ASYMPTOTIC EXPANSION FOR LAYER SOLUTIONS OF A SINGULARLY PERTURBED REACTION-DIFFUSION SYSTEM

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ABSTRACT. For a singularly perturbed n-dimensional system of reaction–diffusion equations, assuming that the 0th order solutions possess boundary and internal layers and are stable in each regular and singular region, we construct matched asymptotic expansions for formal solutions in all the regular, boundary, internal and initial layers to any desired order in ϵ. The formal solution shows that there is an invariant manifold of wave-front-like solutions that attracts other nearby solutions. We also give conditions for the wave-front-like solutions to converge slowly to stationary solutions on that manifold.

1. INTRODUCTION

This is the first of a series of papers devoted to studying internal, boundary and initial layers for singularly perturbed n-dimensional systems of reaction-diffusion equations. By a formal asymptotic method, we derive matched expansions of layer solutions to any desired order in ϵ. We give general conditions for existence and stability of the formal solutions. The formal expansion shows how the initial profile quickly converges to a manifold of slow moving wave-front-like solutions. We also give an analytic condition (see Hypothesis H6) for the formal solution to converge slowly towards a stable stationary solution. In the next paper we will show that under the same set of conditions there is a unique genuine solution that is near the formal series solution. These results have been obtained by other authors for $n = 1$, or $n = 2$ with small diffusion on only one variable. Our goal is to generalize their results to any finite n.

Obtaining matched asymptotic expansion has always been an indispensable part of the complete treatment of singular perturbation problems, for it often provides easily computable and highly accurate approximations to the exact solutions. However, a rigorous treatment of the asymptotic expansion to the layer solutions of the general systems has remained incomplete for almost twenty years since first raised by Fife [12, 13]. We have also noted that new tools need to be developed when moving from scalar equations to systems. Recent advances using the Melnikov integrals in the functional analytic method of homoclinic bifurcations are crucial to this paper, see [4, 26, 22, 23].

Consider the following reaction-diffusion equation

(1.1) $\epsilon u_t = \epsilon^2 u_{xx} + f(u, x, \epsilon), \quad u \in \mathbb{R}^n, \, a < x < b,$

with Neumann boundary conditions at $x = a, b$

(1.2) $u_x(a, t) = u_x(b, t) = 0,$

and initial condition

$$u(x, 0, \epsilon) = \overline{u}(x, \epsilon)$$

at $t = 0$. Here $f : \mathbb{R}^n \times [a, b] \times \mathbb{R} \to \mathbb{R}^n$ is C^∞ with the following expansion:

(1.3) $f(u, x, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j f_j(u, x).$

Due to the presence of the small parameter $\epsilon > 0$, solutions of (1.1) may have internal, boundary and initial layers. Those are the regions of x-t space where u_{xx} and/or u_t are large so that the solutions do not converge uniformly as $\epsilon \to 0$. For the moment we ignore boundary layers, and give a short introduction to spatially regular and internal layers. The following is motivated by a discussion in [12, 13].

In the regular layers, a stationary solution $u(x, \epsilon)$ of (1.1) approaches solutions of

(1.4) $f_0(u, x) = 0$

as $\epsilon \to 0$. Assume that $u = p^i(x)$, $i = 1, 2$, $a \leq x \leq b$, are two solutions of (1.4), and as $\epsilon \to 0$,

(1.5) $u(x, \epsilon) \to \begin{cases} p^1(x), & a < x < \eta, \\ p^2(x), & \eta < x < b. \end{cases}$

There is an internal layer at $x = \eta$. Using stretched variables $\xi = \frac{x - \eta}{\epsilon}$, $\tau = \frac{t}{\epsilon}$, we write (1.1) as (when $\epsilon = 0$)

(1.6) $u_\tau = u_{\xi\xi} + f_0(u, \eta).$

Suppose that $\eta = \eta^0$ can be chosen such that (1.6) possesses a stationary solution $q(\xi)$ that satisfies

(1.7) $0 = u_{\xi\xi} + f_0(u, \eta)$

and approaches $p^1(\eta^0)$ as $\xi \to -\infty$ $(p^2(\eta^0)$ as $\xi \to \infty)$, $q'(\xi) \to 0$ as $\xi \to \pm\infty$. The functions $\{p^1(x), q(\xi), p^2(x)\}$ are the 0th order expansion of a formal solution in regular and internal layers. The position of the stationary internal layer $\eta = \eta_0$ is determined by the existence of a heteroclinic solution to (1.7). The condition $q(\xi) \to p^i(\eta^0)$ as $\xi \to \pm\infty$ is the 0th matching condition between regular and singular layers. Higher order matching conditions will be specified later in this paper when higher order formal expansions are computed.

Under some general conditions, which will be stated in §3, it was proved [22, 23] that there is an exact stationary solution u to (1.1) near the 0th order expansions. Similar results were obtained in [21].

When η changes, generically the heteroclinic solution of (1.7) breaks. However the time dependent equation (1.6) may have a traveling wave solution $u(\xi, \tau) = q(\xi - V\tau, \eta)$ where $q(\xi, \eta)$ satisfies

(1.8) $u_{\xi\xi} + V u_\xi + f_0(u, \eta) = 0.$

Here η serves as a parameter, and the wave speed $V = V(\eta)$ depends on η. The function $q(\xi, \eta)$ approaches one of the $p^i(\eta)$, and $\frac{\partial}{\partial \xi} q$ approaches zero, as $\xi \to \pm\infty$ due to the matching of the internal and regular layers. The wave speed V and the wave front position η do not depend on the stretched time τ, but they depend on the slow time t. To see this, let $\eta = \eta(t)$, $\xi = \frac{x - \eta(t)}{\epsilon}$, and $u(\frac{x - \eta(t)}{\epsilon}, t) = u(\xi, t)$ be a solution to (1.1). Then (when $\epsilon = 0$)

$$(1.9) \qquad 0 = u_{\xi\xi} + \eta'(t) u_\xi + f_0(u, \eta(t)).$$

At each $t \geq 0$, we look for a heteroclinic solution of the above connecting $p^1(\eta(t))$ to $p^2(\eta(t))$. Comparing this with (1.8), we have

$$(1.10) \qquad \frac{d\eta(t)}{dt} = V(\eta(t)).$$

We can see that (1.10) determines $\eta(t)$. One should not be surprised to see that the wave speed V is the same in both x-t and ξ-τ coordinates, since the scaling by ϵ cancels.

We have just described the wave-front-like solutions to the 0th order. Recursive formulas for computing higher order expansions of u and η are also presented in this paper. In particular, we have found that the higher order expansions in the internal layers are uniquely determined by growth conditions of the solutions while the matching of such solutions with those in adjacent regular layers can be proved as a consequence of that. See [22] for a similar case.

Let $0 < \beta < 1$ be a constant. Let the width of the internal layer be $O(\epsilon^\beta)$. The variable $x = \epsilon^\beta$ is $o(1)$ as $\epsilon \to 0$. But in the stretched variable, $\xi = x/\epsilon = \epsilon^{\beta - 1} \to \infty$ as $\epsilon \to 0$. See [6, 7] for discussion of such intermediate variables. Define a piecewise smooth function $W(x, t, \epsilon)$ by

$$(1.11) \qquad W(x, t, \epsilon) = \begin{cases} p^1(x), & a < x < \eta(t) - \epsilon^\beta, \\ p^2(x), & \eta(t) + \epsilon^\beta < x < b, \\ q((x - \eta(t))/\epsilon), & \eta(t) - \epsilon^\beta < x < \eta(t) + \epsilon^\beta. \end{cases}$$

At the interior of each subinterval, W satisfies (1.1) with an error $O(\epsilon^\beta)$. At the points $\eta(t) \pm \epsilon^\beta$, W has a jump discontinuity of size $O(\epsilon^\beta)$. Such a function is called a pseudo-solution to (1.1). In our next paper we will show that there is an exact solution to the original equation (1.1) that is near $W(x, t, \epsilon)$. A function is said to have a wave-front-like profile or to be a wave-front-like function if it approaches the solutions of (1.4) at regular layers, but approaches heteroclinic solutions in stretched variable at internal layers. The function W obviously has a wave-front-like profile. It follows that the exact solution of (1.1) near W has a wave-front-like profile.

Recall that $V(\eta_0) = 0$. The wave speed $V(\eta)$ generally changes sign when passing $\eta = \eta^0$. The case $V < 0$ if $\eta > \eta^0$ is especially interesting. It shows that the wave-front-like solution approaches a stationary wave-front-like solution as time evolves. In the other case $V > 0$ when $\eta > \eta^0$, the stationary wave-front-like solution is not stable among the wave-front-like solutions.

Suppose now the initial condition $\overline{u}(x, \epsilon)$ also has a wave-front-like profile. That is, $\overline{u}(x, 0)$ has a jump at $x = \eta$ and is continuous on $[a, \eta)$ and $(\eta, b]$, and using the stretched variable $\xi = \frac{x - \eta}{\epsilon}$, the limit $\lim_{\epsilon \to 0^+} \overline{u}(\epsilon\xi + \eta, \epsilon) = \overline{\overline{u}}(\xi, 0)$ exists. We assume that as initial data for the ODE $u_\tau = f_0(u, x)$, where x serves as a parameter, $\overline{u}(x, 0)$ is attracted by $p^i(x)$, $i = 1$ for $x \in [a, \eta)$, $i = 2$ for $x \in (\eta, b]$, and

as initial data for (1.6), $\overline{\overline{u}}(\xi, 0)$ is attracted by $q(\xi, \eta)$ modulo a spatial shift. $q(\xi, \eta)$ is stable in the sense of Evans, [8, 9, 10, 11]. The problem of determining when $q(\xi, \eta)$ is stable is very important, but is not the concern of this paper, [20, 2, 25, 19].

Under the above assumptions, using stretched time $\tau = t/\epsilon$, we also derive formal series solutions in the initial layers. Special care has to be exercised to ensure that these solutions match with wave-front-like slow solutions, since the linearized equation has a zero eigenvalue.

In this paper, the intermediate spaces $D_A(\theta)$ are used to study parabolic equations. These spaces are powerful tools to treat fully nonlinear equations. Since our system is semi-linear, we only use some weaker results. All the results in this paper are valid with essentially the same proof if $D_A(\theta)$ is replaced by $D(A^\theta)$. An important feature in this paper is to use weighted norms in function spaces. Weighted function spaces have been used in [28, 29] to study the stability of travelling waves.

The outline of this paper is as follows. We introduce notations and some basic lemmas in §2. In §3 we state a result from [22] that concerns the stationary wave-front-like formal series solution (Theorem 3.1). The result in §3 is a special case of the result in §4. Having a separate section helps to show what new hypotheses are needed to study the slowly moving wave-front-like solutions. In §4, we study wave-front-like formal series solution (Theorems 4.3, 4.4, and Corollary 4.5). We show that these solutions have a slowly moving front and form a manifold that depends on parameters $\{\overline{\eta}_j^i\}_{j=0}^\infty$, $1 \leq i \leq r-1$. Here $\sum_0^\infty \epsilon^j \overline{\eta}_j^i$, is the formal series expansion of the initial wave front, $r - 1$ is the number of internal layers. The result agrees with other publications where the slow manifold is also parameterized by layer positions, [3, 17, 19, 1]. In §5 we study the formal series solutions in the initial layer (Theorems 5.3, 5.5). We show that $\overline{\eta}_0^i$ is determined by the initial condition of (1.1), and $\overline{\eta}_j^i$, $j \geq 1$, are determined by the matching of initial and regular (in time) expansions. In §6, we prove that the formal series solutions in adjacent layers, obtained in §4 and §5, match with each other (Theorems 6.1–6.4). We also construct a pseudo-solution of any prescribed accuracy based on the matched series solutions (Theorem 6.5). Most of the technical lemmas are proved in §7.

Internal and boundary layers in singular perturbation problems have been an active area of current research. Various powerful methods have been developed to treat the layer solutions [14, 16, 15, 25]. Our approach is different from the others. We follow the pattern "matched formal series expansions–pseudo solutions–Newton's method" which has been used to treat singularly perturbed ODEs, see [22, 23]. The last step uses a lemma similar to the shadowing lemma in dynamical system theory. In our next paper we will introduce such a shadowing lemma for parabolic systems where the pseudo-solutions have jumps in both x and t directions. A brief look at such a result reveals that the inverse operator of the parabolic system has a norm of $O(\epsilon^{-k})$, $k > 1$. A good initial approximation with residual smaller than $O(\epsilon^k)$ is needed which is naturally supplied by the matched asymptotic expansions.

2. Notations and basic lemmas

2.1. **Notations.** As $\epsilon \to 0$, the solution $u(x, t, \epsilon)$ of (1.1) may not converge uniformly at regions where u_{xx} and u_t are large. These regions are called singular layers

t	$(SR)^0$	$(RR)^1$	$\ldots$	$(RR)^i$	$(SR)^i$	$(RR)^{i+1}$	$\ldots$	$(SR)^r$
τ	$(SS)^0$	$(RS)^1$	$\ldots$	$(RS)^i$	$(SS)^i$	$(RS)^{i+1}$	$\ldots$	$(SS)^r$
	ξ	x	$\ldots$	x	ξ	x	$\ldots$	ξ

FIGURE 1

with respect to space or time. In particular, u_t may be large near $t = 0$. That region is also called the fast (or initial) layer where the stretched time $\tau = t/\epsilon$ is appropriate to express the solutions. u_x and u_{xx} may be large near some $x = \eta^i, 0 \leq i \leq r$. These regions are called boundary (internal) layers if $i = 0, r$ (or $1 \leq i \leq r - 1$), where the stretched space variable $\xi = \frac{x - \eta^i}{\epsilon}$ is used. Regions that are not singular with respect to space or time are called regular layers. We use S or R to denote singular or regular layers. The symbol related to space is put before that related to time since in the dictionary order space is before time. Thus $(SR)^i, (RR)^i, (SS)^i$ and $(RS)^i$ are used to denote the ith spatially singular, temporally regular region, etc. Figure 1 shows relative locations of all the possible layers. Superscripts on a solution are used to show the type of layers where the solution is expressed by the appropriate variables.

$$\begin{aligned}
u^{RRi}(x, t, \epsilon) &= u(x, t, \epsilon), & \text{for } (x, t) \in (RR)^i, \\
u^{SRi}(\xi, t, \epsilon) &= u(\epsilon\xi + \eta^i, t, \epsilon), & \text{for } (x, t) \in (SR)^i, \\
u^{RSi}(x, \tau, \epsilon) &= u(x, \epsilon\tau, \epsilon), & \text{for } (x, t) \in (RS)^i, \\
u^{SSi}(\xi, \tau, \epsilon) &= u(\epsilon\xi + \eta^i, \epsilon\tau, \epsilon), & \text{for } (x, t) \in (SS)^i.
\end{aligned}$$

Each layers is further expanded in powers of ϵ, $u^{RRi}(x, t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^i u_j^{RRi}(x, t)$, etc.

The notation $\overset{*}{u}(\tau)$ is used to denote the expansion of $u(t)$ in the variable τ. $\tilde{u}(\xi)$ is used to denote the expansion of $u(x)$ in the variable ξ. $\overline{u}$ is used to denote the initial condition for a solution u.

Let $C_{bu}(\mathbb{R}, \mathbb{R}^n), C_{bu}(\mathbb{R}^+, \mathbb{R})$ and $C_{bu}(\mathbb{R}^-, \mathbb{R}^n)$ be the Banach spaces of uniformly continuous and bounded functions with super norms. Let $C_{bu}^m = \{u | u, u', \ldots, u^{(m)} \in C_{bu}\}$ with the norm

$$(2.1) \qquad \|u\|_{C_{bu}^m} = \sum_{i=0}^{m} \|u^{(i)}\|_{C_{bu}}.$$

Here C_{bu}^m denotes $C_{bu}^m(\mathbb{R}, \mathbb{R}^n)$ or $C_{bu}^m(\mathbb{R}^\pm, \mathbb{R}^n)$. We can show that $C_{bu}^m, m \geq 1$, is dense in C_{bu}^{m-1}.

For a continuous function $w(\xi) > 0$, let $E_{\mathbb{R}}(w)$ be the Banach space of functions with the weight $w(\xi)$.

$$\begin{aligned}
E_{\mathbb{R}}(w) &= \{u : \mathbb{R} \to \mathbb{R}^n | u(\cdot)/w(\cdot) \in C_{bu}(\mathbb{R}, \mathbb{R}^n)\}. \\
\|u\|_{E(w)} &= \sup\{|u(\xi)/w(\xi)|, \xi \in \mathbb{R}\}. \\
E_{\mathbb{R}}^m(w) &= \{u | u, \ldots, u^{(m)} \in E(w)\}. \\
\|u\|_{E^m(w)} &= \sum_{j=0}^{m} \|u^{(j)}\|_{E(w)}.
\end{aligned}$$

Similarly, $E_{\mathbb{R}^+}^m(w)$ and $E_{\mathbb{R}^-}^m(w)$ are Banach spaces of weighted functions that are defined on $\mathbb{R}^+$ and $\mathbb{R}^-$. We use $E^m(w)$ to denote $E_{\mathbb{R}}^m(w)$ or $E_{\mathbb{R}^\pm}^m(w)$ if no confusion

should arise. One of the most often used weights is

$$(2.2) \qquad w(\xi) = (1 + |\xi|^j)e^{-\gamma\xi}, \quad \gamma \in \mathbb{R}, \, j \geq 0.$$

Let

$$\begin{aligned}
B_{\mathbb{R}^+}^m &= \{u \in C_{bu}^m(\mathbb{R}^+, \mathbb{R}^n), \lim_{\xi \to \infty} D_\xi^\alpha u(\xi) = u^{(\alpha)}(+\infty) \text{ exists for } \alpha \leq m\}, \\
B_{\mathbb{R}^-}^m &= \{u \in C_{bu}^m(\mathbb{R}^-, \mathbb{R}^n), \lim_{\xi \to -\infty} D_\xi^\alpha u(\xi) = u^{(\alpha)}(-\infty) \text{ exists for } \alpha \leq m\}. \\
B_{\mathbb{R}}^m &= \{u \in C_{bu}^m(\mathbb{R}, \mathbb{R}^n), \lim_{\xi \to \pm\infty} D_\xi^\alpha u(\xi) = u^{(\alpha)}(\pm\infty) \text{ exists for } \alpha \leq m\}.
\end{aligned}$$

One can easily verify that $u^{(\alpha)}(\pm\infty) = 0$ if $\alpha \geq 1$.

Let $w(\xi) = (1 + |\xi|^j)e^{-\gamma|\xi|}$, $\gamma > 0$. Define,

$$\begin{aligned}
B_{\mathbb{R}^+}^m(w) &= \{u \in B_{\mathbb{R}^+}^m, \quad u(\cdot) - u(+\infty) \in E_{\mathbb{R}^+}^m(w)\}. \\
B_{\mathbb{R}^-}^m(w) &= \{u \in B_{\mathbb{R}^-}^m, \quad u(\cdot) - u(-\infty) \in E_{\mathbb{R}^-}^m(w)\}. \\
B_{\mathbb{R}}^m(w) &= \{u \in B_{\mathbb{R}}^m, \quad u(\cdot) - u(\pm\infty) \in E_{\mathbb{R}\pm}^m(w)\}.
\end{aligned}$$

The general notation $B^m(w)$ will be used if no confusion should arise. Let the norms of $B_{\mathbb{R}^+}^m, B_{\mathbb{R}^-}^m, B_{\mathbb{R}}^m$ be induced from C_{bu}^m respectively. It is clear that $B_{\mathbb{R}^+}^m, B_{\mathbb{R}^-}^m$ and $B_{\mathbb{R}}^m$ are Banach spaces with these norms. Let

$$(2.3)$$
$$\|u\|_{B_{\mathbb{R}}^m(w)} = \|u\|_{C_{bu}^m} + \|u - u(+\infty)\|_{E_{\mathbb{R}^+}^m(w)} + \|u - u(-\infty)\|_{E_{\mathbb{R}^-}^m(w)}$$

Similarly definitions are given to $\|u\|_{B_{\mathbb{R}^+}^m}(w)$ and $\|u\|_{B_{\mathbb{R}^-}^m}(w)$. It can be verified that $B_{\mathbb{R}}^m(w), B_{\mathbb{R}^-}^m(w)$ and $B_{\mathbb{R}^+}^m(w)$ are all Banach spaces with the specified norms.

