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ISOMORPHISMS OF ADJOINT CHEVALLEY GROUPS OVER INTEGRAL DOMAINS

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ABSTRACT. It is shown that every automorphism of an adjoint Chevalley group over an integral domain containing the rational number field is uniquely expressible as the product of a ring automorphism, a graph automorphism and an inner automorphism while every isomorphism between simple adjoint Chevalley groups can be expressed uniquely as the product of a ring isomorphism, a graph isomorphism and an inner automorphism. The isomorphisms between the elementary subgroups are also found having analogous expressions.

1. Introduction and main theorems

Let G and G' be simple Chevalley-Demazure group schemes of adjoint type. Suppose R and R' are commutative integral domains containing the rational number field $\mathbb{Q}$. The main purpose of this paper is to determine the isomorphisms between Chevalley groups $G(R)$ and $G'(R')$, as well as the isomorphisms between their elementary subgroups $E(R)$ and $E'(R')$. When G is semisimple, the automorphisms of $G(R)$ and $E(R)$ are also determined in this paper. When R is a field, the automorphisms of simple adjoint Chevalley groups over R have been determined by Steinberg [9] and Humphreys [8]. Our main results are as follows.

Theorem 1.1. *Let R and R' be commutative integral domains containing $\mathbb{Q}$. Suppose G and G' are simple adjoint Chevalley-Demazure group schemes whose ranks are greater than 1. Then*

(i) *every isomorphism between $E(R)$ and $E'(R')$ can be extended uniquely to an isomorphism between $G(R)$ and $G'(R')$;*
(ii) *if α is an isomorphism from $G(R)$ to $G'(R')$, the restriction of α to $E(R)$ is an isomorphism from $E(R)$ to $E'(R')$.*

Suppose G and G' are adjoint Chevalley-Demazure group schemes. Let Φ (resp. Φ') be a root system of G (resp. G') and let Δ (resp. Δ') be a fundamental root system of Φ (resp. Φ'). We refer to [7] for the properties of Chevalley-Demazure group schemes. In particular, for each root $a \in \Phi$ and for each commutative ring R with a unit, there is a canonical monomorphism (cf. [7, XXII])

$$u_{a,R} : R^+ \to G(R).$$

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We rewrite this monomorphism simply as u_a for whatever commutative ring with a unit. We denote by $U_a(R)$ the subgroup consisting of the elements $u_a(r)$ for all $r \in R$ and $a \in \Phi$. The elementary subgroup $E(R)$ of $G(R)$ is generated by $U_a(R)$ for all $a \in \Delta$ or $-\Delta$, where $-\Delta$ is the set of the negative fundamental roots of Φ.

Since both G and G' are of adjoint type, it follows from Demazure's fundamental theorem (see [7, XXIII,§5.1]) that every isomorphism of root systems between Φ and Φ' implies an isomorphism between G and G'. Hence, if $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems such that $\gamma(\Delta) = \Delta'$, it gives rise canonically to an isomorphism $\tilde{\gamma} : G(R) \to G'(R)$ satisfying

$$\tilde{\gamma}(u_a(r)) = u_{\gamma(a)}(r) \text{ for } a \in \Delta \text{ or } -\Delta,\ r \in R.$$

We call this $\tilde{\gamma}$ a *graph isomorphism* related to γ. It is obvious that

$$\tilde{\gamma}(E(R)) = E'(R).$$

This identity allows us to define a *graph isomorphism* from $E(R)$ to $E'(R)$ related to γ to be an isomorphism $\hat{\gamma} : E(R) \to E'(R)$ which satisfies

$$\hat{\gamma}(u_a(r)) = u_{\gamma(a)}(r) \text{ for } a \in \Delta \text{ or } -\Delta,\ r \in R.$$

Suppose G is a simple Chevalley-Demazure group scheme. Let R and R' be commutative rings with units. Since G is a covariant group functor on the category of commutative rings with units, every isomorphism $\varphi : R \to R'$ gives rise to an isomorphism $\tilde{\varphi} : G(R) \to G(R')$ in a canonical way, which is called the *ring isomorphism* related to φ. In particular, we have

$$\tilde{\varphi}(u_a(r)) = u_a(\varphi(r)) \text{ for } a \in \Delta \text{ or } -\Delta,\ r \in R.$$

Thus

$$\tilde{\varphi}(E(R)) = E(R').$$

We shall also call an isomorphism $\hat{\varphi} : E(R) \to E(R')$ to be a *ring isomorphism* related to φ if it satisfies

$$\hat{\varphi}(u_a(r)) = u_a(\varphi(r)) \text{ for } a \in \Delta \text{ or } -\Delta,\ r \in R.$$

Theorem 1.2. *Let R, R', G and G' be as in Theorem 1.1. If α is an isomorphism from $G(R)$ to $G'(R')$, then there exist an element $g \in G'(R')$, an isomorphism of root system $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and an isomorphism of rings $\varphi : R \to R'$ such that*

$$\alpha = Int\, g \cdot \tilde{\gamma} \cdot \tilde{\varphi}.$$

Moreover, g, γ and α are uniquely determined by α.

Remark 1.3. The isomorphisms between $E(R)$ and $E'(R')$ have similar expressions where $\tilde{\gamma}$ and $\tilde{\varphi}$ are replaced by $\hat{\gamma}$ and $\hat{\varphi}$ respectively (see Theorem 3.9).

Theorem 1.4. *Let R be a commutative integral domain which contains $\mathbb{Q}$ and let G be an adjoint Chevalley-Demazure group scheme which has no simple component of type A_1. Then*

(i) *every automorphism of $E(R)$ can be extended uniquely to an automorphism of $G(R)$;*
(ii) *the restriction of each automorphism of $G(R)$ to $E(R)$ is an automorphism of $E(R)$.*

In particular, $Aut\, G(R) \cong Aut\, E(R)$.

Let G be an adjoint Chevalley-Demazure group scheme and let $\{G_i\}_{i=1}^n$ be its simple components. Since $G = \prod_{i=1}^n G_i$ (cf. [7, XXIV,§5.5]), G is a covariant group functor on the category of which the objects are of form $\prod_{i=1}^n R$ for some commutative ring R with a unit. Hence, if $\varphi_i : R \to R$ is an automorphism for $1 \le i \le n$, then the automorphism $\prod_{i=1}^n \varphi_i \in Aut \prod_{i=1}^n R$ gives rise canonically to an automorphism of $G(R)$, which is easily seen to be $\prod_{i=1}^n \tilde{\varphi}_i$ and is called the *ring automorphism* of $G(R)$ related to $\prod_{i=1}^n \varphi_i$. The automorphism $\prod_{i=1}^n \hat{\varphi}_i \in Aut\, E(R)$ is also called the *ring automorphism* of $E(R)$ related to $\prod_{i=1}^n \varphi_i$.

Theorem 1.5. *Let R and G be as in Theorem* 1.4*, then every automorphism α of $G(R)$ has an expression*

$$\alpha = Int\, g \cdot \tilde{\gamma} \cdot \prod_{i=1}^{n} \tilde{\varphi}_i \tag{1.5.1}$$

where $g \in G(R)$, $\gamma : \Phi \to \Phi$ is an automorphism of root system which keeps the fundamental root system Δ invariant and $\varphi_i \in Aut\, R$ for all $1 \le i \le n$. Moreover, g,γ and $\varphi_i\, (1 \le i \le n)$ are uniquely determined by α.

Remark 1.6. The automorphisms of $E(R)$ have similar expressions where $\tilde{\gamma}$ and $\prod_{i=1}^n \tilde{\varphi}_i$ are replaced by $\hat{\gamma}$ and $\prod_{i=1}^n \hat{\varphi}_i$ respectively (see Theorem 4.2).

2. Preliminaries

Let H be a group. If M and P are subgroups of H, we denote by $\mathcal{C}_P(M)$ and $\mathcal{N}_P(M)$ the centralizer and the normalizer of M in P respectively. The centre of H is denoted by $\mathcal{C}(H)$. A subgroup of H generated by subsets $M_1, M_2, \dots$ is written as $\langle M_1, M_2, \dots \rangle$ and $[M_1, M_2]$ stands for the subgroup of H generated by the elements of the form $xyx^{-1}y^{-1}$ for all $x \in M_1, y \in M_2$. If H is an algebraic group, we denote by $L(H)$ the Lie algebra of H. Suppose M is an abstract subgroup of H, we denote by $\overline{M}$ the Zariski closure of M in H and by $\overline{M}^\circ$ the connected component of $\overline{M}$ which contains the identity element of H. Throughout this paper we fix a universal domain K of $\mathbb{Q}$ and R (resp. R') stands for a subring of K which contains $\mathbb{Q}$.

In this section we give some preliminary properties of algebraic groups and Chevalley groups over a ring which are needed in the development of our discussion.

