

NEGATIVE FLOWS OF THE POTENTIAL KP-HIERARCHY

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ABSTRACT. We construct a Grassmannian-like formulation for the potential KP-hierarchy including additional “negative” flows. Our approach will generalize the notion of a τ -function to include negative flows. We compare the resulting hierarchy with results by Hirota, Satsuma and Bogoyavlenskii.

1. INTRODUCTION

It is a well known fact, that the modified KdV-hierarchy allows an extension to a larger set of commuting differential equations. These are in fact the members of the sine-Gordon hierarchy, as stated for example in [DS]. The additional commuting flows are still connected with the action of an infinite dimensional abelian group on an infinite dimensional manifold as it is used in the classical works of Adler and van Moerbeke [AvM] and Segal and Wilson [SW]. In the latter work, a part of a commutative group Γ of multiplication operators acts nontrivially on the quotient $\text{Gl}_{\text{res}}/\text{Gl}_{\text{res}+}$ of two infinite dimensional groups by left multiplication. The resulting quotient can be interpreted as an infinite dimensional Grassmannian manifold. From this viewpoint Sato and later Segal and Wilson derived exact analytic results for a certain class of solutions.

In this paper we investigate the flows of the potential KP-equation in this setting, giving a formulation in terms of τ -functions and group actions on an infinite dimensional group. The extension of the hierarchy is obtained by changing the investigated action of Γ on Gl_{res} to conjugation, which extends the group of flows acting nontrivially on a certain splitting subgroup of Gl_{res} .

Due to the natural \mathbb{Z} -grading in the investigated hierarchy, we call the additional flows “negative” flows.

The outline of this work follows the paper [DNS], where the case of the potential KP-equation is investigated:

In section 2 we take as configuration space of the extended integrable system the whole group Gl_{res} [SW], which replaces the Grassmannian. This group is split into three subgroups represented as lower triangular, diagonal and upper triangular block matrices.

Letting the abelian group Γ of Segal and Wilson act by conjugation we obtain in section 3 a nontrivial action of the *whole* group Γ , not only its positive part w.r.t. the splitting defined in section 2

The next two sections are strictly analogous to [DNS] extending it to the case of negative flows.

Received by the editors June 22, 1994 and, in revised form, March 8, 1995.

1991 *Mathematics Subject Classification*. Primary 35Q53; Secondary 58F07.

Supported by KITCS grant OSR-9255223.

In section 4 we obtain differential equations relating the matrix elements of the block matrices, thereby reducing the configuration space of the extended hierarchy of integrable equations considerably.

With a suitably defined τ -function (section 5) we derive in section 6 from this action equations for scalar functions. These equations extend the potential KP-hierarchy.

We also give some thought to the set of solutions, which in section 7 is shown to be much more complicated than in the potential KP case, which was investigated in [Do2].

In the last chapter we compute one of these flows in the loop group reduction of the group $\mathrm{Gl}_{\mathrm{res}}$ [PS]. This gives an equation of Hirota and Satsuma, which commutes with the potential KdV-equation. This equation was also investigated by Bogoyavlenskii.

2. THE CONFIGURATION SPACE

We start out by recalling the setting of Segal and Wilson [SW]. As in their paper the flows considered will be defined in an infinite dimensional Lie group $\mathrm{Gl}_{\mathrm{res}}$.

Let H be an infinite dimensional separable complex Hilbert space together with a splitting

$$(2.1) \quad H = H_+ \oplus H_-$$

into the orthogonal direct sum of two infinite dimensional closed subspaces H_+ and H_- .

As in [SW] we use the following definition.

Definition 2.1. Let $\mathrm{Gl}_{\mathrm{res}}$ denote the set of invertible operators on the Hilbert space H which take w.r.t. the splitting (2.1) the form

$$(2.2) \quad \begin{pmatrix} a & r \\ s & d \end{pmatrix}$$

where a and d are Fredholm operators, and r and s are compact.

We also define the following subgroups of $\mathrm{Gl}_{\mathrm{res}}$;

$$(2.3) \quad \mathrm{Gl}_+ = \left\{ g \in \mathrm{Gl}_{\mathrm{res}} \mid g = \begin{pmatrix} \mathbf{1} & b \\ 0 & \mathbf{1} \end{pmatrix} \right\},$$

$b : H_- \rightarrow H_+$ compact,

$$(2.4) \quad \mathrm{Gl}_- = \left\{ g \in \mathrm{Gl}_{\mathrm{res}} \mid g = \begin{pmatrix} \mathbf{1} & 0 \\ c & \mathbf{1} \end{pmatrix} \right\},$$

$c : H_+ \rightarrow H_-$ compact, and

$$(2.5) \quad \mathrm{Gl}_0 = \left\{ g \in \mathrm{Gl}_{\mathrm{res}} \mid g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\},$$

where $a : H_+ \rightarrow H_+$, $d : H_- \rightarrow H_-$ are invertible operators.

Throughout the paper we will use the block matrix decomposition of operators on the Hilbert space H introduced above. The connected components of $\mathrm{Gl}_{\mathrm{res}}$ are labeled by the Fredholm index of the upper (or lower) diagonal block. We now look at a factorization of elements of $\mathrm{Gl}_{\mathrm{res}}$ into elements of Gl_- , Gl_0 and Gl_+ .

Lemma 2.1. *For an open dense subset ('the big cell') S of $\mathrm{Gl}_{\mathrm{res}}$ there exists a splitting*

$$(2.6) \quad S \cong \mathrm{Gl}_- \mathrm{Gl}_0 \mathrm{Gl}_+.$$

Proof. On elements of S the splitting is explicitly given by

$$(2.7) \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ c & \mathbf{1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1} & b \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} a & ab \\ ca & cab + d \end{pmatrix}.$$

Here the blocks b and c are again compact and the diagonal blocks are invertible Fredholm.

The elements of $\mathrm{Gl}_{\mathrm{res}}$ which are splittable are those for which the upper diagonal block is invertible. They form a dense open subset S of $\mathrm{Gl}_{\mathrm{res}}$. \square

Similar to [SW] and [DNS], the existence of this splitting will later on motivate the definition of the τ -function in our setting without reference to a Grassmannian.

3. THE FLOWS

The representation of $\mathrm{Gl}_{\mathrm{res}}$ as a space of block matrices inherits a choice of basis for the separable vector space H . We number the basis vectors by integer numbers. Let e_j , $j \geq 0$, and e_j , $j < 0$, be the basis vectors of H_+ and H_- , respectively.

We let $\mathrm{Gl}_{\mathrm{res}}$ act on itself by conjugation and choose the following generators of action: Let Λ be the double right shift, i.e. the operator mapping e_j to e_{j+1} . It is represented by the matrix

$$(3.1) \quad \Lambda = \sum_{j \in \mathbb{Z}} e_{j,j+1},$$

where $e_{j,k}$ are the matrix units $(e_{j,k})_{l,m} = \delta_{j,l} \cdot \delta_{k,m}$.