Let X be one of the Banach spaces: $E_{\mathbb{R}^+}^m(w)$, $E_{\mathbb{R}^-}^m(w)$, $B_{\mathbb{R}^+}^m(w)$ or $B_{\mathbb{R}^-}^m(w)$, with $m \geq 1$. For C^1 functions defined on $\mathbb{R}^+$ or $\mathbb{R}^-$, denote $BC = \{u \in C^1 : u_\xi(0) = 0\}$. $X \cap (BC)$ is a closed subspace of X, and a Banach space with the norm induced from X.

The following lemma is useful when working with these weighted spaces and can be verified easily.

Lemma 2.1. *Let $\alpha > 0$, $|\gamma| < \alpha$ be real constants, $j \geq 0$ be an integer. Then there exists a constant K_1 such that*

$$\int_{-\infty}^{\xi} e^{-\alpha(\xi-s)}(1 + |s|^j)e^{-\gamma s}\, ds + \int_{\xi}^{\infty} e^{-\alpha(s-\xi)}(1 + |s|^j)e^{-\gamma s}\, ds$$

$$\leq \frac{K_1 e^{-\gamma\xi}}{(\alpha - |\gamma|)^{j+1}}(1 + |\xi|^j).$$

2.2. Properties of elliptic equations. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞, and V is a real constant. Equation

$$(2.4) \qquad u_{\xi\xi} + Vu_\xi + f(u) = 0$$

is equivalent to a system in $\mathbb{R}^{2n}$:

$$(2.5) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= -Vv - f(u). \end{aligned}$$

Therefore the phase space for (2.4) is $\mathbb{R}^{2n}$, comprised of points (u, u_ξ). We say p is a hyperbolic equilibrium for (2.4) if $(p, 0)$ is a hyperbolic equilibrium for (2.5). We say equation

$$(2.6) \qquad u_{\xi\xi} + Vu_\xi + A(\xi)u = 0$$

has exponential dichotomy on an interval $I \subset \mathbb{R}$ if the system

$$(2.7) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= -Vv - A(\xi)u \end{aligned}$$

has an exponential dichotomy on I. Here $A(\cdot) : I \to \mathbb{R}^{n \times n}$ is a continuous matrix valued function.

We say $u(\xi)$ is a heteroclinic solution of (2.4) if $(u(\xi), u_\xi(\xi))$ is a heteroclinic solution for the equivalent system (2.5).

Lemma 2.2. *Assume that $p \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ and there exists $\sigma_0 > 0$*

$$(2.8) \qquad f(p) = 0, \quad R_e\sigma\{Df(p)\} \leq -\sigma_0.$$

Then

$$(2.9) \qquad u_{\xi\xi} + Vu_\xi + Df(p)u = 0$$

has an exponential dichotomy on $\mathbb{R}$ with n–dimensional stable and unstable spaces. Let $0 < \alpha < \sqrt{V^2 + 4\sigma_0} - |V|$. Then the decay rate on the stable (or unstable) subspace is bounded by $Ke^{-\alpha\xi}$, $\xi \geq 0$ (or $Ke^{\alpha\xi}$, $\xi \leq 0$) respectively.

Let $p^i, i = 1, 2$, satisfy (2.8). Let $q(\xi)$ be a solution to (2.4) and is defined on $\mathbb{R}^-$ with $q(\xi) \to p^1$ as $\xi \to -\infty$, and/or is defined on $\mathbb{R}^+$ with $q(\xi) \to p^2$ as $\xi \to \infty$. Then

$$(2.10) \qquad u_{\xi\xi} + Vu_\xi + Df(q(\xi))u = 0$$

has exponential dichotomies on $\mathbb{R}^-$ or $\mathbb{R}^+$ respectively, with $\mathcal{R}P_s(t)$ and $\mathcal{R}P_u(t)$ being n-dimensional subspaces in $\mathbb{R}^{2n}$. Here $P_u(t) + P_s(t) = I, t \in \mathbb{R}^-$ or $t \in \mathbb{R}^+$, are the projections to the unstable and stable subspaces. Moreover, the decay rate $\alpha > 0$ is the same as that of (2.9). In the case that q is a heteroclinic solution connecting p^1 and p^2, $\mathcal{R}P_u(0^-) \cap \mathcal{R}P_s(0^+)$ is at least one dimensional, containing $(q_\xi(0), q_{\xi\xi}(0))$.

Lemma 2.3. *(i) Let $p \in \mathbb{R}^n$ satisfy (2.8), α be the constant as in Lemma 2.2. Let $w(\xi)$ be the weight function in (2.2) where $|\gamma| < \alpha$, $X = E_\mathbb{R}^m(w)$ and $g \in X$. Then there exists a unique solution $u \in E_\mathbb{R}^{m+2}(w)$ to the equation*

$$(2.11) \qquad u_{\xi\xi} + Vu_\xi + Df(p)u = g.$$

Moreover,

$$(2.12) \qquad \|u\|_{E^{m+2}(w)} \leq C\|g\|_{E^m(w)}.$$

(ii) Let X be $E_{\mathbb{R}^+}^m(w)$ or $E_{\mathbb{R}^-}^m(w)$, and $g \in X$. Assume that p^i, $i = 1, 2$, and $q(\xi)$ are as in Lemma 2.2. Assume that $u_\xi(0) \neq 0$ for all nontrivial bounded solutions u to the equation $u_{\xi\xi} + Vu_\xi + Df(q)u = 0$. Then there exists a unique solution $u \in E_{\mathbb{R}^+}^{m+2}(w)$ or $E_{\mathbb{R}^-}^{m+2}(w)$ to the boundary value problem

$$\begin{aligned} u_{\xi\xi} + Vu_\xi + Df(q)u &= g, \\ u_\xi(0) &= \phi, \end{aligned}$$

Moreover,

$$(2.13) \qquad \|u\|_{E^{m+2}(w)} \leq C(\|g\|_{E^m(w)} + \|\phi\|_{\mathbb{R}^n}).$$

Let $p^i \in \mathbb{R}^n, i = 1, 2$, satisfy (2.8). Let $q(\xi)$ be a heteroclinic solution to (2.4) connecting p^1 to p^2. Let $X = E_{\mathbb{R}}^m(w)$ where $w(\xi)$ is as in Lemma 2.3. Define $L_q : X \to X$ with $D(L_q) = E_{\mathbb{R}}^{m+2}(w)$ by

$$(2.14) \qquad\qquad L_q u = u_{\xi\xi} + V u_\xi + Df(q(\xi))u.$$

Lemma 2.4. L_q *is a Fredholm operator with Fredholm index zero. Assume that* $\dim Ker(L_q) = 1$ *then* $Ker(L_q) = span\{q_\xi\}$ *and* $Range(L_q) = \{\Psi\}^\perp$. *Here* Ψ *is the unique nontrivial bounded solution for the adjoint equation, up to a scalar multiple,*

$$(2.15) \qquad\qquad L_q^*\Psi \overset{def}{=} \Psi_{\xi\xi} - V\Psi_\xi + Df^\tau(q(\xi))\Psi = 0.$$

$$(2.16) \qquad\qquad \{\Psi\}^\perp = \{u \in X | \int_{-\infty}^{\infty} \Psi^\tau(\xi)u(\xi)d\xi = 0\}.$$

2.3. **Properties of Parabolic Equations.** Let A be a closed densely defined linear operator in a Banach space X. Suppose that A is sectorial and generates a C^0 analytic semigroup e^{At} in X. For $0 \le \theta < 1$, let $D_A(\theta)$ be the intermediate space between D_A and X.

$$\begin{aligned}
D_A(\theta) \quad &= \quad \{x \in X | \lim_{t\to 0} t^{1-\theta} Ae^{At}x = 0\}, \\
\|x\|_\theta \quad &= \quad \sup_{0 < t \le 1} |t^{1-\theta} Ae^{At}x|_X + |x|_X, \\
D_A(\theta + 1) \quad &= \quad \{x | x \in D_A, Ax \in D_A(\theta)\}.
\end{aligned}$$

Let $D_A(1) = D_A$. Let $0 \le \beta \le \theta \le 1$ and $0 \le \theta - \beta < 1$. Let $F : D_A(\theta) \to D_A(\beta)$ be a nonlinear, Lipschitz continuous function and $x \in D_A(\theta)$, $0 \le \theta \le 1$. Then there exist $t_0 > 0$ and a unique classical solution u defined on $[0, t_0]$ such that

$$\begin{aligned}
u_t &= Au + F(u), \\
u(0) &= x,
\end{aligned}$$

where

$$(2.17) \qquad u \in C([0, t_0] : D_A(\theta)) \cap C^1((0, t_0] : X) \cap C((0, t_0] : D_A).$$

Denote the solution by $U(t)$. Then $U : [0, t_0] \to D_A(\theta)$ is Hölder continuous with Hölder exponent $1 + \beta - \theta$. Assume that $F \in C^1$ with DF being Lipschitz continuous. Then $DF(U(\cdot)) : [0, t_0] \to \mathcal{L}(D_A(\theta) : D_A(\beta))$ is Hölder continuous. Let $g : [0, t_0] \to D_A(\beta)$ be locally Hölder continuous, then

$$(2.18) \qquad \begin{aligned}
u_t &= Au + DF(U(t))u + g(t), \\
u(0) &= u_0,
\end{aligned}$$

has a unique classical solution u that also satisfies (2.17). Moreover there exists an evolution operator $T(t, s)$ such that

$$u(t) = T(t, 0)u_0 + \int_0^t T(t, s)g(s)ds.$$

These well known facts can be found in [5], [31] and [24].

The linear equation (2.18) is said to be exponentially stable if $T(t, s)$ is defined for all $0 \le s \le t$ and if there exist constants K, $\alpha > 0$, $0 \le \theta - \beta < 1$, such that

$$\|T(t, s)\|_\theta \le Ke^{-\alpha(t-s)}, \quad t > s;$$

$$\|T(t, s)\|_{\mathcal{L}(D_A(\beta):D_A(\theta))} \le K[1 + (t - s)^{\beta-\theta}], \quad t > s.$$

Lemma 2.5. *Assume that (2.18) is exponentially stable and $g : [0, \infty) \to D_A(\beta)$ is locally Hölder continuous and for some integer $k \geq 0$,*

$$\|g(t)\|_{D_A(\beta)} \leq C(1 + t^k), \quad t \geq 0.$$

Then for each $u_0 \in D_A(\theta)$, (2.18) has a unique solution that satisfies (2.17) for all $t_0 > 0$ and

$$\|u(t)\|_\theta \leq C(1 + t^k), \quad t \geq 0.$$

Let $w(\xi)$ be a weight function in (2.2). Let $X = E_\mathbb{R}(w)$, $E_{\mathbb{R}^+}(w)$, $E_{\mathbb{R}^-}(w)$, $B_\mathbb{R}(w)$, $B_{\mathbb{R}^+}(w)$ or $B_{\mathbb{R}^-}(w)$. Let $A : X \to X$ be defined as

$$Au = u_{\xi\xi}$$

with $D_A = E_\mathbb{R}^2(w)$ or $B_\mathbb{R}^2(w)$ if $X = E_\mathbb{R}(w)$ or $B_\mathbb{R}(w)$, $D_A = E_{\mathbb{R}^\pm}^2(w) \cap BC$ or $B_{\mathbb{R}^\pm}^2(w) \cap BC$ if $X = E_{\mathbb{R}^\pm}(w)$ or $B_{\mathbb{R}^\pm}(w)$.

Lemma 2.6. *A is a sectorial operator in X with D_A dense in X.*

Lemma 2.7. *Let X and $A : D_A \to X$ be as in Lemma 2.6. Then depending on the choice of X, $E_\mathbb{R}^1(w)$, or $B_\mathbb{R}^1(w)$, or $E_{\mathbb{R}^\pm}^1(w) \cap BC$, or $B_{\mathbb{R}^\pm}^1(w) \cap BC \subset D_A(\frac{1}{2})$.*

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^∞. Assume that $f(p^1) = f(p^2) = 0$ and there exists $V \in \mathbb{R}$ such that equation (2.4) admits a heteroclinic solution $q(\xi)$ connecting p^1 to p^2. Let $X = E(w)$ or $B(w)$ and $A : D_A \to X$ be defined as $Au = u_{\xi\xi}$ as in Lemma 2.6. It can be verified that $f : D_A \to D_A$ and $f : X \to X$ are both C^∞. Therefore, $f : D_A(\theta) \to D_A(\theta)$ for any $0 < \theta < 1$ is also C^∞.

Consider

$$(2.19) \qquad \begin{aligned} &u_\tau = u_{\xi\xi} + V u_\xi + f(u), \\ &u(0) = \overline{u}_0, \ \overline{u}_0 \in D_A(\theta), \qquad 0 \leq \theta \leq 1. \end{aligned}$$

For each $\overline{u}_0$, there exists $T > 0$ such that a unique classical solution exists in $[0, T]$. Also, $q(\xi)$ is a stationary solution to (2.19). Consider the linear variational equation around $q(\xi)$.

$$(2.20) \qquad \begin{aligned} &u_\tau = u_{\xi\xi} + V u_\xi + Df(q(\xi))u, \\ &u(0) = \overline{u}_0, \ \overline{u}_0 \in D_A(\theta). \end{aligned}$$

Define $L_q u = u_{\xi\xi} + V u_\xi + Df(q(\xi))u$. As a perturbation to A, L_q is also a sectorial operator in X, cf. [27, page 80].

Definition 2.1. The solution q to equation (2.19) is said to be asymptotically (exponentially) stable modulo spatial shifts if there exists an open set $O \subset D_A(\theta)$, $q \in O$, such that for every $\overline{u}_0 \in O$, there is a constant $c \in \mathbb{R}$, such that

$$|u(\tau) - q(\cdot + c)|_{D_A(\theta)} \leq Ce^{-\gamma\tau}.$$

The zero solution to equation (2.20) is said to be asymptotically stable modulo q_ξ if for every $\overline{u}_0 \in D_A(\theta)$, there is a constant $c \in \mathbb{R}$, $|u(\tau) - cq_\xi|_{D_A(\theta)} \leq Ce^{-\gamma\tau}$.

Lemma 2.8. *(Evans) (a) The stationary solution q of (2.19) is asymptotically stable modulo spatial shifts if and only if the zero solution of (2.20) is asymptotically stable modulo q_ξ.*

(b) The zero solution of (2.20) is asymptotically stable modulo q_ξ if and only if there exists $\alpha > 0$ such that

$$\sigma\{L_q\} \cap \{Re\lambda > -\alpha\} = \{0\},$$

and $\lambda = 0$ is a simple eigenvalue with the eigenspace spanned by q_ξ.

The following lemma puts a strong restriction on the essential spectrum of L_q.

Lemma 2.9. (a) *Let M be a constant $n \times n$ matrix, $\gamma \in \mathbb{R}$ be the constant in (2.2) and $Lu = u_{\xi\xi} + Vu_\xi + Mu$. Then L is a sectorial operator on $X = E_{\mathbb{R}}(w)$, $E_{\mathbb{R}^\pm}(w)$, $B_{\mathbb{R}}(w)$, or $B_{\mathbb{R}^\pm}(w)$ with $D_L = E_{\mathbb{R}}^2(w), E_{\mathbb{R}^\pm}^2(w) \cap BC, B_{\mathbb{R}}^2(w)$ or $B_{\mathbb{R}^\pm}^2(w) \cap BC$ respectively. Assume furthermore that $Re(\sigma(M)) \leq -\sigma_0 < 0$ and γ satisfy that $\eta_0 = (\gamma^2 + 2|V\gamma|)/4 < \sigma_0$, then*

$$Re(\sigma(L)) \leq -\sigma_0 + \eta_0$$

(b) *Let p^i, $i = 1, 2$, satisfy (2.8) and $q(\xi)$ be a heteroclinic solution connecting p^1 to p^2. Let X, γ and η_0 be as in part (a). Then $\sigma\{L_q\} \cap \{Re\lambda > -\sigma_0 + \eta_0\}$ consists of only isolated eigenvalues, each is of finite algebraic multiplicity.*

Corollary 2.10. *The zero solution $u_\tau = u_{\xi\xi} + Vu_\xi + f_u(p^i)u, i = 1, 2$, is asymptotically stable in the space X. The stationary solution $u = p^i, i = 1, 2$, for $u_\tau = u_{\xi\xi} + Vu_\xi + f(u)$ is asymptotically stable.*

3. STATIONARY SOLUTIONS THAT HAVE LAYER STRUCTURES

The stationary solutions of (1.1) satisfy

$$(3.1) \qquad\qquad \epsilon^2 u_{xx} + f(u, x, \epsilon) = 0.$$

The first set of assumptions are used to construct a stationary solution to system (1.1) that exhibits internal and boundary layers and to obtain the asymptotic expansion of such a solution to any desired order in ϵ.

Assume that there is a partition of $[a, b]$:

$$x^0 = a < x^1 < \cdots < x^r = b.$$

On each $[x^{i-1}, x^i]$, a C^∞ function $p^i(x)$, $1 \leq i \leq r$, is defined with $f_0(p^i(x), x) = 0$.

H1. $Re\,\sigma\{f_{0u}(p^i(x), x)\} < 0$ for $x^{i-1} \leq x \leq x^i$, $i = 1, \cdots, r$.

We introduce a stretched variable $\xi = (x - x^i)/\epsilon$ in a neighborhood of each x^i, $0 \leq i \leq r$. The 0-th expansion of (3.1) can be written as

$$(3.2) \qquad\qquad u_{\xi\xi} + f_0(u, x^i) = 0.$$

Assume that a C^∞ function $q^i(\xi)$ is defined for $\xi \in \mathbb{R}$ if $1 \leq i \leq r - 1$, $\xi \in \mathbb{R}^+$ if $i = 0$ and $\xi \in \mathbb{R}^-$ if $i = r$, such that $u = q^i(\xi)$ satisfies (3.2). Also, $q^i(\xi) \to p^i(x^i)$ as $\xi \to -\infty$ for $1 \leq i \leq r$ and $q^i(\xi) \to p^{i+1}(x^i)$ as $\xi \to \infty$ for $0 \leq i \leq r - 1$. Moreover, $q^i_\xi(0) = 0$ for $i = 0, r$, cf. (1.2).

The linear homogeneous equation

$$(3.3) \qquad\qquad \phi_{\xi\xi} + f_{0u}(q^i(\xi), x^i)\phi = 0,$$

and its adjoint equation

$$(3.4) \qquad\qquad \psi_{\xi\xi} + f_{0u}^\tau(q^i(\xi), x^i)\psi = 0,$$

are important in our study. (Here τ denotes the transpose.)

H2. $q^i_\xi(\xi)$, $\xi \in \mathbb{R}$, $1 \leq i \leq r-1$, is the only bounded solution of (3.3) up to constant multiples.

Because of H2, we can show that (3.4) has a unique bounded solution $\psi_i(\xi)$, $\xi \in \mathbb{R}$, $1 \leq i \leq r - 1$, up to constant multiples. See [26]. Moreover, ψ_i decays exponentially as $\xi \to \pm\infty$.

H3. $\displaystyle\int_{-\infty}^{\infty} \psi_i^\tau(\xi) f_{0x}(q^i(\xi), x^i) d\xi \neq 0$, $1 \leq i \leq r-1$.

Hypothesis H3 is expressed by a Melnikov type integral that replaces a hypothesis expressed by potential functions in [12, 13]. If $n = 1$, (3.3) is self adjoint. Thus $\psi = q_\xi^i$. The integral in H3 is equal to $D_x J(x^i)$ where $J(x) = \int_{p^i(x)}^{p^{i+1}(x)} f_0(u, x) du$ as in [12, 13].

H4. Let $\phi^i(\xi)$, $\xi \in \mathbb{R}^+$ for $i = 0$ and $\xi \in \mathbb{R}^-$ for $i = r$, be any nontrivial bounded solution for (3.3). Then $\phi_\xi^i(0) \neq 0$ for $i = 0, r$.

