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A CONSTRUCTION OF THE LEVEL 3 MODULES FOR THE AFFINE LIE ALGEBRA $A_2^{(2)}$ AND A NEW COMBINATORIAL IDENTITY OF THE ROGERS-RAMANUJAN TYPE

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Abstract. We obtain a vertex operator construction of level 3 standard representations for the affine Lie algebra $A_2^{(2)}$. As a corollary, we also get new conbinatorial identities.

0. Introduction

The identities of Rogers and Ramanujan have a famous and interesting history. They also seem to appear at the crossroads of various mathematical disciplines, as is symptomatic of an important piece of mathematics. These identities play an important role in the representation theory of affine algebras and have also made their appearance in an important work of R.J. Baxter on the hard hexagon model in statistical mechanics [Ba].

Some years ago, it was observed in [LM] that the character of certain representations of affine Kac-Moody algebras, when suitably specialized, coincided with one member of the classical Rogers-Ramanujan identities. The explanation of this fact was given by J. Lepowsky and R.L. Wilson in a series of papers [LW1-4] They constructed the standard representations of the affine Lie algebra $A_1^{(1)}$. The representations of level 3, in particular, yielded the interpretation and proof of the Rogers-Ramanujan identities. The main new concept used in their work was that of "vertex operator" which has proved to be extremely fruitful, and has been also used, for example, in the construction of a natural representation of the Monster simple group (see [FLM]).

Lepowsky and Wilson showed the remarkable interaction between the representation theory of affine Lie algebras and combinatorial identities and provided a general framework for the study of this relationship. This approach has been used successfully in a number of works, [LP1-2],[Mi1-3], [MP], [Ma]. Among these, the work of A.Meurman and M.Primc [MP], gave a construction of all standard $A_1^{(1)}$-modules and a Lie-theoretic proof of the generalizations to all moduli of the Rogers-Ramanujan identities due to G.E. Andrews, B. Gordon, D. Bressoud, H. Gölnitz, [A],[G], [Br], [Gö]. This line of research, however, had not produced any combinatorial identities that were unknown to the specialists.

In a previous work, [C2], we had stated certain identities which we believed to underlie the standard representations of level 3 for the affine algebra $A_2^{(2)}$. The Lie

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theoretic proof of such identities is the object of this paper. Here we give a vertex operator construction of the level 3 standard $A_2^{(2)}$- modules and give a proof of the independence of the set of vectors provided in [C2].

While this work was in progress, Andrews gave an interesting combinatorial proof of one of these identities,[A2]. His method of proof is based on certain techniques that he developed in his study of Schur's theorem. It is extremely interesting to note that Andrews' motivation for such a study was the search of a combinatorial interpretation of the q-trinomial coefficients, that he and Baxter had developed during their research about certain generalizations of the hard hexagon model.

It is clear that a whole infinite family of possibly new combinatorial identities corresponding to the higher standard $A_2^{(2)}$-modules remains to be investigated. This is presently under study. It will be important to investigate also the possible applications to the study of statistical mechanics models. Recently in [MP2] our combinatorial identities have appeared again in the study of a different specialization of the fundamental $A_1^{(1)}$-modules.

1. The algebra $A_2^{(2)}$

We construct the affine algebra $A_2^{(2)}$ using the results of [L1].

Let Φ be an A_2 root system, $\Delta = \{\alpha_1, \alpha_2\}$ a base of Φ, and L the corresponding root lattice. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate form on L normalized so that $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in \Phi$. Let σ be the automorphism induced by the Dynkin diagram automorphism determined by $\sigma\alpha_1 = \alpha_2$, $\sigma\alpha_2 = \alpha_1$. Denote by σ_1 the reflection with respect to α_1 and consider

$$\nu = \sigma_1 \sigma. \tag{1.1}$$

Then ν has order 6 and is a "twisted Coxeter element" (see [Sp], [Fi]). It is known that twisted and untwisted Coxeter elements satisfy the properties

$$\Sigma \nu^p \alpha = 0, \tag{1.2}$$

$$\Sigma p \langle \nu^p \alpha, \beta \rangle \equiv 0 \quad \text{modulo} \quad M \tag{1.3}$$

for all $\alpha, \beta \in L$, M the order of ν, summations ranging over $\mathbf{Z}_M$.

In this setting the central extension

$$1 \longrightarrow \langle \kappa | \kappa^6 = 1 \rangle \longrightarrow \hat{L} \xrightarrow{\ \bar{}\ } L \longrightarrow 1 \tag{1.4}$$

determined by the commutator map

$$C(\alpha, \beta) = (-1)^{\langle \Sigma \nu^p \alpha, \beta \rangle} \omega^{\langle \Sigma p \nu^p \alpha, \beta \rangle} \tag{1.5}$$

splits since $C(\alpha, \beta) = 1$, ω being a primitive sixth-root of 1.

Hence $\hat{L}$ is the direct product $L \times \langle \kappa \rangle$.

Set

$$Q = (1 - \nu) L. \tag{1.6}$$

Observe that since $1-\nu=\nu^{-1}$ we have $Q=L$. Now Theorem 6.2 in [L1] gives a unique homomorphism $\psi:\hat{L}\to\mathbf{C}^*$ such that $\psi(\kappa)=\omega$ and L acts as the identity.

Set $T=\mathbf{C}_\psi$, the one-dimensional $\hat{L}$-module affording ψ. Let $\underline{h}$ be the complexification of L, i.e.

$$\underline{h}=L\otimes_{\mathbf{Z}}\mathbf{C} \tag{1.7}$$

and extend ν to $\underline{h}$ by linearity. Then $\underline{h}$ splits into two ν-eigenspaces

$$\underline{h}=\underline{h}_{(1)}\oplus\underline{h}_{(-1)}. \tag{1.8}$$

Form the affine algebra

$$\tilde{\underline{h}}[\nu]=\coprod_{n\in\mathbf{Z}}\underline{h}_{(n)}\otimes t^n\oplus\mathbf{C}c\oplus\mathbf{C}d \tag{1.9}$$

with brackets

$$\begin{aligned}[x\otimes t^i,y\otimes t^j]&=\langle x,y\rangle i\delta_{i+j,0}c,\\ [d,x\otimes t^i]&=ix\otimes t^i\end{aligned} \tag{1.10}$$

for $i,j\in\mathbf{Z},x\in\underline{h}_{(i)},y\in\underline{h}_{(j)}$, and c central. Denote by $\underline{s}$ its commutator subalgebra

$$\underline{s}=\coprod_{n\neq0}\underline{h}_{(n)}\otimes t^n\oplus\mathbf{C}c \tag{1.11}$$

and consider the subalgebras

$$\underline{s}_\pm=\coprod_{\pm n>0}\underline{h}_{(n)}\otimes t^n, \tag{1.12}$$

$$\underline{b}=\underline{b}[\nu]=\underline{s}_+\oplus\mathbf{C}c\oplus\mathbf{C}d \tag{1.13}$$

Then we may observe that the commutator subalgebra of $\underline{s}$ is one-dimensional and coincides with $\mathbf{C}c$. Hence $\underline{s}$ is a Heisenberg Lie algebra.

It is well-known that the induced $\tilde{\underline{h}}[\nu]$-module

$$\mathcal{U}(\tilde{\underline{h}}[\nu])\otimes_{\mathcal{U}(\underline{b})}\mathbf{C}, \tag{1.14}$$

where $\underline{b}$ acts on $\mathbf{C}$ trivially except for c which acts as the identity, is an irreducible representation of $\underline{s}$. As vector spaces, (1.14) is isomorphic to

$$S=S(\underline{s}_-), \tag{1.15}$$

the symmetric algebra on $\underline{s}_-$.

The action of d defines a grading on S. For $n\in\mathbf{Z},x\in\underline{h}_{(n)}$, write $x(n)$ for the operator on S corresponding to $x\otimes t^n$. For $\alpha\in\underline{h}$ denote by $\alpha_{(n)}$ the projection of α onto $\underline{h}_{(n)}$. Define, for $\alpha\in\underline{h}$,

$$E^\pm(\alpha,z)=\exp\Big(\sum_{\pm n>0}\alpha_{(n)}(n)\frac{z^{-n}}{n}\Big) \tag{1.16}$$

where $n \in \mathbf{Z}$, a pair of formal Laurent series with coefficients in $\operatorname{End} S$. Set

$$V_T = S \otimes \mathbf{C}. \tag{1.17}$$

We can finally define, for any $a \in \hat{L}$, the vertex operator

$$X(a, z) = 6^{-\langle \bar{a}, \bar{a} \rangle} \sigma(\bar{a}) E^{-}(-\bar{a}, z) E^{+}(-\bar{a}, z) a \tag{1.18}$$

where $\sigma(\bar{a})$ is a (nonzero) normalization constant,

$$\sigma(\alpha) = 2^{-\langle \alpha, \alpha \rangle} (1 - \omega^{-1})^{\langle \nu\alpha, \alpha \rangle} (1 - \omega^{-2})^{\langle \nu^2 \alpha, \alpha \rangle} \tag{1.19}$$

for $\alpha \in L$.