Lemma 2.1. *The Zariski closure of every infinite abstract simple subgroup of an algebraic group is connected.*

Proof. Suppose H is an infinite abstract simple subgroup of an algebraic group. Let ι be the natural embedding of H into its Zariski closure $\overline{H}$ and let π be the natural homomorphism from $\overline{H}$ to its quotient group $\overline{H}/\overline{H}^\circ$. Consider a composition of homomorphisms

$$H @>\iota>> \overline{H} @>\pi>> \overline{H}/\overline{H}^\circ.$$

Since $\overline{H}/\overline{H}^\circ$ is a finite group, $|H/\ \ker \pi\iota| < \infty$. This yields, since H is infinite and simple,

$$H = \ker \pi\iota = H \cap \overline{H}^\circ \subseteq \overline{H}.$$

Taking Zariski closures of the above groups, we obtain immediately the connectedness of $\overline{H}$. □

Let G be an adjoint Chevalley-Demazure group scheme with its root system Φ and fundamental root system Δ. We denote by $U(R)$ (resp. $U^-(R)$) the subgroup of $G(R)$ generated by $U_a(R)$ for all $a \in \Phi^+$ (resp. $-a \in \Phi^+$), where Φ^+ is the subset of positive roots of Φ. Let B (resp. B^-) be the normalizer of $U(K)$ (resp. $U^-(K)$) in $G(K)$, which is a Borel subgroup of $G(K)$ and let T be the maximal torus of $G(K)$ which is contained in both B and B^-. If G' is also an adjoint Chevalley-Demazure group scheme, we denote analogically by Φ', Δ', $U'_a(R)$ $(a \in \Phi')$, $U'(R)$ (resp. $U'^-(R)$), T' and B' for related root systems and subgroups. Suppose $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems such that $\gamma(\Delta) = \Delta'$, then γ gives rise to a homomorphism of algebraic groups $\bar{\gamma} : G(K) \to G'(K)$ defined by

$$\bar{\gamma}(u_a(k)) = u_{\gamma(a)}(k) \ for\ a \in \Delta \ or\ -\Delta,\ k \in K,$$

which is called the isogeny related to γ (cf. [6, exp.18]).

Lemma 2.2. *Suppose ε is an isogeny from $G(K)$ to $G'(K)$, then there exist an element $g \in G'(K)$ and an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that*

$$\varepsilon = Int\, g \cdot \bar{\gamma} \tag{2.2.1}$$

where $\bar{\gamma}$ is the isogeny related to γ.

Proof. Since $\varepsilon(B)$ is a Borel subgroup of $G'(K)$, there exists an element $g_1 \in G'(K)$ such that $Int\, g_1\varepsilon(B) = B'$. Moreover, $Int\, g_1\varepsilon(T)$ is a maximal torus of B' since T is contained in B. Hence there exists an element $g_2 \in B'$ such that

$$Int\,(g_2g_1)\varepsilon(T) = T'.$$

This, together with the fact that $\mathrm{char} K = 0$ and

$$Int\,(g_2g_1)\varepsilon(B) = B',$$

gives rise to an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that (cf. [6, exp.18])

$$Int\,(g_2g_1)\varepsilon(u_a(k)) = u_{\gamma(a)}(q_ak) \ for\ a \in \Delta \ or\ -\Delta,\ k \in K,$$

where $q_a \in K^*$. Since the fundamental roots in Δ' are linearly independent, there exists an element $t \in T'$ such that

$$\gamma(a)(t) = {q_a}^{-1} \ for\ a \in \Delta \ or\ -\Delta.$$

Let $g = (tg_2g_1)^{-1}$, we then have for each root $a \in \Delta$ or $-\Delta$

$$Int\, g\varepsilon(u_a(k)) = u_{\gamma(a)}(k) \ for\ k \in K,$$

from which follows (2.2.1) immediately. □

Lemma 2.3. (i) *$\overline{U(R)} = U(K)$; $\overline{U^-(R)} = U^-(K)$;*
(ii) *$\overline{T \cap E(R)} = T$;*
(iii) *$\overline{B \cap E(R)} = B$.*

Proof. (i) Suppose a is a positive root of Φ. Since $U_a(R)$ is an infinite group while $\overline{U_a(R)}/\overline{U_a(R)}^\circ$ is a finite group, $\overline{U_a(R)}^\circ$ must be infinite. In other words, $\dim \overline{U_a(R)}^\circ \geq 1$. On the other hand, since

$$\overline{U_a(R)}^\circ \subseteq \overline{U_a(R)} \subseteq U_a(K), \tag{2.3.1}$$

we have

$$\dim \overline{U_a(R)}^{\circ} \leq \dim U_a(K) = 1.$$

Hence $\dim \overline{U_a(R)}^{\circ} = 1$ and (2.3.1) implies that

$$\overline{U_a(R)} = U_a(K). \tag{2.3.2}$$

Therefore

$$\overline{U(R)} = \overline{\langle U_a(R) \mid \forall a \in \Phi^+ \rangle} = \langle \overline{U_a(R)} \mid \forall a \in \Phi^+ \rangle = U(K).$$

By taking negative roots instead of positive roots and by following a similar argument as above, we obtain also the Zariski density of $U^-(R)$ in $U^-(K)$.

(ii) Let $\{a_1, a_2, \ldots, a_n\}$ be the fundamental roots of Φ and write T_i for the one dimensional torus $T \cap \langle U_{a_i}(K), U_{-a_i}(K)\rangle$ for all $1 \leq i \leq n$. Then

$$T = \prod_{i=1}^{n} T_i. \tag{2.3.3}$$

Let $T_i(R)$ be the R-rational points of T_i for $1 \leq i \leq n$, then we have

$$T \cap E(R) \supseteq \prod_{i=1}^{n} T_i(R).$$

Note that $T_i(R)$ is Zariski dense in T_i by [1, Ch.V,Cor.18.3] since R contains rational field $\mathbb{Q}$. Hence we obtain from (2.3.3) that

$$T \supseteq \overline{T \cap E(R)} \supseteq \prod_{i=1}^{n} \overline{T_i(R)} = T.$$

This means that $T \cap E(R)$ is Zariski dense in T.

(iii) We have

$$B \supseteq B \cap E(R) \supseteq (T \cap E(R)) \cdot U(R).$$

This yields

$$B \supseteq \overline{B \cap E(R)} \supseteq \overline{(T \cap E(R)) \cdot U(R)} = T \cdot U = B.$$

Hence $B \cap E(R)$ is Zariski dense in B. □

Recall that the semisimple complex Lie algebra $L(G(\mathbb{C}))$ has a $\mathbb{Z}$-form $\mathfrak{g}$ with a Chevalley basis related to the root system Φ (cf. [10]). We denote by $\mathfrak{g}_R$ the R-Lie algebra $\mathfrak{g} \otimes_{\mathbb{Z}} R$ and let $ad : \mathfrak{g}_K \to M_n(K)$ be the adjoint representation of $\mathfrak{g}_K$, where n is the dimension of the Lie algebra $\mathfrak{g}_K$ over K and $M_n(K)$ is the algebra of $n \times n$ matrices over K.

Lemma 2.4. *Suppose z is an element of $\mathfrak{g}_K$ such that $ad(z) \in M_n(R)$, then z lies in $\mathfrak{g}_R$.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a Chevalley basis of $L(G(\mathbb{C}))$ related to Φ. Then $ad(e_i \otimes 1) \in M_n(\mathbb{Z})$ for all $1 \leq i \leq n$. Suppose z has an expression $\sum_{i-1}^{n} e_i \otimes k_i$, where $k_i \in K$ for all $1 \leq i \leq n$. Then

$$ad(z) = \sum_{i=1}^{n} ad(e_i \otimes k_i). \tag{2.4.1}$$

On the other hand, we way assume $ad(z) = (z_{pq}) \in M_n(R)$, where $z_{pq} \in R$ for all $1 \leq p \leq n, 1 \leq q \leq n$ and suppose $ad(e_i \otimes 1) = (e_{pq}^{(i)}) \in M_n(\mathbb{Z})$, where $e_{pq}^{(i)} \in \mathbb{Z}$ for all $1 \leq i \leq n, 1 \leq p \leq n$ and $1 \leq q \leq n$. Then the equation (2.4.1) implies the following n^2 equations:

$$\begin{aligned} z_{11} &= k_1 e_{11}^{(1)} + k_2 e_{11}^{(2)} + \cdots + k_n e_{11}^{(n)} \\ z_{12} &= k_1 e_{12}^{(1)} + k_2 e_{12}^{(2)} + \cdots + k_n e_{12}^{(n)} \\ &\cdots \\ z_{nn} &= k_1 e_{nn}^{(1)} + k_2 e_{nn}^{(2)} + \cdots + k_n e_{nn}^{(n)}. \end{aligned}$$

Since $ad(e_1 \otimes 1), ad(e_2 \otimes 1), \ldots, ad(e_n \otimes 1)$ are linearly independent, there are n linearly independent equations in the above system. Therefore the unique solution for $k_1, k_2, \ldots, k_n$ in the above equations is given by Cramer's rule as the quotient of the determinant of a matrix in $M_n(R)$ factored by the determinant of a matrix in $M_n(\mathbb{Z})$. Consequently, k_i lies in R for all $1 \leq i \leq n$, which implies that z belongs to $\mathfrak{g}_R$. □