We look for the position of the internal layers $x^i(\epsilon) = \sum_{j=0}^\infty \epsilon^j x_j^i$, formal series solution $\sum_{j=0}^\infty \epsilon^j u_j^{Ri}(x)$ to (3.1) in regular layer $(x^{i-1}(\epsilon), x^i(\epsilon))$, $1 \leq i \leq r$, and formal series solution $\sum_{j=0}^\infty \epsilon^j u_j^{Si}(\xi)$, $0 \leq i \leq r$, to the equation

$$(3.5) \qquad u_{\xi\xi} + f(u, x^i(\epsilon) + \epsilon\xi, \epsilon) = 0,$$

where $\xi = (x - x^i(\epsilon))/\epsilon$. Let the superscripts "$R$" and "$S$" stand for spatial regular and singular layers. Each $u_j^{Si}(\xi)$ satisfies a growth condition

$$(3.6) \qquad |u_j^{Si}(\xi)| \leq C(1 + |\xi|^j)$$

as $|\xi| \to \infty$ and a boundary condition

$$(3.7) \qquad u_{j\xi}^{Si}(0) = 0$$

if $i = 0, r$. Let the inner expansions of the outer solutions be

$$\sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,1}^i(\xi) = \sum_{j=0}^\infty \epsilon^j u_j^{R,i+1}(\epsilon\xi + x^i(\epsilon)),$$

$$\sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,2}^i(\xi) = \sum_{j=0}^\infty \epsilon^j u_j^{Ri}(\epsilon\xi + x^i(\epsilon)).$$

The right hand sides in the above are formal expressions. Their expansions in ϵ are well defined even if $x^i(\epsilon)$ is a formal series and $u_j^{Ri}(x)$ is not defined for $x \in \mathbb{R}$. See [22]. Each $\tilde{u}_{j,h}^i$, $h = 1, 2$, is in fact a polynomial of degree j. We say that the solution in the singular layer $\sum \epsilon^j u_j^{Si}(\xi)$ matches solution $\sum \epsilon^j u_j^{Ri}(x)$ or $\sum \epsilon^j u_j^{R,i+1}(x)$ if

$$(3.8) \qquad \begin{aligned} |u_j^{Si}(\xi) - \tilde{u}_{j,1}^i(\xi)| &\leq C((1 + |\xi|^j) e^{\gamma\xi}), \quad \xi \leq 0, \\ |u_j^{Si}(\xi) - \tilde{u}_{j,2}^i(\xi)| &\leq C((1 + |\xi|^j) e^{-\gamma\xi}), \quad \xi \geq 0. \end{aligned}$$

To construct those series solutions, we shall use the result from [22]. We verify that for an equivalent first order system in $\mathbb{R}^{2n}$, all the conditions in [22] are satisfied. From Lemma 2.2 and H1, $u = p^i(x)$ is a hyperbolic equilibrium for the equation

$$(3.9) \qquad u_{\xi\xi} + f_0(u, x) = 0,$$

for $x^{i-1} \leq x \leq x^i$. That is, $(p^i(x), 0)$ is a hyperbolic equilibrium for the equivalent system

$$(3.10) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= -f_0(u, x). \end{aligned}$$

Also from Lemma 2.2, the unstable spaces are n-dimensional for all $1 \leq i \leq r - 1$, and x. Thus the hypothesis H1 in [22] is satisfied.

On the other hand, when $x = x^i$, $1 \leq i \leq r - 1$. Since $q^i(\xi) \to p^i(x^i)$ or $p^{i+1}(x^i)$ as $\xi \to -\infty$ or $+\infty$, based on Lemma 2.2 again, the linearized equation

$$(3.11) \qquad \begin{aligned} \phi_\xi &= \overline{\phi}, \\ \overline{\phi}_\xi &= -f_{0u}(q^i(\xi), x^i)\phi, \end{aligned}$$

has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. From H2, $(q^i_\xi(\xi), q^i_{\xi\xi}(\xi))$ is the only bounded solution of (3.11) up to a constant factor. From [26], there exists a unique bounded solution $(\psi_{i\xi}(\xi), \psi_i(\xi))$, up to a constant factor, to the adjoint equation

$$(3.12) \qquad \begin{aligned} \psi_\xi &= f_{0u}^\tau(q_i(\xi), x^i)\overline{\psi}, \\ \overline{\psi}_\xi &= -\psi. \end{aligned}$$

Cf. (3.3) and (3.4). Denote the right hand side of (3.10) by $F(u, v, x)$. Then

$$(3.13) \qquad \frac{\partial F}{\partial x}(u, v, x) = \begin{pmatrix} 0 \\ -f_{0x}(u, v) \end{pmatrix}.$$

It is now clear that H3 implies [22, H3)].

Finally, it is clear that H4 implies that [22, H2)]. The result from [22] yields:

Theorem 3.1. *Under the hypotheses* H1 *to* H4*, there exist unique formal series:*

$$(3.14) \qquad \sum_{j=0}^\infty \epsilon^j x_j^i, \quad 0 \leq i \leq r, \quad x_0^i = x^i, x_j^0 = x_j^r = 0 \text{ for all } j \geq 1,$$

$$(3.15) \qquad \sum_{j=0}^\infty \epsilon^j u_j^{Ri}(x), \quad u_0^{Ri}(x) = p^i(x), \quad 1 \leq i \leq r,$$

$$(3.16)$$

$$\sum_{j=0}^\infty \epsilon^j u_j^{Si}(\xi), \quad u_0^{Si}(\xi) = q^i(\xi), \quad \text{with} \begin{cases} \xi \in \mathbb{R}, & 1 \leq i \leq r - 1, \\ \xi \in \mathbb{R}^+, & i = 0, \\ \xi \in \mathbb{R}^-, & i = r, \end{cases}$$

such that (3.14) is the position of the singular layer $x^i(\epsilon)$, (3.15) satisfies (3.1), (3.16) satisfies (3.5), (3.6) and (3.7) if $i = 0, r$. The functions u_j^{Ri} and constants x_j^i are computable by systems of recursive linear algebraic equations. The functions u_j^{Si} are computable by a system of recursive linear nonhomogeneous differential equations. Moreover, the series solutions obtained above satisfy the matching condition (3.8).

4. EXISTENCE OF WAVE-FRONT-LIKE SOLUTIONS

4.1. **Hypotheses and lemmas.** To study time evolution solutions of (1.1), some more hypotheses will be made. Let $L_{q^i} : C_{bu} \to C_{bu}$ be an unbounded operator defined by

$$L_{q^i}(u)(\xi) = u_{\xi\xi} + f_{0u}(q^i(\xi), x^i)u.$$

The domain $D(L_{q^i}) = C_{bu}^2(\mathbb{R}, \mathbb{R}^n)$ for $1 \leq i \leq r - 1$ and $D(L_{q^i}) = C_{bu}^2 \cap BC$, for $i = 0, r$.

H5. There exists a constant α_0 such that all the eigenvalues of L_{q^0} and L_{q^r} satisfy $Re\lambda < -\alpha_0$. The operator L_{q^i}, $1 \leq i \leq r - 1$, has a simple eigenvalue $\lambda = 0$ with an eigenvector $q_\xi^i(\xi)$, all the other spectra satisfy $Re\,\lambda < -\alpha_0$.

Hypothesis H5 ensures the stability of $q^i(\xi)$ as a solution to $u_\tau = u_{\xi\xi} + f_0(u, x^i)$ in C_{bu}.

Since $\lambda = 0$ is a simple eigenvalue of L_{q^i}, $1 \leq i \leq r - 1$, we have

$$(4.1) \qquad \int_{-\infty}^{\infty} \psi_i^\tau(\xi) q_\xi^i(\xi) d\xi \neq 0, \quad 1 \leq i \leq r - 1,$$

where $\psi_i(\xi)$ is the bounded solution for (3.4). The proof of (4.1) uses Lemma 2.4. If (4.1) were not valid, then q_ξ^i is in the range of L_{q^i}. This contradicts the fact $\lambda = 0$ is simple. Assume now

H6. $\displaystyle \int_{-\infty}^{\infty} \psi_i^\tau(\xi) q_\xi^i(\xi) d\xi \cdot \int_{-\infty}^{\infty} \psi_i^\tau(\xi) f_{0x}(q^i(\xi), x^i) d\xi > 0 \quad$ for $1 \leq i \leq r - 1$.

Due to H3 and (4.1), H6 is only a sign condition. We shall see that H6 implies that the position of the wave front near x^i moves towards x^i.

It is useful to note that (4.1) is always valid if $n = 1$. In that case, L_{q^i} is self adjoint, thus $\psi_i(\xi) = q_\xi^i(\xi)$.

We look for wave front positions that are in open intervals O^i containing x^i, $1 \leq i \leq r - 1$. We state the following properties of O^i.

P1. $p^i(x)$ and $p^{i+1}(x)$ can be extended smoothly to O^i with

$$(4.2) \qquad f_0(p^j(x), x) = 0,$$

$$(4.3) \qquad Re\{\sigma(f_{0u}(p^j(x), x))\} < 0, \quad j = i, i + 1.$$

P2. There exists a C^∞ function $V^i : O^i \to \mathbb{R}$ such that for each $x \in O^i$, equation

$$(4.4) \qquad u_{\xi\xi} + V^i(x) u_\xi + f_0(u, x) = 0$$

admits a heteroclinic solution $q^i(\xi, x)$, connecting $p^i(x)$ to $p^{i+1}(x)$, with $(u(0) - q^i(0)) \perp q_\xi^i(0)$. In particular, $V^i(x^i) = 0$ and $q^i(\xi, x^i) = q^i(\xi)$. Moreover, $D_x^k q^i(\cdot, x) \in C_{bu}^j$ for all $j, k \geq 0$.

P3. The linear equation

$$(4.5) \qquad \phi_{\xi\xi} + V^i(x) \phi_\xi + f_{0u}(q^i(\xi, x), x) \phi = 0$$

has a unique bounded solution $q_\xi^i(\xi, x)$, up to constant multiples. And the adjoint equation

$$(4.6) \qquad \psi_{\xi\xi} - V^i(x) \psi_\xi + f_{0u}^\tau(q^i(\xi, x), x) \psi = 0$$

has a unique bounded solution $\psi_i(\xi, x)$, $|\psi_i(0, x)| = 1$, up to constant multiples. Furthermore $\psi(., x)$ is a C^∞ function of x in the space C_{bu}.

P4. In the Banach space $E_\mathbb{R}(w)$, $L_x^i u = u_{\xi\xi} + V^i(x) u_\xi + f_{ou}(q^i(\xi, x), x) u$, $1 \leq i \leq r - 1$, has $\lambda = 0$ as a simple eigenvalue with eigenvector $q_\xi^i(\xi, x)$. All the other spectra of L_x^i satisfy $Re\lambda < -\alpha_0$ for some $\alpha_0 > 0$.

P5. For $x \in O^i$, we have

$$(4.7)$$
$$\{\int_{-\infty}^{\infty} \psi_i^\tau(\xi, x) q_\xi^i(\xi, x) d\xi\}\{\int_{-\infty}^{\infty} \psi_i^\tau(\xi, x) f_{0x}(q^i(\xi, x), x) d\xi\} > 0, \quad 1 \leq i \leq r - 1.$$

Also $V^i(x) > 0 (= 0, < 0)$ if $x < x^i (= x^i, > x^i)$. And

$$\frac{\partial V^i(x^i)}{\partial x} = \frac{-\int_{-\infty}^{\infty} \psi_i^\tau(\xi) f_{0x}(q^i(\xi), x^i) d\xi}{\int_{-\infty}^{\infty} \psi_i^\tau(\xi) q_\xi^i(\xi) d\xi} < 0.$$

Lemma 4.1. *Assume that* H1*,* H2 *and* H5 *are satisfied. Then there exist open intervals* O^i *containing* x^i, $1 \le i \le r-1$ *such that Properties* P1–P4 *hold. Moreover, if* H3 *and* H6 *are also satisfied, then Property* P5 *also holds.*

The proof of Lemma 4.1 will be given in §7.
Consider the nonhomogeneous equation, $1 \le i \le r-1$,:

$$(4.8) \qquad u_{\xi\xi} + V^i(x)u_\xi + f_{0u}(q^i(\xi, x), x)u + V_1 q_\xi^i(\xi, x) = g(\xi).$$

Here $V^i(x)$ and $q^i(\xi, x)$ are the functions in P3, $V_1 \in \mathbb{R}$ is a parameter. Suppose that $g \in E_{\mathbb{R}}^m(1 + |\xi|^k), m, k \ge 0$.

Lemma 4.2. *Assume that* H1*,* H2 *and* H5 *are satisfied. Let* $1 \le i \le r-1$. *Then there exists a unique* C^∞ *function* $V_*^i : O^i \times E_{\mathbb{R}}^m(1 + |\xi|^k) \to \mathbb{R}$ *such that if* $V_1 = V_*^i(x, g)$, *then there is a unique solution* $u(\xi, x, g)$ *of (4.8) with* $u \in E_{\mathbb{R}}^{m+2}(1 + |\xi|^k)$ *and* $u(0) \perp q_\xi^i(0, x)$. *Moreover,* $u(\cdot, x, g)$ *is* C^∞ *in* (x, g) *with respect to the indicated norms.*

In the rest of this section, we always assume P1–P4. Sometime P5 is also assumed as will be indicated in the text.

4.2. **Formal power series solutions in** $(RR)^i$**.** Let the position of the i-th internal layer be

$$(4.9) \qquad \eta^i(t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \eta_j^i(t), \quad 1 \le i \le r-1.$$

For convenience, let $\eta^0(t, \epsilon) = a$, $\eta^r(t, \epsilon) = b$.
Assume that $\eta_0^i(t) \in O^i$ and $\eta_0^{i-1}(t) < \eta_0^i(t)$. In the interval $(\eta_0^{i-1}(t), \eta_0^i(t))$, $1 \le i \le r$, we seek formal series solution

$$u^{RRi}(x, t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{RRi}(x)$$

that satisfies (1.1). Since f does not depend on t, by induction, we can show that u_j^{RRi} does not depend on t. Expanding in powers of ϵ, we have (drop the superscripts):

$$(4.10) \qquad\qquad 0 = f_0(u_0(x), x),$$
$$(4.10_1) \qquad\qquad 0 = f_{0u}(u_0(x), x)u_1 + f_1(u_0(x), x),$$
$$\cdots$$

$$(4.10_k) \qquad 0 = f_{0u}(u_0(x), x)u_k + u_{k-2,xx} + \sum_{\alpha,\delta} C_{\alpha\delta} D_u^{|\alpha|} f_\delta(u_0(x), x)u^\alpha.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$ is a multi-index, $u^\alpha = u_1^{\alpha_1} \ldots u_{k-1}^{\alpha_{k-1}}$, $\sum_{j=1}^{k-1} j\alpha_j + \delta = k$, $C_{\alpha\delta}$

is a constant.

Let the solution of (4.10) be $u_0(x) = p^i(x), 1 \leq i \leq r$. Since O^i satisfies P1, then $f_{0u}(u_0(x), x)$ is nonsingular. Thus, u_j^{RRi}, $j \geq 1$, can be solved successively from system (4.10)– (4.10$_k$), $k \geq 1$.

Theorem 4.3. *Assume that O^i satisfies* P1. *Let $\eta_0^i(t) \in O^i$ and $\eta_0^{i-1}(t) < \eta_0^i(t)$ for $1 \leq i \leq r - 1$. Then there exists a unique formal series solution $u^{RRi}(x,t,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{RRi}(x)$ to*

$$0 = \epsilon^2 u_{xx} + f(u, x, \epsilon), \quad \eta_0^{i-1}(t) < x < \eta_0^i(t).$$

With $u_0^i = p^i(x)$, it can be obtained from system (4.10)–(4.10$_k$), $k \geq 1$, recursively.

4.3. Formal solutions in $(SR)^i$.

First consider the internal layers, $1 \leq i \leq r - 1$. Let the position of the internal layer, at $t = 0$, be $\eta^i(0, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overline{\eta}_j^i$, i.e., $\eta_j^i(0) = \overline{\eta}_j^i$.

Assume in this section that $\{\overline{\eta}_j^i\}_0^{\infty}$ is given. The problem of determining $\{\overline{\eta}_j^i\}_0^{\infty}$ will be discussed in §5. Assume that

(4.11) $\qquad \overline{\eta}_0^i \in O^i,\ 1 \leq i \leq r - 1 \quad \text{and} \quad \overline{\eta}_0^{i-1} < \overline{\eta}_0^i,\ 2 \leq i \leq r - 1.$

Let $\xi = [x - \eta^i(t, \epsilon)]/\epsilon$. We seek the layer position $\eta^i(t, \epsilon)$ and the formal solution $u^{SRi}(\xi, t, \epsilon)$ near the singular layer at $x = \eta^i(t, \epsilon)$. Since

$$u(x, t, \epsilon) = u^{SRi}((x - \eta^i(t, \epsilon))/\epsilon, t, \epsilon),$$

from (1.1), $u^{SRi}(\xi, t, \epsilon)$ satisfies (drop the super-indices):

(4.12) $\qquad \epsilon u_t = u_{\xi\xi} + D_t \eta(t, \epsilon) u_\xi + f(u, \epsilon\xi + \eta(t, \epsilon), \epsilon).$

Let $u(\xi, t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j(\xi, t)$, $\eta(t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \eta_j(t)$. Expanding in powers of ϵ, we have ($\eta'(t)$ denotes $\frac{d}{dt}\eta(t)$):

(4.13) $\qquad\qquad 0 = \eta_0'(t)u_{0\xi} + u_{0\xi\xi} + f_0(u_0, \eta_0(t)),$

$$u_{0t} = \eta_1'(t)u_{0\xi} + f_{0x}(u_0, \eta_0(t))\eta_1(t)$$

(4.13$_1$) $\qquad\qquad + u_{1\xi\xi} + \eta_0'(t)u_{1\xi} + f_{0u}(u_0, \eta_0(t))u_1$

$$+ \{f_{0x}(u_0, \eta_0(t))\xi + f_1(u_0, \eta_0(t))\},$$

$$\cdots$$

$$u_{k-1,t} = \eta_k'(t)u_{0\xi} + f_{0x}(u_0, \eta_0(t))\eta_k(t)$$

(4.13$_k$)

$$+ u_{k\xi\xi} + \eta_0'(t)u_{k\xi} + f_{0u}(u_0, \eta_0(t))u_k$$

$$+ \sum_{j=1}^{k-1} \eta_j'(t)u_{k-j,\xi} + \sum C_{\alpha\beta\gamma\delta} D_u^{|\alpha|} D_x^{|\beta|+\gamma} f_\delta(u_0, \eta_0(t))u^\alpha \eta^\beta \xi^\gamma.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$, $\beta = (\beta_1, \ldots, \beta_{k-1})$ are multi-indices, δ and γ are non-negative integers, $u^\alpha = u_1^{\alpha_1} \ldots u_{k-1}^{\alpha_{k-1}}$, $\eta^\beta = \eta_1^{\beta_1} \ldots \eta_{k-1}^{\beta_{k-1}}$, $\delta + \gamma + \sum_{j=1}^{k-1}(\alpha_j + \beta_j) \cdot j = k$, $C_{\alpha\beta\gamma\delta}$ is a constant.

From P2, there exists a unique heteroclinic solution

(4.14) $\qquad\qquad\qquad u_0(\xi, t) = q^i(\xi, \eta_0(t))$

to (4.13) connecting $p^i(\eta_0^i(t))$ to $p^{i+1}(\eta_0^i(t))$ with $(u_0(0,t) - q^i(0)) \perp q_\xi^i(0)$, provided that

$$(4.15) \qquad\qquad \eta_0'(t) = V^i(\eta_0(t)).$$

With the initial condition $\eta_0^i(t) = \overline{\eta}_0^i$, (4.15) uniquely determines $\eta_0^i(t)$ as long as $\eta_0^i \in O^i$. From (4.14) and the last assertion of P2, we see that $D_t^j u_0 \in C_{bu}^m$ for all j, $m \geq 0$.

If we also assume P5, then $x = x^i$ is a stable equilibrium of (4.15) in O^i. Since $\overline{\eta}_0^i \in O^i$, we have $\eta_0^i(t) \in O^i$ for all $t \geq 0$ and approaches x^i as $t \to \infty$.