We define the generator of the m -th flow to be Λ^m . Here m can be any nonzero integer. Then the flows on $\mathrm{Gl}_{\mathrm{res}}$ are defined by

$$(3.2) \quad g(\mathbf{t}) = \exp\left(\sum_{m \neq 0} t_m \Lambda^m\right) g(0) \exp\left(-\sum_{m \neq 0} t_m \Lambda^m\right).$$

\mathbf{t} denotes the vector with coordinates t_j , $j \in \mathbb{Z}$, and $g(0) \in \mathrm{Gl}_{\mathrm{res}}$ is an initial value for the flow.

As long as only finitely many t_j are nonzero, the flow on $\mathrm{Gl}_{\mathrm{res}}$ is obviously continuous, which results in the definition of continuous local flows on the three factors Gl_- , Gl_0 and Gl_+ in (2.7).

$$(3.3) \quad g_-(\mathbf{t}) g_0(\mathbf{t}) g_+(\mathbf{t}) := g(\mathbf{t}).$$

It is easy to see, that none of these flows acts trivially on $\mathrm{Gl}_{\mathrm{res}}$ or either of the splitting subgroups Gl_+ and Gl_- .

Our setting generalizes the Grassmannian flows: Calculating the splitting explicitly we see, that the negative (positive) flows act by simple translation on Gl_- (Gl_+) and therefore negative flows are not of interest on (the big cell of) the Grassmannian $\cong \mathrm{Gl}_{\mathrm{res}}/\mathrm{Gl}_0 \mathrm{Gl}_+$.

For later use we also define the abelian group $\Gamma \subset \mathrm{Gl}_{\mathrm{res}}$ generated by the exponentials of the matrices Λ^m , $m \in \mathbb{Z}$, and its subgroups Γ_+ and Γ_- generated by the exponentials of the matrices Λ^m for $m \geq 0$ and $m \leq 0$, respectively. Obviously $\Gamma = \Gamma_- \Gamma_+$. These groups of course coincide with the groups defined in [SW].

4. MATRIX EQUATIONS

For notational convenience we abbreviate:

$$(4.1) \quad \begin{pmatrix} \alpha(\mathbf{t}) & \beta(\mathbf{t}) \\ \gamma(\mathbf{t}) & \delta(\mathbf{t}) \end{pmatrix} = \exp\left(\sum_m t_m \Lambda^m\right),$$

$$(4.2) \quad \begin{pmatrix} \hat{\alpha}(\mathbf{t}) & \hat{\beta}(\mathbf{t}) \\ \hat{\gamma}(\mathbf{t}) & \hat{\delta}(\mathbf{t}) \end{pmatrix} = \exp\left(-\sum_m t_m \Lambda^m\right).$$

In addition we write $a(\mathbf{t})$, $b(\mathbf{t})$, $c(\mathbf{t})$, $d(\mathbf{t})$ for the matrix elements of the splitting (2.7) of the flow (3.2).

For the beginning we consider only $g(\mathbf{t})$ which are splittable. Then $g(\mathbf{t} + \mathbf{h})$ is splittable for \mathbf{h} (i.e. all h_i) sufficiently small. In fact, as we will see later, $g(\mathbf{t})$ fails to be splittable only for isolated values of \mathbf{t} .

Equations (3.2) and (3.3) lead to the following block matrix equations:

$$(4.3) \quad \begin{pmatrix} \alpha(\mathbf{h}) & \beta(\mathbf{h}) \\ \gamma(\mathbf{h}) & \delta(\mathbf{h}) \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ c(\mathbf{t}) & \mathbf{1} \end{pmatrix} \begin{pmatrix} a(\mathbf{t}) & 0 \\ 0 & d(\mathbf{t}) \end{pmatrix} \begin{pmatrix} \mathbf{1} & b(\mathbf{t}) \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \hat{\alpha}(\mathbf{h}) & \hat{\beta}(\mathbf{h}) \\ \hat{\gamma}(\mathbf{h}) & \hat{\delta}(\mathbf{h}) \end{pmatrix} \\ = \begin{pmatrix} a(\mathbf{t} + \mathbf{h}) & a(\mathbf{t} + \mathbf{h})b(\mathbf{t} + \mathbf{h}) \\ c(\mathbf{t} + \mathbf{h})a(\mathbf{t} + \mathbf{h}) & d(\mathbf{t} + \mathbf{h}) + c(\mathbf{t} + \mathbf{h})a(\mathbf{t} + \mathbf{h})b(\mathbf{t} + \mathbf{h}) \end{pmatrix}.$$

Notice that $\hat{\alpha}(\mathbf{h}) = \alpha(\mathbf{h})^{-1}$ and $\hat{\delta}(\mathbf{h}) = \delta(\mathbf{h})^{-1}$ if either $h_i = 0$ for $i < 0$ or $h_i = 0$ for $i > 0$, i.e. in the case that we have a purely positive or a purely negative flow.

By writing out these equations for the blocks explicitly we easily derive a set of differential equations for the functions $a(\mathbf{t})$, $b(\mathbf{t})$, $c(\mathbf{t})$ and $d(\mathbf{t})$.

To write them in a compact notation, we define the following abbreviation for the blocks of the matrix Λ^m , $m > 0$:

$$(4.4) \quad \Lambda^m = \begin{pmatrix} \Lambda_{++}^m & \Lambda_{+-}^m \\ 0 & \Lambda_{--}^m \end{pmatrix},$$

$$(4.5) \quad \Lambda^{-m} = \begin{pmatrix} \Lambda_{++}^{-m} & 0 \\ \Lambda_{-+}^{-m} & \Lambda_{--}^{-m} \end{pmatrix}.$$

Further on we will also use the subscripts $++$, $+-$, $-+$ and $--$ to denote the blocks of a matrix.

Denoting by ∂_m the partial derivative w.r.t. the parameter t_m , we end up with the equations

$$(4.6) \quad \partial_m b = \Lambda_{++}^m b - b \Lambda_{--}^m - \begin{cases} \Lambda_{+-}^m - a^{-1} \Lambda_{+-}^m d, & m > 0, \\ -b \Lambda_{-+}^m b, & m < 0, \end{cases}$$

$$(4.7) \quad \partial_m c = \Lambda_{--}^m c - c \Lambda_{++}^m - \begin{cases} c \Lambda_{+-}^m c, & m > 0, \\ d \Lambda_{-+}^m a^{-1} - \Lambda_{-+}^m, & m < 0, \end{cases}$$

for the offdiagonal blocks $b(\mathbf{t})$ and $c(\mathbf{t})$.

