem 1.2.

Unfortunately, although the methods of [B, G], §14, seem to work best for the additive part, acyclic model arguments seem to be needed for the multiplicative part. We shall follow as closely as possible the notation of [B, G], §2.

4.1. **Definitions.** Let K be a contravariant functor from the category of cyclic sets to the category of R–modules, where R is a commutative ring with identity. For a cyclic set X let $\widehat{K}(X) = \prod_{n\geq 0} \widetilde{\prod_{x\in X_n}} \{K(\Lambda[n]), x\}$, where $\widetilde{\prod}$ indicates the submodule of the product consisting of elements $\{m_x, x\}$ where (for $x \in X_n$) $m_{t_n x} = \gamma_n^* m_x$, $\gamma_n : \Lambda[n] \longrightarrow \Lambda[n]$ is the cyclic map induced by $\gamma_n(1, \iota_n) = (t_n, \iota_n)$, and $\gamma_n^* = K(\gamma_n)$.

Define $\Phi : K \longrightarrow \widehat{K}$ by $\Phi(X)(u) = \{K(\widetilde{x})(u), x\}$, for a cyclic set X and $u \in K(X)$, where $\widetilde{x} : \Lambda[n] \longrightarrow X$ is the standard map corresponding to $x \in X_n$ (*i.e.* $\widetilde{x}(1, \iota_n) = x$). It follows that $\Phi(X)$ maps $K(X)$ to $\widehat{K}(X)$ since $\widetilde{t_n x} = \widetilde{x}\gamma_n$.

The functor K is said to be corepresentable (with respect to the models $\Lambda[n]$) if there is a natural transformation $\Psi : \widehat{K} \longrightarrow K$ such that $\Psi\Phi = 1$.

4.2. **Lemma.** *The functor S_λ^* is corepresentable (w.r.t. the models $\Lambda[n]$).*

Proof. Given a cyclic set X, $\{m_x, x\} \in \widehat{S_\lambda^n}(X)$ and $y \in X_n$, let

$$\Psi(X)(\{m_x, x\})(y) = m_y(1, \iota_n).$$

Since $m_{t_n y} = \gamma_n^* m_y$, it follows that $\Psi(X)(\{m_x, x\})$ is a cyclic cochain. And it is immediate that $\Psi\Phi = 1$. $\square$

4.3. **Lemma.** *The functors* A^*_{cy}, S^*_λ, $A^*_{cy}\otimes A^*_{cy}$ *and* $S^*_\lambda\otimes S^*_\lambda$ *are acyclic with respect to the models* $\Lambda[n]$ *in the sense of* [B, G], *p.* 9.

Proof. This follows exactly as in [B, G] since $A^*_{cy}(\Lambda[n]) \cong A^*(\Delta[n])$ and $S^*_\lambda(\Lambda[n]) \cong S^*(\Delta[n])$. □

Suppose that K^* is a functor (such as those of Lemma 4.3) which is acyclic with respect to the models $\Lambda[n]$ in the sense of [B, G], p. 9. So, for fixed n and any p, we have a homotopy $h: K^p(\Lambda[n]) \longrightarrow K^{p-1}(\Lambda[n])$ such that $hD+Dh = 1-\eta\epsilon$, where D is the differential in $K^*(\Lambda[n])$, and $\eta:\mathbb{Q}\longrightarrow K^\circ(\Lambda[n])$ and $\epsilon: K^*(\Lambda[n])\longrightarrow\mathbb{Q}$ are the unit and augmentation (so that $\epsilon\eta=1$). In order to apply the obvious cyclic analogues of [B, G], Lemma 2.3 and Proposition 2.4, it is necessary that $h\gamma^*_n = \gamma^*_n h$. But this is easily done by starting with any h and averaging to obtain $\widetilde{h} := \frac{1}{n+1}\sum_{j=0}^{n}(\gamma^*_n)^j h(\gamma^*_n)^{-j}$. One must also average ϵ by putting $\widetilde{\epsilon} = \frac{1}{n+1}\sum_{j=0}^{n}\epsilon\,(\gamma^*_n)^{-j}$. One then has $\widetilde{h}D + D\widetilde{h} = 1-\eta\widetilde{\epsilon}$, provided that the image of η is in the fixed part of $K^\circ(\Lambda[n])$: and that is the case for all functors considered here.