We compute the sequences $\{u_j\}_{j=0}^\infty$ and $\{\eta_j\}_{j=0}^\infty$ by induction. Assume that u_j and η_j, $0 \leq j \leq k-1$, have been obtained, u_j is written as

$$u_j(\xi, t) = U_j(\xi, x, x_1, \dots, x_j)$$

where $x = \eta_0(t)$, $x_1 = \eta_1(t), \dots, x_{k-1} = \eta_{k-1}(t)$, and $D_y^\alpha U_j \in E^m(1 + |\xi|^j)$ where $y = (x, x_1, \dots, x_j)$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{j-1})$, $m \geq 0$, is an arbitrary integer. Also assume that $\eta_j'(t) = V_j^i(x, x_1, \dots, x_j)$ and $\eta_j(t) \to x_j^i$ as $t \to \infty$. Let $\eta_k(t) = x_k$ and $\eta_k'(t) = V_k$. We can write $u_{k-1,t} = \sum_{\ell=0}^{k-1} \dfrac{\partial U_{k-1}}{\partial x_\ell} \cdot V_\ell^i(x, \dots, x_\ell)$ where $x_0 = x$ and $V_0^i(x) = V^i(x)$. Equation (4.13_k) can be written as

$$(4.16) \qquad \begin{aligned} &u_{k\xi\xi} + V^i(x)u_{k\xi} + f_{ou}(q^i(\xi,x),x)u_k + V_k q_\xi^i(\xi,x) \\ &= h_k(\xi, U_0, U_1, \dots, U_{k-1}, x, x_1, \dots, x_k), \end{aligned}$$

where

$$\begin{aligned} h_k =\, &f_0(U_0, x)x_k + \sum_{j=1}^{k-1} V_j^i(x, \dots, x_j)U_{k-j,\xi} \\ &+ \sum C_{\alpha\beta\gamma\delta} D_u^{|\alpha|} D_x^{|\beta|+\gamma} f_\delta(U_0, x) U^\alpha \eta^\beta \xi^\gamma - \sum_{\ell=0}^{k-1} \frac{\partial U_{k-1}}{\partial x_\ell} V_\ell^i(x, \dots, x_\ell), \end{aligned}$$

$$\text{and } \eta^\beta = x_1^{\beta_1}, \dots, x_{k-1}^{\beta_{k-1}}.$$

From the induction assumptions, we can verify that $D_y^\alpha h_k \in E^m(1 + |\xi|^k)$ for all $m \geq 0$, where $\alpha = (\alpha_0, \dots, \alpha_k)$ and $y = (x, \dots, x_k)$. Therefore by Lemma 4.2, there exists a unique C^∞ function $V_k^i : O^i \times \mathbb{R}^k \to \mathbb{R}$ such that if $V_k = V_k^i(x, x_1, \dots, x_k)$, then (4.16) has a unique solution $U_k(\xi, x, x_1, \dots, x_k)$, $U_k(0, x, \dots, x_k) \perp q_\xi^i(0, x)$ and

$$D_y^\alpha U_k \in E^m(1 + |\xi|^k)$$

for all $m \geq 0$. With the initial condition $x_k(0) = \overline{\eta}_k^i$, equation

$$(4.17) \qquad\qquad x_k' = V_k^i(x, x_1, \dots, x_k)$$

has a unique solution $x_k = \eta_k(t)$. Let

$$(4.18) \qquad\qquad u_k(\xi, t) = U_k(\xi, \eta_0(t), \dots, \eta_k(t)).$$

This is clearly a solution to (4.13_k) and satisfies

$$(4.19) \qquad\qquad |D_t^\ell u_j^i| \in E^m(1 + |\xi|^j), \quad \ell, m \geq 0.$$

Since h_k is linear in x_k with $\frac{\partial h_k}{\partial x_k} = -f_{0x}(u_0, x)$, and $\{h_k - V_k q_\xi^i(\xi, x)\} \in \{\psi_i(\cdot, x)\}^\perp$, see the proof of Lemma 4.2, thus V_k^i is a linear function of x_k.

If P5 is also assumed, then

$$\frac{\partial V_k^i}{\partial x_k} = -\left\{\int_{-\infty}^{\infty} \psi_i^\tau(\xi,x)q_\xi^i(\xi,x)d\xi\right\}^{-1} \cdot \left\{\int_{-\infty}^{\infty} \psi_i^\tau(\xi,x)f_{0x}(q^i(\xi,x),x)d\xi\right\} < 0.$$

Recall that in this case, $\eta_j(t) \to x_j^i$, $0 \le j \le k-1$ as $t \to \infty$. Also, when $x_j = x_j^i, 0 \le j \le k-1$, $x_k = x_k^i$ is a stable equilibrium solution to (4.17) and $\eta_k(t) \to x_k^i$ as $t \to \infty$. See Theorem 3.1 for $\{x_j^i\}_{j=0}^k$. Finally $u_k(\xi,t) \to U_k(\xi,x_0^i,\dots,x_k^i)$ as $t \to \infty$, the latter is $u_k^{Si}(\xi)$ in Theorem 3.1.

Next, consider the boundary layers, $i = 0, r$. At the boundary layer $(SR)^0$ near $x = a$, let $x = a + \epsilon\xi$, $u(x,t,\epsilon) = u^{SR0}(\xi,t,\epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{SR0}(\xi,t)$. The system for $\{u_j^{SR0}\}$ is simpler than (4.13)–(4.13$_k$) since the layer position does not move. After dropping the super–indices, we have

$$(4.20) \qquad\qquad 0 = u_{0\xi\xi} + f_0(u_0,a),$$

$$(4.20_1) \qquad u_{0t} = u_{1\xi\xi} + f_{0u}(u_0,a)u_1 + f_{0x}(u_0,a)\xi + f_1(u_0,a),$$

$$(4.20_k) \qquad u_{k-1,t} = u_{k\xi\xi} + f_{0u}(u_0,a)u_k + \sum C_{\alpha\gamma\delta}D_u^{|\alpha|}D_x^\gamma f_\delta(u_0,a)u^\alpha\xi^\gamma.$$

Here $\alpha = (\alpha_1,\dots,\alpha_{k-1})$, $u^\alpha = u_1^{\alpha_1}\dots u_{k-1}^{\alpha_{k-1}}$, $\gamma \ge 0$, $\delta + \gamma + \sum_{j=1}^{k-1} j\alpha_j = k$ and $C_{\alpha\gamma\delta}$ is a constant. The boundary conditions

$$(4.21) \qquad\qquad u_{j\xi}(0) = 0$$

are imposed on $\{u_j\}_{j=0}^\infty$.

Since $q^0(\xi)$ satisfies (4.20) and (4.21), set $u_0(\xi,t) = q^0(\xi)$. $u_0 \in C_{bu}^m(\mathbb{R}^+,\mathbb{R}^n)$ for all $m \ge 0$. Observe that the right hand side of (4.20$_j$) does not depend on t. Thus, $u_{jt} = 0$ for all $j \ge 0$. After rewriting (4.20$_j$) and (4.21) to a first order system in $\mathbb{R}^{2n}$, we find they correspond to (5.10j) in [22, §5]. We look for a solution satisfying

$$(4.22) \qquad\qquad u_j \in E_{\mathbb{R}^+}^m(1 + |\xi|^j).$$

This condition correspond to (5.11j) in [22]. Moreover Hypothesis H4 implies assumption H2 in [22]. From the results of [22], we conclude that there exists $\{u_j^{SR0}\}_{j=0}^\infty$ satisfying (4.20)–(4.20$_j$), $j \ge 1$, and (4.21) and the growth condition (4.22). In fact $u_j^{SR0}(\xi,t) = u_j^{Si}(\xi)$ as in Theorem 3.1 of this paper. Similar arguments also apply to $(SR)^r$.

Theorem 4.4. (I) *Assume* P1–P4 *and* $\sum_{j=0}^\infty \epsilon^j \overline{\eta}_j^i$ *is given that satisfies (4.11),* $1 \le i \le r-1$, *and* $\sum \epsilon^j \overline{\eta}_j^0 = a$, $\sum \epsilon^j \overline{\eta}_j^r = b$. *Let* $\overline{t} > 0$ *be such that* $\eta_0^i \in O^i$ *for* $0 \le t < \overline{t}$. *Then there exist unique formal series for* $0 \le t < \overline{t}$,

$$(4.23) \qquad \eta^i(t,\epsilon) = \sum \epsilon^j \eta_j^i(t), \, 0 \le i \le r, \quad \eta_j^i(0) = \overline{\eta}_j^i,$$

$$(4.24)$$
$$u^{SRi}(\xi,t,\epsilon) = \sum \epsilon^j u_j^{SRi}(\xi,t), \, 0 \le i \le r, \quad u_0^{SRi}(\xi,t) = q^i(\xi,\eta_0^i(t)),$$

with u_j^{SRi} defined for $\xi \in \mathbb{R}$ if $1 \le i \le r-1$, $\xi \in \mathbb{R}^+$ if $i = 0$, $\xi \in \mathbb{R}^-$ if $i = r$ such that the followings are satisfied: (4.23) and (4.24) formally satisfy (4.12). $u_j^{SRi}(\xi, t)$ satisfies (4.19), and the boundary condition (4.21) if $i = 0, r$.

The series $\sum \epsilon^j u_j^{SRi}$ is computable recursively from (4.13)–(4.13$_k$), $k \ge 1$, and $\sum \epsilon^j \eta_j^i(t)$ is computable recursively from (4.15) if $j = 0$ or (4.17) if $j \ge 1$.

(II) If P5 is also assumed then $\bar{t} = \infty$. Moreover, $\eta_j^i(t) \to x_j^i$ as $t \to \infty$, where $\sum \epsilon^j x_j^i$ is the stationary front as in Theorem 3.1. Also $u_j^{SRi}(\xi, t) \to u_j^{Si}(\xi)$ as $t \to \infty$, where $\sum \epsilon^j u_j^{Si}(\xi)$ is the formal series solution in the stationary singular layer as in Theorem 3.1.

Corollary 4.5. *Assume* H1, H2, H4 *and* H5, *then the results in Part* (I) *of Theorem 4.4 are valid. If furthermore,* H3 *and* H6 *are also assumed, then results in Part* (II) *of that theorem are also valid.*

Remark . From Corollary 4.5, Hypotheses H3 and H6 imply that the stationary wave-front-like solution in Theorem 3.1 is exponentially stable. If H6 is replaced by

$$\int_{-\infty}^{\infty} \psi_i^\tau(\xi) q_\xi^i(\xi) d\xi \cdot \int_{-\infty}^{\infty} \psi_i^\tau(\xi) f_{0x}(q^i(\xi), x^i) d\xi > 0 \quad \text{for one of } 1 \le i \le r-1,$$

then $\dfrac{\partial V^i(x^i)}{\partial x} > 0$ and the solution is unstable.

5. Solutions in the initial layer

In the initial layer near $t = 0$, we use the stretched time $\tau = t/\epsilon$. (1.1) is now

$$(5.1) \qquad\qquad u_\tau = \epsilon^2 u_{xx} + f(u, x, \epsilon).$$

5.1. **Assumptions on the initial conditions.** We assume that the initial data $\bar{u}(x, \epsilon)$ has a layer structure described as follows. There is a partition of the interval $[a, b]$

$$a = \bar{\eta}^0 < \bar{\eta}^1 < \ldots < \bar{\eta}^r = b$$

with $\bar{\eta}^i \in O^i$, $1 \le i \le r-1$. At each $\bar{\eta}^i$, $1 \le i \le r-1$, using the stretched variable $\xi = (x - \bar{\eta}^i)/\epsilon$, we have $\bar{u}(x, \epsilon) = \bar{u}(\bar{\eta}^i + \epsilon\xi, \epsilon) = \bar{\bar{u}}^{Si}(\xi, \epsilon)$, $(a - \bar{\eta}^i)/\epsilon \le \xi \le (b - \bar{\eta}^i)/\epsilon$. Assume that

H7.

$$(5.2) \qquad\qquad \bar{u}(x, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \bar{u}_j^{Ri}(x), \quad \bar{\eta}^{i-1} < x < \bar{\eta}^i, \ 1 \le i \le r.$$

$$(5.3) \qquad\qquad \bar{\bar{u}}^{Si}(\xi, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \bar{\bar{u}}_j^{Si}(\xi), \quad 0 \le i \le r.$$

$$(5.4) \qquad\qquad \bar{\bar{u}}_{j\xi}^{Si}(0) = 0, \quad j \ge 0, \ i = 0, r.$$

The functions $\bar{u}_j^{Ri}$ is C^∞ and has a C^∞ extension to $[\bar{\eta}^{i-1}, \bar{\eta}^i]$.

Observe that $\bar{\bar{u}}^{Si}(\xi, \epsilon)$ is defined for ξ in an interval of size $O(\epsilon^{-1})$. As the result of a formal expansion, assume that $\bar{\bar{u}}_j^{Si}$ is C^∞, and is defined for $\xi \in \mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$ if $1 \le i \le r-1$, $i = 0$ or $i = r$. For example, $\bar{\bar{u}}_0^{Si}(\xi)$ is the limit of $\bar{\bar{u}}^{Si}(\xi, \epsilon)$ as $\epsilon \to 0$, uniformly in any compact subset of ξ.

Let the inner expansion of the outer power series be

$$\begin{array}{rcl}
\sum_{j=0}^{\infty} \epsilon^j \tilde{\tilde{u}}_{j,1}^{Ri}(\xi) & = & \sum_{j=0}^{\infty} \epsilon^j \overline{u}_j^{Ri}(\overline{\eta}^i + \epsilon\xi), \quad 1 \le i \le r, \\
\sum_{j=0}^{\infty} \epsilon^j \tilde{\tilde{u}}_{j,2}^{Ri}(\xi) & = & \sum_{j=0}^{\infty} \epsilon^j \overline{u}_j^{R(i+1)}(\overline{\eta}^i + \epsilon\xi), \quad 0 \le i \le r-1.
\end{array}$$

Here, $\tilde{\tilde{u}}_{j,\nu}^{Ri}(\xi), \nu = 1, 2$, is a polynomial of degree j. Observe to perform the formal expansion, the right hand sides only have to be defined in a neighborhood of zero.

H8. Each $\overline{\overline{u}}_j^{Si} \in E^m(1 + |\xi|^j)$, for all $m \ge 0$, and the matching conditions with outer expansions are satisfied:

$$\begin{array}{l}
\overline{\overline{u}}_j^{Si}(\xi) - \tilde{\tilde{u}}_{j,1}^{Ri}(\xi) \in E_{\mathbb{R}^-}^m((1 + |\xi|^j)e^{-\gamma|\xi|}), \\
\overline{\overline{u}}_j^{Si}(\xi) - \tilde{\tilde{u}}_{j,2}^{Ri}(\xi) \in E_{\mathbb{R}^+}^m((1 + \xi^j)e^{-\gamma\xi}),
\end{array}$$

where $\gamma > 0$ is a constant.

General discussion of matching conditions can be found in [6, 7].

In the regular region, the 0-th order equation of (5.1) is

(5.5) $$u_\tau = f_0(u, x), \quad x \in [\overline{\eta}^{i-1}, \overline{\eta}^i].$$

Let V^i be the function in P2, (ii) so that the following equation

(5.6) $$u_\tau = u_{\xi\xi} + V^i(\overline{\eta}^i)u_\xi + f_0(u, \overline{\eta}^i)$$

has a stationary solution (heteroclinic solution, $u_\tau = 0$) $q^i(\xi, \overline{\eta}^i)$ connecting $p^i(\overline{\eta}^i)$ to $p^{i+1}(\overline{\eta}^i)$, $1 \le i \le r-1$. For $i = 0, r$, let $q^i(\xi, \overline{\eta}^i) = q^i(\xi)$, $V^i(\overline{\eta}^i) = 0$. Recall that $q^0(\xi), \xi \ge 0$, and $q^r(\xi), \xi \le 0$, are stable stationary solutions of (5.6) satisfying boundary conditions at $x = a$ or $x = b$, approaching $p^1(a)$ as $\xi \to +\infty$ or $p^r(b)$ as $\xi \to -\infty$ respectively.

According to Lemma 2.6, D_ξ^2 is a sectorial operator in $X = \mathbb{B}_\mathbb{R}(w)$, so is its perturbation $Au = u_{\xi\xi} + V^i(\overline{\eta}^i)u_\xi$, $1 \le i \le r-1$. When $i = 0, r$, A is sectorial in $X = B_{\mathbb{R}\pm}(w)$ with $D_A = B_{\mathbb{R}\pm}^2(w) \cap (BC)$. Observe that $u \to f_0(u, \overline{\eta}^i)$ maps D_A to itself. Let $u(0) = u_0 \in D_A$. Local existence for solutions of (5.6) in $B_\mathbb{R}(w), 1 \le i \le r-1$, or $B_{\mathbb{R}\pm}(w)$, $i = 0, r$, has been established. In particular, $u \in C^1([0, t_0] : X) \cap C([0, t_0] : D_A)$. See [5]

H9. (i) For each $x \in [\overline{\eta}^{i-1}, \overline{\eta}^i]$, the equilibrium $p^i(x)$ of (5.5) attracts $\overline{u}_0^i(x)$. (ii) The stationary solution $q^i(\xi)$ of (5.6) attracts $\overline{\overline{u}}_0^{Si}(\xi)$ in the space $B_{\mathbb{R}^+}^2(w)$ or $B_{\mathbb{R}^-}^2(w)$ if $i = 0, r$; the stationary solution $q^i(\xi, \overline{\eta}^i)$ of (5.6) attracts $\overline{\overline{u}}_0^{Si}(\xi)$ in the space $B_\mathbb{R}^2(w)$ modulo a spatial shift if $1 \le i \le r-1$.

H9, (i) is a reasonable assumption since H1 and P1 imply that $p^i(x)$ is a stable solution for (5.5). H9, (ii) is also a reasonable assumption since from H8, we have $\overline{\overline{u}}_0^{Si} \in B^2(w)$ and we can prove the following lemma.

Lemma 5.1. $q^0(\xi, \overline{\eta}^0)$ and $q^r(\xi, \overline{\eta}^r)$, are asymptotically stable stationary solutions in $B_{\mathbb{R}^+}^2(w)$ and $B_{\mathbb{R}^-}^2(w)$ respectively. $q^i(\xi, \overline{\eta}^i)$, $1 \le i \le r-1$, are asymptotically stable in $B_\mathbb{R}^2(w)$ modulo spatial shifts.

For $x \in O^i$, define

$$L_{q^i}u = u_{\xi\xi} + V^i(x)u_\xi + f_{0u}(q^i(\xi, x), x)u, \quad 0 \le i \le r.$$

See P2 for $V^i(x)$ and $q^i(\xi, x)$. L_{q^i} is a closed linear operator if $X = E_{\mathbb{R}}(w)$, $1 \leq i \leq r - 1$, with $D(L_{q^i}) = E_{\mathbb{R}}^2(w)$; or if $X = E_{\mathbb{R}^\pm}(w)$, $i = 0$ or r, with $D(L_{q^i}) = E_{\mathbb{R}^\pm}^2(w) \cap (BC)$. Consider

$$(5.7) \qquad\qquad u_\tau = L_{q^i} u, \quad 0 \leq i \leq r.$$

Also the boundary condition $u_\xi(0) = 0$ is imposed if $i = 0, r$.

Lemma 5.2. *Equation (5.7) is asymptotically stable in $E_{\mathbb{R}^\pm}^2(w)$ if $i = 0, r$. It is asymptotically stable modulo $q_\xi^i(\cdot, x)$ if $1 \leq i \leq r - 1$.*

5.2. **Formal power series solutions in $(RS)^i$.** Let

$$u^{RSi}(x, \tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{RSi}(x, \tau).$$

From (5.1), expanding in powers of ϵ and dropping the super indices, we have for $x \in [\overline{\eta}^{i-1}, \overline{\eta}^i]$,

$$(5.8) \qquad u_{0\tau} = f_0(u_0, x),$$

$$(5.8_1) \qquad u_{1\tau} = f_{0u}(u_0, x)u_1 + f_1(u_0, x),$$

$$(5.8_2) \qquad u_{2\tau} = u_{0xx} + f_{0u}(u_0, x)u_2 + f_{0uu} \cdot u_1^2/2 + f_{1u} \cdot u_1 + f_2,$$

$$\cdots$$

$$(5.8_k) \qquad u_{k\tau} = u_{k-2, xx} + f_{0u}(u_0, x)u_k + \sum C_{\alpha\delta} D_u^{|\alpha|} f_\delta \cdot u^\alpha.$$

Here $\delta \geq 0$, $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$, $u^\alpha = u_1^{\alpha_1} \ldots u_{k-1}^{\alpha_{k-1}}$, $\delta + \sum j\alpha_j = k$, $C_{\alpha\delta}$ is a constant.