For $m > 0$, (4.7) is the Riccati type differential equation already encountered in [DNS], which has a negative counterpart (4.6), for $m < 0$. In coordinates they read

($i < 0, j \geq 0, m > 0$):

$$(4.8) \quad \partial_m c_{i,j} = c_{i-m,j} - c_{i,j+m} - \sum_{k=0}^{m-1} c_{i,k} c_{k-m,j},$$

$$(4.9) \quad \partial_{-m} b_{j,i} = b_{j+m,i} - b_{j,i-m} + \sum_{k=0}^{m-1} b_{j,k-m} b_{k,i}.$$

From this the main result of this section immediately follows:

Theorem 4.1. *Let the flow $g(\mathbf{t})$ be defined by (3.2). Then each of the splitting blocks $b(\mathbf{t})$ and $c(\mathbf{t})$, defined by (2.7), is completely determined by its first row or column.*

5. THE τ -FUNCTION

After defining commuting flows on the infinite dimensional group $\mathrm{Gl}_{\mathrm{res}}$ and its subgroups Gl_+ , Gl_- and Gl_0 , we are ready to look for an expression of these flows as flows on a single scalar valued function, as in the case of the classical potential KP hierarchy.

The essential step due to Sato is the introduction of the so called τ -function which is defined along the flows through an initial splittable element $g_0 \in \mathrm{Gl}_{\mathrm{res}}$. In this section we will follow closely the work of Segal and Wilson, generalizing it to the extended action of the whole group Γ .

In order to motivate the definition of the τ -function in our setting, we consider for every $k \in \mathbb{L}$ the subgroup $\mathrm{Gl}_{\mathrm{res}}(k)$ of $\mathrm{Gl}_{\mathrm{res}}$ of invertible block matrices w.r.t. the splitting (2.1), which are of the form

$$(5.1) \quad \begin{pmatrix} \mathbf{1} + p & r \\ s & \mathbf{1} + q \end{pmatrix}$$

where p and q are operators of trace class, r is in the k -th Schatten ideal in $B(H_-, H_+)$ and s is in the $\frac{k}{k-1}$ -th Schatten ideal in $B(H_+, H_-)$. These groups are proper connected subgroups of $\mathrm{Gl}_{\mathrm{res}}$ for every parameter k .

As the elements of $\mathrm{Gl}_{\mathrm{res}}(k)$ have the form

$$\mathbf{1} + \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the blocks A, B, C, D take values in different ideals of operators, the action (3.2) induces an action of $\mathrm{Gl}_{\mathrm{res}}(k)$ on itself for every k . The splittable elements of $\mathrm{Gl}_{\mathrm{res}}(k)$ are those for which p has no eigenvalue -1 . Also in this case all splitting factors are in $\mathrm{Gl}_{\mathrm{res}}(k)$.

The definition of $\mathrm{Gl}_{\mathrm{res}}(k)$ allows us to define the following family of complex function on Γ :

Definition 5.1. For a fixed splittable element g_0 in $\mathrm{Gl}_{\mathrm{res}}(k)$, the function $\tau_{g_0} : \Gamma \rightarrow \mathbb{C}$ is defined as the quotient

$$\tau_{g_0}(g) := \frac{\det((gg_0g^{-1})_{++})}{\det(g_{++}(g_0)_{++}(g^{-1})_{++})}.$$

By the remark at the end of section 2, this function is nonzero precisely when gg_0g^{-1} is splittable.

If we denote by $\alpha, a_0, \hat{\alpha}, \dots$ the splitting blocks of g, g_0 and g^{-1} as defined in (2.7), then in terms of these the τ -function is given by

$$\begin{aligned}
 \tau_{g_0}(g) &= \det((\alpha + \beta c_0)a_0(\hat{\alpha} + b_0\hat{\gamma}) + \beta d_0\hat{\gamma}) \det(\alpha a_0\hat{\alpha})^{-1} \\
 (5.2) \quad &= \det(\mathbf{1} + (\alpha^{-1}\beta c_0 a_0 + a_0 b_0 \hat{\gamma} \hat{\alpha}^{-1} + \alpha^{-1}\beta(c_0 a_0 b_0 + d_0)\hat{\gamma} \hat{\alpha}^{-1})a_0^{-1}) \\
 &= \det(\mathbf{1} + a_0^{-1}(\alpha^{-1}\beta c_0 a_0 + a_0 b_0 \hat{\gamma} \hat{\alpha}^{-1} + \alpha^{-1}\beta(c_0 a_0 b_0 + d_0)\hat{\gamma} \hat{\alpha}^{-1})),
 \end{aligned}$$

which reduces to the known expression [SW, DNS] if we look only at elements g describing positive flows and to a similar expression for purely negative flows.

The offdiagonal blocks β and $\hat{\gamma}$ of g and g^{-1} and therefore all product terms in the last two lines of (5.2) are of trace class. Thus the last two lines in (5.2) are well defined and equal for all $g \in \Gamma$. We use them to *define* a τ -function for each $g_0 \in S \subset \mathbf{G}|_{\text{res}}$.

If $g = g(\mathbf{t}) = \exp(\sum_m t_m \Lambda^m)$ we also write

$$\tau(\mathbf{t}, g_0) := \tau_{g_0}(g).$$

We will see later on, that this function is analytic as a function from Γ to \mathbb{C} .

From its definition τ inherits an equivariance property w.r.t. the used group action.

Lemma 5.1. *If \mathbf{t} and \mathbf{s} are both negative or both positive flows, then*

$$(5.3) \quad \tau(\mathbf{t} + \mathbf{s}, g_0) = \tau(\mathbf{s}, g(\mathbf{t})) \cdot \tau(\mathbf{t}, g_0).$$

Otherwise

$$(5.4) \quad \tau(\mathbf{t}_+ + \mathbf{s}_-, g_0) = \tau(\mathbf{t}_+, g(\mathbf{s}_-)) \cdot \tau(\mathbf{s}_-, g_0) \cdot c(g_-, g_+)$$

$$(5.5) \quad = \tau(\mathbf{s}_-, g(\mathbf{t}_+)) \cdot \tau(\mathbf{t}_+, g_0) \cdot c(g_-, g_+).$$

Here

$$g_+ = g_+(\mathbf{t}_+) = \exp\left(\sum_{m>0} (\mathbf{t}_+)_m \Lambda^m\right)$$

describes a positive flow,

$$g_- = g_-(\mathbf{s}_-) = \exp\left(\sum_{m<0} (\mathbf{s}_-)_m \Lambda^m\right)$$

a negative flow and

$$(5.6) \quad c(g_-, g_+) = \det(a_+ a_- a_+^{-1} a_-^{-1}),$$

a_+ and a_- being the upper diagonal blocks of g_+ and g_- , respectively.