Lemma 2.5. *Let g be an element of $G(K)$, then g lies in $G(R)$ if $Int\, g(u_a(1))$ belongs to $G(R)$ for all $a \in \Phi$.*

Proof. Let $\{e_a, h_b \,|\, \forall a \in \Phi, b \in \Delta\}$ be a Chevalley basis of the semisimple Lie algebra $L(G(\mathbb{C}))$, where $[e_b, e_{-b}] = h_b$ for $b \in \Delta$. Considering $G(K)$ as a subgroup of $GL_n(\mathfrak{g}_K)$ through the adjoint representation of $G(K)$ where $n = \dim \mathfrak{g}_K$, we obtain that $u_a(1) = \exp ad(e_a \otimes 1)$ for all $a \in \Phi$ (cf. [10]) and

$$Int\, g(u_a(1)) = \exp ad(g(e_a \otimes 1)) \tag{2.5.1}$$

where exp is the canonical exponential map which sends the nilpotent elements of $M_n(K)$ to the unipotent elements of $GL_n(K)$. Recall that the logarithm map log sends the unipotent subset of $M_n(R)$ to the nilpotent subset of $M_n(R)$ and the composite log·exp is the identity map on the nilpotent subset (cf. [3, Ch.II,6.1]). We have by (2.5.1)

$$\log(Int\, g(u_a(1))) = ad(g(e_a \otimes 1)) \ \in M_n(R) \ for\ a \in \Phi.$$

Hence $g(e_a \otimes 1)$ belongs to $\mathfrak{g}_R$ for all $a \in \Phi$ by Lemma 2.4. Moreover, we have

$$g(h_a \otimes 1) = [g(e_a \otimes 1), g(e_{-a} \otimes 1)] \ \in \mathfrak{g}_R \ for\ a \in \Delta.$$

Hence $g \in GL_n(\mathfrak{g}_R) \cap G(K) = G(R)$. □

Remark. Lemma 2.4 and Lemma 2.5 have been shown in [4] for the case when R is a Laurent polynomial ring over the complex number field.

Let a be a root in Φ, we denote by $\mathfrak{g}_a$ the root subspace of $\mathfrak{g}_K$ related to a and by $\mathfrak{u}$ the subalgebra generated by $\mathfrak{g}_a$ for all $a \in \Phi^+$. If $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}_K$, we denote by $\mathcal{C}_{\mathfrak{u}}(\mathfrak{b})$ the centralizer of $\mathfrak{b}$ in $\mathfrak{u}$.

Lemma 2.6. *Let a be a positive root and $I = \{c \in \Phi^+ \,|\, a + b \in \Phi^+ \Rightarrow c + b \in \Phi^+, \forall b \in \Phi^+\}$. Then*

$$\mathcal{C}_{U(K)}(\mathcal{C}_{U(K)}(U_a(K))) = \prod_{c \in I} U_c(K). \tag{2.6.1}$$

Proof. It is easily seen that the Lie algebra $L(U(K))$ of $U(K)$ is $\mathfrak{u}$, hence we have

$$L(\mathcal{C}_{U(K)}(U_a(K))) = \mathcal{C}_{\mathfrak{u}}(L(U_a(K))) = \mathcal{C}_{\mathfrak{u}}(\mathfrak{g}_a).$$

Note that $\mathcal{C}_{U(K)}(U_a(K))$ is connected since it is a T-stable closed subgroup of $U(K)$ (cf. [1, Ch.IV,14.4]). Then we have

$$\begin{aligned} L(\mathcal{C}_{U(K)}(\mathcal{C}_{U(K)}(U_a(K)))) &= \mathcal{C}_{\mathfrak{u}}(L(\mathcal{C}_{U(K)}(U_a(K)))) \\ &= \mathcal{C}_{\mathfrak{u}}(\mathcal{C}_{\mathfrak{u}}(\mathfrak{g}_a)) = \mathcal{C}_{\mathfrak{u}}(\sum_{\substack{b\in\Phi^+\\ a+b\notin\Phi}} \mathfrak{g}_b) = \bigcap_{\substack{b\in\Phi^+\\ a+b\notin\Phi}} \mathcal{C}_{\mathfrak{u}}(\mathfrak{g}_b) \\ &= \bigcap_{\substack{b\in\Phi^+\\ a+b\notin\Phi}} \sum_{\substack{c\in\Phi^+\\ b+c\notin\Phi}} \mathfrak{g}_c = \sum_{c\in I} \mathfrak{g}_c = L(\langle U_c(K)\,|\,\forall c\in I\rangle). \end{aligned}$$

Moreover, it is easily seen that

$$\langle U_c(K)\,|\,\forall c\in I\rangle = \prod_{c\in I} U_c(K).$$

Note that, since $\mathcal{C}_{U(K)}(\mathcal{C}_{U(K)}(U_a(K)))$ is a T-stable closed subgroup of $U(K)$, it is also connected. Thus (2.6.1) follows from the above identities since the correspondence between the connected subgroups of $U(K)$ and the Lie subalgebras of $\mathfrak{u}$ is bijective. □

Lemma 2.7. *Let a be a positive root, then*

$$\overline{\mathcal{C}_{U(R)}(U_a(\mathbb{Q}))} = \mathcal{C}_{U(K)}(U_a(K)). \tag{2.7.1}$$

Proof. It is obvious that $U_b(R) \subseteq \mathcal{C}_{U(R)}(U_a(\mathbb{Q}))$ for all $b\in\Phi^+$ with $a+b\notin\Phi$. Note that $U_a(\mathbb{Q})$ is a Zariski dense subgroup of $U_a(K)$ by (2.3.2). We then have

$$\begin{aligned} \langle U_b(R)\,|\,b\in\Phi^+, a+b\notin\Phi\rangle &\subseteq \mathcal{C}_{U(R)}(U_a(\mathbb{Q})) \\ &= \mathcal{C}_{U(R)}(U_a(K)) \subseteq \mathcal{C}_{U(K)}(U_a(K)). \end{aligned} \tag{2.7.2}$$

Moreover, since $U_b(R)$ is Zariski dense in $U_b(K)$ for all $b\in\Phi^+$ by (2.3.2), we have

$$\begin{aligned} \overline{\langle U_b(R)\,|\,b\in\Phi^+, a+b\notin\Phi\rangle} &= \langle \overline{U_b(R)}\,|\,b\in\Phi^+, a+b\notin\Phi\rangle \\ &= \langle U_b(K)\,|\,b\in\Phi^+, a+b\notin\Phi\rangle = \mathcal{C}_{U(K)}(U_a(K)). \end{aligned}$$

Therefore, taking Zariski closures of the subgroups in (2.7.2), we obtain immediately (2.7.1). □

Proposition 2.8. *Every normal subgroup of $G(R)$ that contains $E(\mathbb{Q})$ must contain the elementary subgroup $E(R)$.*

Proof. For each root $a\in\Phi$ and each element $q\in\mathbb{Q}^*$, let

$$h_a(q) = u_a(q)u_{-a}(-q^{-1})u_a(q)u_{-a}(1)u_a(-1)u_{-a}(1) \ \in T\cap E(\mathbb{Q}).$$

Then (cf. [7])

$$h_a(q)u_a(r)h_a(q)^{-1} = u_a(q^2r) \ \text{for } q\in\mathbb{Q}^*,\ r\in R.$$

Suppose H is a normal subgroup of $G(R)$ which contains $E(\mathbb{Q})$ and let $q\neq\pm1$, then for all $r\in R$ and $a\in\Phi$ we have

$$u_a(r) = h_a(q)u_a((q^2-1)^{-1}r)h_a(q)^{-1}u_a((q^2-1)^{-1}r)^{-1} \ \in H.$$

This implies that H contains $E(R)$. □

Proposition 2.9. *If α is an automorphism of $G(R)$ which fixes each element of $E(R)$, then α is the identity map on $G(R)$.*

Proof. Since $E(R)$ is a normal subgroup of $G(R)$ by [11], we have for all $g \in G(R)$

$$gxg^{-1} = \alpha(gxg^{-1}) = \alpha(g)x\alpha(g)^{-1} \ for\ x \in E(R).$$

This yields

$$(\alpha(g)^{-1}g)x = x(\alpha(g)^{-1}g) \ for\ x \in E(R),\ g \in G(R),$$

which means that, since $E(R)$ is Zariski dense in $G(K)$ (cf. [2]),

$$\alpha(g)^{-1}g \in \mathcal{C}_{G(R)}(E(R)) = \mathcal{C}(G(K)).$$

Note that $\mathcal{C}(G(K))$ is trivial since G is of adjoint type. Then we obtain

$$\alpha(g) = g \ for\ all\ g \in G(R).$$

□

3. Isomorphisms of simple Chevalley groups

In this section we assume that G (resp. G') is a simple adjoint Chevalley-Demazure group scheme with its root system Φ (resp. Φ') and fundamental root system Δ (resp. Δ') whose rank is greater than 1. Let R and R' stand for subrings of K containing $\mathbb{Q}$. The elementary subgroup of $G'(R')$ is denoted by $E'(R')$.