With x as a parameter, (5.8)–(5.8_k), $k \geq 1$, are to be solved recursively with the initial data

$$(5.9) \qquad\qquad u_j(x, 0) = \overline{u}_j(x), \quad j \geq 0,$$

where $\overline{u}_j(x)$ is the $\overline{u}_j^{Ri}(x)$ in H7. Since $p^i(x)$ attracts $\overline{u}_0(x)$, cf. H9, (i), the solution $u_0^i(x, \tau)$ of (5.8) approaches $p^i(x)$ exponentially as $\tau \to \infty$. From a standard perturbation theory, the linear variational equation

$$u_\tau = f_{0u}(u_0, x)u$$

is exponentially stable. We now proceed by induction. Assume that $u_j, 0 \leq j \leq k - 1$, have been solved with $|D_x^\alpha u_j(x, \tau)|_{C^m([0,\infty))} \leq C_{j\alpha m}$, for all $\alpha \geq 0, m \geq 0$. Rewrite (5.8_k) as

$$(5.10) \qquad\qquad u_{k\tau} = f_{0u}(u_0, x)u_k + h_k(u_0, u_1, \ldots, u_{k-1}).$$

It is easy to see that $|D_x^\alpha h_k|_{C^m([0,\infty))} \leq \infty$ for all $\alpha \geq 0, m \geq 0$. Equation (5.10) with initial condition (5.9) then has a unique solution u_k that satisfies

$$(5.11) \qquad\qquad |D_x^\alpha u_k(x, \tau)|_{C^m([0,\infty))} \leq C,$$

for all $\alpha, m \geq 0$, uniformly with respect to x.

For $\alpha = 0$, estimate (5.11) comes from the variation of constant formula and the exponential stability of the evolution operator for (5.10). For $\alpha \neq 0$, differentiate (5.10) with respect to x α-times and consider the equation for $D_x^\alpha u_k$. (5.11) then follows easily. We have proved the following.

Theorem 5.3. *Assume* H1, H7, *and* H9 (i). *Then there exists formal series*

$$u^{RSi}(x, \tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{RSi}(x, \tau), \quad u_j^{RSi}(x, 0) = \overline{u}_j^{Ri}(x),$$

for $\overline{\eta}^{i-1} \le x \le \overline{\eta}^i$, $1 \le i \le r$, *such that* u^{RSi} *formally satisfies (5.1). The series can be obtained by recursively solving the system of ODEs (5.8)–(5.8$_k$), $k \ge 1$, with the initial condition (5.9). Furthermore, each u_j^{RSi} is C^m bounded jointly in x and τ for all $m \ge 0$.*

5.3. **Formal series in $(SS)^i$ and matching of $(SS)^i$ with $(SR)^i$.** The position of the wave front $\eta^i(t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^i \eta_j^i(t)$ depends on its initial condition $\eta^i(0, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overline{\eta}_j^i$. We will show in this section that $\{\overline{\eta}_j^i\}_{j=0}^{\infty}$ is determined by the matching of expansions in $(SS)^i$ and $(SR)^i$. In the fast time variable $\tau = t/\epsilon$, from a formal expansion,

$$(5.12) \qquad \overset{*}{\eta}{}^i(\tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overset{*}{\eta}{}_j^i(\tau) \overset{\text{def}}{=} \eta^i(t, \epsilon) = \eta^i(\epsilon\tau, \epsilon).$$

Each $\overset{*}{\eta}{}_j^i(\tau)$ is a polynomial of degree j. In fact, from $\sum_{j=0}^{\infty} \epsilon^j \overset{*}{\eta}{}_j^i(\tau) = \sum_{j=0}^{\infty} \epsilon^j \eta_j^i(\epsilon\tau)$, we have

$$(5.13) \qquad \overset{*}{\eta}{}_j^i(\tau) = \sum_{\ell=0}^{j} \eta_{j-\ell}^{i(\ell)}(0)\tau^\ell/\ell!$$

where (ℓ) denotes the ℓ-th derivative with respect to t. In particular,

$$(5.14) \qquad \overset{*}{\eta}{}_j^i(0) = \eta_j^i(0) = \overline{\eta}_j^i.$$

The stretched variable $\xi = (x - \sum_{j=0}^{\infty} \epsilon^j \overset{*}{\eta}{}_j^i(\tau))/\epsilon$ is used to express the solution $u^{SSi}(\xi, \tau, \epsilon)$. When $\tau = 0$, in the new variable, the initial data are

$$\overline{u}^{Si}(\xi, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overline{u}_j^{Si}(\xi) \overset{\text{def}}{=} \overline{u}(x, \epsilon) = \overline{u}(\epsilon\xi + \sum_{j=0}^{\infty} \epsilon^j \overline{\eta}_j^i, \epsilon).$$

Let now $\overline{\eta}_0^i = \overline{\eta}^i$. Recall the definition of $\overline{\overline{u}}^{Si}(\xi, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overline{\overline{u}}^{Si}(\xi)$ It is easy to see that

$$\overline{u}^{Si}(\xi, \epsilon) = \overline{\overline{u}}^{Si}(\xi + \sum_{j=0}^{\infty} \epsilon^j \overline{\eta}_{j+1}^i, \epsilon).$$

$$(5.15) \qquad \overline{u}_0(\xi) = \overline{\overline{u}}_0(\xi + \overline{\eta}_1^i),$$

$$(5.15_1) \qquad \overline{u}_1(\xi) = \overline{\overline{u}}_1(\xi + \overline{\eta}_1^i) + \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}_1^i)\overline{\eta}_2^i,$$

$$\cdots$$

$$(5.15_k) \qquad \overline{u}_k(\xi) = \overline{\overline{u}}_k(\xi + \overline{\eta}_1^i) + \ldots + \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}_1^i)\overline{\eta}_{k+1}^i.$$

Observe that in (5.15_k), the ... comprises of terms containing only $\{\overline{\eta}^i_j\}^k_{j=1}$.

Let the solution in $(SS)^i$ be

$$u^{SSi}(\xi, \tau, \epsilon) = u^{SSi}((x - \overset{*}{\eta}{}^i(\tau, \epsilon))/\epsilon, \tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u^{SSi}_j(\xi, \tau)$$

that satisfies

$$(5.16) \qquad u_\tau = u_{\xi\xi} + \frac{1}{\epsilon} D_\tau \overset{*}{\eta}{}^i(\tau, \epsilon) \cdot u_\xi + f(u, \epsilon\xi + \overset{*}{\eta}{}^i(\tau, \epsilon), \epsilon).$$

Observe that

$$(5.17) \qquad \frac{1}{\epsilon} D_\tau \overset{*}{\eta}{}^i(\tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overset{*}{\eta}{}^{i(1)}_{j+1}(\tau) = \sum_{j=0}^{\infty} \epsilon^j (\sum_{\ell=0}^{j} \eta^{i(\ell+1)}_{j-\ell}(0)\tau^\ell/\ell!)$$

where (1) denotes derivatives with respect to τ. Notice that $\overset{*}{\eta}{}^{i(1)}_{j+1}(\tau)$ is a polynomial of degree j. Expanding (5.16) and using (5.13) and (5.17), we have a system for $\sum_{j=0}^{\infty} \epsilon^j u^{SSi}_j$. (Recall $\eta^{i(1)}_0(0) = V^i(\overline{\eta}^i)$ from (4.15), and $V^i(\overline{\eta}^i) = 0$ for $i = 0, r$.)

$$(5.18) \qquad u_{0\tau} = u_{0\xi\xi} + V^i(\overline{\eta}^i)u_{0\xi} + f_0(u_0, \overline{\eta}^i),$$

$$(5.18_1) \qquad \begin{aligned} u_{1\tau} &= u_{1\xi\xi} + V^i(\overline{\eta}^i)u_{1\xi} + f_{0u}(u_0, \overline{\eta}^i)u_1 \\ &+ \overset{*}{\eta}{}^{i(1)}_2(\tau)u_{0\xi} + f_{0x}(u_0, \overline{\eta}^i)(\xi + \overset{*}{\eta}{}^i_1(\tau)) + f_1(u_0, \overline{\eta}^i), \end{aligned}$$

$$\cdots$$

$$(5.18_k) \qquad \begin{aligned} u_{k\tau} &= u_{k\xi\xi} + V^i(\overline{\eta}^i)u_{k\xi} + f_{0u}(u_0, \overline{\eta}^i)u_k \\ &+ \sum_{j=1}^{k} \overset{*}{\eta}{}^{i(1)}_{j+1}(\tau)u_{k-j,\xi} + \sum C_{\alpha\beta\gamma\delta} D_u^{|\alpha|} D_x^{|\beta|+\gamma} f_\delta(u_0, \overline{\eta}^i)u^\alpha \eta^\beta \xi^\gamma. \end{aligned}$$

Here $\alpha = (\alpha_1, \dots, \alpha_{k-1})$, $u^\alpha = u_1^{\alpha_1} \dots u_{k-1}^{\alpha_{k-1}}$, $\beta = (\beta_1, \dots, \beta_k)$, $\eta^\beta = \overset{*}{\eta}{}^{\beta}_1 \dots \overset{*}{\eta}{}^{\beta_k}_k$, $\gamma, \delta \geq 0$ are integers, $\delta + \gamma + \sum_{j=1}^{k-1}(\alpha_j + \beta_j) \cdot j + k\beta_k = k$, $C_{\alpha\beta\gamma\delta}$ is a constant. The initial conditions for $u_0, u_1, \dots, u_k$ are given in (5.15)–(5.15_k), $k \geq 1$.

The existence of local solutions for (5.18)–(5.18_k), $k \geq 1$, follows from the theory of abstract parabolic equations and analytic semigroup. Let $Au = u_{\xi\xi} + V^i(\overline{\eta}^i)u_\xi$. Let $1 \leq i \leq r - 1$ first. Consider (5.18) in $X = B_\mathbb{R}(w)$. Then A is sectorial with $D_A = B_\mathbb{R}^2(w)$. Since $u_0(\xi, 0) = \overline{u}^i_0(\xi) \in D_A$, (5.18) admits a unique solution

$$(5.19) \qquad u \in C^1([0, t_0] : X) \cap C([0, t_0] : D_A)$$

for some $t_0 > 0$. If $i = 0$ or r, the same conclusion hold but $X = B_{\mathbb{R}\pm}(w)$ and $D_A = B_{\mathbb{R}\pm}(w) \cap (BC)$. However, due to H9, the solution $u_0^{SSi}(\xi, \tau)$ exists for any $t_0 > 0$ and approaches $q^i(\xi + c, \overline{\eta}^i)$ as $\tau \to \infty$, for some constant $c \in \mathbb{R}$. When $1 \leq i \leq r - 1$, (5.18_k), $k \geq 1$, is considered in $X = E_\mathbb{R}(w)$ and A is sectorial with $D_A = E_\mathbb{R}^2(w)$. When $i = 0, r$, $X = E_{\mathbb{R}\pm}(w)$ and $D_A = E_{\mathbb{R}\pm}^2 \cap (BC)$. Since $u_k(\xi, 0) = \overline{u}^i_k(\xi) \in D_A$, and it can be seen by induction that the right hand side of (5.18_k) is in $D_A(\frac{1}{2})$, (5.18_k) admits a solution u that satisfies (5.19). Since the evolution operator of (5.18_k) is not asymptotically stable, and the nonhomogeneous terms are of $O((1+\tau)^k)$ in $D_A(\frac{1}{2})$, in general, we can show $|u_k|_{E^2(1+|\xi|^k)} =$

$O((1+\tau)^{k+1})$. However, a better result can be obtained by considering the matching of $(SS)^i$ and $(SR)^i$. In our case, $u_k(\xi, \tau)$ exists for all $\tau \in [0, \infty)$ and satisfies $|D_\tau^\alpha u_k(\xi, \tau)| + |D_\xi^\beta u_k(\xi, \tau)| \leq C(1 + |\xi|^k + \tau^k)$ for $\alpha \leq 1$, $\beta \leq 2$.

For the purpose of matching we expand $u^{SRi}(\xi, t, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j u_j^{SRi}(\xi, t)$ in the fast time $\tau = t/\epsilon$.

$$u^{SRi}(\xi, \epsilon\tau, \epsilon) = \overset{*}{u}^{SRi}(\xi, \tau, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \overset{*}{u}_j^{SRi}(\xi, \tau).$$

Since $u(x, t, \epsilon) = \overset{*}{u}^{SRi}((x - \overset{*}{\eta}(\tau, \epsilon))/\epsilon, \tau, \epsilon)$, $\overset{*}{u}^{SRi}(\xi, \tau, \epsilon)$ formally satisfies the same equations as (5.16), therefore $\sum_{j=0}^{\infty} \epsilon^j \overset{*}{u}_j^{SRi}(\xi, \tau)$ formally satisfy system (5.18), (5.18_1), ..., (5.18_k), ... Also from

$$\sum_{j=0}^{\infty} \epsilon^j u_j^{SRi}(\xi, \epsilon\tau) = \sum_{j=0}^{\infty} \epsilon^j \overset{*}{u}_j^{SRi}(\xi, \tau),$$

and $D_t^\ell u_j^{SRi}(\xi, t) \in E^m(1 + |\xi|^j)$ for all $\ell, m \geq 0$, $\overset{*}{u}_j^{SRi}$ is a polynomial in τ, in the form

$$(5.20) \qquad \overset{*}{u}_j^{SRi} = \sum_{\ell=0}^{j} \overset{*}{u}_{j\ell}, \text{ with } |\overset{*}{u}_{j\ell}|_{E^m(1+|\xi|^\ell)} \leq C\tau^{j-\ell}, \text{ for all } m \geq 0.$$

In particular, $\overset{*}{u}_j^{SRi} \in O(1 + |\xi|^j + \tau^j)$.

Observe here that $\{u_j^{SRi}\}_0^\infty$ has not been determined since $\{\overline{\eta}_j^i\}_1^\infty$ is still unknown. But knowing $\{\overline{\eta}_j^i\}_0^k$ suffices to compute $\{u_j^{SRi}\}_0^k$ and $\{\eta_j^i(t)\}_0^k$.

We now prove by induction that by successively choosing $\{\overline{\eta}_j^i\}_{j=1}^\infty$, ($\overline{\eta}_{j+1}^i$ affects initial condition $\overline{u}_j(\xi)$, cf. (5.15_j)), system (5.18)–(5.18_j) has a unique solution $\{u_j^{SSi}\}_{j=0}^\infty$ such that

$$(5.21) \qquad u_j^{SSi} = \sum_{j=0}^{k} u_{j\ell} \text{ with } \|u_{j\ell}\|_{E^2(1+|\xi|^\ell)} \leq C(1 + \tau^{j-\ell}),$$

and

$$(5.22) \qquad |u_{j\ell} - \overset{*}{u}_{j\ell}|_{E^2(1+|\xi|^\ell)} \leq C(1 + \tau^{j-\ell})e^{-\gamma\tau}.$$

First, if $1 \leq i \leq r - 1$, (5.18) has a stable stationary solution $q^i(\xi, \overline{\eta}^i)$ that attracts $\overline{\overline{u}}_0$ modulo spatial shifts. cf. H9. Thus, there exists a unique $\overline{\eta}_1^i$ such that with $u_0(\xi, 0) = \overline{u}_0(\xi) = \overline{\overline{u}}(\xi + \overline{\eta}_1^i)$,

$$(5.23) \qquad \|u_0^{SSi}(\cdot, \tau) - q^i(\cdot, \overline{\eta}^i)\|_{B_\mathbb{R}^2(w)} \leq Ce^{-\gamma\tau}, \qquad \tau \geq 0,$$

where $w(\xi) = e^{-\gamma|\xi|}$ and $\gamma > 0$. If $i = 0, r$, $q^i(\xi, \overline{\eta}^i)$ is stable. We choose $\overline{\eta}_1^i = 0$, (5.23) is still valid. Now (5.23) is even stronger than (5.22), $j = \ell = 0$. The extra control of the rate of approaching $u_0(\pm\infty, \tau)$ as $\xi \to \pm\infty$ will be used in §6. At this point, $\{u_j^{SRi}\}_{j=0}^1$ and $\{\eta_j^i\}_{j=0}^1$ have also been determined.

Assume that $\{u_j^{SSi}\}_{j=0}^{k-1}$ and $\{\overline{\eta}_j^i\}_{j=0}^k$ have been determined and (5.21), (5.22), $0 \leq j \leq k - 1$, are satisfied. Therefore, $\eta_j^i(t)$, $t \geq 0$, $0 \leq j \leq k$, and all their derivatives at $t = 0$ are determined as well as $\{u_j^{SRi}\}_0^k$ and $\{\overset{*}{u}_j^{SRi}\}_0^k$. From (5.13)

and (5.17), $\overset{*}{\eta}{}^{i}_{j}(\tau)$ and $\overset{*}{\eta}{}^{i(1)}_{j+1}(\tau)$, $0 \le j \le k$, are determined, which will be used in (5.18_k). We now rewrite (5.18_k), which is satisfied by u^{SSi}_k and $\overset{*}{u}{}^{SRi}_k$ as the following:

$$u_{k\tau} - u_{k\xi\xi} - V^i(\overline{\eta}^i)u_{k\xi} - f_{0u}(u_0, \overline{\eta}^i)u_k = g_k(u_0, \dots, u_{k-1}),$$
$$\overset{*}{u}_{kt} - \overset{*}{u}_{k\xi\xi} - V^i(\overline{\eta}^i)\overset{*}{u}_{k\xi} - f_{0u}(\overset{*}{u}_0, \overline{\eta}^i)\overset{*}{u}_k = g_k(\overset{*}{u}_0, \dots, \overset{*}{u}_{k-1}).$$

Since $g_k(u_0, \dots, u_{k-1})$ is a polynomial in $(u_1, \dots, u_{k-1})$ and $(u_{1\xi}, \dots, u_{k-1,\xi})$, using (5.20)–(5.22) we can verify that

$$g_k(u_0, \dots, u_{k-1}) - g_k(\overset{*}{u}_0, \dots, \overset{*}{u}_{k-1}) = \sum_{j=0}^{k} g_{kj}$$

with $\|g_{kj}\|_{E^1(1+\|\xi\|^j)} \le C(1 + \tau^{k-j})e^{-\gamma\tau}$. Let $\Delta u = u_k - \overset{*}{u}_k$.

Denote

$$L^i u = u_{\xi\xi} + V^i(\eta^i)u + f_{ou}(u^i_0, \overline{\eta}^i)u,$$

(5.24)
$$\Delta u_\tau - L^i \Delta u = \sum_{j=0}^{k}\left[f_{0u}(u_0, \overline{\eta}^i) - f_{0u}(\overset{*}{u}_0, \overline{\eta}^i)\right]\overset{*}{u}_{kj} + \sum_{j=0}^{k}g_{kj}$$
$$= \sum_{j=0}^{k}G_{kj}.$$

At this point all the terms in the right hand side of (5.24) are known. Observe that the initial data for Δu has the form $u_k(\xi, 0) - \overset{*}{u}_k(\xi, 0) = \varphi(\xi) + \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}^i_1)\overline{\eta}^i_{k+1}$, where $\varphi \in E^2(1 + |\xi|^k)$. Recall that $\overline{\eta}^0_0 = a$, $\overline{\eta}^r_o = b$, $\overline{\eta}^i_j = 0$ for all $j \ge 1$ if $i = 0, r$.

Lemma 5.4. (i) *Let $1 \le i \le r - 1$. Consider*

(5.25)
$$u_\tau = L^i u + h,$$
$$u(\xi, 0) = \varphi(\xi) + \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}^i_1)\eta,$$

where $h : \mathbb{R}^+ \to E^1_{\mathbb{R}}(1 + |\xi|^j)$ is continuous with $|h(\tau)|_{E^1(1+|\xi|^j)} \le C(1 + \tau^\ell)e^{-\gamma\tau}$ and $\varphi \in E^2_{\mathbb{R}}(1 + |\xi|^j)$. Then there exists a unique $\eta \in \mathbb{R}$ such that there exists a unique solution u to (5.25) with $|u(\tau)|_{E^2_{\mathbb{R}}(1+|\xi|^j)} \le C(1 + \tau^\ell)e^{-\gamma\tau}$.