Also set, for $\alpha \in L$,

$$\alpha(z) = \sum_{n \in \mathbf{Z}} \alpha(n) z^{-n-1}. \tag{1.20}$$

Let $\hat{\nu}$ denote the lifting to $\hat{L}$ of ν with the properties

$$(\hat{\nu} a)^{-} = \nu \bar{a}, \tag{1.21}$$

$$\nu \bar{a} = \bar{a} \Rightarrow \hat{\nu} a = a \tag{1.22}$$

It is important to remark that

$$X(\hat{\nu} a, z) = \lim_{z \to \omega^{-1} z} X(a, z) \tag{1.23}$$

where the limit notation on the right hand side indicates the substitution of $\omega^{-1} z$ for z.

This property is important for the following developments and it constitutes the motivation for the definition of Q in (1.6).

We can now state the theorem that gives the commutation of two vertex operators in this setting. It is a particular case of Theorem 8.1 in [L1].

Theorem 1.1. *Let $a \in \hat{L}$, such that $\bar{a} \in L_2 = \{\alpha \in L; \langle \alpha, \alpha \rangle = 2\}$. Then*

$$\begin{aligned}
[X(a, z_1), X(a, z_2)] = {} & \frac{1}{6} \varepsilon_2(\nu^2 \bar{a}, \bar{a}) X(\hat{\nu} a, z_2) \delta(\omega^{-2} z_2 / z_1) \\
& + \frac{1}{6} \varepsilon_2(\nu^{-2} \bar{a}, \bar{a}) X(\hat{\nu}^{-1} a, z_2) \delta(\omega^{-4} z_2 / z_1) \\
& + \frac{1}{6^2} \varepsilon_2(-\bar{a}, \bar{a}) D \delta(-z_2 / z_1) \\
& - \frac{1}{6} \varepsilon_2(-\bar{a}, \bar{a}) z_2 \bar{a}(z_2) \delta(-z_2 / z_1),
\end{aligned}$$

where

$$\varepsilon_2(\alpha, \beta) = (-1)^{\langle \nu^{-1} \alpha + \nu^{-2} \alpha, \beta \rangle} \omega^{\langle \nu^{-1} \alpha + 2\nu^{-2} \alpha, \beta \rangle}.$$

Remark. Notice that, because of (1.23), this result will determine the commutator of $X(a, z_1)$ and $X(b, z_2)$ for any pair $a, b \in \hat{L}_2$.

Define a nonassociative algebra $(\underline{g}, [\cdot,\cdot])$ over $\mathbf{C}$ as follows:

$$\underline{g} = \underline{h} \oplus \coprod_{a \in \hat{L}_2} \mathbf{C}x_a, \tag{1.24}$$

where $\{x_a\}_{a\in\hat{L}_2}$ is a set of symbols, subject only to the linear relation $x_{\kappa a} = \omega x_a$, and

$$[\underline{h}, \underline{h}] = 0, [h, x_a] = \langle h, \bar{a}\rangle x_a = -[x_a, h],$$

$$[x_a, x_b] = \begin{cases} \varepsilon_2(\bar{a}, -\bar{a})\bar{a} & \text{if } ab = 1 \\ \varepsilon_2(\bar{a}, \bar{b})x_{ab} & \text{if } ab \in \hat{L}_2 \\ 0 & \text{if } ab \notin \hat{L}_2 \cup \{1, k, \ldots, k^5\} \end{cases}$$

for $a, b \in \hat{L}_2, h \in \underline{h}$.

It can be shown that $\underline{g}$ is a Lie algebra (see [FK,Se] and cf. [FLM], Theorem 6.2.1).

We extend the form $\langle\cdot,\cdot\rangle$ to a form on $\underline{g}$:

$$\langle \underline{h}, x_a\rangle = \langle x_a, \underline{h}\rangle = 0, \quad \text{for} \quad a \in \hat{L}_2,$$

$$\langle x_a, x_b\rangle = \begin{cases} \varepsilon_2(\bar{a}, -\bar{a}) & \text{if} \quad ab = 1 \\ 0 & \text{if} \quad ab \notin \{1, k, \ldots, k^5\}. \end{cases}$$

Then $\langle\cdot,\cdot\rangle$ is nonsingular and symmetric. We also extend ν to a linear automorphism ν of $\underline{g}$ by

$$\nu x_a = x_{\hat{\nu} a}.$$

Notice that $\nu^6 = 1$ on $\underline{g}$ and ν preserves $\langle\cdot,\cdot\rangle$ and is a Lie algebra automorphism.

Since L_2 spans $\underline{h}$, $\underline{g}$ is a semisimple Lie algebra.

We can then construct the twisted affine Lie algebra

$$\tilde{\underline{g}}[\nu] = \coprod_{\mu \in \frac{1}{6}\mathbf{Z}} \underline{g}_{(\nu)} \otimes t^n \oplus \mathbf{C}c \oplus \mathbf{C}d \tag{1.25}$$

with brackets

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + \langle x, y\rangle i\delta_{i+j,0}c$$

where $i, j \in \mathbf{Z}, x \in \underline{g}_{(i)}, y \in \underline{g}_{(j)}$.

The Lie algebra $\tilde{\underline{g}}[\nu]$ is isomorphic to the principal realization of the affine Lie algebra of type $A_2^{(2)}$ (cf. [Fi]).

Let $\{h_0, h_1, f_0, f_1, e_0, e_1\}$ be a set of canonical generators for $\tilde{\underline{g}}[\nu]$. Recall that $c = 2h_0 + h_1$. Define also the elements h_0^*, h_1^* such that

$$h_i^*(h_j) = \delta_{ij}.$$

2. Preliminary definitions and notations

Let us denote by $\mathcal{P}$ the set of all sequences of integers $\mu = (m_1, \dots, m_s)$ where $s \geq 0, m_1 \leq \dots \leq m_s < 0$. For $\mu \in \mathcal{P}$ set $|\mu| = m_1 + \dots + m_s$, $\ell(\mu) = s$. We shall give $\mathcal{P}$ an ordering by declaring that $\mu < \nu$ if one of the following conditions holds:

(i) $\ell(\mu) > \ell(\nu)$,
(ii) $\ell(\mu) = \ell(\nu)$ and $|\mu| < |\nu|$,
(iii) $\ell(\mu) = \ell(\nu), |\mu| = |\nu|$ and $m_1 = n_1, \dots, m_{i-1} = n_{i-1}, m_i > n_i$ for some $\ell \leq i \leq \ell(\mu)$, where $\nu = (n_1, \dots, n_t)$.

It is obvious that $\mathcal{P}$ is well ordered and that $\mu = \emptyset$, the empty sequence, is the largest element in $\mathcal{P}$.

For fixed integers n and p, the sequences in $\mathcal{P}$ with $|\mu| = n$ and $\ell(\mu) = p$ form a finite set, therefore there is a smallest sequence, denote it by $(p; n)$, and denote by $\{p; n\}$ the next to the last, if any. It is easy to verify that $(p; n) = (m_1, \dots, m_p)$ is characterized by having

$$0 \leq m_p - m_1 \leq 1$$

while $\{p; n\}$ is characterized by

$$2 \leq m_p - m_1 \leq 3.$$

We work in the setting and with the notation of [C3].

Recall the following

Theorem 2.1 (Cf. Theorem 20 in [C3]). *If $L(\Lambda)$ is a standard module for the affine Lie algebra of type $A_2^{(2)}$ and v_0 is a maximal vector in it, then the set of vectors of the form*

$$\bar{b}(\lambda/6)X(b; \mu/6)v_0$$

such that $\mu \in \mathcal{P}, \lambda \in \mathcal{O}$ and

(i) $m_{i+1} - m_i \geq 2$,
(ii) $2 \leq m_{i+1} - m_i \leq 3 \Rightarrow m_{i+1} + m_i \equiv 0$ *modulo* 3,
(iii) -1 *is not an entry of μ if $\Lambda = 3h_1^*$ or -2 is not an entry of μ if $\Lambda = h_0^* + h_1^*$,*

is a spanning set for $L(\Lambda)$.