Thus one has available the cyclic analogues of [B, G], Lemma 2.3 and Proposition 2.4. Hence one has the following.

4.4. **Corollary.** *There is a natural chain map*

$$\mu_\lambda : S^*_\lambda\otimes S^*_\lambda \longrightarrow S^*_\lambda .$$

which is homotopy associative, homotopy commutative, has a homotopy unit and is unique up to natural chain homotopy.

By naturality μ_λ induces $\widetilde{\mu}_\lambda : \widetilde{S}^*_\lambda\otimes S^*_\lambda \longrightarrow \widetilde{S}^*_\lambda$ and $\widetilde{\widetilde{\mu}}_\lambda : \widetilde{S}^*_\lambda\otimes\widetilde{S}^*_\lambda\longrightarrow\widetilde{S}^*_\lambda$. Let $\mu_A : A^*_{cy}\otimes A^*_{cy}\longrightarrow A^*_{cy}$ be the usual multiplication of forms; and let $\widetilde{\mu}_A$ and $\widetilde{\widetilde{\mu}}_A$ be the corresponding restrictions. Then [B, G], Lemma 2.3 and Proposition 2.4 also give the following.

4.5. **Proposition.** *There are natural chain homotopies* $\rho\mu_A\simeq\mu_\lambda(\rho\otimes\rho)$, $\widetilde{\rho}\widetilde{\mu}_A\simeq\widetilde{\mu}_\lambda(\widetilde{\rho}\otimes\rho)$ *and* $\widetilde{\rho}\widetilde{\widetilde{\mu}}_A\simeq\widetilde{\widetilde{\mu}}_\lambda(\widetilde{\rho}\otimes\widetilde{\rho})$. *Furthermore, if* $\mu: S^*\otimes S^*\longrightarrow S^*$ *is the standard cup product* (*as in* [M], §30) *and if* $i: S^*_\lambda\longrightarrow S^*$ *is the inclusion, then there is a natural chain homotopy* $i\mu_\lambda\simeq\mu(i\otimes i)$.

Proof. The first natural chain homotopy follows from [B, G], Lemma 2.3 and Proposition 2.4. (See also [B, G], Proposition 3.3.) The second and third follow from the first by naturality. The last follows since S^* is corepresentable with respect to the models $\Lambda[n]$ if in the definition of $\widehat{S}^*$ one uses the product $\prod$ instead of the limit $\widetilde{\prod}$. (And one is viewing S^* as a functor on cyclic sets not simplicial sets.) □

4.6. **Corollary.** *For any cyclic set* X, ρ *induces* $\mathbb{Q}$*–algebra homomorphisms* $\rho^*: H(A^*_{cy}(X))\longrightarrow H(S^*_\lambda(X))$ *and* $\widetilde{\rho}^*: H(\widetilde{A}^*_{cy}(X))\longrightarrow H(\widetilde{S}^*_\lambda(X))$, *the second being an isomorphism.*

Remark. The functors $\widetilde{S}^*_\lambda$, A^*_{cy} and $\widetilde{A}^*_{cy}$ cannot be corepresentable (w.r.t. the models $\Lambda[n]$). Otherwise one would get chain equivalences which are clearly impossible. The fact that A^*_{cy} is not corepresentable w.r.t. $\Lambda[n]$ whereas A^*

is corepresentable w.r.t. $\Delta[n]$ ([B, G], Proposition 2.5) corresponds to the facts that $\mathbb{Q} \longrightarrow \nabla^0 \longrightarrow \nabla^1 \longrightarrow \cdots$ is an injective resolution of $\mathbb{Q}$ in the category of simplicial rational vector spaces but not in the category of cyclic rational vector spaces. The former fact gives a quick proof that, for any simplicial set X, $H(A^*(X)) \cong H^*(X; \mathbb{Q})$. (See, e.g., [L], 6.2.)

We now restate and prove Theorem 1.2.

4.8. **Theorem.** *For cyclic sets X there is a natural isomorphism of $\mathbb{Q}$–algebras*

$$H\left(A^*_{cy}(X)\right) \xrightarrow{\sim} H^*(|X|/G; \mathbb{Q}).$$

(*Here, as usual,* $G = S^1$.)