(ii) *Let $i = 0$ or r. Consider (5.25) with $\overline{\eta}^i_1 = \eta = 0$, where $h : \mathbb{R}^+ \to E^1_{\mathbb{R}\pm}(1 + |\xi|^j)$ is continuous with $|h(\tau)|_{E^1_{\mathbb{R}\pm}(1+|\xi|^j)} \le C(1 + \tau^\ell)e^{-\gamma\tau}$ and $\varphi \in E^2_{\mathbb{R}\pm}(1 + |\xi|^j) \cap (BC)$. Then there exists a unique solution u to (5.25) such that $u : \mathbb{R}^+ \to E^2_{\mathbb{R}\pm}(1 + |\xi|^j) \cap (BC)$ is continuous and $|u(\tau)|_{E^2_{\mathbb{R}\pm}(1+|\xi|^j)} \le C(1 + \tau^\ell)e^{-\gamma\tau}$.*

We now write $\Delta u_k = \sum_{j=0}^{k} \Delta u_{kj}$ where Δu_{kj} satisfies

(5.26)
$$u_\tau - L^i u = [f_{0u}(u^i_0, \overline{\eta}^i) - f_{0u}(\overset{*}{u}{}^i_0, \overline{\eta}^i)]\overset{*}{u}_{kj} + g_{kj} = G_{kj}.$$

(5.27)
$$u(\xi, 0) = \begin{cases} \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}^i_1)C_j, & \text{if } 0 \le j \le k - 1, \\ \varphi(\xi) + \overline{\overline{u}}_{0\xi}(\xi + \overline{\eta}^i_1)C_k, & \text{if } j = k. \end{cases}$$

Let $1 \leq i \leq r - 1$ first. Since $\varphi \in E^2(1 + |\xi|^k)$ and $|G_{kj}|_{E^1_\mathbb{R}(1+|\xi|^j)} \leq C(1 + \tau^{k-j})e^{-\gamma\tau}$, from Lemma 5.4, (i), there exists a unique $C_j, 0 \leq j \leq k$, such that the unique solution Δu_{kj} of (5.26), (5.27) satisfies

$$(5.28) \qquad |\Delta u_{kj}|_{E^2_\mathbb{R}(1+|\xi|^j)} \leq C(1 + \tau^{k-j})e^{-\gamma\tau}.$$

Let $i = 0$ or r next. Then $\varphi(\xi) = \overline{\overline{u}}_k(\xi) = \overline{u}_k(\xi) \in E^2_{\mathbb{R}\pm}(1 + |\xi|)^k) \cap (BC)$ and $|G_{kj}|_{E^1_{\mathbb{R}\pm}(1+|\xi|^j)} \leq C(1 + \tau^{k-j})e^{-\gamma\tau}$ and $C_j = 0$, $0 \leq j \leq k$. From Lemma 5.4, (ii), the unique solution Δu_{kj} of (5.26), (5.27) satisfies the Neumann boundary condition at $\xi = 0$ and

$$(5.29) \qquad |\Delta u_{kj}|_{E^2_{\mathbb{R}\pm}(1+|\xi|^j)} \leq C(1 + \tau^{k-j})e^{-\gamma\tau}.$$

In all the cases let $\overline{\eta}^i_{j+1} = \sum_0^k c_j$ and $u^i_{kj} = \Delta u_{kj} + \overset{*}{u}_{kj}$. From (5.20), we have (5.21) for $j = k$. Thus, (5.21) and (5.22) have been proved by the induction.

We summarize the results in

Theorem 5.5. *Assume* H1– H5. *For each* $0 \leq i \leq r$ *assume the initial data* $\overline{\overline{u}}^{Si}(\xi, \epsilon) = \sum \epsilon^j \overline{\overline{u}}^{Si}_j(\xi)$ *satisfies* H7 *and* H9, (ii). *Then there exist formal series*

$$(5.30) \qquad u^{SSi}(\xi, \tau, \epsilon) = \sum \epsilon^j u^{SSi}_j(\xi, \tau), \quad u^{SSi}_j(\xi, 0) = \overline{u}^{Si}_j(\xi),$$

$$(5.31) \qquad \sum \epsilon^j \overline{\eta}^i_j, \quad \overline{\eta}^i_0 = \overline{\eta}^i.$$

Here $\overline{\eta}^i$ *is determined by the layer position of the initial data.* (5.30) *formally satisfies* (5.16) *where* $\overset{*}{\eta}{}^i(\tau, \epsilon) = \eta^i(\epsilon\tau, \epsilon)$ *is the expansion of the layer position in the variable* $\tau = t/\epsilon$, *see* (5.12). *Furthermore* (5.30) *is recursively determined by* (5.18)–(5.18$_k$), $k \geq 1$, *with the initial conditions* (5.15)–(5.15$_k$), $k \geq 1$, *in the space* $u^{SSi}_j \in E^2(1 + |\xi|^j)$, *where* $\overline{\eta}^i_j$, $j \geq 1$, *is chosen such that for any integers* $\alpha \leq 2$, $\beta \leq 1$,

$$|\partial^\alpha_\xi u^{SSi}_j| + |\partial^\beta_\tau u^{SSi}_j| \leq C(1 + |\xi|^j + \tau^j),$$

$$|u^{SSi}_j - \overset{*}{u}^{SRi}_j| \leq C((1 + |\xi|^j + \tau^j)e^{-\gamma\tau},$$

for some $\gamma > 0$.

6. Matching of layer solutions, construction of pseudo solutions

6.1. The matching of $(RS)^i$ and $(RR)^i$.

Theorem 6.1. *There exists* $\gamma > 0$ *such that for any integers* α, β, $j \geq 0$,

$$|\partial^\alpha_\tau \partial^\beta_x (u^{RSi}_j(x, \tau) - u^{RRi}_j(x))| \leq C_{\alpha\beta j} e^{-\gamma\tau},$$

uniformly for all $x \in [\overline{\eta}^{i-1}, \overline{\eta}^i]$, $1 \leq i \leq r$.

When $j = 0$, Theorem 6.1 is a consequence of the exponential stability of $p^i(x)$, $1 \leq i \leq r$, as a stationary solution to (5.8). By induction, we can prove that the nonhomogeneous term and the coefficient $f_{0u}(u_0, x)$ in (5.8$_k$) approach the corresponding terms in (4.10$_k$) exponentially as $\tau \to \infty$. The desired result then follows easily. Details are omitted.

6.2. **The matching of** $(SS)^i$ **and** $(SR)^i$. The matching of expansions in $(SS)^i$ and $(SR)^i$ has been obtained in §5.3. For convenience, we state the result in the following

Theorem 6.2. *Let* $\Delta u_j^i = u_j^{SSi} - \overset{*}{u}{}_j^{SRi}$, *where* $\sum_0^\infty \epsilon^j \overset{*}{u}{}_j^{SRi}$ *is the expansion of* $\sum_0^\infty \epsilon^j u_j^{SRi}(\xi, t)$ *in the variable* $\tau = t/\epsilon$. *Then there exists* $\gamma > 0$ *such that for any nonnegative integers* $\alpha \leq 2$, $\beta \leq 1$,

$$\left\{ |\partial_\xi^\alpha \Delta u_j^i| + |\partial_\tau^\beta \Delta u_j^i| \right\} \leq C_j (1 + |\xi|^j + \tau^j) e^{-\gamma\tau}.$$

6.3. **The matching of** $(SR)^i$ **with** $(RR)^i$ **and** $(RR)^{i+1}$. Let the inner expansion of the outer formal solutions in $(RR)^i$ be

$$
(6.1) \quad
\begin{aligned}
\sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,1}^{RRi}(\xi, t) &= \sum_{j=0}^\infty \epsilon^j u_j^{RRi}\left(\epsilon\xi + \sum_{\ell=0}^\infty \epsilon^\ell \eta_\ell^i(t)\right), \quad 1 \leq i \leq r, \\
\sum_{j=0}^\infty \epsilon^i \tilde{u}_{j,2}^{RRi}(\xi, t) &= \sum_{j=0}^\infty \epsilon^j u_j^{RR(i+1)}\left(\epsilon\xi + \sum_{\ell=0}^\infty \epsilon^\ell \eta_\ell^i(t)\right), \quad 0 \leq i \leq r-1.
\end{aligned}
$$

Recall that $\eta_0^0(t) = a$, $\eta_0^r(t) = b$ and $\eta_\ell^0(t) = \eta_\ell^r(t) = 0$ for all $\ell \geq 1$. From (6.1), we find that $\tilde{u}_{j,\nu}^{RRi}, \nu = 1, 2$, is a polynomial in ξ of degree j, with its coefficients depending smoothly on $\eta_\ell^i(t)$, $\ell \leq j$. Thus,

$$\tilde{u}_{k,\nu}^{RRi} \in E_{\mathbb{R}}^m (1 + |\xi|^k), \qquad \nu = 1, 2,$$

with the norm bounded uniformly with respect to $t \geq 0$.

Concerning the matching of $(SR)^i$ and $(RR)^i$, we want to show that there exists $\gamma > 0$ such that the following estimates hold.

(6.2)
$$
\begin{aligned}
|\partial_t^\alpha (u_j^{SRi}(.,t) - \tilde{u}_{j,1}^{RRi}(.,t))|_{E_{\mathbb{R}^-}^m ((1+|\xi|^j)e^{-\gamma|\xi|})} &\leq C_{\alpha m j}, \quad 1 \leq i \leq r, \\
|\partial_t^\alpha (u_j^{SRi}(.,t) - \tilde{u}_{j,2}^{RRi}(.,t))|_{E_{\mathbb{R}^+}^m ((1+\xi^j)e^{-\gamma\xi})} &\leq C_{\alpha m j}, \quad 0 \leq i \leq r-1,
\end{aligned}
$$

for all α, m, $j \geq 0$ uniformly uniformly with respect to $t \geq 0$ in the weighted norms. Only the proof of the second estimate, $0 \leq i \leq r - 1$, will be presented since the proof of the first is similar.

Notice that $\sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,2}^{RRi}(\xi, t)$ formally satisfies

$$\epsilon \tilde{u}_t = \tilde{u}_{\xi\xi} + D_t \eta(t, \epsilon) \tilde{u}_\xi + f(\tilde{u}, \epsilon\xi + \eta(t, \epsilon), \epsilon).$$

This has the same form as (4.12). Expanding in power series of ϵ, the equations for $\{\tilde{u}_{j,2}^{RRi}\}_{j=0}^\infty$ are precisely (4.13), (4.13_1), ..., (4.13_k), ... When $j = 0$, the equation is

$$(6.3) \qquad\qquad 0 = V^i(x)\tilde{u}_{0\xi} + \tilde{u}_{0\xi\xi} + f_0(\tilde{u}_0, x).$$

The above has a solution $\tilde{u}_0(\xi, t) = p^i(\eta_0(t))$ (which is in fact independent of ξ). We can prove by induction that $\tilde{u}_j(\xi, t) = \tilde{U}_j(\xi, x, x_1, \ldots, x_j)$ where $x = \eta_0^i(t), \ldots, x_j = \eta_j^i(t)$. Also $D_y^\nu \tilde{U}_j \in E^m(1 + |\xi|^j)$ where $y = (x, \ldots, x_j)$ and $\nu = (\nu_0, \nu_1, \ldots, \nu_j)$. These properties are similar to those of U_j, see §5.

The assertion is clearly valid if $j = 0$. Suppose it is valid for $0 \leq j \leq k-1$. Then $\tilde{u}_k$ satisfies the following equation, similar to (4.16),

(6.4)
$$\tilde{u}_{k\xi\xi} + V^i(x)\tilde{u}_{k\xi} + f_{0u}(\tilde{u}_0, x)\tilde{u}_k + V_k\tilde{u}_{0\xi} = \tilde{h}_k(\xi, \tilde{U}_0, ...\tilde{U}_{k-1}, x_1, ..., x_k),$$

where $\tilde{h}$ has the same form as h_k in (4.16) except U_j is replaced by $\tilde{U}_j$. Notice that the homogeneous equation associated to (6.4) has an exponential dichotomy in $\mathbb{R}$, and the nonhomogeneous terms are in $E^m(1 + |\xi|^k)$. Thus (6.4) has a unique solution $\tilde{U}_k(., x, \ldots, x_k) \in E^m(1 + |\xi|^k)$ that depends smoothly on $(x, \ldots, x_k)$, see Lemma 2.3, (i). Since $\tilde{u}_k \in E^m(1 + |\xi|^k)$ is a solution to (6.4), thus $\tilde{u}_k = \tilde{U}_k(., x, \ldots, x_k)$. The proof is complete.

For $0 \leq i \leq r-1$, let $\Delta u_k \overset{\text{def}}{=} u_k - \tilde{u}_k$. We claim that $\Delta u_j = \Delta U_j \overset{\text{def}}{=} U_j(\xi, x, \ldots, x_j) - \tilde{U}_j(\xi, x, \ldots, x_j)$ satisfies

(6.5)
$$D_y^\nu \Delta U_j = O((1 + \xi^j)e^{-\gamma\xi}),$$

where $\gamma > 0$ is a constant, $D_y^\nu = D_x^{\nu_0} D_{x_1}^{\nu_1} \cdots D_{y_j}^{\nu_j}$. For $k = 0$ and $\nu = 0$, $\Delta u_0 = q^i(\xi, x) - p^{i+1}(x) = U_0(\xi, x) - \tilde{U}_0(\xi, x)$. Obviously we have $\Delta u_0 = O(e^{-\gamma\xi})$. When $k = 0$ and $\nu = 1$, $Z \overset{\text{def}}{=} D_x \Delta U_0$ satisfies

$$0 = V^i(x)Z_\xi + Z_{\xi\xi} + f_{0u}(p^{i+1}(x), x)Z + f_{0x}(q^i(\xi, x), x) - f_{0x}(p^i(x), x)$$
$$+ [f_{0u}(q^i(\xi, x), x) - f_{0u}(p^i(x), x)]\partial_x q^i(\xi, x) + \partial_x V^i(x)\Delta U_0.$$

The associated homogeneous equation for Z has an exponential dichotomy in $\mathbb{R}$ and the nonhomogeneous terms are of $O(e^{-\gamma\xi})$. If γ is smaller than the exponential coefficient of the dichotomy, then $Z = O(e^{-\gamma\xi})$, see Lemma 2.3, (i). Assuming now $k = 0$ and $\nu > 1$, we can write a similar equation for $Z \overset{\text{def}}{=} D_x^\nu \Delta U_0$ and prove (6.5) by induction on ν.

Suppose now (6.5) has been proved for $0 \leq j \leq k-1$, $k \geq 1$. Then Δu_k satisfies

(6.6)
$$Z_{\xi\xi} + V^i(x)Z_\xi + f_{0u}(\tilde{u}_0, x)Z$$
$$= h_k(\xi, U_0, \ldots, U_{k-1}, x, \ldots, x_k) - \tilde{h}_k(\xi, \tilde{U}_0, \ldots, \tilde{U}_{k-1}, x, \ldots, x_k)$$
$$+ [f_{0u}(\tilde{u}_0, x) - f_{0u}(q^i(\xi, x), x)]u_k + V_k[\tilde{u}_{0\xi} - q_\xi^i(\xi, x)].$$

For $1 \leq i \leq r-1$, (6.6) is considered for $\xi \in \mathbb{R}$. The right hand side and its derivatives with respect to $y = (x, x_1, \ldots, x_k)$ are in $E^m(1 + |\xi|^k)$ and $E^m((1 + |\xi|^k)e^{-\gamma\xi})$, due to the induction assumption. Recall that $\tilde{u}_0 = p^{i+1}$ and $L_{p^{i+1}}$ has an exponential dichotomy on $\mathbb{R}$. From Lemma 2.3 (i), (6.6) has a unique solution Z that is in both $E^{m+2}(1 + |\xi|^k)$ and $E^{m+2}((1 + |\xi|^k)e^{-\gamma\xi})$ and is differentiable with respect to y. However, it is known that $U_k, \tilde{U}_k$, therefore $\Delta U_k \in E^{m+2}(1 + |\xi|^k)$. We then have $D_y^\nu \Delta U_k = D_y^\nu Z \in E^{m+2}((1 + |\xi|^k)e^{-\gamma\xi})$. When $1 \leq i \leq k-1$, (6.5) has been proved by induction.

For $i = 0$, Δu_k satisfies (6.6) for $\xi \in \mathbb{R}^+$ and the boundary condition

(6.7)
$$Z_\xi(0, y) = -\tilde{U}_{k\xi}(0, y),$$

where $y = (x, x_1, \ldots, x_k)$. From Lemma 2.3 (ii), (6.6) and (6.7) have a unique solution Z that is in $E_{\mathbb{R}^+}^{m+2}(1 + |\xi|^k)$ and $E_{\mathbb{R}^+}^{m+2}((1 + |\xi|^k)e^{-\gamma\xi})$ together with all its derivatives. However, we know that ΔU_k is a solution of (6.6) and (6.7) and is in $E^{m+2}(1 + |\xi|^k)$. Thus $\Delta U_k = Z$. By induction, (6.5) has been proved for $i = 0$.

Since equation (6.6) and (6.7) depend continuously on y that is in a compact subset of $\mathbb{R}^{k+1}$, the norms of ΔU and all its derivatives in the function spaces are uniformly bounded with respect to y. When $\alpha = 0$, the second estimate of (6.2) has already been proved. Recall that $\Delta u_k(\xi, t) = \Delta U_k(\xi, y)$. Since the derivatives of $\eta_j^i(t)$ are bounded uniformly with respect to $t \geq 0$, when $\alpha \neq 0$, the desired assertion follows from (6.5) and the chain rule of differentiation.

Theorem 6.3. *Let $\Delta u_j^i = u_j^{SRi} - \tilde{u}_{j,1}^{RRi}$, $1 \leq i \leq r$, or $u_j^{SRi} - \tilde{u}_{j,2}^{RRi}$, $0 \leq i \leq r-1$, where $\tilde{u}_{j,1}^{RRi}$ and $\tilde{u}_{j,2}^{RRi}$ are defined in (6.1). Then there exists $\gamma > 0$ such that for all integers $\alpha, \beta, j \geq 0$,*

$$|\partial_t^\alpha \partial_\xi^\beta \Delta u_j^i| \leq C_{\alpha\beta j}(1 + |\xi|^j)e^{-\gamma|\xi|}$$

uniformly with respect to $t \geq 0$.

6.4. **Matching of $(SS)^i$ with $(RS)^i$ and $(RS)^{i+1}$.** We use the inner variable $\xi = (x - \overset{*}{\eta}(\tau, \epsilon))/\epsilon$ to expand the outer solutions

(6.8)
$$\sum_{j=0}^\infty \epsilon^j u_j^{RSi}(\epsilon\xi + \overset{*}{\eta}{}^i(\tau, \epsilon), \tau) = \sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,1}^{RSi}(\xi, \tau), \qquad 1 \leq i \leq r,$$

$$\sum_{j=0}^\infty \epsilon^j u_j^{RS,i+1}(\epsilon\xi + \overset{*}{\eta}{}^i(\tau, \epsilon), \tau) = \sum_{j=0}^\infty \epsilon^j \tilde{u}_{j,2}^{RSi}(\xi, \tau), \qquad 0 \leq i \leq r-1.$$

Both expansions formally satisfy (5.16) as does $u^{SSi}(\xi, \tau, \epsilon)$. Therefore, both satisfy (5.18)– (5.18$_k$), $k \geq 1$, just as $\displaystyle\sum_{j=0}^\infty \epsilon^j u_j^{SSi}$. It suffices to show the matching of expansions in $(SS)^i$ and $(RS)^{i+1}$, $0 \leq i \leq r-1$. Let the indices be dropped so that $\tilde{u}_j$ denotes $\tilde{u}_{j,2}^{RSi}$. Then

(6.9) $$\tilde{u}_{0\tau} = f_0(\tilde{u}_0, \overline{\eta}^i), \quad \tilde{u}(0) = \overline{u}_0^{R,i+1}(\overline{\eta}^i),$$

(6.10) $$u_{0\tau}^{SSi} = u_{0\xi\xi}^{SSi} + V^i(\overline{\eta}^i)u_{0\xi}^{SSi} + f_0(u_0^{SSi}, \overline{\eta}^i).$$

Let $\xi \to +\infty$ in (6.10). Since $u_0 \in B_{\mathbb{R}^+}^2(e^{-\gamma\xi})$, $u_{0\xi}$ and $u_{0\xi\xi} \to 0$ as $\xi \to \infty$. We have $u_{0\tau}(+\infty, \tau) = f_0(u_0(+\infty, \tau), \overline{\eta}^i)$ with $u_0(\infty, 0) = \overline{\overline{u}}_0(\infty) = \overline{u}_0^{R,i+1}(\overline{\eta}^i)$. This is the same as (6.9). Therefore

$$u_0^{SSi}(\infty, \tau) = \tilde{u}_0(\overline{\eta}^i, \tau), \quad \tau \geq 0.$$

Since $u_0^{SSi} : \mathbb{R}^+ \to B_{\mathbb{R}}^2(e^{-\gamma\xi})$ is bounded, we have

(6.11) $$|u_0^{SSi}(\cdot, \tau) - \tilde{u}_0(\cdot, \tau)|_{E_{\mathbb{R}^+}^2(e^{-\gamma\xi})} \leq C$$

uniformly with respect to $\tau \geq 0$.