Recall that $\mathcal{O}$ is the subset of $\mathcal{P}$ consisting of those sequences whose entries are congruent to ± 1 modulo 6.

The aim of this work is to prove that this set of vectors is actually a basis of a level 3 standard module.

To prove this we consider the Verma module $V = M(\Lambda)$, with $\Lambda = 3h_1^*$ or $\Lambda = h_1^* + h_0^*$ and in this module we consider the span V' of a set of vectors parametrized by the elements of $\mathcal{P}$ that do not satisfy the conditions of Theorem 2.1.

If we prove that this subspace V' coincides with the maximal submodule $W(\Lambda)$ of V then our conclusion follows.

Indeed, $L(\Lambda) \simeq V/W(\Lambda)$, and Theorem 2.1 implies that, up to the factor $\bar{b}(\lambda)$, for any integer n, the dimension of the component of degree n of the standard module is less than or equal to the number of partitions of n satisfying the conditions stated. If our conclusion is true then we would have also that the dimension of the component of degree n of the maximal proper submodule of the Verma module is less than or equal to the number of partitions of n that do not satisfy the conditions.

On the other hand, the dimensions of the standard module and of the maximal proper submodule add up to the dimension of the Verma module which is parametrized by the partitions of n. Hence we must have that, for any n the dimension of $L(\Lambda)_n$ equals the number of partitions of n satisfying the conditions of the theorem. So we have independence.

Therefore all we have to do is to show that a set of vectors parametrized by partitions not satisfying the conditions of Theorem 2.1 spans $W(\Lambda)$.

For this, it is enough to prove that

1. The span V' of such a set of vectors is a submodule of V;
2. It is not the zero submodule (see Lemma 4.10);
3. It is contained in $W(\Lambda)$ (immediate);
4. It contains the generators of $W(\Lambda)$ (Prop. 4.11).

If 1-4 are true then $V' = W(\Lambda)$. The most difficult point is (1). So we concentrate on that.

Consider the set $\mathcal{B}$ of all sequences

$$\mu = (i_1, \dots, i_r, n_1, n_2, j_1, \dots, j_s) \tag{2.1}$$

such that $i_1 \le i_2 \le \dots \le j_s \le 0$ and one and only one pair (n_1, n_2) is colored differently, either "red" or "blue", according to whether the colored pair (n_1, n_2) is the minimal partition of $n_1 + n_2$ into two parts (as for example $(-8, -7)$ is the minimal partition of -15 into two parts) or it is the next to the minimal (e.g. $(-9, -6)$). If it is necessary, we indicate a red pair by including it into brackets and a blue pair by including it into braces (e.g. $[-8, -7]$, and $\{-9, -6\}$).

So $\mathcal{B}$ is obtained from $\mathcal{P}$ by coloring a pair of consecutive entries. Then we define a map

$$f_\Lambda : \mathcal{P} \longrightarrow \mathcal{B}$$

as follows.

If $\mu = (m_1, \dots, m_s) \in \mathcal{P}$ does not satisfy the conditions (i) and (ii) of Theorem 2.1, let (m_i, m_{i+1}) be the first pair (i.e. with smallest index i), where we find a violation, then set $f_\Lambda(\mu)$ equal to μ where we color the pair (m_i, m_{i+1}). Hence $f_\Lambda(\mu) \in \mathcal{B}$.

On the other hand, if μ satisfies the conditions of Theorem 2.1, then set

$$f_\Lambda(\mu) = (m_1, \dots, m_s, 0, 0)$$

where $(0, 0)$ is colored red.

Finally, if $\Lambda = 3h_1^*$ and if $m_s = -1$, then either this is the only entry equal to -1 or not. If m_s is the only entry equal to -1 then set

$$f_\Lambda(\mu) = (m_1, \dots, m_{s-1}, m_s, 0)$$

where $(m_s, 0)$ is red; if not then $f_\Lambda(\mu)$ is defined already since $(-1, -1)$ is a violation.

Instead, if $\Lambda = h_1^* + h_0^*$ and $m_i = -2$ for some i, then either $i < s$ or $i = s$. If $i < s$ then μ does not satisfy the conditions (i) and (ii) of Theorem 2.1, and so $f_\Lambda(\mu)$ is already defined. If $i = s$ is the only index for which $m_i = -2$ then set $f_\Lambda(\mu) = (m_1, \dots, m_{s-1}, m_s, 0)$ where $(m_s, 0)$ is colored blue. Then f_Λ is clearly injective. We now define a good ordering on $\mathcal{B}$. Let $\mu, \nu \in \mathcal{B}$, $\mu =$

$(m_1, \dots, m_r, M_{r+1}, M_{r+2}, \dots, m_s)$, $\nu = (n_1, \dots, n_t, N_{t+1}, N_{t+2}, \dots, n_u)$, where (M_{r+1}, M_{r+2}) and (N_{t+1}, M_{t+2}) are the colored pairs.

We shall write $\mu < \nu$ if $\mu \neq \nu$ and one of the following statements is true:

(i) $\ell(\mu) > \ell(\nu)$,

(ii) $\ell(\mu) = \ell(\nu), |\mu| < |\nu|$,

(iii) $\ell(\mu) = \ell(\nu), |\mu| = |\nu|$, and $m_1 = n_1, \dots, m_{i-1} = n_{i-1}, m_i > n_i$, for some $1 \leq i \leq l(\mu)$,

(iv) $\ell(\mu) = \ell(\nu), |\mu| = |\nu|, m_i = n_i$, for $i = 1, \dots, l(\nu)$ and $r > t$.

Set $\mathcal{B}_\Lambda$ to be Imf_Λ.

Observe that $\mathcal{B}_\Lambda$ is in a one-to-one correspondence, via f_Λ, with those partitions that do not satisfy the conditions of Theorem 2.1.

If $\mu \in \mathcal{B}$ is as in (2.1), we agree to denote by $X(\mu)$ the monomial

$$X(i_1) \cdots X(i_r) T(n_1 + n_2) X(j_1) \cdots X(j_s)$$

where for an integer k, $X(k)$ is the coefficient of z^{-k} in $X(b, z)$ and $T(n_1 + n_2)$ is the coefficient of $z^{-(n_1+n_2)}$ in one of the two "annihilating elements" (see [C2],[C3] and [C4]):

$$R_2(z) = R_2(b, z) = \frac{1}{2\omega} X(b, z)^{(2)} - 12(1+\omega^{-1}) E^-(-b, z) X(b, -z) E^+(-b, z), \tag{2.2}$$

$$R(z) = R(b, \hat{\nu} b; z) = X(b, \hat{\nu} b; z) - E^-(-b, z) X(b, \nu^{-1} b; -z) E^+(-b, z) \tag{2.3}$$

depending on whether the pair (n_1, n_2) is minimal or next to minimal respectively (red or blue in our terminology).

Remark. The formal power series (2.2) and (2.3) have coefficients in a suitable completion of the universal enveloping algebra (cf. [MP]). This pair of series were used in [C3] to prove the spanning result of Theorem 2.1. In particular, (2.2) gives the "difference-two condition" and (2.3) gives the "congruence condition" of Theorem 2.1.

Finally, if $T(z)$ is a relation such as (2.2) or (2.3) for example, then by $\tilde{T}(z)$ we mean the "conjugate" relation:

$$\tilde{T}(z) = -E^-(-\bar{b}, z) T(-z) E^+(-\bar{b}, z).$$

3. The commutation relations

To continue we need to know the commutation relations of various elements.

Recall the notations of [L],[C2], ω is a primitive sixth root of unity.