Proof. Equation (5.3) is proved in [DNS] for the case of positive flows. The case of negative flows is proved in the same way. Equations (5.4) and (5.5) can be easily calculated explicitly using (5.2) and (4.3). \square

$c(g_-, g_+)$ is the “projective multiplier” introduced by Segal and Wilson [SW, Prop. 3.6]. We give here the simple proof of the following result stated in [SW]:

Proposition 5.1. *$c(g_-, g_+) : \Gamma_- \times \Gamma_+ \rightarrow \mathbb{C}$ is a homomorphism in every argument.*

Proof. Since elements of Γ_- and Γ_+ commute we have $a_+ a_- = a_- a_+ - b_+ c_-$ and $c(g_-, g_+) = \det(\mathbf{1} - b_+ c_- a_+^{-1} a_-^{-1})$. Therefore the determinant exists and c is defined on $\Gamma_- \times \Gamma_+$.

Let $g_-^{(1)}, g_-^{(2)} \in \Gamma_-$, $g_+ \in \Gamma_+$, and $a_-^{(1)}, a_-^{(2)}, \dots$ denote the corresponding blocks of $g_-^{(1)}$ and $g_-^{(2)}$, respectively. We have

$$\begin{aligned}
 c(g_-^{(1)} g_-^{(2)}, g_+) &= \det(a_+ a_-^{(1)} a_-^{(2)} a_+^{-1} (a_-^{(2)})^{-1} (a_-^{(1)})^{-1}) \\
 (5.7) \quad &= \det((a_-^{(1)})^{-1} a_+ a_-^{(1)} a_+^{-1}) \det(a_+ a_-^{(2)} a_+^{-1} (a_-^{(2)})^{-1}) \\
 &= c(g_-^{(1)}, g_+) c(g_-^{(2)}, g_+)
 \end{aligned}$$

and similarly for the second argument. \square

Therefore, if we write $g_- = e^{f_-}$, $f_- = \sum_{m>0} a_m \Lambda^{-m}$ and $g_+ = e^{f_+}$, $f_+ = \sum_{m>0} b_m \Lambda^m$, we get

$$(5.8) \quad c(g_-, g_+) = e^{-\sum_m a_m b_m}.$$

If we complexify the flow variables, it follows from the definition and the known theorems on the τ -function in the Grassmannian setting, that it is holomorphic as a function from Γ_+ to \mathbb{C} and as a function of Γ_- of \mathbb{C} . Obviously, if we work in the setting of Segal and Wilson, $\tau(\mathbf{t}, g_0)$ is also locally bounded and thus holomorphic as a map from Γ , being the product of the abelian groups Γ_- and Γ_+ , endowed with the operator norm on the diagonal and the trace norm on the offdiagonal blocks. The zeros of τ are therefore complex hypersurfaces of (real) codimension 2 in \mathbb{C}^2 .

The most important result of this section is

Theorem 5.1. *Let $g(\mathbf{t})$ be defined by the flow (4.3) for an initial element $g_0 \in S \subset \text{Gl}_{\text{res}}$, and let $b(\mathbf{t})$ and $c(\mathbf{t})$ be the block matrices defined by the splitting (2.7) of $g(\mathbf{t})$. Then the first row of $c(\mathbf{t})$ and the first column of $b(\mathbf{t})$, and therefore the whole matrices $c(\mathbf{t})$ and $b(\mathbf{t})$ are determined by $\tau_{g_0}(g(\mathbf{t})) = \tau(\mathbf{t}, g_0)$.*

Proof. This is clear for $c(\mathbf{t})$, as all calculations reduce to those in [DNS] for positive flows: Introducing the special element $n_\zeta = \mathbf{1} - \Lambda/\zeta = \exp(\sum_{m>0} (\mathbf{n}_\zeta)_m \Lambda^m)$, $|\zeta| > 1$, in Γ_+ , $(\mathbf{n}_\zeta)_k = -k^{-1} \zeta^{-k}$, $k > 0$, we get

$$(5.9) \quad \tau(\mathbf{n}_\zeta, g(\mathbf{t})) = 1 - \sum_{k=1}^{\infty} \zeta^{-k} c_{-1, k-1}(\mathbf{t}).$$

For negative flows everything translates analogously. We choose the special element $m_\xi = (1 - \xi/\Lambda)^{-1} = \exp(\sum_{m<0} (\mathbf{m}_\xi)_m \Lambda^m)$, $|\xi| < 1$, in Γ_- . Then, with $(\mathbf{m}_\xi)_k = k^{-1} \xi^k$, $k < 0$,

$$\begin{aligned}
 \tau(\mathbf{m}_\xi, g(\mathbf{t})) &= \det(\mathbf{1} + \hat{\alpha}^{-1}(\mathbf{m}_\xi) b(\mathbf{t}) \hat{c}(\mathbf{m}_\xi)) \\
 (5.10) \quad &= 1 - \sum_{k=1}^{\infty} \xi^k b_{k-1, -1}(\mathbf{t}).
 \end{aligned}$$

\square

Now we use the equivariance property (5.5) of τ_{g_0} for mixed flows to get additional differential equations. Let $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$ be such that $(\mathbf{t}_+)_k = 0$ for $k \leq 0$ and $(\mathbf{t}_-)_k = 0$ for $k \geq 0$. A simple calculation yields ($m > 0$):

$$(5.11) \quad \partial_{-m} \tau(\mathbf{t}, g_0) = \partial_{t_{-m}}|_{t_{-m}=0} (\tau(t_{-m}, g(\mathbf{t})) \tau(\mathbf{t}, g_0) c(g_-(t_{-m}), g_+(\mathbf{t}_+))).$$

It follows by the same reasoning as before,

$$(5.12) \quad \partial_{-m} \ln(\tau(\mathbf{t}, g_0)) = \partial_{t_{-m}}|_{t_{-m}=0} (\text{tr}(b(\mathbf{t}) \hat{\gamma}(t_{-m})) - c(g_-(t_{-m}), g_+(\mathbf{t}_+))).$$

In terms of matrix entries this reads:

$$(5.13) \quad \partial_{-m} \ln(\tau(\mathbf{t}, g_0)) = - \sum_{k=1}^m b_{m-k, -k}(\mathbf{t}) - mt_m, \quad m > 0,$$

analogous to the well known formula [SW, DNS]

$$(5.14) \quad \partial_m \ln(\tau(\mathbf{t}, g_0)) = \sum_{k=1}^m c_{-k, m-k}(\mathbf{t}) - mt_{-m}, \quad m > 0.$$

Especially, together with equation [DNS, (5.8.1)–(5.8.2)] we have

$$(5.15) \quad \partial_{-1} c_{-1,0}(\mathbf{t}) = \partial_{-1} \partial_1 \ln \tau + 1 = -\partial_1 b_{0,-1}(\mathbf{t}).$$

This connects the two offdiagonal blocks and at the same time proves, that the scalar functions $c_{-1,0}(\mathbf{t})$ and $b_{0,-1}(\mathbf{t})$ are meromorphic w.r.t. the variables t_1 and t_{-1} , being quotients of holomorphic functions. We will see later on how far $\tau(\mathbf{t}, g_0)$ is determined by $c_{-1,0}(\mathbf{t})$ and $b_{0,-1}(\mathbf{t})$.