Let $\Delta u_j = u_j^{SSi} - \tilde{u}_{j,2}^{RSi}$. We now show by induction that

(6.12)
$$\Delta u_j = \sum_{i=0}^j \Delta u_{j\ell}, \text{ with } |\Delta u_{j\ell}|_{E_{\mathbb{R}^+}^2((1+\xi^\ell)e^{-\gamma\xi})} \leq C(1 + \tau^{j-\ell}).$$

Suppose (6.12) has been proved for $0 \leq j \leq k-1$. For $1 \leq i \leq r-1$, Δu_k satisfies

$$
\begin{aligned}
(6.13) \qquad \Delta u_{k\tau} = {}& \Delta u_{k\xi\xi} + V^i(\overline{\eta}^i)\Delta u_{k\xi} + f_{0u}(\tilde{u}_0, \overline{\eta}^i)\Delta u_k \\
& + [f_{0u}(u_0^{SSi}, \overline{\eta}^i) - f_{0u}(\tilde{u}_0, \overline{\eta}^i)]u_k^{SSi} \\
& + g_k(u_0, \ldots, u_{k-1}) - g_k(\tilde{u}_0, \ldots, \tilde{u}_{k-1}).
\end{aligned}
$$

For $i = 0$, Δu_k satisfies the equations

$$
\begin{aligned}
(6.14) \qquad \Delta u_{k\tau} = {}& \Delta u_{k\xi\xi} + V^i(\overline{\eta}^i)\Delta u_{k\xi} + f_{0u}(u_0^{SSi}, \overline{\eta}^i)\Delta u_k \\
& + [f_{0u}(u_0^{SSi}, \overline{\eta}^i) - f_{0u}(\tilde{u}_0, \overline{\eta}^i)]\tilde{u}_k \\
& + g_k(u_0, \ldots, u_{k-1}) - g_k(\tilde{u}_0, \ldots, \tilde{u}_{k-1}),
\end{aligned}
$$

$$
(6.14_1) \qquad \Delta u_{k\xi}(0, \tau) = -\tilde{u}_{j,2}^{RSi}(0, \tau), \quad \text{at } \xi = 0,
$$

where $V^i = 0$, $\overline{\eta}^i = a$. Observe that g_k is a polynomial on $u_1, \ldots, u_{k-1}, u_{1\xi}, \ldots, u_{k-1,\xi}$ and $\overset{*}{\eta}{}^i_1, \ldots, \overset{*}{\eta}{}^i_k$. By the induction assumption, the nonhomogeneous term of (6.13) has the form

$$
(6.15) \qquad G_k = \sum_{j=0}^{k} G_{kj}, \quad \text{with } |G_{kj}|_{E^1_{\mathbb{R}}((1+\xi^j)e^{-\gamma\xi})} \leq C(1 + \tau^{k-j}),
$$

if $1 \leq i \leq r-1$. The nonhomogeneous term in (6.14) has the same form as in (6.15), but the norm has to be replaced by the norm in $E^1_{\mathbb{R}^+}((1+\xi^j)e^{-\gamma\xi})$.

We need similar decompositions of $\{\tilde{u}_j\}_{j=0}^\infty$ and $\{\Delta u_j(\xi, 0)\}_{j=0}^\infty$. The expansion of $\sum \epsilon^j u_j^{RS,i+1}$ into $\sum \epsilon^j \tilde{u}_j$ can be divided into two steps. Let

$$
\sum_{j=0}^\infty \epsilon^j \tilde{\tilde{u}}_j(\xi, \tau) = \sum_{j=0}^\infty \epsilon^j u_j^{RS,i+1}(\epsilon\xi + \overset{*}{\eta}{}^i_0, \tau),
$$

$$
\sum_{j=0}^\infty \epsilon^j \tilde{u}_j(\xi, \tau) = \sum_{j=0}^\infty \epsilon^j \tilde{\tilde{u}}_j(\xi + \sum_{\ell=0}^\infty \epsilon^\ell \overset{*}{\eta}{}^i_{\ell+1}(\tau), \tau).
$$

We then have

$$
(6.16) \qquad \tilde{u}_k(\xi, \tau) = \sum_{\delta, \alpha} C_{\alpha\delta} D_x^{|\alpha|} \tilde{\tilde{u}}_\delta(\xi + \overset{*}{\eta}{}^i_1(\tau))[\overset{*}{\eta}]^\alpha \overset{\text{def}}{=} \sum_{\delta=0}^{k} \tilde{u}_{k\delta}.
$$

Here $\delta \geq 0$ is an integer, $[\overset{*}{\eta}]^\alpha = [\overset{*}{\eta}{}^i_2(\tau)]^{\alpha_1} \ldots [\overset{*}{\eta}{}^i_{k+1}(\tau)]^{\alpha_k}$, $C_{\alpha\delta}$ is a constant and $\delta + \sum_{j=1}^{k} j\alpha_j = k$. $\tilde{u}_{k\delta}$ consists of derivatives of $\tilde{\tilde{u}}_\delta$ only and

$$
(6.17) \qquad |\tilde{u}_{k\delta}|_{E^m((1+|\xi|^\delta))} \leq C_{mk\delta}(1 + \tau^{k-\delta}), \qquad m \geq 0.
$$

In comparison, $\sum \epsilon^j \overline{u}_j(\xi)$ is also defined in a similar way

$$
\sum_{j=0}^\infty \epsilon^j \overline{u}_j(\xi) = \sum_{j=0}^\infty \epsilon^j \overline{\overline{u}}_j(\xi + \sum_{\ell=0}^\infty \epsilon^\ell \overline{\eta}^i_{\ell+1}).
$$

Exactly like (6.16), we have

$$(6.18) \qquad \overline{u}_k(\xi) \;=\; \sum_{\delta,\alpha} C_{\alpha\delta} D_x^{|\alpha|} \overline{\overline{u}}_\delta(\xi + \overline{\eta}_1^i)[\overline{\eta}]^\alpha \overset{\text{def}}{=} \sum_{\delta=0}^k \overline{u}_{k\delta},$$

$$|\overline{u}_{k\delta}|_{E^m(1+|\xi|^\delta)} \le \overline{C}_{mk\delta}.$$

Recall that $\overset{*}{\eta}{}_j^i(0) = \overline{\eta}_j^i$ and $|\overline{\overline{u}}_\delta(\xi) - \tilde{\tilde{u}}_\delta(\xi,0)|_{E^m((1+|\xi|^\delta)e^{-\gamma\xi})} \le C$, cf. **H8**. We have

$$(6.19) \qquad \Delta u_k(\xi,0) = \sum_{j=0}^k \Delta u_{kj}(\xi,0) \overset{\text{def}}{=} \sum_{j=0}^k [\overline{u}_{kj}(\xi) - \tilde{u}_{kj}(\xi,0)],$$

with

$$(6.20) \qquad |\Delta u_{kj}(\cdot,0)|_{E^m((1+|\xi|^j)e^{-\gamma\xi})} \le C.$$

For $1 \le i \le r-1$ we solve

$$(6.21) \qquad \begin{aligned} &\Delta u_{kj\tau} = \Delta u_{kj\xi\xi} + V^i(\overline{\eta}^i)\Delta u_{kj\xi} + f_{0u}(\tilde{u}_0,\overline{\eta}^i)\Delta u_{kj} + G_{kj}, \\ &\Delta u_{kj}(\xi,0) = \overline{u}_{kj}(\xi) - \tilde{u}_{kj}(\xi,0). \end{aligned}$$

For $i = 0$, we solve

$$(6.22) \qquad \begin{aligned} &\Delta u_{kj\tau} = \Delta u_{kj\xi\xi} + V^i(\overline{\eta}^i)\Delta u_{kj\xi} + f_{0u}(u_0^{SSi},\overline{\eta}^i)\Delta u_{kj} + G_{kj}, \\ &\Delta u_{kj}(\xi,0) = \overline{u}_{kj}(\xi) - \tilde{u}_{kj}(\xi,0), \end{aligned}$$

$$(6.22_1) \qquad \Delta u_{kj\xi}(0,\tau) = -\tilde{u}_{kj\xi}(0,\tau).$$

Let $1 \le i \le r-1$ first. Since $\tilde{u}_0 \to p^{i+1}(\overline{\eta}^i)$ as $\tau \to \infty$, equation (6.21) is exponentially stable in the space $E_{\mathbb{R}}^2((1+|\xi|^j)e^{-\gamma\xi})$. Therefore, from Lemma 2.5, with $\theta = 1$, $\beta = \frac{1}{2}$,

$$|\Delta u_{kj}|_{E_{\mathbb{R}}^2((1+|\xi|^j)e^{-\gamma\xi})} \le C(1 + \tau^{k-j}).$$

Thus (6.12) has been proved for $1 \le i \le r-1$.

Let $i = 0$. We consider (6.22) in $\xi \ge 0$. The estimate in (6.15) is replaced by

$$|G_{kj}|_{E_{\mathbb{R}+}^1((1+\xi^j)e^{-\gamma\xi})} \le C(1 + \tau^{k-j}).$$

To solve (6.22) and (6.22$_1$), let $\tilde{u}_{kj\xi}(0,\tau) = \varphi_j(\tau)$. We have $|D^\alpha \varphi_j(\tau)| \le C(1 + \tau^{k-j})$, $\alpha = 0, 1$. Let $\Phi(\xi,\tau)$ be the solution to the elliptic system with a boundary condition at $\xi = 0$,

$$\begin{aligned} &\Phi_{\xi\xi} + V^i(\overline{\eta}^i)\Phi_\xi + f_{0u}(q^i(\xi,\overline{\eta}^i),\overline{\eta}^i)\Phi = 0, \\ &\Phi_\xi(0,\tau) = -\varphi_j(\tau). \end{aligned}$$

Then from Lemma 2.3(ii), there exists a unique solution Φ such that

$$|\partial_\tau^\alpha \Phi|_{E_{\mathbb{R}+}^2((1+\xi^j)e^{-\gamma\xi})} \le C(1 + \tau^{k-j}), \quad \alpha = 0, 1.$$

The solution $\Delta u_{kj} = \Phi + \Psi$, with

$$\begin{aligned} \Psi_\tau \;&=\; \Psi_{\xi\xi} + V^i(\overline{\eta}^i)\Psi_\xi + f_{0u}(u_0^{SSi},\overline{\eta}^i)\Psi + G_{kj} - \Phi_\tau \\ &\quad + \; [f_{0u}(u_0^{SSi},\overline{\eta}^i) - f_{0u}(q^i(\xi,\overline{\eta}^i),\overline{\eta}^i)]\Phi, \\ \Psi_\xi(0,\tau) \;&=\; 0, \\ \Psi(\xi,0) \;&=\; \Delta u_{kj}(\xi,0) - \Phi(\xi,0). \end{aligned}$$

Since the linear equation for Ψ is asymptotically stable, and the forcing terms are in $E^1_{\mathbb{R}^+((1+|\xi|^j)e^{-\gamma\xi})}$ with norms bounded by $C(1+\tau^{k-j})$, from Lemma 2.5 again,

$$|\Psi|_{E^2_{\mathbb{R}^+}((1+\xi^j)e^{-\gamma\xi})} \leq C(1+\tau^{k-j}).$$

This proves the case $i=0$. The matching of $(SS)^i$ with $(RS)^{i+1}$, $0 \leq i \leq r-1$, has been proved by induction. The matching of $(SS)^i$ with $(RS)^i$, $1 \leq i \leq r$, can be treated similarly.

Theorem 6.4. *Let $\Delta u^i_j = u^{SSi}_j - \tilde{u}^{RSi}_{j,1}$, $1 \leq i \leq r$, or $u^{SSi}_j - \tilde{u}^{RSi}_{j,2}$, $0 \leq i \leq r-1$, where $\tilde{u}^{RSi}_{j,1}$ and $\tilde{u}^{RSi}_{j,2}$ are defined in (6.8). Then there exists $\gamma > 0$ such that for all integers $\alpha \leq 2$, $\beta \leq 1$,*

$$|\partial^\alpha_\xi \Delta u^i_j| + |\partial^\beta_\tau \Delta u^i_j| \leq C_j(1+|\xi|^j+\tau^j)e^{-\gamma|\xi|}.$$

6.5. **Constructing a pseudo solution.** A pseudo solution is a piecewise smooth function that almost satisfies (1.1) with a small residual error

$$\epsilon u_t - \epsilon^2 u_{xx} - f(u,x,\epsilon),$$

in the interior of each subregion where the function is smooth, and a small jump error at each of their common boundaries. By truncating the formal series, we can construct pseudo solutions with arbitrary accuracy.

Let

$$u^{RRi,m}(x,\epsilon) = \sum_{j=0}^m \epsilon^j u^{RRi}_j(x), \quad u^{SRi,m}(\xi,t,\epsilon) = \sum_{j=0}^m \epsilon^j u^{SRi}_j(\xi,t),$$

$$u^{RSi,m}(x,\tau,\epsilon) = \sum_{j=0}^m \epsilon^j u^{RSi}_j(x,\tau), \quad u^{SSi.m}(\xi,\tau,\epsilon) = \sum_{j=0}^m \epsilon^j u^{SSi}_j(\xi,\tau),$$

$$\eta^{i,m}(t,\epsilon) = \sum_{j=0}^m \epsilon^j \eta^i_j(t), \quad \overset{*}{\eta}{}^{i,m}(\tau,\epsilon) = \sum_{j=0}^m \epsilon^j \overset{*}{\eta}{}^i_j(\tau).$$

We use $\eta^{i,m}$ and $\overset{*}{\eta}{}^{i,m}$ as abbreviations for $\eta^{i,m}(t,\epsilon)$ and $\overset{*}{\eta}{}^{i,m}(\tau,\epsilon)$. Let $0 < \beta < 1$ be a constant. Let the width of the initial and internal/boundary layers be $O(\epsilon^\beta)$. Define the subregions

$$(RR)^{i,m} = \{t > \epsilon^\beta, \quad x \in (\eta^{i-1,m}+\epsilon^\beta, \eta^{i,m}-\epsilon^\beta)\}, \quad 1 \leq i \leq r,$$

$$(SR)^{i,m} = \{t > \epsilon^\beta, \quad x \in (\eta^{i,m}-\epsilon^\beta, \eta^{i,m}+\epsilon^\beta) \cap [a,b]\}, \quad 0 \leq i \leq r,$$

$$(RS)^{i,m} = \{0 \leq t < \epsilon^\beta, \quad x \in (\overset{*}{\eta}{}^{i-1,m+1}+\epsilon^\beta, \overset{*}{\eta}{}^{i,m+1}-\epsilon^\beta)\}, \quad 1 \leq i \leq r,$$

$$(SS)^{i,m} = \{0 \leq t < \epsilon^\beta, \quad x \in (\overset{*}{\eta}{}^{i,m+1}-\epsilon^\beta, \overset{*}{\eta}{}^{i,m+1}+\epsilon^\beta)\}, \quad 0 \leq i \leq r.$$

Theorem 6.5. *For all $m \geq 0$, let*

(6.23)

$$U^m(x,t,\epsilon) = \begin{cases} u^{RRi,m}(x,t,\epsilon), & \text{if } (x,t) \in (RR)^{i,m}, \\ u^{SRi,m}(\dfrac{x-\eta^{i,m}}{\epsilon},t,\epsilon), & \text{if } (x,t) \in (SR)^{i,m}, \\ u^{RSi,m}(x,\dfrac{t}{\epsilon},\epsilon), & \text{if } (x,t) \in (RS)^{i,m}, \\ u^{SSi,m}(\dfrac{x-\overset{*}{\eta}{}^{i,m+1}}{\epsilon},\dfrac{t}{\epsilon},\epsilon), & \text{if } (x,t) \in (SS)^{i,m}. \end{cases}$$

Then U^m is a pseudo solution to (1.1) with residual error being $O(\epsilon^{\beta(m+1)})$ in each layer where U^m is smooth. The jump errors of U^m, $\partial_t U^m$, $\partial_x U^m$ and $\partial_x^2 U^m$ are of $O(\epsilon^{\beta(m+1)})$ along the common boundaries of adjacent layers. The function U^m satisfies Neumann boundary conditions, and almost satisfies the initial condition with an error $U^m(x,0,\epsilon) - \overline{u}(x,\epsilon) = O(\epsilon^{\beta(m+1)})$.

Proof. The residual errors can be evaluated by substituting U^m into (1.1) and expanding in powers of ϵ. For example, in $(SS)^{i,m}$, the error is bounded by

$$C\,\epsilon^{m+1}\,\sup\{1 + |\xi|^{m+1} + |\tau|^{m+1} \ : \ |\xi| < \epsilon^{\beta-1}, |\tau| < \epsilon^{\beta-1}\} \leq C\,\epsilon^{\beta(m+1)}.$$

The estimates of jump errors use the matching of adjacent layers. For example, the jump between $(SS)^{i,m}$ and $(RS)^{i+1,m}$ at the boundary

$$\begin{aligned}
\Gamma &= \{x = \overset{*}{\eta}{}^{i,m+1} + \epsilon^{\beta}, 0 \leq \tau \leq \epsilon^{\beta-1}\} \\
&= \{\xi = \epsilon^{\beta-1}, 0 \leq \tau \leq \epsilon^{\beta-1}\},
\end{aligned}$$

where $\xi = \frac{x - \overset{*}{\eta}{}^{i,m+1}}{\epsilon}$, is

(6.24)

$$|\sum_0^m \epsilon^j (u_j^{SSi}(\xi,\tau) - u_j^{RS,i+1}(x,\tau))|$$

$$\leq |\sum_0^m \epsilon^j (u_j^{SSi}(\xi,\tau) - \tilde{u}_{j,2}^{RSi}(\xi,\tau))| + |\sum_0^m \epsilon^j (\tilde{u}_{j,2}^{RSi}(\xi,\tau) - u_j^{RS,i+1}(x,\tau))|.$$

The first term of the above is bounded by

$$O(\sum_0^m \epsilon^j \sup\{(1 + |\xi|^j + |\tau|^j)e^{-\gamma\xi}\}) = O(e^{-\gamma\epsilon^{\beta-1}})$$

where $\gamma > 0$ is a constant, due to the matching condition Theorem 6.4. For the second term, observe that

$$\begin{aligned}
\sum_0^m \epsilon^j u_j^{RS,m+1}(\epsilon\xi + \overset{*}{\eta}{}^{i,m+1},\tau) &= \sum_0^m \epsilon^j u_j^{RS,m+1}(\epsilon\xi + \overset{*}{\eta}{}^i,\tau) + O(\epsilon^{\beta(m+1)}) \\
&= \sum_0^m \epsilon^j \tilde{u}_{j,2}^{RSi}(\xi,\tau) + O(\epsilon^{\beta(m+1)}).
\end{aligned}$$

Here the $O(\epsilon^{\beta(m+1)})$ terms are caused by truncating $\overset{*}{\eta}{}^i$. This gives the desired estimate on the jump error. The other jump errors can be estimated similarly. $\square$

Finally we mention that using composite expansion techniques we can construct pseudo solutions with only residual error but no jump error in the entire region $t \geq 0$, $x \in [a, b]$. See [22] and [6, 7] for more details.

7. Proof of the lemmas

Proof of Lemma 2.2. Consider a system in $\mathbb{R}^{2n}$ that is equivalent to (2.9):

(7.1)
$$\begin{aligned}
u_\xi &= v, \\
v_\xi &= -Df(p)u - Vv.
\end{aligned}$$

Let λ be an eigenvalue for $J = \begin{pmatrix} 0 & I \\ -Df(p) & -V \end{pmatrix}$. Then,

$$\det(\lambda I - J) = \prod_{i=1}^{n} (\lambda^2 + V\lambda + \mu_i),$$

where μ_i, $1 \leq i \leq n$, are the eigenvalues for $Df(p)$. Since $Re\mu_i \leq -\sigma_0$, it is elementary to show that $\lambda^2 + V\lambda + \mu_i = 0$ has two roots $\{\lambda_{i1}, \lambda_{i2}\}$ with $Re\lambda_{i1} \leq -V - \sqrt{V^2 + 4\sigma_0} < \sqrt{V^2 + 4\sigma_0} - V \leq Re\lambda_{i2}$. Therefore (7.1) has exponential dichotomy with stable and unstable spaces both n-dimensional, so does (2.9).