Theorem 3.1. *The following relation among formal power series is true*

$$\begin{aligned}
&[X(z), R_2(w)] = \\
&\omega^2 \sigma(\bar{b})^{-1} R(w) \delta(\omega^{-2} z/w) \\
&- \omega^2 \sigma(\bar{b})^{-1} \tilde{R}(w) \delta(\omega^{-4} z/w) \\
&+ 4\sigma(\bar{b}) \tilde{R}_2(w) D\delta(-z/w) \\
&- 12\sigma(\bar{b}) : \bar{b}(w) \tilde{R}_2(w) : \delta(-z/w) \\
&- 2\sigma(\bar{b}) D_w \tilde{R}_2(w) \delta(-z/w).
\end{aligned}$$

Proof. We start by computing the bracket of $X(b,z_1)$ with $X(b,z_2)^{(2)}$. Here the symbol "lim" signifies to set the variables w_1 and w_2 equal to z_2.

$$
\begin{aligned}
&[X(b,z_1),X(b,z_2)^{(2)}] = \lim P(w_1,w_2)[X(b,z_1),X(b,w_1)X(b,w_2)]\\
&= \lim P\{[X(b,z_1),X(b,w_1)]X(b,w_2) + X(b,w_1)[X(b,z_1)X(b,w_2)]\}\\
&= \lim P(\{\frac{\omega^2}{6}X(\hat{\nu}b,w_1)\delta(\omega^{-2}w_1/z_1)\\
&- \frac{\omega^2}{6}X(\hat{\nu}^{-1}b,w_1)\delta(\omega^{-4}w_1/z_1)\\
&- \frac{\omega}{6}\bar{b}(w_1)\delta(-w_1/z_1)\\
&+ \frac{\omega}{6^2}cD\delta(-w_1/z_1)\}X(b,w_2)\\
&+ X(b,w_1)\{\frac{\omega^2}{6}X(\hat{\nu}b,w_2)\delta(\omega^{-2}w_2/z_1)\\
&- \frac{\omega^2}{6}X(\hat{\nu}^{-1}b,w_2)\delta(\omega^{-4}w_2/z_1)\\
&- \frac{\omega}{6}\bar{b}(w_2)\delta(-w_2/z_1)\\
&+ \frac{\omega}{6^2}cD\delta(-w_2/z_1)\}).
\end{aligned}
$$

We shall examine and compute separately four different summands of this expression, the computation of (a), (b), and (d) being immediate consequences of the definitions.

(a)

$$
\begin{aligned}
&\lim P\frac{\omega^2}{6}\{X(\hat{\nu}b,w_1)X(b,w_2)\delta(\omega^{-2}w_1/z_1)\\
&+ X(b,w_1)X(\hat{\nu}b,w_2)\delta(\omega^{-2}w_2/z_1)\}\\
&= \frac{\omega^2}{6}2X(b,\hat{\nu}b,z_2)\delta(\omega^{-2}z_2/z_1)
\end{aligned}
$$

(b)

$$
\begin{aligned}
&-\lim P\frac{\omega^2}{6}\{X(\hat{\nu}^{-1}b,w_1)X(b,w_2)\delta(\omega^{-4}w_1/z_1)\\
&+ X(b,w_1)X(\hat{\nu}^{-1}b,w_2)\delta(\omega^{-4}w_2/z_1)\}\\
&= -\frac{\omega^2}{6}2X(b,\hat{\nu}^{-1}b,z_2)\delta(\omega^{-4}z_2/z_1)
\end{aligned}
$$

(c)

$$
\begin{aligned}
&-\lim P\frac{\omega}{6}\{\bar{b}(w_1)X(b,w_2)\delta(-w_1/z_1)\\
&+ X(b,w_1)\bar{b}(w_2)\delta(-w_2/z_1)\}
\end{aligned}
$$

To compute this limit we split each $\bar{b}(w_1)$ and $\bar{b}(w_2)$ into the positive and negative part:

$$
\begin{aligned}
&-\lim P\frac{\omega}{6}\{\bar b^-(w_1)X(b,w_2)\delta(-w_1/z_1)\\
&+\bar b^+(w_1)X(b,w_2)\delta(-w_1/z_1)\\
&+X(b,w_1)\bar b^-(w_2)\delta(-w_2/z_1)\\
&+X(b,w_1)\bar b^+(w_2)\delta(-w_2/z_1)\}
\end{aligned}
$$

and then we normal order them. We get:

$$
\begin{aligned}
&-\lim P\frac{\omega}{6}\{\bar b^-(w_1)X(b,w_2)\delta(-w_1/z_1)\\
&+X(b,w_2)\bar b^+(w_1)\delta(-w_1/z_1)\\
&+[\bar b^+(w_1),X(b,w_2)]\delta(-w_1/z_1)\\
&+[X(b,w_1),\bar b^-(w_2)]\delta(-w_2/z_1)\\
&+\bar b^-(w_2)X(b,w_1)\delta(-w_2/z_1)\\
&+X(b,w_1)\bar b^+(w_2)\delta(-w_2/z_1)\}
\end{aligned}
$$

Recalling the results of [LW], we can compute the brackets in this expression:

$$
\begin{aligned}
&-\lim P\frac{\omega}{6}\{[\bar b^+(w_1),X(b,w_2)]\delta(-w_1/z_1)\\
&+[X(b,w_1),\bar b^-(w_2)]\delta(-w_2/z_1)\}\\
&=-\frac{\omega}{6}\lim\ P\{\sum_{n>0}(\frac{w_2}{w_1})^n\delta(-w_1/z_1)X(b,w_2)\\
&-\sum_{n<0}(\frac{w_1}{w_2})^n\delta(-w_2/z_2)X(b,w_1)\}
\end{aligned}
$$

(where the index n in each summation ranges over those integers congruent to ± 1 modulo 6)

$$
=-\frac{\omega}{6}\lim\ P(\sum_{n<0}(\frac{w_1}{w_2})^n)\{X(b,w_2)\delta(-w_1/z_1)-X(b,w_1)\delta(-w_2/z_1)\}
$$

Observe that

$$
\begin{aligned}
&\sum_{n<0}(w_1/w_2)^n=\frac{1}{6}\sum_{p\in Z_6}\langle\nu^p\bar b,\bar b\rangle\delta^-(\omega^p w_1/w_2)\\
&=\frac{1}{6}\sum_{p\in Z_6}\langle\nu^p\bar b,\bar b\rangle\frac{\omega^p w_1/w_2}{1-\omega^p w_1/w_2}.
\end{aligned}
$$

The Laurent polynomial P has been chosen so that it contains a factor $(1-\omega^p w_1/w_2)$ for each $p\neq 0$; therefore the limit of each summand where $p\neq 0$ is zero, since the limit of the expression inside the braces is zero. The only case that needs extra care is when $p=0$; in such case we have

$$
\frac{1}{6}P\langle\bar b,\bar b\rangle\frac{w_1/w_2}{1-w_1/w_2}\{X(b,w_2)\delta(-w_1/z_1)-X(b,w_1)\delta(-w_2/z_1)\}
$$

To compute the limit of this expression we need to observe that the limits of both the numerator and denominator exist and are zero, hence the limit can be computed by using a formal analogue of L'Hopital's Theorem.

So if we set

$$h_{12} = \frac{1}{3}P(\frac{w_1}{w_2})\{X(b,w_2)\delta(-w_1/z_1) - X(b,w_1)\delta(-w_2/z_1)\}$$

we have (recall that $M = \lim P = 2 \cdot 6^3$ and denote by D_2 the derivation with respect to the second variable)

$$\lim \frac{h_{12}}{1 - w_1/w_2} = -\lim D_2 h_{12}$$
$$= \frac{1}{3}M\delta(-z_2/z_1)D_{z_2}X(b,z_2) + \frac{1}{3}MD\delta(-z_2/z_1)X(b,z_2)$$

So finally we can conclude the computation of (c):

$$-\frac{\omega}{6}M : \bar{b}(z_2)X(b,z_2) : \delta(-z_2/z_1)$$
$$-\frac{\omega}{6}\frac{1}{3}M\{D_{z_2}X(b,z_2)\delta(-z_2/z_1)$$
$$+ D\delta(-z_2/z_1)X(b,z_2)\}$$

(d)

$$\lim P\frac{\omega}{6^2}\{cX(b,w_2)D\delta(-w_1/z_1)$$
$$+ cX(b,w_1)D\delta(-w_2/z_1)\}$$
$$= \frac{\omega}{6^2}2cMX(b,z_2)D\delta(-z_2/z_1)$$

Finally, we observe that

$$\frac{\omega}{6^2}cM - \frac{\omega}{6 \cdot 3}M = \frac{\omega}{6^2}2M(c-1)$$

So we have the result. Q.E.D.

The Heisenberg subalgebra acts diagonally on the elements $X(z)$ by Theorem 2.4 of [LW]. The same is true for the relation (2.2) and (2.3) as we are going to show.