Lemma 3.1. *If there exists a nontrivial homomorphism from $E(\mathbb{Q})$ to $G'(K)$, then*

$$\dim G(K) = \dim G'(K).$$

Proof. See [5, Cor.2.4]. □

Lemma 3.2. *Let H be a connected algebraic group, then*

- (i) $\dim G(K) \leq \dim H$ *if there exists a nontrivial homomorphism from $E(\mathbb{Q})$ to H;*
- (ii) *the image of a nontrivial homomorphism from $E(\mathbb{Q})$ to H is Zariski dense in H if $\dim G(K)$ is equal to $\dim H$.*

Proof. (i) Let $\alpha : E(\mathbb{Q}) \to H$ be a nontrivial homomorphism. Since $E(\mathbb{Q})$ is a simple group, $\overline{\alpha(E(\mathbb{Q}))}$ is a connected and non-solvable subgroup of H by Lemma 2.1. Therefore, if $\Re$ is the solvable radical of $\overline{\alpha(E(\mathbb{Q}))}$, the quotient group $\overline{\alpha(E(\mathbb{Q}))}/\Re$ is a semisimple algebraic group of positive dimension. Let $\{H_i\}_{i=1}^m$ be the family of the simple components of $\overline{\alpha(E(\mathbb{Q}))}/\Re$ and let H_i^{ad} be an adjoint simple algebraic group of the same type as H_i for all $1 \leq i \leq m$. Then there exists an isogeny $\varepsilon : \overline{\alpha(E(\mathbb{Q}))}/\Re \to \prod_{i=1}^m H_i^{ad}$. Let π be the natural homomorphism from $\overline{\alpha(E(\mathbb{Q}))}$ to $\overline{\alpha(E(\mathbb{Q}))}/\Re$ and let p_j be the canonical projection of $\prod_{i=1}^m H_i^{ad}$ to the j-th factor H_j^{ad} for $1 \leq j \leq m$. Note that, since p_j $(1 \leq j \leq m)$, ε and π are homomorphisms which preserve the Zariski density, so does their composite $p_j\varepsilon\pi$. In particular we have for all $1 \leq j \leq m$

$$\overline{p_j\varepsilon\pi\alpha(E(\mathbb{Q}))} = p_j\varepsilon\pi(\overline{\alpha(E(\mathbb{Q}))}) = H_j^{ad},$$

which means that the composite $p_j\varepsilon\pi\alpha$ is a homomorphism from $E(\mathbb{Q})$ to H_j^{ad} with a Zariski dense image. It follows from Lemma 3.1 that for all $1 \leq j \leq m$

$$\dim G(K) = \dim H_j^{ad} = \dim H_j.$$

Hence

$$\dim G(K) \leq \overline{\alpha(E(\mathbb{Q}))}/\Re \leq \overline{\alpha(E(\mathbb{Q}))} \leq \dim H. \tag{3.2.1}$$

(ii) Suppose $G(K)$ and H have the same dimension, then it follows from (3.2.1) that for a nontrivial homomorphism $\alpha : E(\mathbb{Q}) \to H$ we have

$$\dim \overline{\alpha(E(\mathbb{Q}))} = \dim H.$$

Since H is connected, this implies by Lemma 2.1 that $\overline{\alpha(E(\mathbb{Q}))} = H$ as required. □

Corollary 3.3. *If $E(R)$ and $E'(R')$ are isomorphic to each other, then*

$$\dim G(K) = \dim G'(K).$$

Proof. This comes directly from Lemma 3.2(i). □

Proposition 3.4. *Suppose α is an isomorphism from $E(R)$ (resp. $G(R)$) to $E'(R')$ (resp. $G'(R')$), then there exist an element $g \in G'(R')$ and an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that*

$$Int\ g\alpha(u_a(q)) = u_{\gamma(a)}(q)\ for\ a \in \Delta\ or\ -\Delta,\ q \in \mathbb{Q}. \tag{3.4.1}$$

In particular

$$Int\ g\alpha(E(\mathbb{Q})) = E'(\mathbb{Q}). \tag{3.4.2}$$

Proof. Since $E(R)$ (resp. $G(R)$) and $E'(R')$ (resp. $G'(R')$) are isomorphic to each other, $G(K)$ and $G'(K)$ have the same dimension by Corollary 3.3. Hence the restriction of α to $E(\mathbb{Q})$, which is a nontrivial homomorphism from $E(\mathbb{Q})$ to $G'(K)$, has a Zariski dense image by Lemma 3.2. It follows from the Borel-Tits theorem [2, Th.A] that there exist a homomorphism of fields $\varphi : \mathbb{Q} \to K$ and an isogeny ε from ${}^{\varphi}G(K)$, the group obtained from the base change through φ, to $G'(K)$ such that

$$\alpha(x) = \varepsilon\varphi^{\circ}(x)\ for\ x \in E(\mathbb{Q})$$

where φ° is the canonical homomorphism from $G(K)$ to ${}^{\varphi}G(K)$ induced by φ (see [2] for the notation). Note that there is no other possibility for φ but of the natural embedding, which implies that φ° is the identity map. This yields

$$\alpha(x) = \varepsilon(x)\ for\ x \in E(\mathbb{Q}). \tag{3.4.3}$$

It follows from Lemma 2.2 that there exist an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and an element $g \in G'$ such that

$$\varepsilon = Int\, g^{-1} \cdot \bar{\gamma} \tag{3.4.4}$$

where $\bar{\gamma}$ is the isogeny from $G(K)$ to $G'(K)$ related to γ. Hence the identity (3.4.1) comes from the definition of $\bar{\gamma}$ and the fact that

$$Int\, g\alpha(u_a(q)) = Int\, g\varepsilon(u_a(q)) = \bar{\gamma}(u_a(q))\ for\ a \in \Delta\ or\ -\Delta,\ q \in \mathbb{Q}.$$

We claim that g lies in $G'(R')$. This is because, for each root $a' \in \Phi'$, we have by (3.4.3) and (3.4.4)

$$\begin{aligned} Int\, g^{-1}(u_{a'}(1)) &= Int\, g^{-1}\bar{\gamma}(\bar{\gamma}^{-1}(u_{a'}(1))) = \varepsilon(\bar{\gamma}^{-1}(u_{a'}(1))) \\ &= \alpha(\bar{\gamma}^{-1}(u_{a'}(1))) = \alpha(u_{\gamma^{-1}(a')}(1))\ \in G'(R'), \end{aligned}$$

which implies by Lemma 2.5 that g^{-1}, hence also g, lies in $G'(R')$. □

Lemma 3.5. *Suppose $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems with $\gamma(\Delta) = \Delta'$. If $\alpha : E(R) \to E'(R')$ is an isomorphism such that*

$$\alpha(u_a(q)) = u_{\gamma(a)}(q) \ for\ a \in \Delta \ or\ -\Delta,\ q \in \mathbb{Q},$$

then $\alpha(U(R)) = U'(R')$.

Proof. It follows from the definition of $\bar{\gamma}$ that $\bar{\gamma}(B) = B'$ and

$$\alpha(g) = \bar{\gamma}(g) \ for\ g \in E(\mathbb{Q}). \tag{3.5.1}$$

Therefore

$$\alpha(B \cap E(\mathbb{Q})) = \bar{\gamma}(B \cap E(\mathbb{Q})) = B' \cap E(\mathbb{Q}).$$

Hence

$$\alpha(B \cap E(R)) \supseteq \alpha(B \cap E(\mathbb{Q})) = B' \cap E(\mathbb{Q}).$$

By taking the Zariski closures of the above groups, we obtain from Lemma 2.3(iii)

$$\overline{\alpha(B \cap E(R))} \supseteq \overline{B' \cap E'(\mathbb{Q})} = B'.$$

However, since $\overline{\alpha(B \cap E(R))}$ is a solvable group, we have

$$\overline{\alpha(B \cap E(R))} = B'.$$

In particular, we obtain that

$$\alpha(U(R)) \subseteq \alpha(B \cap E(R)) \subseteq B' \cap E'(R'). \tag{3.5.2}$$

Let a be a positive root in Φ, we can choose an element $t \in T \cap E(\mathbb{Q})$ such that $a(t) \neq 1$ since $T \cap E(\mathbb{Q})$ is Zariski dense in T by Lemma 2.3(ii). Note that $a(t)$ lies in $\mathbb{Q}$, we have

$$u_a(r) = t u_a((a(t)-1)^{-1} r) t^{-1} u_a((a(t)-1)^{-1} r)^{-1} \ for\ r \in R. \tag{3.5.3}$$

This implies that

$$u_a(R) \subseteq [T \cap E(\mathbb{Q}),\, U(R)] \ for\ a \in \Phi^+.$$

Thus $U(R)$ is contained in $[T \cap E(\mathbb{Q}), U(R)]$. Hence we obtain by (3.5.1) and (3.5.2)

$$\begin{aligned} \alpha(U(R)) &\subseteq [\alpha(T \cap E(\mathbb{Q})),\, \alpha(U(R))] = [\tilde{\gamma}(T \cap E(\mathbb{Q})),\, \alpha(U(R))] \\ &= [T' \cap E'(\mathbb{Q}),\, \alpha(U(R))] \subseteq [T',\, B'] \cap E'(R') \\ &= U' \cap E'(R') = U'(R'). \end{aligned}$$