6. EXTENSION OF THE POTENTIAL KP-HIERARCHY

In this section we show, that the equations (4.8) and (4.9) can be expressed as equations for a single complex valued function. To this end we use again the special elements n_ζ and m_ξ of Γ introduced above.

From [SW, DNS] we expect $c_{-1,0}$ to be the function satisfying the potential KP-equation. This is stated in the following theorem.

Theorem 6.1. *Let $c(\mathbf{t})$ and $b(\mathbf{t})$ be defined as in Theorem 5.1. The derivatives of $c_{-1,j}(\mathbf{t})$ and $b_{j,-1}$, $j \geq 0$, can be expressed as differential polynomials in $c_{-1,0}$ and $b_{0,-1}$. Derivatives of $c_{-1,j}$ for positive flows only depend on $c_{-1,0}$, derivatives of $b_{0,-1}$ for negative flows only depend on $b_{0,-1}$.*

Proof. We first look at the equations for $c(\mathbf{t} + \mathbf{n}_\zeta)$ and $b(\mathbf{t} + \mathbf{m}_\xi)$.

$$(6.1) \quad c(\mathbf{t} + \mathbf{n}_\zeta)(\alpha(\mathbf{n}_\zeta) + \beta(\mathbf{n}_\zeta)c(\mathbf{t})) = \delta(\mathbf{n}_\zeta)c(\mathbf{t}),$$

$$(6.2) \quad (\hat{\alpha}(\mathbf{m}_\xi) + b(\mathbf{t})\hat{\gamma}(\mathbf{m}_\xi))b(\mathbf{t} + \mathbf{m}_\xi) = b(\mathbf{t})\hat{\delta}(\mathbf{m}_\xi).$$

To get differential equations, we develop $b(\mathbf{t} + \mathbf{n}_\zeta)$ and $c(\mathbf{t} + \mathbf{m}_\xi)$ w.r.t. the parameters ζ and ξ , respectively.

$$(6.3) \quad c(\mathbf{t} + \mathbf{n}_\zeta) = \sum_{k=0}^{\infty} \zeta^{-k} P_k c(\mathbf{t}),$$

$$(6.4) \quad b(\mathbf{t} + \mathbf{m}_\xi) = \sum_{k=0}^{\infty} \xi^k P_{-k} b(\mathbf{t}).$$

P_k , $k \in \mathbb{Z}$, are differential operators determined by the special structure of the elements n_ζ and m_ξ . For $k > 0$ they satisfy:

$$P_{-k}(\partial_{-1}, \dots, \partial_{-k}) = -P_k(\partial_1, \dots, \partial_k).$$

With $b(\mathbf{n}_\zeta) = -\zeta^{-1}\Lambda_{+-}$, $a(\mathbf{n}_\zeta) = \mathbf{1} - \zeta^{-1}\Lambda_{++}$, $\hat{c}(\mathbf{m}_\xi) = -\xi\Lambda_{-+}^{-1}$ and $\hat{a}(\mathbf{m}_\xi) = \mathbf{1} - \xi\Lambda_{++}^{-1}$ we get

$$(6.5) \quad \sum_{k \geq 0} \zeta^{-k} P_k c (\mathbf{1} - \zeta^{-1}\Lambda_{++} - \zeta^{-1}\Lambda_{+-}c) = c - \zeta^{-1}\Lambda_{--}c,$$

$$(6.6) \quad (\mathbf{1} - \xi\Lambda_{++}^{-1} - \xi b\Lambda_{-+}^{-1}) \sum_{k \geq 0} \xi^k P_{-k} b = b - \xi b\Lambda_{--}^{-1},$$

or in coordinates:

$$(6.7) \quad P_{k+1}c_{-1,j} - P_k c_{-1,j+1} - (P_k c_{-1,0})c_{-1,j} = 0,$$

$$(6.8) \quad P_{-k-1}b_{i,-1} - P_{-k}b_{i+1,-1} - b_{i,-1}P_{-k}b_{0,-1} = 0,$$

for $k \geq 1$, $j \geq 0$. Multiplying the first equation with ζ^{-j-1} , the second with ξ^{i+1} , we end up after summation with the following equations ($k \geq 1$):

$$(6.9) \quad (P_{k+1} - \zeta P_k - (P_k c_{-1,0}))\tau(\mathbf{n}_\zeta, g(\mathbf{t})) = 0,$$

$$(6.10) \quad (P_{-k-1} - \xi^{-1}P_{-k} - (P_{-k}b_{0,-1}))\tau(\mathbf{m}_\xi, g(\mathbf{t})) = 0.$$

Equations (6.7) and (6.8) make it possible to express $\partial_k c_{-1,j}$ and $\partial_{-k} b_{j,-1}$ for $j, k \geq 1$ as differential polynomials in $c_{-1,0}$ and $b_{0,-1}$, respectively.

Using the commutativity of the flows, we can evaluate $\tau(m_\xi + n_\zeta, g(\mathbf{t}))$ in two different ways. We get by (5.5) and Taylor expansion the following set of equations ($i, j \geq 1$):

$$(6.11) \quad P_j b_{i-1,-1}(\mathbf{t}) - \sum_{k=1}^{j-1} c_{-1,k-1}(\mathbf{t}) P_{j-k} b_{i-1,-1}(\mathbf{t}) \\ = P_{-i} c_{-1,j-1}(\mathbf{t}) - \sum_{k=1}^{i-1} b_{k-1,-1}(\mathbf{t}) P_{k-i} c_{-1,j-1}(\mathbf{t}).$$

This doesn't allow us to express the negative derivatives of $c_{-1,j+1}$ by derivatives of $c_{-1,0}$, however, it gives us new relations expressing for example $\partial_{-k} c_{-1,j}$ as a polynomial in $c_{-1,i}$, $i < j$, and positive derivatives of $b_{0,-1}$, which completes the proof. \square

Remark 6.1. Using the equations

$$(6.12) \quad a(\mathbf{t} + \mathbf{m}_\zeta) = a(\mathbf{m}_\zeta)a(\mathbf{t})(\hat{a}(\mathbf{m}_\zeta) + b(\mathbf{t})\hat{c}(\mathbf{m}_\zeta)),$$

$$(6.13) \quad a(\mathbf{t} + \mathbf{n}_\xi) = (a(\mathbf{n}_\xi) + b(\mathbf{n}_\xi)c(\mathbf{t}))a(\mathbf{t})\hat{a}(\mathbf{n}_\xi),$$

which follow from (4.3), we get as a by-product in the same way explicit expressions for $P_k a$ ($k \geq 1$):

$$(6.14) \quad P_k a = -\partial_1 a \Lambda_{++}^{k-1},$$

and

$$(6.15) \quad P_{-k} a = \Lambda_{++}^{1-k} \partial_{-1} a, \quad k \geq 1.$$

It follows that the evolution of the upper diagonal block w.r.t. an arbitrary flow is determined by its evolution w.r.t. the first negative and the first positive flow.