Lemma 3.2. *On any highest weight module of level* 3 *we have*

$$[\bar{b}(z_1), X(z_2)^{(2)}] = 2X(z_2)^{(2)}\frac{1}{6}\sum_{p \in Z_6}\langle \nu^p \bar{b}, \bar{b}\rangle \delta(\omega^{-p}z_1/z_2)$$

Proof.

$$[\bar{b}(z_1), X(z_2)^{(2)}] = \lim P(w_1,w_2)[\bar{b}(z_1), X(w_1)X(w_2)]$$
$$= \lim P(w_1,w_2)\{[\bar{b}(z_1), X(w_1)]X(w_2) + X(w_1)[\bar{b}(z_1), X(w_2)]\}$$
$$= \lim P(w_1,w_2)\{\frac{1}{6}\sum \langle \nu^p \bar{b}, \bar{b}\rangle \delta(\omega^{-p}z_1/w_2)X(w_1)X(w_2)$$
$$+ X(w_1)\frac{1}{6}\sum \langle \nu^p \bar{b}, \bar{b}\rangle \delta(\omega^{-p}z_1/w_2)X(w_2)\}$$
$$= 2\frac{1}{6}\sum \langle \nu^p \bar{b}, \bar{b}\rangle \delta(\omega^{-p}z_1/z_2)X(z_2)^{(2)}$$

Q.E.D.

Lemma 3.3. *On any highest weight module we have*

$$[\bar{b}(z_1), E^-(-z_2)X(-z_2)E^+(-z_2)]$$
$$= (c-1)E^-(-z_2)X(-z_2)E^+(-z_2) \sum_{n\in Z, n\equiv\pm1 \bmod 6} (z_1/z_2)^n$$

where c is the canonical central element of the affine Lie algebra.

Proof.

$$[\bar{b}(z_1), E^-(-z_2)X(-z_2)E^+(-z_2)] = [\bar{b}(z_1), E^-(-z_2)]X(-z_2)E^+(-z_2)$$
$$+ E^-(-z_2)[\bar{b}(z_1), X(-z_2)]E^+(-z_2) + E^-(-z_2)X(-z_2)[\bar{b}(z_1), E^+(-z_2)]$$
$$= (-cE^-(-z_2) \sum_{n>0, n\equiv\pm1 \bmod 6} (z_1/z_2)^n \langle \bar{b}_{(n)}, -\bar{b}_{(-n)}\rangle)X(-z_2)E^+(-z_2)$$
$$+ E^-(-z_2)(\frac{1}{6}\sum \langle \nu^p\bar{b}, \bar{b}\rangle \delta(-\omega^{-p}z_1/z_2))X(-z_2)E^+(-z_2)$$
$$+ E^-(-z_2)X(-z_2)(-cE^+(-z_2) \sum_{n<0, n\equiv\pm1 \bmod 6} (z_1/z_2)^n \langle \bar{b}_{(n)}, -\bar{b}_{(-n)}\rangle)$$

Now recall that

$$\frac{1}{6}\sum \langle \nu^p\bar{b}, \bar{b}\rangle \delta(-\omega^{-p}z_1/z_2) = -\sum_{n\in Z, n\equiv\pm1 \bmod 6} (z_1/z_2)^n$$

and $\langle \bar{b}_{(n)}, \bar{b}_{(-n)}\rangle = 1$. So we have the result. Q.E.D.

The following theorem now follows:

Theorem 3.4. *On a highest weight module of level 3 the following relation holds*

$$[\bar{b}(z_1), R_2(z_2)] = 2R_2(z_2)\frac{1}{6}\sum_{p\in Z_6} \langle \nu^p\bar{b}, \bar{b}\rangle \delta(\omega^{-p}z_1/z_2)$$

In a completely similar fashion we can prove

Theorem 3.5. *On a highest weight module of level 3 the following relation holds*

$$[\bar{b}(z_1), R(z_2)] = R(z_2)\sum_{n\equiv\pm1} (z_1/z_2)^n(1+\omega^{-n})$$

Theorem 3.6. *The following relation among formal power series with coefficients in $EndM(\Lambda)$ is true*

$$\prod_{p\in Z}\left(1-\omega^{-p}\frac{z}{w}\right)^{\langle\nu^p\bar{b},\bar{b}\rangle} X(z)\tilde{R}_2(w)$$
$$-\prod_{p\in Z}\left(1-\omega^{-p}\frac{w}{z}\right)^{\langle\nu^p\bar{b},\bar{b}\rangle} \tilde{R}_2(w)X(z) =$$
$$\omega^2\sigma(\bar{b})^{-1}\tilde{R}(w)\delta(\omega z/w)$$
$$-\omega^2\sigma(\bar{b})^{-1}R(w)\delta(\omega^{-1}z/w)$$
$$+4\sigma(\bar{b})R_2(w)D\delta(z/w)$$
$$-2\sigma(\bar{b})D_wR_2(w)\delta(z/w).$$

Proof.

$$
\begin{aligned}
&- E^-(-w)[X(z),R_2(-w)]E^+(-w) = -E^-(-w)X(z)R_2(-w)E^+(-w)\\
&+ E^-(-w)R_2(-w)X(z)E^+(-w) = -\prod_{p\in Z}\Big(1-\omega^{-p}\frac{z}{w}\Big)^{\langle \nu^p\bar b,\bar b\rangle}\\
&X(z)E^-(-w)R_2(-w)E^+(-w)\\
&+ E^-(-w)R_2(-w)E^+(-w)X(z)\prod_{p\in Z}\Big(1-\omega^{-p}\frac{w}{z}\Big)^{\langle \nu^p\bar b,\bar b\rangle}\\
&= \prod_{p\in Z}\Big(1-\omega^{-p}\frac{z}{w}\Big)^{\langle \nu^p\bar b,\bar b\rangle} X(z)\tilde R_2(w)\\
&- \prod_{p\in Z}\Big(1-\omega^{-p}\frac{w}{z}\Big)^{\langle \nu^p\bar b,\bar b\rangle} \tilde R_2(w)X(z)
\end{aligned}
$$

On the other hand the first member, using Theorem 3.1, gives

$$
\begin{aligned}
&- E^-(-w)\{\omega^2\sigma(\bar b)^{-1}R(-w)\delta(\omega z/w)\\
&- \omega^2\sigma(\bar b)^{-1}\tilde R(-w)\delta(\omega^{-1}z/w)\\
&+ 4\sigma(\bar b)\tilde R_2(-w)D\delta(z/w)\\
&- 12\sigma(\bar b) : \bar b(-w)\tilde R_2(-w) : \delta(z/w)\\
&- 2\sigma(\bar b)D_w\tilde R_2(-w)\delta(z/w)\}E^+(-w)\\
&= \omega^2\sigma(\bar b)^{-1}\tilde R(w)\delta(\omega z/w)\\
&- \omega^2\sigma(\bar b)^{-1}R(w)\delta(\omega^{-1}z/w)\\
&+ 4\sigma(\bar b)R_2(w)D\delta(z/w)\\
&- 12\sigma(\bar b) : \bar b(-w)R_2(w) : \delta(z/w)\\
&- 2\sigma(\bar b)\{-E^-(-w)D_w\tilde R_2(w)E^+(-w)\}\delta(z/w)\\
&= \omega^2\sigma(\bar b)^{-1}\tilde R(w)\delta(\omega z/w)\\
&- \omega^2\sigma(\bar b)^{-1}R(w)\delta(\omega^{-1}z/w)\\
&+ 4\sigma(\bar b)R_2(w)D\delta(z/w)\\
&- 2\sigma D_wR_2(w)\delta(z/w).
\end{aligned}
$$

Q.E.D.

4. The proof of independence

Recall the notion of $\underline{s}$-filtration of the universal enveloping algebra $\mathcal{U}$ of $\underline{\tilde g}[\nu]$ (see [LW3]): for $j \in \mathbf{Z}$ set

(4.1) $$\mathcal{U}_{(j)} = (0) \quad \text{if} \quad j < 0$$
$$\mathcal{U}_{(0)} = \mathcal{U}(\underline{s})$$

and, if $j > 0$, denote by $\mathcal{U}_{(j)}$ the linear span of all elements $x_1 \cdots x_n \in \mathcal{U}$, where each $x_i \in \underline{\tilde{g}}[\nu]$ and at most j of the elements x_r lie outside the subalgebra $\underline{s}_-$. We clearly have

$$(0) = \mathcal{U}_{(-1)} \subset \mathcal{U}_{(0)} \subset \mathcal{U}_{(1)} \subset \cdots \subset \mathcal{U}.$$

and

$$\mathcal{U} = \bigcup_{j \geq 0} \mathcal{U}.$$

From the commutation relations of the algebra $A_2^{(2)}$ it follows that, for any permutation π of the indices,

$$x(b; m_{\pi(1)}, \ldots, m_{\pi(n)}) - x(b; m_1, \ldots, m_n) \in \mathcal{U}_{(n-1)}.$$

If v_0 is a maximal vector of the Verma module $V = M(\Lambda)$, set

$$V' = \operatorname{span}\{\bar{b}(\lambda)X(\mu)v_0 | \lambda \in \mathcal{O}, \mu \in \mathcal{B}_\Lambda\}.$$

and

$$V'' = \operatorname{span}\{aT(n)bv_0 | a, b \in \mathcal{U}(\tilde{g}), T = R \text{ or } T = R_2, n \in \mathbf{Z}\}.$$

where Z is the set of integers, $\mathcal{O}$ and $\mathcal{B}_\Lambda$ were defined in the previous section.