Replacing α by α^{-1} and following a similar argument as above, we obtain on the other hand that $\alpha(U(R)) \supseteq U'(R')$. Hence $\alpha(U(R))$ is equal to $U'(R')$ as required. □

Lemma 3.6. *Let α and γ be as in Lemma 3.5, then*

$$\alpha(U_a(R)) = U_{\gamma(a)}(R') \ for\ a \in \Phi. \tag{3.6.1}$$

Proof. We first show (3.6.1) for the case where a is a positive root. Using Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned} \mathcal{C}_{U(R)}(\mathcal{C}_{U(R)}(U_a(\mathbb{Q}))) &= U(R) \cap \mathcal{C}_{U(K)}(\mathcal{C}_{U(R)}(U_a(\mathbb{Q}))) \\ &= U(R) \cap \mathcal{C}_{U(K)}(\mathcal{C}_{U(K)}(U_a(K))) = U(R) \cap \prod_{c \in I} U_c(K), \end{aligned}$$

where I is as in Lemma 2.6. Moreover, since γ is an isomorphism of root systems, we also have by Lemma 2.6 that

$$\mathcal{C}_{U'(R')}(\mathcal{C}_{U'(R')}(U_{\gamma(a)}(\mathbb{Q}))) = U'(R') \cap \prod_{c \in I} U_{\gamma(c)}(K).$$

Hence, applying Lemma 3.5, we obtain that

$$\begin{aligned} \alpha(U_a(R)) &\subseteq \alpha(\mathcal{C}_{U(R)}(\mathcal{C}_{U(R)}(U_a(\mathbb{Q})))) \\ &= \mathcal{C}_{U'(R')}(\mathcal{C}_{U'(R')}(\alpha(U_a(\mathbb{Q})))) = U'(R') \cap \prod_{c \in I} U_{\gamma(c)}(K). \end{aligned} \tag{3.6.2}$$

Suppose $I = \{c_1, c_2, \ldots, c_m\}$ where $c_1 = a$. If $m = 1$, then

$$\alpha(U_a(R)) \subseteq U'(R') \cap U_{\gamma(a)}(K) = U_{\gamma(a)}(R'),$$

from which follows (3.6.1) since α is an isomorphism. Suppose $m \geq 2$. Then $(\ker c_m)^\circ - \ker a$ is an open subset of $(\ker c_m)^\circ$. Note that, since $(\ker c_m)^\circ$ splits over $\mathbb{Q}$ (cf. [1, Ch.III, Cor.8.7]), $(\ker c_m)^\circ \cap E(\mathbb{Q})$ is Zariski dense in $(\ker c_m)^\circ$ by [2, Cor.6.8]. Therefore

$$\{(\ker c_m)^\circ - \ker a\} \cap E(\mathbb{Q}) = \{(\ker c_m)^\circ \cap E(\mathbb{Q})\} \cap \{(\ker c_m)^\circ - \ker a\} \neq \emptyset.$$

Let $t \in \{(\ker c_m)^\circ - \ker a\} \cap E(\mathbb{Q})$. Then the coincidence of the restrictions of α and $\bar{\gamma}$ to $E(\mathbb{Q})$ implies that $\alpha(t)$ lies in T' since $\bar{\gamma}(T) = T'$ (see §2 for the notation). Moreover, for any root $b \in \Phi$, t lies in $\ker b$ if and only if $\alpha(t)$ lies in $\ker \gamma(b)$ because

$$\begin{aligned} u_{\gamma(b)}(b(t)) &= \alpha(u_b(b(t))) = \alpha(t u_b(1) t^{-1}) \\ &= \alpha(t) u_{\gamma(b)}(1) \alpha(t)^{-1} = u_{\gamma(b)}(\gamma(b)(\alpha(t))). \end{aligned}$$

Therefore, $\alpha(t)$ lies in $\{\ker \gamma(c_m) - \ker \gamma(a)\} \cap E'(\mathbb{Q})$. This yields that

$$[\alpha(t), \prod_{i=1}^{m} U_{\gamma(c_i)}(K)] \subseteq \prod_{i=1}^{m-1} U_{\gamma(c_i)}(K).$$

Note that $U_a(R) = [t, U_a(R)]$ by (3.5.3). We then have by (3.6.2)

$$\begin{aligned} \alpha(U_a(R)) &= [\alpha(t), \alpha(U_a(R))] \\ &\subseteq [\alpha(t), U'(R') \cap \prod_{i=1}^{m} U_{\gamma(c_i)}(K)] \subseteq U'(R') \cap \prod_{i=1}^{m-1} U_{\gamma(c_i)}(K). \end{aligned}$$

This results in (3.6.1) if $m = 2$. When $m \geq 3$, (3.6.1) follows from the repetitions of analogous arguments as above.

We show now that (3.6.1) holds also for all $-a$, where $a \in \Phi^+$. Let $w_a = u_a(1) u_{-a}(-1) u_a(1)$ for $a \in \Phi^+$. Then we have $Int\, w_a(U_a(R)) = U_{-a}(R)$. Note that for all $a \in \Phi^+$

$$\alpha(w_a) = \tilde{\gamma}(w_a) = w_{\gamma(a)}.$$

This yields

$$\alpha(U_{-a}(R)) = Int\, w_{\gamma(a)}(\alpha(U_a(R))) = Int\, w_{\gamma(a)}(U_{\gamma(a)}(R')) = U_{-\gamma(a)}(R').$$

Hence (3.6.1) holds for all $a \in \Phi$. □

Let α and γ be as in Lemma 3.5. Thanks to Lemma 3.6, we can assign a map $\varphi_a : R \to R'$ to each root $a \in \Phi$ satisfying

$$\alpha(u_a(r)) = u_{\gamma(a)}(\varphi_a(r)) \ for\ r \in \Phi.$$

It is easily seen that φ_a is an isomorphism between the additive groups R^+ and R'^+.

Lemma 3.7. *For each root a in Φ, φ_a is an isomorphism of rings and*

$$\varphi_a = \varphi_b \ for\ b \in \Phi. \tag{3.7.1}$$

Proof. We first consider the case where a is a fundamental root. Since G is not of type A_1, there exists a positive root b such that $a + b \in \Phi$. We have by the commutator formula [7, Exp.XXII,§5]

$$u_a(r)u_b(s)u_a(r)^{-1}u_b(s)^{-1} = u_{a+b}(n_{a,b}rs) \prod_{\substack{c\in\Phi^+\\ h(c)>h(a+b)}} u_c(r_c) \ for\ r, s \in R,$$

where $n_{a,b}$ is an integer determined uniquely by a and b while $r_c \in R$, and h is the height function of Φ. Applying α on both sides, we obtain that

$$u_{\gamma(a)}(\varphi_a(r))u_{\gamma(b)}(\varphi_b(s))u_{\gamma(a)}(\varphi_a(r))^{-1}u_{\gamma(b)}(\varphi_b(s))^{-1} = u_{\gamma(a+b)}(n_{a,b}\varphi_{a+b}(rs))u$$

where u is a product of elements of the form $u_{\gamma(c)}(p)$ for some positive root c such that $h(c) > h(a + b)$ and for some $p \in R'$. On the other hand, it follows from the commutator formula that

$$u_{\gamma(a)}(\varphi_a(r))u_{\gamma(b)}(\varphi_b(s))u_{\gamma(a)}(\varphi_a(r))^{-1}u_{\gamma(b)}(\varphi_b(s))^{-1} = u_{\gamma(a+b)}(n_{a,b}\varphi_a(r)\varphi_b(s))u_1$$

where u_1 is also a product of elements of the form $u_{\gamma(c)}(p)$ for some $c \in \Phi^+$ with $h(c) > h(a + b)$ and $p \in R'$. Note that, if $h' : \Phi' \to \mathbb{Z}$ is the height function of Φ', then $h'(\gamma(c)) > h'(\gamma(a + b))$ for all the factors $u_{\gamma(c)}(p)$ of u (resp. u_1). Thus, comparing these two identities, we have

$$\varphi_{a+b}(rs) = \varphi_a(r)\varphi_b(s) \quad for\ r, s \in R. \tag{3.7.2}$$

Taking r and s to be 1 alternately, we obtain that $\varphi_{a+b} = \varphi_a = \varphi_b$. Note that for each fundamental root $c \in \Delta$ there exists a sequence of fundamental roots

$$a = a_1, a_2, \ldots, a_m = c$$

such that $a_i + a_{i+1} \in \Phi$ for all $1 \le i \le m - 1$. Hence we have, by following similar arguments as above, that

$$\varphi_a = \varphi_{a_2} = \cdots = \varphi_{a_m} = \varphi_c.$$

Thus we may simply write φ in stead of φ_a for all $a \in \Delta$. It follows from (3.7.2) that $\varphi(rs) = \varphi(r)\varphi(s)$ for all $a \in \Delta$, which means that φ is a homomorphism of rings and therefore is an isomorphism of rings.