Proof. From [J] one has natural isomorphisms $H(S^*_\lambda(X)) \cong H^*_G(|X|; \mathbb{Q})$ and $H\left(\widetilde{S}^*_\lambda(X)\right) \cong H^*_G\left(|X|, |X|^G; \mathbb{Q}\right)$. In addition there is a natural isomorphism $H^*_G\left(|X|, |X|^G; \mathbb{Q}\right) \longrightarrow H^*\left(|X|/G, |X|^G; \mathbb{Q}\right)$. (See, e.g., [A, P], Proposition (3.10.9), and Corollary 2.6 above.)

Thus, for $n \geq 2$, one has the sequence of isomorphisms $H^n\left(A^*_{cy}(X)\right) \longrightarrow H^n(\widetilde{A}^*_{cy}(X)) \longrightarrow H^n(\widetilde{S}^*_\lambda(X)) \longrightarrow H^n_G(|X|, |X|^G; \mathbb{Q}) \longrightarrow H^n(|X|/G, |X|^G; \mathbb{Q}) \longrightarrow H^n(|X|/G; \mathbb{Q})$, using the fact that $|X|^G$ is discrete.

The cases where $n = 0$ or 1 are straightforward. The multiplicativity also follows easily thanks to Proposition 4.5. □

5. Proof of Theorem 1.3.

In this section we define ∇C^* and A^*_G, and prove Theorem 1.3.

5.1. **Definitions.** (1) Let R_n be the rational cgda $\mathbb{Q}[u_n] \otimes \Lambda(v_n)$, where $\deg(v_n) = 1$, $\deg(u_n) = 2$ and $dv_n = u_n$.

(2) Let R be the simplicial cgda which is R_n in simplicial dimension n, and in which the simplicial operators are defined by requiring that $u_n = s_0^n u_0$ and $v_n = s_0^n v_0$, the n–fold degeneracies.

(3) The cyclic rational cgda ∇C^* is defined as follows. Let $\nabla C^*_n = R_n \otimes \nabla^*_n$. The simplicial operators are the tensors of those on R_n with those on ∇^*_n. (E.g., for $\varphi \in \nabla^*_n$ and $0 \leq i \leq n$, $d_i(u_n \otimes \varphi) = u_{n-1} \otimes d_i\varphi$.) The cyclic group operators are given as before on ∇^* (Definitions 1.1) and by requiring that $t_n u_n = u_n$ and $t_n v_n = v_n - dt_{n0}$, for all $n \geq 0$.

It is easy to check that ∇C^* is a cyclic rational cgda. (E.g., $s_0 t_n v_n = v_{n+1} - dt_{n+1\,0} - dt_{n+1\,1} = t^2_{n+1} v_{n+1} = t^2_{n+1} s_n v_n$.)

(4) If X is a cyclic set, then let

$$A^*_G(X) = \mathrm{Mor}_{\Delta C^{op}}(X, \nabla C^*).$$

It is also easy to check the following.

5.2. **Lemma.** *For each degree q, the cyclic rational vector space ∇C^q is acyclic in the sense of Lemma* 2.7.

Hence (cf. Lemma 3.3) one has the next corollary.

5.3. **Corollary.** *If Y is a cyclic set and $X \subseteq Y$ is a cyclic subset, then the restriction homomorphism $A^*_G(Y) \longrightarrow A^*_G(X)$ is surjective.*

Since $\nabla C^\circ = \mathbb{Q} \otimes \nabla^\circ \cong \nabla^\circ$, one has that $A^\circ_G = A^\circ_{cy}$. Thus one has $\eta : \mathbb{Q} \longrightarrow A^\circ_G(\Lambda[n])$ as before. (See Lemma 4.3 and the comments below it.)

5.4. **Lemma.** *The functors* A_G^* *and* $A_G^* \otimes A_G^*$ *are acyclic with respect to the models* $\Lambda[n]$.