Clearly V'' is a submodule of the Verma module V and $V'' \subset W$.

For $\mu \in \mathcal{B}$ and t a natural number, set

$$V_{[\mu]} = \operatorname{span}\{\bar{b}(\lambda)X(\nu)v_0 | \lambda \in \mathcal{O}, \nu \in \mathcal{B}, \nu > \mu\}.$$

and

$$V_t = \operatorname{span}\{\bar{b}(\lambda)X(\nu)v_0 | \lambda \in \mathcal{O}, \nu \in \mathcal{B}, |\nu| < t\}.$$

Analogously, set V_t'' to be the span of the set of elements $aT(n)bv_0$ where $a \in \mathcal{U}_{(r)}$, $b \in \mathcal{U}_{(s)}$, T is either R or R_2, $n \in \mathbf{Z}$, and $r + s + 2 \leq t$.

Clearly

$$V'' = \bigcup_{t \geq 2} V_t''.$$

We can extract the coefficients of $z^{-m}w^{-n}$ in the relation of Theorem 3.1 to obtain

$$\begin{aligned}
&[X(m), R_2(n)] = \omega^2\sigma(\bar{b})^{-1}R(n+m)\omega^{-2m} - \omega^2\sigma(\bar{b})^{-1}\tilde{R}(n+m)\omega^{-4m} \\
&+ 4\sigma(\bar{b})\tilde{R}_2(n+m)(-1)^m m \\
&- 12\sigma \sum_{i+j=n+m} : \bar{b}(i)\tilde{R}_2(j) : (-1)^m - 2\sigma(-1)^m\tilde{R}_2(n+m)(n+m).
\end{aligned}$$

Recall that

$$\tilde{T}(n) = -(-1)^n T(n)$$

So

$$\begin{aligned}
&[X(m), R_2(n)] = \omega^2\sigma(\bar{b})^{-1}R(n+m)\omega^{-2m} - \omega^2\sigma(\bar{b})^{-1}R(n+m)(-1)^{n+m+1}\omega^{-4m} \\
&+ 4\sigma(\bar{b})R_2(n+m)(-1)^{n+m+1}(-1)^m m \\
&- 12\sigma(-1)^m \sum_{i+j=n+m} : \bar{b}(i)\tilde{R}_2(j) : -2\sigma(-1)^m R_2(n+m)(-1)^{n+m+1}(n+m).
\end{aligned}$$

Therefore we can state

Proposition 4.1. *For some coefficient A*

$$\begin{aligned}[X(m), R_2(n)] &= \omega^2\sigma(\bar{b})^{-1}\omega^{-2m}(1-\omega^{-2m}(-1)^{n+m+1})R(n+m) + AR_2(n+m) \\ &- 12\sigma(-1)^m \sum_{i+j=n+m} :\bar{b}(i)\tilde{R}_2(j): -2\sigma(-1)^m R_2(n+m)(-1)^{n+m+1}(n+m).\end{aligned} \tag{4.2}$$

Analogously we can extract the coefficients from Theorem 3.6 and get

Proposition 4.2.

$$\begin{aligned}(-1)^{n+1}[X(m), R_2(n)] &= \omega^2\sigma(\bar{b})^{-1}((-1)^{n+1}\omega^m - \omega^{-m})R(n+m) \\ &+ A'R_2(n+m) + \sum_{i\geq 1} a_i X(m-i)R_2(n+i) - \sum_{i\geq 1} a_i R_2(n-i)X(m+i)\end{aligned} \tag{4.3}$$

for some coefficients A', a_i.

Provided we choose m appropriately we can eliminate $[X(m), R_2(n)]$ among relations (4.2) and (4.3) and obtain

Proposition 4.3. *With a suitable choice of* m,

$$R(n+m) = A''R_2(n+m) + \sum_{i\geq 1} a'_i X(m-i)R_2(n+i) + \sum_{i\geq 1} a'_i R_2(n-i)X(m+i)$$

(Cf. Lemma 9.8 in [MP].) We can now prove

Theorem 4.4. *The submodule* V'' *of* V *is the linear span of the elements*

$$\bar{b}(\lambda)X(m_1, \ldots, m_r)T(n)X(m_{r+1}, \ldots, m_s)v_0$$

where $\lambda \in \mathcal{O}$, $T = R$ *or* $T = R_2$, $s \geq r \geq 0$, $m_i, n \in \mathbf{Z}$ *and* $m_1 \leq \ldots \leq m_r \leq n_1 \leq n_2 < m_{r+1} \leq \ldots \leq m_s \leq 0$.

Proof. Observe that $V_{[\mu]} \subset V''_{l(\mu)}$. We shall prove by induction on the good order of $\mathcal{B}$ that

$$V_{[\mu]} = V''_{l(\mu)} \tag{4.4}$$

By using the commutation relations of the Heisenberg elements (Theorems 3.4 and 3.5) we may assume that any element $v = aT(n)bv_0$ of V''_{t+1} is actually of the form

$$v = a'X(i_1, \ldots, i_{r-1})X(i_r)T(n)X(j_1, \ldots, j_s)v_0 \tag{4.5}$$

where $a' \in \mathcal{U}(\underline{s}_-)$. Also notice that (4.4) holds for $\mu = (2;0)$, the minimal partition of 0 into 2 parts. (Start of the induction.) Notice also that, because $[X(n), X(m)] \in \mathcal{U}_{(1)}$, we may assume that in (4.5) we have

$$i_1 \leq \ldots \leq i_r \text{ and } j_1 \leq \ldots \leq j_s.$$

Now assume first that $T(n) = R_2(n)$. If we have $i_r > n_1$, then using the commutation relation of Theorem 3.1 we get

$$v = a' X(i_1, \dots, i_{r-1}) R_2(n) X(i_r) X(j_1, \dots, j_s) v_0 + V''_t$$

Similarly, in the case $n_2 \geq j_1$. Hence in a finite number of steps, we see that v lies in $V_{[\mu]}$. Now assume that $T(n) = R(n)$. If $i_r > n_1$ then we can use the relation of Proposition 4.3 and replace $R(n)$ with $R_2(n)$, and then use the first part of this proof to conclude (cf. proof of Lemma 9.11 in [MP]). Analogously, we proceed if $j_1 \leq n_2$. Therefore $V''_{t+1} \subset V_{[\mu]}$ and the result follows. Q.E.D.

In other words we have shown that

$$\text{span}\{\bar{b}(\lambda) X(\mu) v_0; \lambda \in \mathcal{O}, \mu \in \mathcal{B}\} \tag{4.6}$$

is a submodule of the Verma module $V = M(\Lambda)$. It remains to show that V' is a submodule.

From the definitions we can easily deduce the following

Proposition 4.5 (cf. Lemma 9.9 in [MP]). *For $v \in V$ and $n \in Z$, there exists $a > 0$ such that*

$$R_2(n) v \in a X(2; n) v + \sum_{\mu > (2:n)} \mathbf{C} X(\mu) v$$

and $a' > 0$ such that

$$R_2(n) v \in a' X(2; n) v + \sum_{\mu > (2:n)} \mathbf{C} X(\mu) v$$

For the computations that follow we need a "straightening lemma"

Straightening Lemma. *Let ν be a sequence of the form*

$$\nu = (n_1, \dots, n_t, N_{t+1}, N_{t+2}, \dots, n_u)$$

ν not necessarily in $\mathcal{B}$ and let $\mu \in \mathcal{B}$

$$\mu = (m_1, \dots, m_r, M_{r+1}, M_{r+2}, \dots, m_s).$$

Moreover, let one of the following conditions hold

(i) $\ell(\mu) > \ell(\nu)$,

(ii) $\ell(\mu) = \ell(\nu), |\mu| < |\nu|$,

(iii) $\ell(\mu) = \ell(\nu), |\mu| = |\nu|$, *and* $m_1 = n_1, \dots, m_{i-1} = n_{i-1}, m_i > n_i$, *for some* $1 \leq i \leq l(\mu)$,

Then

$$X(\nu) v_0 \in V_{[\mu]}$$

Proof. If $\ell(\mu) > \ell(\nu)$, the proof of Theorem 4.4 shows that the monomials $X(\nu) v_0 \in \mathbf{C} X(\nu') v_0 + V''_{\ell(\nu)}$ where ν' is obtained from ν by reordering its entries in non decreasing order. Therefore $X(\nu) v_0 \in V_{[\mu]}$. Analogously we proceed if $\ell(\mu) = \ell(\nu)$ and $|\mu| < |\nu|$. Suppose instead that $\ell(\mu) = \ell(\nu)$ and $|\mu| = |\nu|$ and $m_1 = n_1, \dots, m_{i-1} = n_{i-1}, m_i > n_i$, for some $1 \leq i \leq l(\mu)$, and both ν and μ have a red subpair. Then Theorem 3.1 and Theorem 1.1 , (cf. also [C3]) imply that $X(\nu) v_0 \in \mathbf{C} X(\nu') v_0 + V_{[\mu]}$ and $\nu' > \mu$.