Let us abbreviate $u := c_{-1,0}$, $v := b_{0,-1}$. If we assign the following degrees to the matrix entries and derivation operators:

$$(6.16) \quad \begin{aligned} \deg(b_{j,k}) &= \deg(c_{j,k}) = k - j, \\ \deg(\partial_m) &= m. \end{aligned}$$

then the equations (4.8), (4.9), (6.8) and (6.7) respect this gradation.

The picture that develops here is the following: We have in principle two copies of the KP-hierarchy, given by the functions $u = c_{-1,0}(\mathbf{t})$ together with the positive flows and $v = b_{0,-1}(\mathbf{t})$ and the negative flows, respectively. They are not completely independent, but are coupled by the equation (5.15). Therefore the PKP-hierarchy for v can be pulled back to yield additional differential equations in u and v , commuting with the PKP-hierarchy for u . These equations are belonging to the t_k flows for negative k . In addition, all equations of the extended hierarchy are homogenous w.r.t. the gradation (6.16).

For example the t_{-3} -flow gives the PKP-equation for v :

$$(6.17) \quad \frac{3}{4}\partial_{-2}^2 v = \partial_{-1}(\partial_{-3}v - \frac{1}{4}\partial_{-1}^3 v + \frac{3}{2}(\partial_{-1}v)^2).$$

For u this yields:

$$(6.18) \quad \partial_{-1}\partial_{-2}^2 u = \partial_{-1}\partial_1(\frac{4}{3}\partial_{-3}v - \frac{1}{3}\partial_{-1}^3 v + (\partial_{-1}v)^2).$$

Neglecting an integration constant we get the equation in u and v :

$$(6.19) \quad \frac{3}{4}\partial_{-2}^2 u = -\partial_{-1}\partial_{-3}u + \frac{1}{4}\partial_{-1}^4 u - \frac{3}{2}(\partial_{-1}^2 u \partial_{-1}v).$$

7. THE SET OF SOLUTIONS

Up to now it may not be clear, why we did not start with a group like $\mathrm{Gl}_{\mathrm{res}}(k)$ in the first place, thereby avoiding the analytic difficulties we encounter with the infinite determinant and the definition of the τ -function.

First we will see in the next section, that the reduction to 1 + 1-dimensional equations like the KdV-equation requires the full group $\mathrm{Gl}_{\mathrm{res}}$ or at least a dense subset in it.

Second it is hard to see, which solutions we throw away by looking at the restricted subgroups. We will see that, in this setting unexpectedly, we encounter the appearance of another splitting, the analog of the well known Birkhoff factorization of $\mathrm{Gl}_{\mathrm{res}}$.

The question we are primarily interested in is, what initial conditions give the same τ -function. This amounts to the question how far the Gl_0 splitting part of $g(\mathbf{t})$, i.e. $a(\mathbf{t})$ and $d(\mathbf{t})$, is determined by the parts in Gl_- and Gl_+ , i.e. $b(\mathbf{t})$ and $c(\mathbf{t})$. The latter ones are given by the τ -function. On the other hand for given $c_{-1,0}$ and $b_{0,-1}$ all derivatives $\partial_k c_{-1,j}$ and $\partial_{-k} b_{j,-1}$, $k \geq 1$ and $j \geq 1$ are given by the equations (6.7) and (6.8) as differential polynomials in $c_{-1,0}$ and $b_{0,-1}$. The Riccati equations (4.8) and (4.9) then give the whole offdiagonal blocks, and therefore all derivatives of $\ln \tau$ are determined up to integration constants. We can fix these by requiring

$$(7.1) \quad \partial_k \ln \tau(0, g_0) = 0$$

for all k with $|k| > 1$, and all splittable g_0 .

If we restrict ourselves to the subgroup $\mathrm{Gl}_{\mathrm{res}}^{(2)}$ of $\mathrm{Gl}_{\mathrm{res}}$, for which the offdiagonal blocks are Hilbert-Schmidt, then we may actually show, that for all splittable g_0 the condition (7.1) can be enforced by acting with the group $\Gamma_- \times \Gamma_+$: Denoting with L_{γ_-} and R_{γ_+} the left multiplication with $\gamma_- = \exp \sum_{k < 0} t_k \Lambda^k \in \Gamma_-$ and right multiplication with $\gamma_+ = \exp \sum_{k > 0} t_k \Lambda^k \in \Gamma_+$, respectively, we see immediatly:

$$(7.2) \quad \tau(g_+, L_{\gamma_-} g_0) = c(\gamma_-, g_+) \tau(g_+, g_0),$$

$$(7.3) \quad \tau(g_-, R_{\gamma_+} g_0) = c(g_-, \gamma_+) \tau(g_-, g_0),$$

and therefore for $k > 1$:

$$(7.4) \quad \partial_k \ln \tau(0, L_{\gamma_-} g_0) = \partial_k \ln \tau(0, g_0) - k t_k,$$

$$(7.5) \quad \partial_{-k} \ln \tau(0, R_{\gamma_+} g_0) = \partial_{-k} \ln \tau(0, g_0) - k t_{-k}.$$

Choosing t_k appropriately for all k , $|k| > 1$, we can reach (7.1) from any initial condition by acting with a uniquely but formally defined element $\gamma_- \times \gamma_+$ in $\Gamma_- \times \Gamma_+$ in the prescribed way. In the case, where the offdiagonal blocks of g_0 are Hilbert-Schmidt, the γ_+ and γ_- obtained by this procedure converge in Γ_+ and Γ_- . The same holds, as was shown in [Do2], if one works with certain weighted ℓ_1 Banach structures instead of $\mathrm{Gl}_{\mathrm{res}}$. For compact offdiagonal blocks, on the other hand, one can employ the example of Pressley and Segal [PS, (8.3.4)] to show, that the offdiagonal blocks of $\gamma_- \times \gamma_+$ may not even be bounded operators.

The ambiguity left is an overall multiplicative constant, which we can set to be one, since it doesn't affect the logarithmic derivatives of τ .