We show now that

$$\varphi_a = \varphi \ for\ a \in \Phi^+. \tag{3.7.3}$$

We use induction on the height of the roots. Suppose a is not a fundamental root and $\varphi_c = \varphi$ for all $c \in \Phi^+$ such that $h(c) < h(a)$. Since a can be written as the sum of two positive roots, say b and c, with $h(b) < h(a)$ and $h(c) < h(a)$, we have

$$u_b(r)u_c(s)u_b(r)^{-1}u_c(s)^{-1} = u_a(n_{b,c}rs)v \ for \ r, s \in R, \tag{3.7.4}$$

where $n_{b,c}$ is an integer which depends only on b and c, while v is a product of elements of the form $u_d(p)$ for some $p \in R$ and $d \in \Phi^+$ such that $h(d) > h(a)$, or equivalently $h'(\gamma(d)) > h'(\gamma(a))$. Applying α on both sides of (3.7.4) and using the induction hypothesis, we obtain

$$u_{\gamma(b)}(\varphi(r))u_{\gamma(c)}(\varphi(s))u_{\gamma(b)}(\varphi(r))^{-1}u_{\gamma(c)}(\varphi(s))^{-1} = u_{\gamma(a)}(n_{b,c}\varphi_a(rs))v_1$$

where v_1 is a product of elements that involves only those positive roots of which the height is greater than $h'(\gamma(a))$. On the other hand, we have by the commutator formula that

$$u_{\gamma(b)}(\varphi(r))u_{\gamma(c)}(\varphi(s))u_{\gamma(b)}(\varphi(r))^{-1}u_{\gamma(c)}(\varphi(s))^{-1} = u_{\gamma(a)}(n_{b,c}\varphi(r)\varphi(s))v_2$$

where v_2 is also a product of elements involving only the positive roots of which the height is greater than $h'(\gamma(a))$. Comparing these two identities, we obtain immediately that

$$\varphi_a(rs) = \varphi(r)\varphi(s) \ for \ r, s \in R.$$

This yields (3.7.3) when s is the identity element.

Finally we show that $\varphi_a = \varphi$ for all negative root a, from which follows (3.7.1). Suppose a is a negative root, then for all $r \in R$

$$u_a(r) = w_{-a}u_{-a}(-r){w_{-a}}^{-1}.$$

Applying α on both sides of the identity, we have

$$u_{\gamma(a)}(\varphi_a(r)) = w_{-\gamma(a)}u_{\gamma(a)}(-\varphi(r)){w_{-\gamma(a)}}^{-1} = u_{\gamma(a)}(\varphi(r))$$

which implies that $\varphi_a = \varphi$. This completes our proof. □

Corollary 3.8. *Let α and γ be as in Lemma* 3.5*, then there exists an isomorphism of rings $\varphi : R \to R'$ such that $\alpha = \hat{\varphi}$.*

Proof. This is a consequence of Lemma 3.6, Lemma 3.7 and the definition of $\hat{\varphi}$. □

Theorem 3.9. *If $\alpha : E(R) \to E'(R')$ is an isomorphism, then there exist an element $g \in G'(R')$, an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and an isomorphism of rings $\varphi : R \to R'$ such that*

$$\alpha = Int\, g \cdot \hat{\gamma} \cdot \hat{\varphi}. \tag{3.9.1}$$

Moreover, g, γ and φ are uniquely determined by α.

Proof. It follows from Proposition 3.4 that there exist an element $g \in G'(R')$ and an isomorphism of root systems $\gamma : \Phi \to \Phi'$ such that

$$Int\, g^{-1}\, \alpha(u_a(q)) = u_{\gamma(a)}(q) = \hat{\gamma}(u_a(q)) \ for \ a \in \Delta \ or \ -\Delta,\ q \in \mathbb{Q}.$$

Since $E'(R')$ is a normal subgroup of $G'(R')$ (cf. [11]), $Int\, g^{-1}\alpha$ is an isomorphism from $E(R)$ to $E'(R')$. Hence $\hat{\gamma}^{-1} \cdot Int\, g^{-1} \cdot \alpha$ is also an isomorphism from $E(R)$ to $E'(R')$. Therefore by Corollary 3.8 there exists an isomorphism of rings $\varphi : R \to R'$ such that

$$\hat{\gamma}^{-1} \cdot Int\, g^{-1} \cdot \alpha = \hat{\varphi}$$

from which follows (3.9.1). Suppose there exist an element $g_1 \in G'(R')$, an isomorphism of root systems $\gamma_1 : \Phi \to \Phi'$ with $\gamma_1(\Delta) = \Delta'$ and an isomorphism of rings $\varphi_1 : R \to R'$ such that

$$\alpha = Int\, g \cdot \hat{\gamma} \cdot \hat{\varphi} = Int\, g_1 \cdot \hat{\gamma}_1 \cdot \hat{\varphi}_1,$$

then

$$Int\, g_1^{-1} g = \hat{\gamma}_1 \cdot \hat{\varphi}_1 \cdot \hat{\varphi}^{-1} \cdot \hat{\gamma}^{-1}. \tag{3.9.2}$$

Let $U'(R')$ (resp. $U'^-(R')$) be the subgroup of $G'(R')$ generated by $u_a(r)$ for all $a \in \Phi'$ (resp. $-a \in \Phi'$) and $r \in R'$. Since

$$\hat{\gamma}(U(R')) = \hat{\gamma}_1(U(R')) = U'(R')$$

and

$$\hat{\varphi}(U(R)) = \hat{\varphi}_1(U(R)) = U(R'),$$

we have by (3.9.2)

$$Int\, g_1^{-1} g(U'(R')) = U'(R').$$

Similarly we also have

$$Int\, g_1^{-1} g(U'^-(R')) = U'^-(R').$$

Therefore, if we denote by B'^- the opposite Borel subgroup of B', then

$$\begin{aligned} g_1^{-1} g &\in \mathcal{N}_{G'(R')}(U'(R')) \cap \mathcal{N}_{G'(R')}(U'^-(R')) \\ &\subseteq \mathcal{N}_{G'(R')}(\overline{U'(R')}) \cap \mathcal{N}_{G'(R')}(\overline{U'^-(R')}) = \mathcal{N}_{G'(R')}(U'(K)) \cap \mathcal{N}_{G'(R')}(U'^-(K)) \\ &= G'(R') \cap \mathcal{N}_{G'(K)}(U'(K)) \cap \mathcal{N}_{G'(K)}(U'^-(K)) \\ &= G'(R') \cap B' \cap B'^- = G'(R') \cap T'. \end{aligned}$$

This yields that, for each fundamental root $a \in \Delta'$,

$$Int\, g_1^{-1} g(u_a(1)) = u_a(a(g_1^{-1} g)).$$

On the other hand, we have

$$\hat{\gamma}_1 \hat{\varphi}_1 \hat{\varphi}^{-1} \hat{\gamma}^{-1}(u_a(1)) = u_{\gamma_1 \gamma^{-1}(a)}(1) \ for\ a \in \Delta'.$$

Comparing these two identities, we obtain that $\gamma_1 = \gamma$ and $a(g_1^{-1} g) = 1$ for all $a \in \Delta'$, which means that

$$g_1^{-1} g \in \bigcap_{a \in \Delta'} \ker a = \mathcal{C}(G'(K)).$$

This implies immediately that $g_1 = g$ and that, by (3.9.2), $\varphi_1 = \varphi$. Hence the expression (3.9.1) of α is unique. □

Proof of Theorem 1.1. (i) Suppose $\alpha : E(R) \to E'(R')$ is an isomorphism. it follows from Theorem 3.9 that α has an expression of the form $Int\, g \cdot \hat{\gamma} \cdot \hat{\varphi}$ where $g \in G'(R')$, $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems with $\gamma(\Delta) = \Delta'$ and $\varphi : R \to R'$ is an isomorphism of rings. It is evident from the definitions that $\hat{\gamma}$ can be extended to the graph isomorphism $\tilde{\gamma}$ from $G(R')$ to $G'(R')$ and that $\hat{\varphi}$ can be extended to the ring isomorphism $\tilde{\varphi}$ from $G(R)$ to $G(R')$. Hence α can be extended to an isomorphism $\tilde{\alpha}$ from $G(R)$ to $G'(R')$ in an obvious way. If $\bar{\alpha} : G(R) \to G'(R')$ is an isomorphism which is also an extension of α, then $\tilde{\alpha} \cdot \bar{\alpha}^{-1}$ is an automorphism of $G(R)$ which fixes each element of $E(R)$ and, therefore, $\tilde{\alpha} = \bar{\alpha}$ by Proposition 2.9. Thus the extension of α to an isomorphism between $G(R)$ and $G'(R')$ is unique.