Lemma 4.6. *If $\mu = (m_1, \dots, m_r, M_{r+1}, M_{r+2}, \dots, m_s)$ and, for $i \le r$, there is a a pair (m_i, m_{i+1}) that does not satisfy one of the conditions of Theorem* 2.1 *then*

$$X(\mu)v_0 \in V' + V_{[\mu]}.$$

Proof. Assume first that $i < r$. We distinguish four cases according to whether the color of the pairs (m_i, m_{i+1}) and (M_{r+1}, M_{r+2}) are respectively: red-red; red-blue; blue-red; blue-blue. Let $n = m_i + m_{i+1}$ and $N = M_{r+1} + M_{r+2}$. Case red-red.

We can write

$$\begin{aligned}
&X(\mu)v_0 = X(m_1, \dots, m_{i-1})(a^{-1}R_2(n) - a^{-1}bX(\{2;n\}) - \dots)X(\dots, m_s)v_0 \\
&= a^{-1}a'X(m_1, \dots, m_{i-1})R_2(n) \dots X(2;N) \dots X(m_s)v_0 \\
&\quad + a^{-1}b'X(m_1, \dots, m_{i-1})R_2(n) \dots X(\{2;N\}) \dots X(m_s)v_0 \\
&\quad + \dots - a^{-1}bX(m_1, \dots, m_{i-1}\{2;n\} \dots R_2(N) \dots X(m_s)v_0 + \dots
\end{aligned}$$

Each of these monomials is of the type $X(\nu)v_0$ where ν is not necessarily in $\mathcal{B}$, however $|\nu| \ge |\mu|$, $\ell(\nu) \le \ell(\mu)$ or if $|\nu| = |\mu|$, and $\ell(\nu) = \ell(\mu)$ then $m_1 = n_1, \dots, m_{k-1} = n_{k-1}, m_k = n_k$ for some k.

In each of these cases the proof of Theorem 4.4 shows that $X(\nu)v_0 = X(\nu')v_0 + V_{[\nu']}$ where ν' is obtained from ν by reordering its entries in a nondecreasing order. From this it follows that $\nu' > \mu$ and so $X(\nu)v_0 \in V_{[\mu]}$. In the only case when ν and μ have the same entries in the same order, then $\nu > \mu$ because the colored pair of ν appears to the left of the one in μ. So even in this case $X(\nu)v_0 \in V_{[\mu]}$.

Case red-blue.

Proceeding as in the previous case we get for $X(\mu)v_0$ a linear combination of monomials of the form

$X(m_1 \dots m_{i-1})R_2(n) \dots X(2;N) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})R_2(n) \dots X(\{2;N\}) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})X(\{2;n\}) \dots R(N) \dots X(m_s)v_0$

etc. Monomials such as the first two above are either in V' already or in $V_{[\mu]}$ arguing as in the case red-red. For the monomials like the third above we use Proposition 4.3 and so we reduce to the previous case.

Case blue-red.

$X(\mu)v_0$ is a linear combination of monomials of the form
$X(m_1 \dots m_{i-1})R(n) \dots X(2;N) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})R(n) \dots X(\{2;N\}) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})X(2;n) \dots R_2(N) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})X(\{2;n\}) \dots R_2(N) \dots X(m_s)v_0$

Monomials such as the first two above are in $V_{[\mu]}$. Those such as the last two are reduced to the case red-red with the help of Proposition 4.3.

Case blue-blue.

We get monomials of the form
$X(m_1 \dots m_{i-1})R(n) \dots X(2;N) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})R(n) \dots X(\{2;N\}) \dots X(m_s)v_0$
$X(m_1 \dots m_{i-1})X(2;n) \dots R(N) \dots X(m_s)v_0$

The monomials such as the first two are in V' or in $V_{[\mu]}$ while the third one is of the red-blue case.

Finally assume $i = r$.

If $m_r = M_{r+1} = M_{r+2}$ then we get the conclusion by Theorem 3.1 (cf. proof of Lemma 9.17, first case, in [MP]).

In the case where $[M_{r+1}, M_{r+2}]$ is a red pair then we use Proposition 4.3 (backwards) with a linear combination of monomials of the form $R(n)$, $R_2(m)$, $X(k)R_2(h)$, $R_2(k)X(h)$ which are in V' or in $V_{[\mu]}$ (using the Straightening Lemma).

In the case where $\{M_{r+1}, M_{r+2}\}$ is a blue pair the we use Proposition 4.3 to replace the blue pair with a linear combination of monomials containing only red pairs and so we reduce to the previous cases.

This completes the proof.

We also need another lemma.

Lemma 4.7. *If* $\mu = (m_1, \dots, m_r, M_{r+1}, M_{r+2}, \dots, m_s) \in \mathcal{B}$ *and* $M_{r+1} + M_{r+2} \equiv 0 \text{ mod. } 3$ *then*

$$X(\mu) \in V_{[\mu]}$$

Proof. Since $N = M_{r+1} + M_{r+2} \equiv 0$ mod. 3 then $R(M_{r+1} + M_{r+2}) \equiv 0$ modulo $W_{[\{2;N\}]}$ (see Proposition 16 of [C3]).

Theorem 4.8. *V' is a submodule of V.*

Proof. We have seen that V'' is a submodule of V. Also, clearly $V' \subset V''$. Since

$$V'' = \bigcup_{\mu \in \mathcal{B}} V_{[\mu]}$$

it is enough to prove that $V_{[\mu]} \subset V'$ for all $\mu \in \mathcal{B}$. This is done by induction using the previous lemmas. Indeed, since $\mu = (2; n) \in \mathcal{B}$ the vector $X(\mu)v_0 \in V'$. So the theorem is true for this kind of monomials.

Suppose that $V_{[\mu]} \subset V'$ for all $\mu \geq \nu$. Let μ be the biggest element of $\mathcal{B}$ smaller than ν (this exists because $\mathcal{B}$ is well ordered), and consider $v = \bar{b}(\lambda)X(\mu)v_0$. If $\mu \in \mathcal{B}_\Lambda$ then $v \in V'$. If μ is not in $\mathcal{B}_\Lambda$ then either μ satisfies the hypotheses of Lemma 4.6 or it satisfies the hypotheses of Lemma 4.7. In both cases $X(\mu)v_0 \in V' + V_{[\mu]}$. By the induction hypothesis $V_{[\mu]} \subset V'$ so we can conclude.

We shall also need

Lemma 4.9. *Let* $\Lambda = 3h_1^*$ *or* $\Lambda = h_1^* + h_0^*$ *, let* $V = M(\Lambda)$ *be the Verma module of highest weight* Λ*, and let* v_0 *be a maximal vector in it, then* $R_2(0)v_0 = 0$ *and* $R(0)v_0 = 0$.

Proof. It is clear that V' is a submodule of the maximal submodule $W(\Lambda)$ of V. Now,

$$W(\Lambda) = \mathcal{U}(n_-)f_0 v_0 + \mathcal{U}(n_-)f_1^4 v_0$$

or

$$W(\Lambda) = \mathcal{U}(n_-)f_0^2 v_0 + \mathcal{U}(n_-)f_1^2 v_0$$

in both cases the homogeneous component of degree 0 is zero and so the result follows.

Lemma 4.10. *If V is a Verma module as above then $V' \neq 0$.*

Proof. $R_2(-1)v_0 = aX(-1)X(0)v_0+$ etc, and this is a nonzero vector because of the universal property of the Verma module.