We look at the differential equations (4.6) and (4.7) for the offdiagonal blocks b and c , respectively. It is obvious, that they, and therefore the solutions, don't change, if the last terms containing a and d do not. This gives us conditions on the stabilizer of a solution. In the following we will consider only splittable initial conditions. We write out the conditions on the last terms in (4.6) and (4.7) as matrix equations:

Let \bar{a} , \bar{d} and a , d be pairs of matrices, for which

$$(7.6) \quad \bar{a}^{-1} \Lambda_{+-}^m \bar{d} = a^{-1} \Lambda_{+-}^m d,$$

$$(7.7) \quad \bar{d} \Lambda_{-+}^m \bar{a}^{-1} = d \Lambda_{-+}^m a^{-1}.$$

We may write

$$D = \bar{d}^{-1} d, \quad A = \bar{a}^{-1} a,$$

and thus obtain from equation (7.7) the condition

$$(7.8) \quad D \Lambda_{-+}^{-m} = \Lambda_{-+}^{-m} A.$$

In coordinates this is

$$(7.9) \quad (D \Lambda_{-+}^{-m})_{ij} = \begin{cases} D_{i,j-m}, & j = 0, \dots, m-1 \\ 0, & j \geq m \end{cases}, \quad i < 0,$$

$$(7.10) \quad (\Lambda_{-+}^{-m} A)_{ij} = \begin{cases} A_{i+m,j}, & i = -m, \dots, -1 \\ 0, & i < -m \end{cases}, \quad j > 0.$$

Therefore A and D are lower echelon matrices, which are determined by each other.

The notion "triangular" here means that the entries on the diagonal are all 1, an echelon matrix is a product of an arbitrary diagonal and a triangular matrix.

In the same way we derive from equation (7.6) the conditions

$$(7.11) \quad \tilde{A}\Lambda_{+-}^m = \Lambda_{+-}^m \tilde{D}$$

for the matrices

$$\tilde{D} = d\bar{d}^{-1}, \quad \tilde{A} = a\bar{a}^{-1},$$

which, because of

$$(7.12) \quad (\Lambda_{+-}^m \tilde{D})_{ij} = \begin{cases} \tilde{D}_{i-m,j}, & i = 0, \dots, m-1 \\ 0, & i \geq m \end{cases}, j < 0,$$

$$(7.13) \quad (\tilde{A}\Lambda_{+-}^m)_{ij} = \begin{cases} \tilde{A}_{i,j+m}, & j = -m, \dots, -1 \\ 0, & j < -m \end{cases}, i > 0,$$

amount to \tilde{A} and \tilde{D} being upper echelon matrices, which are determined by each other.

We end up with the following condition: The set of $g \in \text{Gl}_{\text{res}}$ yielding the same τ -function as

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

and for which (7.1) holds, is parametrized by invertible upper and lower triangular matrices \tilde{A} and A , for which

$$(7.14) \quad a = \tilde{A}aA^{-1}.$$

We showed before, that if we look at $\text{Gl}_{\text{res}}^{(2)}$, $\tilde{\Gamma} = \Gamma_+ \times \Gamma_-$ acts freely on the matrices g_0 , which give the same solution. Let's denote by \mathcal{S} the quotient of the flow action of Γ and the above described action of $\tilde{\Gamma}$ on $\text{Gl}_{\text{res}}^{(2)}$. A solution then determines an element of \mathcal{S} up to a transformation of the upper triangular block by (7.14). I.e. the fibre in \mathcal{S} over a solution, which has an initial condition with upper diagonal block a is given by the stabilizer of a w.r.t. the action (7.14) of invertible upper and lower echelon matrices.

Notice that the stabilizer of a may vary in a very complicated way over Gl_{res} . We want to investigate the structure of the stabilizer in a special case a little further. We employ the Gauss algorithm in order to split the matrix a into an upper triangular, a diagonal and a lower triangular matrix.

The Gauss algorithm allows us to write down matrices \tilde{a}_L , a_D and a_U , s.t.

$$a\tilde{a}_L = a_U a_D,$$

for a in a dense subset of $B(H_+)$, to which we restrict ourselves. Even for infinite matrices it is no problem to multiply a lower triangular matrix like $a_L = \tilde{a}_L^{-1}$ from the right with an arbitrary matrix, as all matrix elements of the product consist of sums of finitely many products of matrix entries. Therefore we have a kind of Birkhoff factorization

$$(7.15) \quad a = a_U a_D a_L,$$

where the factors are uniquely determined matrices, but not necessarily bounded operators on H_+ .

Writing down (7.14) we get

$$a_U^{-1} \tilde{A} a_U a_D a_L A^{-1} a_L^{-1} = a_D.$$

The only freedom we have, is to choose the diagonal part of $a_U^{-1} \tilde{A} a_U$, because of the uniqueness of the splitting (7.15). However, as \tilde{A} has to be a bounded operator, we get additional highly nontrivial conditions on this diagonal matrix. To be precise, if

$$a_U^{-1} \tilde{A} a_U = U \cdot D,$$

U being upper triangular and D being diagonal, then we have to choose D in such a way, that $a_U U D a_U^{-1}$ is a bounded operator.

If a contains a permutation matrix in its Birkhoff decomposition, then the stabilizer gets bigger, including upper triangular matrices, which are transformed by a to lower triangular matrices. It is an open problem, if there is any “solution manifold” as in the Grassmannian case [Do2].

8. A COMMENT ON THE CONNECTION WITH THE HIROTA-SATSUMA EQUATION

We want to link the negative PKP-hierarchy to work done by Bogoyavlenskii and others [Bo, Sz, HS, AKNS].

To this end we consider the usual reduction from the KP to the KdV-hierarchy [SW]: Let Gl_{KdV} be the subgroup of Gl_{res} of all matrices, which commute with the square of the double shift Λ^2 . Taken as initial conditions of the flows (3.2), these are precisely the ones on which all even numbered flows act trivially.

This subgroup can easily be identified with the loop group $L\text{GL}(2, \mathbb{C})$, where the identification in terms of matrix units is

$$(8.1) \quad e_{ij} \longrightarrow \lambda^{\lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor} e_{i \bmod 2, j \bmod 2},$$

$[x]$ denoting the greatest integer less than x . This way the double shift Λ is identified with the matrix

$$(8.2) \quad p = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

The identification, however, does not respect the splitting (2.7) as it identifies offdiagonal with diagonal blocks. As we are not really interested in the behaviour of the diagonal blocks we include them into the positive part.

This produces a well known formulation [Do1] of the potential KdV-equation (PKdV) in terms of a loop group splitting, the PKdV thus being the standard loop group reduction of the potential KP-hierarchy.

Again, we let the flows act by conjugation. The subgroups of the splitting are now

$$(8.3) \quad \begin{aligned} G_+ &= \{g(\lambda) \in L\text{GL}(2, \mathbb{C}) | g(\lambda) \text{ is analytic inside the unit circle}\}, \\ G_- &= \{g(\lambda) \in L\text{GL}(2, \mathbb{C}) | g(\lambda) \text{ is analytic outside the unit circle,} \\ &\quad g(0) = \mathbf{1}\}, \end{aligned}$$

The product $G_- G_+$ is a dense open subset in $L\text{GL}(2, \mathbb{C})$ [PS]. Since the independent function $u = c_{-1,0}$ which satisfies the extended potential KP-hierarchy is mapped to the upper right corner of the λ^{-1} coefficient of g_- , the latter is also the function which satisfies the reduced negative KP-hierarchy.