Proof. One has that $A_G^*(\Lambda[n]) = \nabla C_n^* = R_n \otimes \nabla_n$. One begins by defining h on R_n by $h(1) = 0$, $h(u_n^j) = v_n u_n^{j-1}$ for $j \geq 1$ and $h(v_n u_n^i) = 0$. And h is defined on ∇_n in the usual way ([B, G], Proposition 1.3). One defines ϵ on R_n by $\epsilon(v_n) = 0$, $\epsilon(u_n) = 0$. Then h is defined on $R_n \otimes \nabla_n$ in the standard way (*i.e.*, $h(x \otimes y) = \frac{1}{2}\{h(x) \otimes (y + \eta\epsilon(y)) + (-1)^m(x + \eta\epsilon(x)) \otimes h(y)\}$, where $\deg(x) = m$). Then one averages as in the comments below Lemma 4.3. □

Everything is now in place to apply the cyclic analogues of [B, G], Lemma 2.3 and Proposition 2.4, which give the following.

5.5. **Proposition.** *There is a natural chain map* $\rho_G : A_G^* \longrightarrow S_\lambda^*$ *such that*

(1) $\rho_G \eta = \eta : \mathbb{Q} \longrightarrow S_\lambda^\circ$;
(2) *in degree* 0, $\rho_G = \rho : A_G^\circ = A_{cy}^\circ \longrightarrow S_\lambda^\circ$ *(see Lemma* 3.1);
(3) *there is a natural chain homotopy* $\rho_G i_{\nabla^*} \simeq \rho : A_{cy}^* \longrightarrow S_\lambda^*$, *where* $i_{\nabla^*} : A_{cy}^* \longrightarrow A_G^*$ *is induced by the obvious inclusion* $i_\nabla : \nabla^* \longrightarrow \nabla C^*$ *of cyclic cgdas; and*
(4) *there is a natural chain homotopy*

$$\rho_G \mu_G \simeq \mu_\lambda(\rho_G \otimes \rho_G) : A_G^* \otimes A_G^* \longrightarrow S_\lambda^* ,$$

where μ_G *is the multiplication on* A_G^* *and* μ_λ *is a multiplication on* S_λ^* *given by Corollary* 4.4.

Finally we are ready to restate and prove Theorem 1.3.

5.6. **Theorem.** *For cyclic sets* X, ρ_G *of Proposition* 5.5 *induces a natural isomorphism of rational commutative graded algebras*

$$\rho_G^* : H(A_G^*(X)) \xrightarrow{\simeq} H_G^*(|X|; \mathbb{Q}) .$$

Proof. As in the proof of Proposition 3.7, we mimic the proof in [B, G], 14.5. The multiplicative part of the theorem follows from Proposition 5.5(4).

Step 1 of [B, G] follows for the cyclic sets $\Lambda[n]/K_r$, because, arguing as in Lemma 3.6, one has that

$$A_G^*(\Lambda[n]/K_r) \cong (\nabla C_n^*)^{K_r} ;$$

and, as before, the homology of the latter is $\mathbb{Q}$ concentrated in degree 0. There is, however, the crucial question of what happens on the trivial cyclic set $\Delta[0]$. We shall postpone this to last.

The remaining steps of [B, G], 14.5 follow just as in the proof of Proposition 3.7, but using Corollary 5.3 instead of Lemma 3.3.

So, returning to $\Delta[0]$, one has that

$$A_G^*(\Delta[0]) \cong (\nabla C^*)_0^f = \mathbb{Q}[u_0] ,$$

the polynomial ring. And $S_\lambda^*(\Delta[0]) = \mathbb{Q}[w]$, the polynomial ring on $w \in S_\lambda^2(\Delta[0])$ defined by $w(s_0^2(0)) = 1$. The map $\rho_G : A_G^*([0]) \longrightarrow S_\lambda^*(\Delta[0])$ is uniquely determined. In order to show that it is an isomorphism, because of the multiplicative structure, it is enough to show that $\rho_G(u_0) \neq 0$. Calculating (*i.e.*, going through the details of [B, G], Lemma 2.3 and Proposition 2.4, and not forgetting to average the homotopies) one finds that $\rho_G(u_0) = -\frac{1}{2}w$. □

6. Appendix

Proof of Lemma 2.3. The case where $k = 0$ is clear; and so we proceed by induction, assuming that $k > 0$ and that the result is proven up to $k-1$. Without loss of generality we can write the relation in the form

$$s_{i_1} \dots s_{i_k} x = t^m_{n+k} s_{j_1} \dots s_{j_k} x$$

where $i_1 > \dots > i_k$.