Proposition 4.11. *With the above hypotheses, if $\Lambda = 3h_1^*$ then*

$$f_0 v_0 \text{ and } f_1^4 v_0 \in V'',$$

while if $\Lambda = h_1^* + h_0^*$ then

$$f_0^2 v_0 \text{ and } f_1^2 v_0 \in V''.$$

Proof. In all cases $V' = V'' \subset W$. Hence the proof is obtained by counting the dimensions of the top component of the modules and checking that they are equal.

Since all the points 1-4 of Section 2 have been shown to be true we can conclude, with

Theorem 4.12. *The set of vectors of Theorem* 2.1 *is a basis of the standard module $L(\Lambda)$ of level* 3.

As a corollary of this result we get a partition identity

Theorem 4.13.

A. *The number of partitions $(m_1, \ldots, m_r)$ of an integer n into parts different from* 1 *and such that the difference of two consecutive parts is at least* 2 *(i.e. $m_i - m_{i+1} \geq 2$), and is exactly* 2 *or* 3 *only if their sum is a multiple of* 3 *(i.e. $2 \leq m_i - m_{i+1} \leq 3$ implies $m_i + m_{i+1} \equiv 0 \mod 3$) is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 3$ modulo* 12.

B. *The number of partitions $(m_1, \ldots, m_r)$ of an integer n into parts different from* 2 *and such that the difference of two consecutive parts is at least* 2 *(i.e. $m_i - m_{i+1} \geq 2$), and is exactly* 2 *or* 3 *only if their sum is a multiple of* 3 *(i.e. $2 \leq m_i - m_{i+1} \leq 3$ implies $m_i + m_{i+1} \equiv 0 \mod 3$) is the same as the number of partitions of n into distinct parts congruent to* $1, 3, 5, 6$ *modulo* 6.

These partition identities have been generalized in [AAG], however there must be a natural extension of these combinatorial identities that would allow one to construct all standard modules for the algebra $A_2^{(2)}$. Such a construction would possibly yield a generalization of the classical Rogers-Ramanujan identities in a direction different from the $A_1^{(1)}$-direction that corresponds to the identities of Andrews, Bressoud and Göllnitz-Gordon. This problem is currently under investigation.

References

[AAG] K. Alladi, G. E. Andrews, B. Gordon, *Refinements and generalizations of Capparelli's conjecture on partitions*, in the Journal of Algebra.

[A1] G. E. Andrews, *The Theory of Partitions*, in "Encyclopedia of Mathematics and Its Applications" (G.C. Rota, Ed.), Vol. 2, Addison-Wesley, Massachusetts, 1976. MR **58:**27738

[A2] G. E. Andrews, *Schur's theorem, Capparelli's conjecture, and q-trinomial coefficients*, in Proc. Rademacher Centenary Conf. (1992), Contemporary Math. **167**, 1994, pp. 141–154.

[A3] G. E. Andrews, *The hard-hexagon model and the Rogers-Ramanujan type identities*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 5290–5292. MR **82m:**82005

[A4] G. E. Andrews, *q-trinomial Coefficients and Rogers-Ramanujan Type Identities*, in "Analytic Number Theory, Proceedings of a Conference in Honor of Paul T. Bateman", B.C. Berndt, et al. editors, Birkhäuser, (1990), 1-11. MR **92e:**11109

[A5] G. E. Andrews, R.J. Baxter, *Lattice gas generalization of the hard hexagon model. III. q-trinomial coefficients*, J. Stat. Phys. **47** (1987), 297–330. MR **88h:**82069

[A6] G. E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory*, CBMS Regional Conf. Ser. in Math., no. 66, AMS, Providence, RI (1986). MR **88b:**11063

[Ba] R.J. Baxter, *Rogers-Ramanujan identities in the hard hexagon model*, J. Stat. Phys. **26** (1981), 427–452. MR **84m:**82104

[Br] D. Bressoud, *Analytic and combinatorial generalizations of the Rogers-Ramanujan identities*, Mem. Amer. Math. Soc. **24** (1980). MR **81i:**10019

[C1] S. Capparelli, *Vertex Operator Relations for Affine Algebras and Combinatorial Identities*, Ph.D. Thesis, Rutgers University, 1988.

[C2] S. Capparelli, *Elements of the annihilating ideal of a standard module*, J. Algebra **145** (1992), 32–54. MR **93b:**17072

[C3] S. Capparelli, *On some representations of twisted affine Lie algebras and combinatorial identities*, J. Algebra **154** (1993), 335–355. MR **94d:**17031

[C4] S. Capparelli, *Relations for generating functions associated to an infinite dimensional Lie algebra*, Boll. UMI, (7), 6-B (1992), 733–793. MR **94a:**17018

[C5] S. Capparelli, *A combinatorial proof of a partition identity related to the level 3 representations of a twisted affine Lie algebra*, Comm. Algebra **23** (1995), 2959–2969.

[Fi] L. Figueiredo, *Calculus of principally twisted vertex operators*, Mem. Amer. Math. Soc. **69** (1987). MR **89c:**17029

[FK] I. B. Frenkel, V. G. Kac, *Basic representations of affine Lie algebras and dual resonance modules*, Invent. Math. **62** (1980), 23–66. MR **84f:**17004

[FLM] I. B. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, New York, Academic Press, 1989. MR **90h:**17026

[G] B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399. MR **23:**A809

[Gö] H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967), 154–190. MR **35:**2848

[K] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1985. MR **87c:**17023

[L1] J. Lepowsky, *Calculus of twisted vertex operators*, Proc. Natl. Acad. Sci. USA **82** (1985), 8295–8299. MR **88f:**17030

[LM] J. Lepowsky, S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. in Math. **29** (1978), 15–59. MR **82f:**17005

[LP1] J. Lepowsky, M. Primc, *Standard modules for type one affine Lie algebras, Number Theory*, Lecture Notes in Math., vol. 1052, Springer-Verlag, New York, 1984, pp. 194–251. MR **86f:**17015

[LP2] J. Lepowsky, M. Primc, *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, Contemporary Math. **46** (1985). MR **87g:**17021

[LW1] J. Lepowsky, R. L. Wilson, *Construction of the affine Lie algebra* $A_1^{(1)}$, Comm. Math. Phys. **62** (1978), 43–53. MR **58:**28089

[LW2] J. Lepowsky, R. L. Wilson, *A Lie-theoretic interpretation and proof of the Rogers-Ramanujan identities*, Adv. in Math. **45** (1982), 21–72. MR **84d:**05021

[LW3] J. Lepowsky, R. L. Wilson, *The structure of standard modules I: Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. **77** (1984), 199–290. MR **85m:**17008

[LW4] J. Lepowsky, R. L. Wilson, *The structure of standard modules II: The case* $A_1^{(1)}$, *principal gradation*, Invent. Math. **79** (1985), 417–442. MR **86g:**17014

[Ma] M. Mandia, *Structure of the level one standard modules for the affine Lie algebras* $B_\ell^{(1)}$, $F_4^{(1)}$ *and* $G_2^{(1)}$, Mem. Amer. Math. Soc. **362** (1987). MR **88h:**17023

[MP] A. Meurman, M. Primc, *Annihilating ideals of standard modules of* $s\ell(2,\mathbf{C})^\sim$ *and combinatorial identities*, Adv. in Math. **64** (1987), 177–240. MR **89c:**17031

[MP2] A. Meurman, M. Primc, *Annihilating fields of standard modules of* $s\ell(2,\mathbf{C})^\sim$ *and combinatorial identities*, Preprint.

[Mi1] K. C. Misra, *Structure of certain standard modules for $A_n^{(1)}$ and the Rogers-Ramanujan identities*, J. Algebra **88** (1984), 196–227. MR **85i:**17018

[Mi2] K. C. Misra, *Structure of some standard modules for $C_n^{(1)}$*, J. Algebra **90** (1984), 385–409. MR **86h:**17021

[Mi3] K. C. Misra, *Constructions of fundamental representations of symplectic affine Lie algebras*, In Topological and Geometrical Methods in Field Theory (J. Hietarinata and J. Westerholm, eds.), World Scientific, Singapore, 1986, pp. 147–169. MR **90j:**17045

[S] I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917), 302–321; reprinted in: *I. Schur, Gesammelte Abhandlungen, Vol. 2* (1973), Springer-Verlag, Berlin. MR **57:**2858b

[Se] G. Segal, *Unitary representations of some infinite dimensional groups*, Comm. Math. Phys. **80** (1981), 301–342. MR **82k:**22004

[Sp] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–198. MR **50:**7371

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