The first equation plays the same role for the KdV-equation as the sine-Gordon equation does for the modified KdV-equation.

The flow matrices are mapped to the matrix

$$(8.4) \quad \Lambda \longrightarrow p_1 = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix},$$

$$(8.5) \quad \Lambda^{-1} \longrightarrow p_{-1} = \lambda^{-1} p_1.$$

Therefore the flows read

$$(8.6) \quad g(x, t) = \exp(xp_1 + tp_{-1})g_0 \exp(-xp_1 - tp_{-1}) = g_-(x, t)g_+(x, t)^{-1}.$$

To reproduce a zero curvature condition for the negative flows we adopt the double group formulation of negative flows [Wu, Ha] This amounts to the following: If we look at (8.6) we may write the splitting in a slightly different way, $g_0 = g_{0-}g_{0+}^{-1}$,

$$(8.7) \quad \begin{aligned} g(x, t) &= (\exp(xp_1 + tp_{-1}), \exp(xp_1 + tp_{-1}))(g_{0-}, g_{0+}) \\ &= (g_-(x, t), g_+(x, t))(v(x, t), v(x, t)), \end{aligned}$$

where we have identified the set of splittable matrices in $L\mathbf{GL}(2, \mathbb{C})$ to the subset $G_- \times G_+$ of the product group $L\mathbf{GL}(2, \mathbb{C}) \times L\mathbf{GL}(2, \mathbb{C})$ by

$$(8.8) \quad g_- g_+^{-1} \longrightarrow (g_-, g_+).$$

The additional factor (v, v) occurs due to the fact, that the flows do not stay in the subgroup $G_- \times G_+$. This amounts to a splitting of the double group $\mathcal{G} = L\mathbf{GL}(2, \mathbb{C}) \times L\mathbf{GL}(2, \mathbb{C})$ into the subgroups $\mathcal{G}_- = G_- \times G_+$ and $\mathcal{G}_+ = \text{diag}(\mathcal{G}) = \{(x, y) \in \mathcal{G} | x = y\}$. The flows then act by left multiplication with

$$(\exp(xp_1 + tp_{-1}), \exp(xp_1 + tp_{-1})) \in \mathcal{G}_+$$

and reproduce the well known formulation of an integrable system with only positive flows.

We explicitly write down $v(x, t)$:

$$(8.9) \quad \begin{aligned} v(x, t) &= g_-(x, t)^{-1} \exp(xp_1 + tp_{-1})g_{0-} \\ &= g_+(x, t)^{-1} \exp(xp_1 + tp_{-1})g_{0+}. \end{aligned}$$

It follows that the matrices

$$(8.10) \quad U = \partial_x v v^{-1},$$

$$(8.11) \quad V = \partial_t v v^{-1},$$

satisfy the zero curvature condition

$$(8.12) \quad \partial_t U - \partial_x V + [U, V] = 0.$$

W.l.o.g. we restrict the initial condition g_0 to the loop group $LSL(2, \mathbb{C})$ and write

$$(8.13) \quad g_- = \exp(\lambda^{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}) + \text{lower order terms}.$$

We therefore get

$$(8.14) \quad U(x, t) = (\text{Ad}(g_-^{-1})p_1)_+ = \begin{pmatrix} -b & 1 \\ \lambda + 2a & b \end{pmatrix}$$

and by

$$(8.15) \quad V(x, t)_- = (\text{Ad}(g_+^{-1})p_{-1})_- = \lambda^{-1}V_{-1},$$

with V_{-1} independent of λ , also

$$(8.16) \quad \begin{aligned} V(x, t) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (-g_-^{-1} \partial_t g_- + g_-^{-1} p_{-1} g_-) - \\ &= \lambda^{-1} U(x, t) - \begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix} \lambda^{-1}. \end{aligned}$$

Here the subscripts $+$ and $-$ indicate projection to the Lie algebras of \mathfrak{gl}_+ and \mathfrak{gl}_- , respectively.

In addition we have by evaluating the upper right corner of the λ^{-1} -coefficient of

$$(8.17) \quad g_-^{-1} \partial_x g_- - \text{Ad}(g_-^{-1}) p_1 = g_+^{-1} \partial_x g_+ - \text{Ad}(g_+^{-1}) p_1$$

$$a = -\frac{1}{2}(b_x + b^2).$$

In the reduction to the loop group, equation (8.17) allows us to derive scalar differential equations from the flows in the group. In this respect it substitutes the Riccati equation (4.8) which came up in the general case.

Evaluating (8.12) yields the first negative equation in the KdV-hierarchy for the matrix element b :

$$(8.18) \quad b_{xx} - \frac{1}{4} b_{xxx} - 2b_x b_{tx} - b_t b_{xx} = 0,$$

or for $u(x, t) = -4(b(x, t) - t)$:

$$(8.19) \quad u_{xxx} = u_t u_{xx} + 2u_x u_{xt},$$

which is Bogoyavlenskii's version of the Hirota-Satsuma equation [HS] for long waves in a medium with nonlinear dispersion. Bogoyavlenskii [Bo] investigated it as an integrable equation with overturning (breaking) solitons. In the setting of this section it was derived as the first negative flow of the potential KdV-equation by Szmigielski [Sz].

In the case of the Grassmannian formulation of the potential KP equation, one recovers the elements of the potential KdV hierarchy as reductions of equations of the potential KP-hierarchy, i.e. simply by setting derivatives w.r.t. even numbered variables to zero.

If one does the reduction on the group level, as in this chapter, the τ -function gets lost and is replaced by the zero curvature condition. This yields the fact, that due to the loss of the equations (6.7) and (6.8) in the reduced case, we can no more express the offdiagonal blocks merely by one corner entry $c_{-1,0}$ or $b_{0,-1}$, respectively.

This reduction corresponds to the symmetry reduction of the standard KP equation [DKLW], which yields among other systems the Boussinesq equation and a differentiated KdV equation.

It works even though the splitting in the reduced and the unreduced case is quite different, i.e. in the reduced case we split along the diagonal, in the unreduced case we leave whole block matrices on the diagonal.

Instead of an equation for one scalar function we end up in the reduction to $LSL(n, \mathbb{C})$ with $n - 1$ seemingly independent functions, which describe the offdiagonal blocks. The Riccati equations (4.8) and (4.9) are still applicable and occur in the form of equation (8.17).

Obviously all this applies also to the extension of the KP hierarchy introduced in this paper. We plan to investigate the connection between the extended potential KdV-hierarchy and the extended potential KP-hierarchy in a separate publication.

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