(ii) It follows from Proposition 3.4 that there exists an element $g \in G'(R')$ such that

$$\alpha(E(\mathbb{Q})) = Int\, g(E'(\mathbb{Q})).$$

Thus H is a normal subgroup of $G(R)$ which contains $E(\mathbb{Q})$ if and only if $\alpha(H)$ is a normal subgroup of $G'(R')$ containing $E'(\mathbb{Q})$. This implies that α induces a bijection between the set of normal subgroups of $G(R)$ containing $E(\mathbb{Q})$, which is denoted by N, and the set of normal subgroups of $G'(R')$ containing $E'(\mathbb{Q})$, which is denoted by N'. Note that by Proposition 2.8

$$E(R) = \bigcap_{H \in N} H.$$

Hence we have

$$\alpha(E(R)) = \bigcap_{H \in N} \alpha(H) = \bigcap_{H' \in N'} H' = E'(R').$$

.

Proof of Theorem 1.2. It follows from Theorem 1.1(ii) and Theorem 3.9 that the restriction $\alpha|_{E(R)}$ of α to $E(R)$ is an isomorphism between $E(R)$ and $E'(R')$ which has an expression of the form $Int\, g \cdot \hat{\gamma} \cdot \hat{\varphi}$ where $g \in G'(R')$, $\hat{\gamma}$ is a graph isomorphism from $E(R')$ to $E'(R')$ related to an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and $\hat{\varphi}$ is a ring isomorphism from $E(R)$ to $E(R')$ related to an isomorphism of rings $\varphi : R \to R'$. Thus $\alpha|_{E(R)}$ can be extended to an isomorphism from $G(R)$ to $G'(R')$ by extending $\hat{\gamma}$ (resp. $\hat{\varphi}$) to $\tilde{\gamma}$ (resp. $\tilde{\varphi}$). This extension of $\alpha|_{E(R)}$ has the form $Int\, g \cdot \tilde{\gamma} \cdot \tilde{\varphi}$ and is equal to α by Theorem 1.1(i). The uniqueness of the elements g, γ and φ comes directly from Theorem 3.9.

4. Automorphisms of $G(R)$ and $E(R)$

In this section, we assume that G is an adjoint Chevalley-Demazure group scheme that has no simple component of type A_1. Let $\{G_i\}_{i=1}^n$ be the simple components of G and Φ_i (resp. Δ_i) be the root (resp. fundamental root) system of G_i for all $1 \le i \le n$. Denote by $E_i(R)$ the elementary subgroup of $G_i(R)$ for all $1 \le i \le n$.

Proposition 4.1. *Suppose H is either $E(R)$ or $G(R)$ and α is an automorphism of H. Then there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that*

(i) $\alpha(E_i(R)) = E_{\sigma(i)}(R)$ *for* $1 \le i \le n$;
(ii) $\alpha(G_i(R)) = G_{\sigma(i)}(R)$ *if* $\alpha \in Aut\, G(R)$.

Proof. We show first that $\alpha(E(\mathbb{Q}))$ is a Zariski dense subset of $G(K)$. If $n = 1$, we obtain by Lemma 2.1 and Lemma 3.2(i) that $\dim\, G \le \dim\, \overline{\alpha(E(\mathbb{Q}))}$. Since $\overline{\alpha(E(\mathbb{Q}))}$ is a subgroup of $G(K)$, this implies that

$$\dim\, G = \dim \overline{\alpha(E(\mathbb{Q}))}.$$

Thus we have immediately the Zariski density of $\alpha(E(\mathbb{Q}))$ in $G(K)$ since $\overline{\alpha(E(\mathbb{Q}))}$ is connected by Lemma 2.1. Suppose $n > 1$, then

$$E_i(\mathbb{Q}) \subseteq \mathcal{C}_{E(R)}(E_j(\mathbb{Q})) \; for\; 1 \le i \ne j \le n. \tag{4.1.1}$$

Hence

$$\overline{\alpha(E_i(\mathbb{Q}))} \subseteq \mathcal{C}_{G(K)}(\overline{\alpha(E_j(\mathbb{Q}))}) \ for\ 1 \leq i \neq j \leq n. \tag{4.1.2}$$

Therefore, since $E(\mathbb{Q})$ is the direct product of $E_i(\mathbb{Q})$ for all $1 \leq i \leq n$, we obtain

$$\overline{\alpha(E(\mathbb{Q}))} = \overline{\alpha(E_1(\mathbb{Q}))} \cdot \overline{\alpha(E_2(\mathbb{Q}))} \ldots \overline{\alpha(E_n(\mathbb{Q}))}. \tag{4.1.3}$$

Hence $\overline{\alpha(E(\mathbb{Q}))}$ is connected since each $\overline{\alpha(E_i(\mathbb{Q}))}$ is connected for $1 \leq i \leq n$ by Lemma 2.1. Let $\Re$ be the solvable radical of $\overline{\alpha(E(\mathbb{Q}))}$ and let Y_i be the quotient group of $\Re \cdot \overline{\alpha(E_i(\mathbb{Q}))}$ modulo $\Re$ for all $1 \leq i \leq n$, then

$$\overline{\alpha(E(\mathbb{Q}))}/\Re = Y_1 \cdot Y_2 \ldots Y_n. \tag{4.1.4}$$

It is obvious that Y_i is a semisimple normal subgroup of $\overline{\alpha(E(\mathbb{Q}))}/\Re$ for all $1 \leq i \leq n$. Moreover $[Y_i, Y_j]$ is trivial for all $1 \leq i \neq j \leq n$ since

$$[\overline{\alpha(E_i(\mathbb{Q}))}, \overline{\alpha(E_j(\mathbb{Q}))}] = \{1\}.$$

This implies that $|Y_i \cap Y_j| < \infty$ for all $1 \leq i \neq j \leq n$ since $\overline{\alpha(E(\mathbb{Q}))}/\Re$ is semisimple. Thus (4.1.4) yields

$$\dim \overline{\alpha(E(\mathbb{Q}))}/\Re = \sum_{i=1}^{n} \dim Y_i. \tag{4.1.5}$$

Let $\Re_i$ be the solvable radical of $\overline{\alpha(E_i(\mathbb{Q}))}$ for $1 \leq i \leq n$. Note that

$$\Re_i = \Re \cap \overline{\alpha(E_i(\mathbb{Q}))};\ Y_i \cong \overline{\alpha(E_i(\mathbb{Q}))}/\Re_i.$$

We obtain from (4.1.5) that

$$\dim \overline{\alpha(E(\mathbb{Q}))}/\Re = \sum_{i=1}^{n} \dim \overline{\alpha(E_i(\mathbb{Q}))}/\Re_i. \tag{4.1.6}$$

Let π_i $(1 \leq i \leq n)$ be the natural homomorphism from $\overline{\alpha(E_i(\mathbb{Q}))}$ to its quotient group $\overline{\alpha(E_i(\mathbb{Q}))}/\Re_i$. Note that the restriction of $\pi_i \cdot \alpha$ to $E_i(\mathbb{Q})$ is nontrivial. We obtain from Lemma 3.2(i)

$$\dim G_i \leq \dim \overline{\alpha(E_i(\mathbb{Q}))}/\Re_i \ for\ 1 \leq i \leq n.$$

Thus we have from (4.1.6) that

$$\dim G \leq \sum_{i=1}^{n} \dim \overline{\alpha(E(\mathbb{Q}))}/\Re_i \leq \dim \overline{\alpha(E(\mathbb{Q}))} \leq \dim G.$$

This forces

$$\overline{\alpha(E(\mathbb{Q}))} = G(K).$$

We show now that for each $i \in \{1, 2, \ldots, n\}$, $\overline{\alpha(E_i(\mathbb{Q}))}$ is a simple component of $G(K)$. From the above identity and (4.1.3) we have

$$G(K) = \overline{\alpha(E_1(\mathbb{Q}))} \cdot \overline{\alpha(E_2(\mathbb{Q}))} \ldots \overline{\alpha(E_n(\mathbb{Q}))}.$$

Then (4.1.2) implies that $\overline{\alpha(E_i(\mathbb{Q}))}$ is a normal subgroup of $G(K)$ for all $1 \leq i \leq n$ and

$$\overline{\alpha(E_i(\mathbb{Q}))} \cap \overline{\alpha(E_j(\mathbb{Q}))} \subseteq \mathcal{C}(G(K)) \ for\ 1 \leq i \neq j \leq n. \tag{4.1.7}$$

Note that $\overline{\alpha(E_i(\mathbb{Q}))}$ is of positive dimension for all $1 \le i \le n$. Hence each $\overline{\alpha(E_i(\mathbb{Q}))}$ contains at least one simple component of $G(K)$ and, meanwhile, is the direct product of all those simple components which are contained in $\overline{\alpha(E_i(\mathbb{Q}))}$. Moreover, (4.1.7) implies that each simple component $G_k(K)$ $(1 \le k \le n)$ lies in at most one $\overline{\alpha(E_i(\mathbb{Q}))}$ for some $1 \le i \le n$. Since $G(K)$ has exact n different simple components, each $\overline{\alpha(E_i(\mathbb{Q}))}$ is in fact a simple component of $G(K)$ for all $1 \le i \le n$. In other words, there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that for all $1 \le i \le n$

$$\overline{\alpha(E_i(\mathbb{Q}))} = G_{\sigma(i)}(K). \tag{4.1.8}$$

Now we come to show (ii). Note that for all $1 \le i \ne j \le n$, $[G_i(R), E_j(\mathbb{Q})]$ is trivial and we have

$$G_i(R) \subseteq \mathcal{C}_{G(K)}(\prod_{j\ne i}^{n} E_j(\mathbb{Q})) \ for \ 1 \le i \le n.$$