Now in ΔC^{op} one has that (for $0 \le m \le n+k$)

$$t^m_{n+k} s_j = \begin{cases} s_{j+m} t^m_{n+k-1} & \text{if } j+m < n+k, \\ s_{j+m-(n+k+1)} t^{m-1}_{n+k-1} & \text{if } j+m > n+k, \\ t_{n+k} s_{n+k-1} t^{m-1}_{n+k-1} & \text{if } j+m = n+k. \end{cases}$$

Case 1. $j_1 + m \neq n+k$. So we have a relation of the form

$$s_{i_1} \dots s_{i_k} x = s_{j'} t^{m'}_{n+k-1} s_{j_2} \dots s_{j_k} x.$$

If $j' > i_1$, applying $d_{j'+1}$ gives $s_{i_1} \dots s_{i_k} d_{j'+1-k} x = t^{m'}_{n+k-1} s_{j_2} \dots s_{j_k} x$. Whence $x = d_{j_k} \dots d_{j_2} t^{-m'}_{n+k-1} s_{i_1} \dots s_{i_k} d_{j'+1-k} x$. Now writing the operator on the right in standard form (TSD, where T is a cyclic group operator, S is a sequence of degeneracies and D is a sequence of boundaries) shows that some $t^s_n x$ is degenerate—a contradiction.

If $j' \le i_1$, first suppose that no index $i_t = j'$ or $j'-1$. Then applying $d_{j'}$ gives a contradiction as before. If some $i_t = j'$, then applying $d_{j'}$, the inductive assumption and $s_{j'}$ gives the result. If no $i_t = j'$ but some $i_t = j'-1$, then applying $d_{j'}$, the inductive assumption and $s_{j'}$ gives $s_{i_1} \dots s_{j'} \dots s_{i_k} z = t^m_{n+k} s_{j_1} \dots s_{j_k} z$ and $s_{i_1} \dots s_{j'} \dots s_{i_k} x = s_{i_1} \dots s_{j'-1} \dots s_{i_k} x$. Applying d_{i_1}, the inductive assumption and s_{i_1} to the latter gives the result.

Case 2. $j_1 + m = n+k$. First suppose that $k \ge 2$. If $j_1 \le j_2$, then $s_{j_1} s_{j_2} = s_{j_2+1} s_{j_1}$, and we are back in Case 1. If $j_1 > j_2$, then $s_{j_1} s_{j_2} = s_{j_2} s_{j_1 - 1}$, and again we are back in Case 1.

Thus we are left with the case where $k = 1$ and $j_1 + m = n+1$. The relation is $s_{i_1} x = t^m_{n+1} s_{j_1} x = t_{n+1} s_n t^{m-1}_n x$. Applying d_0 gives a contradiction of the non–degeneracy unless $i_1 = 0$.

So now we are left with $s_0 x = t_{n+1} s_n t^{m-1}_n x$. Applying d_0 gives $x = t^{m-1}_n x$; and so $s_0 x = t_{n+1} s_n x$. Now applying d_1 gives a contradiction of the non–degeneracy unless $n = 0$. But, if $n = 0$, we have $s_0 x = t_1 s_0 x$, and hence $x \in Y^f_0$—a contradiction. □

Proof of Corollary 2.4. Suppose that

$$\begin{array}{ccc} \dot\Lambda[n]/K_r & \xrightarrow{\bar g} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \Lambda[n]/K_r & \xrightarrow[h]{} & Z \end{array}$$

is a commutative diagram of cyclic sets. Define $\varphi : X \cup \langle x \rangle \longrightarrow Z$ by $\varphi | X = f$ and $\varphi(x) = h([1, \iota_n])$. We must check that φ is extendable as a map of cyclic sets. (Clearly φ is unique.)

Let TSD be a cyclic operator, where T is a cyclic group operator, S is a sequence of degeneracies and D is a sequence of boundaries. If D is non-trivial, then $TSDx \in X$; and so $\varphi(TSDx) = f(TSDx) = TSf(Dx) = TSf\overline{g}(D([1, \iota_n])) = TShiD([1, \iota_n]) = TSDh([1, \iota_n]) = TSD\varphi(x)$.

If D is trivial, and $TSx = T'S'D'x$, then D' is trivial since TS has a left inverse and $x \notin X$. So $TSx = T'S'x$. Hence, by Lemma 2.3, $TS\varphi(x) = T'S'\varphi(x)$. Thus φ extends as a map of cyclic sets. □

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