Hence

$$\begin{aligned}\alpha(G_i(R)) \subseteq \mathcal{C}_{G(K)}(\prod_{j\ne i}^{n} \alpha(E_j(\mathbb{Q}))) &= \mathcal{C}_{G(K)}(\prod_{j\ne i}^{n} \overline{\alpha(E_j(\mathbb{Q}))})\\ &= \mathcal{C}_{G(K)}(\prod_{j\ne i}^{n} G_{\sigma(j)}(K)) = G_{\sigma(i)}(K).\end{aligned}$$

Consequently

$$\alpha(G_i(R)) \subseteq G(R) \cap G_{\sigma(i)}(K) = G_{\sigma(i)}(R) \ for \ 1 \le i \le n.$$

By taking α^{-1} instead of α and by following a similar argument as above, we obtain on the other hand that $\alpha(G_{\sigma(i)}(R)) \subseteq G_i(R)$ for all $1 \le i \le n$. Hence $\alpha(G_i(R)) = G_{\sigma(i)}(R)$ for all $1 \le i \le n$.

Finally we show (i). If α is an automorphism of $G(R)$, then (i) comes as a consequence of the above (ii) and Theorem 1.1(ii). Suppose α is an automorphism of $E(R)$. Note that

$$E_i(R) \subseteq \mathcal{C}_{E(R)}(\prod_{j\ne i}^{n} E_j(\mathbb{Q})) \ for \ 1 \le i \le n.$$

We have, by using the identity (4.1.8),

$$\begin{aligned}\alpha(E_i(R)) \subseteq \mathcal{C}_{E(R)}(\prod_{j\ne i}^{n} \alpha(E_j(\mathbb{Q}))) &= \mathcal{C}_{E(R)}(\prod_{j\ne i}^{n} \overline{\alpha(E_j(\mathbb{Q}))})\\ &= E(R) \cap \mathcal{C}_{G(K)}(\prod_{j\ne i}^{n} G_{\sigma(j)}(K)) = E(R) \cap G_{\sigma(i)}(K) = E_{\sigma(i)}(R).\end{aligned}$$

Since α is an automorphism, we obtain that $\alpha(E_i(R)) = E_{\sigma(i)}(R)$ for all $1 \le i \le n$ as required. □

Theorem 4.2. *Suppose α is an automorphism of $E(R)$, then there exist an element $g \in G(R)$, an automorphism of root system $\gamma : \Phi \to \Phi$ which keeps fundamental* *root system Δ invariant and an automorphism $\varphi_i \in Aut\, R$ for each $1 \le i \le n$ such that*

$$\alpha = Int\, g \cdot \hat{\gamma} \cdot \prod_{i=1}^{n} \hat{\varphi}_i. \tag{4.2.1}$$

Moreover, g, γ and φ_i $(1 \le i \le n)$ are uniquely determined by α.

Proof. It is known from Proposition 4.1 that for each $1 \le i \le n$, the restriction of α to $E_i(R)$ is an isomorphism from $E_i(R)$ to $E_{\sigma(i)}(R)$ for some permutation σ of $\{1, 2, \dots, n\}$. Hence by theorem 3.9 there exist an element $g_i \in G_{\sigma(i)}(R)$, an isomorphism of root system $\gamma_i : \Phi_i \to \Phi_{\sigma(i)}$ with $\gamma_i(\Delta_i) = \Delta_{\sigma(i)}$ and an automorphism $\varphi_i \in Aut\, R$ such that the restriction of α to $E_i(R)$ has an expression

$$\alpha|_{E_i(R)} = Int\, g_i \cdot \hat{\gamma}_i \cdot \hat{\varphi}_i \ for\ 1 \le i \le n. \tag{4.2.2}$$

Since $\Phi = \bigcup_{i=1}^{n} \Phi_i$, it is easily seen that the isomorphisms $\gamma_1, \gamma_2, \dots, \gamma_n$, being pieced together, induce an automorphism of root system $\gamma : \Phi \to \Phi$ defined by

$$\gamma(a) = \gamma_i(a)\ for\ a \in \Phi_i,\ 1 \le i \le n,$$

which keeps the fundamental root system Δ invariant. Moreover, we have by the definition of the graph automorphism that for all $1 \le i \le n$

$$\hat{\gamma}(x) = \hat{\gamma}_i(x)\ for\ x \in E_i(R). \tag{4.2.3}$$

Suppose x is an arbitrary element of $E(R)$, we may assume that $x = x_1 x_2 \dots x_n$, where $x_i \in E_i(R)$ for all $1 \le i \le n$. Then we have by (4.2.2)

$$\alpha(x) = \prod_{i=1}^{n} Int\, g_i \hat{\gamma}_i \hat{\varphi}_i(x_i). \tag{4.2.4}$$

Note that, since $G(R)$ is the direct product of $G_i(R)$ for all $1 \le i \le n$, we have for each $i \in \{1, 2, \dots, n\}$

$$Int\, g_1 Int\, g_2 \dots Int\, g_n \hat{\gamma}_i \hat{\varphi}_i(x_i) = Int\, g_i \hat{\gamma}_i \hat{\varphi}_i(x_i).$$

Let $g = \prod_{i=1}^{n} g_n$, then the identities (4.2.3) and (4.2.4) yield

$$\alpha(x) = Int\, g \hat{\gamma}(\prod_{i=1}^{n} \hat{\varphi}_i(x_i)) = Int\, g \hat{\gamma}(\prod_{i=1}^{n} \hat{\varphi}_i)(x)\ for\ x \in E(R),$$

from which follows immediately (4.2.1). □

Proof of Theorem 1.4. (i) It follows from Theorem 4.2 that every automorphism α of $E(R)$ has an expression of the form $Int\, g \cdot \hat{\gamma} \cdot \prod_{i=1}^{n} \hat{\varphi}_i$ for some $g \in G(R), \gamma \in Aut\, \Phi$ with $\gamma(\Delta) = \Delta$ and $\varphi_i \in Aut\, R$ for $1 \le i \le n$. Since $\hat{\gamma}$ and $\prod_{i=1}^{n} \hat{\varphi}_i$ have the extensions $\tilde{\gamma}$ and $\prod_{i=1}^{n} \tilde{\varphi}_i$ in $Aut\, G(R)$ respectively, α can be extended to an automorphisms of $G(R)$ in an obvious way.

Suppose $\tilde{\alpha}$ and $\breve{\alpha}$ are automorphisms of $G(R)$ and the both are extensions of α, then $\tilde{\alpha} \cdot \breve{\alpha}^{-1}$ is an automorphism of $G(R)$ which fixes every element of $E(R)$. Hence $\tilde{\alpha} = \breve{\alpha}$ by Proposition 2.9. Thus the extension of α is unique.

(ii) Suppose α is an automorphism of $G(R)$, then by Proposition 4.1(i) there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $\alpha(E_i(R)) = E_{\sigma(i)}(R)$. Hence

$$\alpha(E(R)) = \prod_{i=1}^{n} \alpha(E_i(R)) = \prod_{i=1}^{n} E_{\sigma(i)}(R) = E(R).$$

Proof of Theorem 1.5. Since the restriction of α to $E(R)$ induces an automorphism of $E(R)$ by Theorem 1.4(ii), it follows from Theorem 4.2 that there exist an element $g \in G(R)$, a graph automorphism $\hat{\gamma} \in Aut\, E(R)$ related to an automorphism of root system $\gamma : \Phi \to \Phi$ with $\gamma(\Delta) = \Delta$ and a ring automorphism $\hat{\varphi}_i$ of $E(R)$ related to an automorphism $\varphi_i \in Aut\, R$ for each $1 \leq i \leq n$ such that

$$\alpha|_{E(R)} = Int\, g \cdot \hat{\gamma} \cdot \prod_{i=1}^{n} \hat{\varphi}_i.$$

It is easily seen from the definitions that the graph automorphism $\tilde{\gamma}$ of $G(R)$ is an extension of $\hat{\gamma}$ while the ring automorphism $\prod_{i=1}^{n} \tilde{\varphi}_i$ is an extension of $\prod_{i=1}^{n} \hat{\varphi}_i$, hence the automorphism $Int\, g \cdot \tilde{\gamma} \cdot \prod_{i=1}^{n} \tilde{\varphi}_i$ is an extension of $\alpha|_{E(R)}$. Since the extension of $\alpha|_{E(R)}$ is unique by Theorem 1.4(i), we obtain immediately the expression (1.5.1). Moreover, we have the uniqueness of g,γ and φ_i $(1 \leq i \leq n)$ because, by Theorem 4.2 and Theorem 1.4(ii), all of them are uniquely determined by the restriction $\alpha|_{E(R)}$ which is, as a consequence of Theorem 1.4(i), uniquely determined by α. This completes our proof.

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