Since $Df(q(\xi)) \to Df(p^i)$, $i = 1$ or 2 as $\xi \to -\infty$ or $+\infty$, by a perturbation theory of exponential dichotomy, cf. [26], (2.10) has exponential dichotomies on R^- and $\mathbb{R}^+$ respectively. The stable and unstable spaces, $\mathcal{R}P_s(t)$ and $\mathcal{R}P_u(t)$ are n-dimensional. The rate of decay on $\mathcal{R}P_s(t)$ and $\mathcal{R}P_u(t)$ can be any constant $0 < \alpha_1 < \alpha$. Since α can be arbitrary close to $\sqrt{V^2 + 4\sigma_0} - |V|$, so is α_1. $\quad\square$

We now consider the following systems

$$(7.2) \qquad \begin{aligned} u_\xi &= v, \\ u_\xi &= -Vv - Df(p)u + g, \end{aligned}$$

$$(7.3) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= -Vv - Df(q(\xi))u + g. \end{aligned}$$

Let $T(\xi, s)$ be the solution operator for (7.3).

Proof of Lemma 2.3. (i) Let $g \in X = E_\mathbb{R}^m(w)$. Let $L_p u = u_{\xi\xi} + Vu_\xi + Df(p)u$, with $D(L_p) = E_\mathbb{R}^{m+2}(w)$. From Lemma 2.2, system (7.2) that is equivalent to $L_p u = g$ has an exponential dichotomy on $\mathbb{R}$. Let P_u and P_s be the projection to the unstable and stable spaces. Let

(7.4)

$$\begin{pmatrix} u \\ v \end{pmatrix}(\xi) = \int_{-\infty}^{\xi} e^{J(\xi-s)} P_s \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds + \int_{\infty}^{\xi} e^{J(\xi-s)} P_u \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds.$$

Using the exponential estimates on $\|e^{J(\xi-s)^\infty} P_s\|$ and $\|e^{J(\xi-s)} P_u\|$, and Lemma 2.1, we can verify that $(u, v) \in E_\mathbb{R}^{m+2}(w) \times E_\mathbb{R}^{m+1}(w)$ and (u, v) solves (7.2). Details will be omitted. Therefore $\mathcal{R}(L_p) = X$. On the other hand, if u is a solution to $L_p u = g$, then $(u, u_\xi) \in E_\mathbb{R}^{m+2}(w) \times E_\mathbb{R}^{m+1}(w)$ is a solution to (7.2). It is standard to show (u, u_ξ) is given by (7.4). This proves that $Ker(L_p) = \{0\}$.

(ii) Let $g \in X = E_{\mathbb{R}^+}^m(w)$. All the solutions of (7.3) with $u \in E_{\mathbb{R}^+}^{m+2}(w)$ are given by

$$\begin{pmatrix} u \\ v \end{pmatrix}(\xi) = T(\xi, 0) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^\xi T(\xi, s) P_s \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds$$

$$+ \int_\infty^\xi T(\xi, s) P_u \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds,$$

where $(u_0, v_0) \in \mathcal{R}P_s(0)$. Let $\Pi(u, v) = v$ be the projection from $\mathbb{R}^{2n} \sim \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$. We can show that $\Pi : \mathcal{R}P_s \to \mathbb{R}^n$ is a homeomorphism. If not, then there exists a nontrivial $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{R}P_s(0)$, such that $v_0 = 0$. Thus, there exists a nontrivial

bounded solution to (2.10) with $u_\xi(0) = 0$. This is a contradiction. Based on what have been proved, there exists a unique $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{R}P_s$ such that

$$(7.5) \qquad \Pi \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = v_0 = \phi - \Pi \int_\infty^0 T(0,s)P_u \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds.$$

The desired solution can be obtained by such $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

The case $X = E_{\mathbb{R}^-}^m(w)$ can be treated similarly.

Finally the estimates on $\|u\|_{E^{m+2}(w)}$ in both cases (i) and (ii) come from Banach's closed graph theorem. □

Proof of Lemma 2.4. Consider (7.3) but with $\xi \in \mathbb{R}$, $g \in X = E_{\mathbb{R}}^m(w)$. Equation $L_q u = g$ is then equivalent to (7.3). From Lemma 2.2, (7.3) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. Also if $g = 0$, $(q_\xi, q_{\xi\xi})$ is the only bounded solution to (7.3), up to constant multiples. It follows from the same argument as in [26], which treats the case $w \equiv 1$, that (7.3) has a solution $(u,v) \in E_{\mathbb{R}}^{m+2}(w) \times E_{\mathbb{R}}^{m+1}(w)$ if and only if $\langle \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rangle_{L^2} = 0$, where (ψ_1, ψ_2) is a unique (up to constant multiples) bounded solution to the adjoint equation of (7.3)

$$(7.6) \qquad \begin{aligned} \psi_{1\xi} &= Df^\tau(q(\xi))\psi_2, \\ \psi_{2\xi} &= -\psi_1 + V\psi_2. \end{aligned}$$

It is now clear that $\langle g, \psi_2 \rangle = 0 \Leftrightarrow g \in \mathcal{R}(L_q)$. The equation for ψ_2 is $\psi_{2\xi\xi} - V\psi_{2\xi} + Df^\tau(q(\xi))\psi_2 = 0$. □

Proof of Lemma 2.5. Using the definition of the exponential stability right before Lemma 2.5, we have

$$\|T(t,0)u_0\|_\theta \le Ke^{-\alpha t}\|u_0\|_\theta \le C,$$

$$\|\int_0^t T(t,s)g(s)\,ds\|_\theta \le \int_0^t \|T(t,\frac{t+s}{2})T(\frac{t+s}{2},s)g(s)\|_\theta\,ds$$

$$\le \int_0^t Ke^{-\alpha(t-s)/2}K(1+(\frac{t-s}{2})^{\beta-\theta})\cdot C(1+s^k)\,ds$$

$$\le C(1+t^k).$$

The desired estimate of $u(t)$ follows from the variation of constant formula. □

Proof of Lemma 2.6. For any $\gamma \in \mathbb{R}$, the locus of $\mathcal{P} = \{\lambda^2 | Re\lambda = 2|\gamma| + 1\}$ is a parabola. Let $\Sigma = \{|arg(\lambda - \sigma_1)| < \frac{\pi}{2} + \delta\}$, $0 < \delta < \frac{\pi}{2}$, be a sector. Let $\sigma_1 > 0$ be sufficiently large such that $\Sigma \cap \mathcal{P} = \emptyset$. Then $\lambda \in \Sigma$ implies that $Re\sqrt{\lambda} > 2|\gamma| + 1$, where $\sqrt{\lambda}$ is in the branch with $|\arg\sqrt{\lambda}| < \pi/2$.

(i) Let $g \in X = E_{\mathbb{R}}(w)$, $w = (1 + |\xi|^j)e^{-\gamma\xi}$, $\lambda \in \Sigma$. Consider $u_{\xi\xi} - \lambda u = g$, and its equivalent system

$$(7.7) \qquad \begin{aligned} u_\xi &= \sqrt{\lambda}v, \\ v_\xi &= \sqrt{\lambda}u + g/\sqrt{\lambda}. \end{aligned}$$

The eigenvalues for $H = \begin{pmatrix} 0 & \sqrt{\lambda}I \\ \sqrt{\lambda}I & 0 \end{pmatrix}$ are $\mu = \pm\sqrt{\lambda}$, each is of multiplicity n. Also $|Re\mu| > 2|\gamma| + 1$. Therefore (7.7) has an exponential dichotomy on $\mathbb{R}$, with n-dimensional stable and unstable subspaces.

For the matrix H, $\mathcal{R}P_u = \left\{ \begin{pmatrix} w \\ w \end{pmatrix} | w \in \mathbb{R}^n \right\}$ that is associated to eigenvalue $\sqrt{\lambda}$ and $\mathcal{R}P_s = \left\{ \begin{pmatrix} w \\ -w \end{pmatrix} | w \in \mathbb{R}^n \right\}$ that is associated to eigenvalue $-\sqrt{\lambda}$. Thus,

$P_s \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u-v)/2 \\ (v-u)/2 \end{pmatrix}$ and $P_u \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (u+v)/2 \\ (u+v)/2 \end{pmatrix}$ with $|P_u| + |P_s| \leq C$.

The constant C is independent of $\lambda \in \Sigma$. Thus, in the following, the constant K is independent of $\lambda \in \Sigma$.

$$
(7.8) \qquad \begin{aligned}
|e^{H\xi}P_s| &\leq Ke^{-Re\sqrt{\lambda}\xi}, \quad \xi \geq 0, \\
|e^{H\xi}P_u| &\leq Ke^{Re\sqrt{\lambda}\xi}, \quad \xi \leq 0.
\end{aligned}
$$

Using (7.8), we can show that the integrals in (7.9) converge and define a solution to (7.7).

(7.9)
$$
\begin{pmatrix} u \\ v \end{pmatrix}(\xi) = \int_{-\infty}^{\xi} e^{H(\xi-s)}P_s \begin{pmatrix} 0 \\ g(s)/\sqrt{\lambda} \end{pmatrix} ds + \int_{\infty}^{\xi} e^{H(\xi-s)}P_u \begin{pmatrix} 0 \\ g(s)/\sqrt{\lambda} \end{pmatrix} ds.
$$

In fact, using the estimates (7.8) and Lemma 2.1, we have

$$
\begin{aligned}
\left| \begin{pmatrix} u \\ v \end{pmatrix}(\xi) \right| &\leq \int_{-\infty}^{\xi} Ke^{-Re\sqrt{\lambda}(\xi-s)}\|g\|_{E(w)} \cdot \frac{1}{|\sqrt{\lambda}|}(1+|s|^j)e^{-\gamma s}ds \\
&\quad + \int_{\xi}^{\infty} Ke^{-Re\sqrt{\lambda}(s-\xi)}\|g\|_{E(w)}\frac{1}{|\sqrt{\lambda}|}(1+|s|^j)e^{-\gamma s}ds \\
&\leq \frac{K_1\|g\|_{E(w)}}{(Re\sqrt{\lambda}-|\gamma|)^{j+1} \cdot |\sqrt{\lambda}|} \cdot (1+|\xi|^j)e^{-\gamma\xi}.
\end{aligned}
$$

Here K_1 is a constant that depends on j. From $Re\sqrt{\lambda} > 2|\gamma| + 1$, we have $Re\sqrt{\lambda} - |\gamma| > 1$ and $Re\sqrt{\lambda} - |\gamma| > \frac{1}{2}Re\sqrt{\lambda}$. Therefore the above is bounded by

$$
\frac{2K_1\|g\|_{E(w)}}{Re\sqrt{\lambda} \cdot |\sqrt{\lambda}|} \cdot (1+|\xi|^j)e^{-\gamma\xi}.
$$

Since $\sigma_1 > 0$, $|arg\lambda| \leq |arg(\lambda - \sigma_1)| \leq \frac{\pi}{2} + \delta$, and $|arg\sqrt{\lambda}| \leq \frac{\pi}{4} + \frac{\delta}{2} < \frac{\pi}{2}$. Thus, $Re\sqrt{\lambda} > |\sqrt{\lambda}| \cos(\frac{\pi}{4} + \frac{\delta}{2}) = C_0|\sqrt{\lambda}|$. From this

$$
\left| \begin{pmatrix} u \\ v \end{pmatrix}(\xi) \right| \leq \frac{2K_1\|g\|_{E(w)}}{C_0|\lambda|}(1+|\xi|^j)e^{-\gamma\xi}.
$$

Thus $\|u\|_{E(w)} \leq \frac{c}{|\lambda|}\|g\|_{E(w)} \leq \frac{c}{|\lambda-\sigma_1|}\|g\|_{E(w)}$. The solution in (7.9) is also unique in $E_{\mathbb{R}}(w)$ since (7.7) has an exponential dichotomy on $\mathbb{R}$. This proves the Lemma when $X = E_{\mathbb{R}}(w)$.

(ii) Let $g \in X = E_{\mathbb{R}^+}(w)$, $\lambda \in \Sigma$. Consider $u_{\xi\xi} - \lambda u = g$ with the boundary condition $u_{\xi}(0) = 0$. If $\begin{pmatrix} u \\ v \end{pmatrix} \in E_{\mathbb{R}^+}^m(w)$ is a solution for (7.7), it can be expressed

as

$$\begin{pmatrix} u \\ v \end{pmatrix}(\xi) = e^{H\xi}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^\xi e^{H(\xi-s)}P_s\begin{pmatrix} 0 \\ g(s)/\sqrt{\lambda} \end{pmatrix}ds$$
$$+ \int_\infty^\xi e^{H(\xi-s)}P_u\begin{pmatrix} 0 \\ g(s)/\sqrt{\lambda} \end{pmatrix}ds.$$

Here $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{R}P_s$ has to be determined so that $v(0) = 0$. To this end, choose

$$v_0 = -\Pi \int_\infty^0 e^{-Hs}P_u\begin{pmatrix} 0 \\ g(s)/\sqrt{\lambda} \end{pmatrix}ds$$

and $u_0 = -v_0$, then $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{R}P_s$. Similar to (i), $\left|\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right|_{\mathbb{R}^{2n}} \le C\|g\|_{E_\mathbb{R}^+(w)}/|\lambda|$.
With this $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, we have

$$\left|\begin{pmatrix} u \\ v \end{pmatrix}\right|_{E_{\mathbb{R}+}(w)} \le C\|g\|_{E(w)/|\lambda|} \le C\|g\|_{E_{\mathbb{R}+}(w)}/|\lambda - \sigma_1|.$$

This proves the case $X = E_{\mathbb{R}+}(w)$. The same argument can be used for the case $X = E_{\mathbb{R}-}(w)$.

(iii) Let $g \in X = B_\mathbb{R}(w), \lambda \in \Sigma$. If $\begin{pmatrix} u \\ v \end{pmatrix} \in B_\mathbb{R}(w)$ is a solution to (7.7), it is expressed by (7.9). Let $g = g(\infty) + g_1(\xi)$, where $g(\infty) = \lim_{\xi\to\infty} g(\xi)$. Then $g_1 \in E_\mathbb{R}(w)$. It follows that $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, where $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -g(\infty)/\lambda \\ 0 \end{pmatrix}$ is a constant solution and

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}(\xi) = \int_{-\infty}^\xi e^{H(\xi-s)}P_s\begin{pmatrix} 0 \\ g_1(s)/\sqrt{\lambda} \end{pmatrix}ds + \int_\infty^\xi e^{H(\xi-s)}P_u\begin{pmatrix} 0 \\ g_1(s)/\sqrt{\lambda} \end{pmatrix}ds.$$

From (i), $\|u_1\|_{E_\mathbb{R}(w)} \le C\|g_1\|_{E_\mathbb{R}(w)/|\lambda|}$. Also, $|u_2| \le |g(\infty)|/|\lambda|$. Thus, $\|u\|_{B_\mathbb{R}(w)} \le C\|g\|_{B_\mathbb{R}(w)}/|\lambda| \le C\|g\|_{B_\mathbb{R}(w)}/|\lambda - \sigma_1|$. This proves the case $X = B_\mathbb{R}(w)$. The cases $X = B_{\mathbb{R}\pm}(w)$ can be handled similar to (ii). $\qquad\square$

Proof of Lemma 2.7. (i) Let $w = 1 + |\xi|^j$, $X = E_\mathbb{R}(w)$ and $D_A = E_\mathbb{R}^2(w)$ first. Let $u_0 \in E_\mathbb{R}^1(w), v = D_\xi u_0 \in E_\mathbb{R}(w)$. Let

$$I(t) = t^{1/2}Ae^{At}u_0 = t^{1/2}D_\xi^2 e^{At}u_0 = t^{1/2}D_\xi e^{At}v.$$

Using the fundamental solution for the heat equation

$$I(\xi, t) = t^{1/2}D_\xi \int_{-\infty}^\infty (2\sqrt{\pi t})^{-1}\exp(-\frac{x^2}{4t})v(\xi - x)dx$$
$$= (2\sqrt{\pi})^{-1}\int_{-\infty}^\infty \exp(-\frac{x^2}{4t})v(\xi - x)(-\frac{2x}{4t})dx.$$

Let $\frac{x^2}{4t} = \eta$, $x = \pm\sqrt{4t\eta}$.

$$|2\sqrt{\pi}I(\xi,t)| \leq |-\int_{\infty}^{0} e^{-\eta}v(\xi+\sqrt{4t\eta})d\eta$$

$$-\int_{0}^{\infty} e^{-\eta}v(\xi-\sqrt{4t\eta})d\eta|$$

$$= |\int_{0}^{\infty} e^{-\eta}[v(\xi+\sqrt{4t\eta})-v(\xi)]d\eta.$$

$$-\int_{0}^{\infty} e^{-\eta}[v(\xi-\sqrt{4t\eta})-v(\xi)]d\eta|$$

$$|2\sqrt{\pi}(1+|\xi|^j)^{-1}I(\xi,t)| \leq |\int_{0}^{M}+\int_{M}^{\infty} e^{-\eta}|v(\xi+\sqrt{4t\eta})-v(\xi)|(1+|\xi|^j)^{-1}d\eta|$$

$$+|\int_{0}^{M}+\int_{M}^{\infty} e^{-\eta}|v(\xi-\sqrt{4t\eta})-v(\xi)|(1+|\xi|^j)^{-1}d\eta|$$

$$= I_1(\xi,t) + I_2(\xi,t).$$

We show $\lim\limits_{t\to0^+}\sup\limits_{\xi} I_1(\xi,t) = 0$. Similar arguments will show $\lim\limits_{t\to0^+}\sup\limits_{\xi} I_2(\xi,t) = 0$, thus $\lim\limits_{t\to0^+} |I(t)|_{E(w)} = 0$. For any $\epsilon > 0$, choose M so large such that

$$(7.10) \qquad \int_{M}^{\infty} e^{-\eta}|v(\xi)|(1+|\xi|^j)^{-1}d\eta \leq \int_{M}^{\infty} e^{-\eta}d\eta \cdot |v|_{E(w)} < \frac{\epsilon}{4},$$

$$(7.11) \quad \int_{M}^{\infty} e^{-\eta}|v(\xi+\sqrt{4t\eta})|(1+|\xi+\sqrt{4t\eta}|^j)^{-1}\frac{1+|\xi+\sqrt{4t\eta}|^j}{1+|\xi|^j}d\eta$$

$$\leq C\int_{M}^{\infty} e^{-\eta}|v|_{E(w)}(1+(\sqrt{4t\eta})^j)d\eta < \frac{\epsilon}{4},$$

For that fixed M, let t_ϵ be small such that for $0 < t < t_\epsilon$,

$$(7.12)$$

$$\int_{0}^{M} e^{-\eta}|v(\xi+\sqrt{4t\eta})(1+|\xi+\sqrt{4t\eta}|^j)^{-1} - v(\xi)(1+|\xi|^j)^{-1}|d\eta < \frac{\epsilon}{4}.$$

This is possible since $v(\xi)w^{-1}(\xi)$ is uniformly continuous. Also if t_ϵ is even smaller, then

$$(7.13)$$

$$\int_{0}^{M} e^{-\eta}|v(\xi+\sqrt{4t\eta})||(1+|\xi+\sqrt{4t\eta}|^j)^{-1} - (1+|\xi|^j)^{-1}|d\eta$$

$$\leq \int_{0}^{M} e^{-\eta}|v|_{E(w)}(1-\frac{1+|\xi+\sqrt{4t\eta}|^j}{1+|\xi|^j})d\eta < \frac{\epsilon}{4},$$

since $\frac{|\xi+\alpha|^j-|\xi|^j}{1+|\xi|^j} \to 0$ uniformly with respect to ξ as $\alpha \to 0$.

From (7.10)–(7.13), we see $\lim\limits_{t\to0^+}\sup\limits_{\xi} I_1(\xi,t) = 0$.

(ii) Let $X = E_{\mathbb{R}^+}(w)$, $D_A = E_{\mathbb{R}^+}^2(w) \cap BC$, and $u_0 \in E_{\mathbb{R}^+}^1(w) \cap BC$. Let $\tilde{u}_0$ be the even extension of u_0 to $\xi \in \mathbb{R}$. Then $\tilde{u}_0 \in E_{\mathbb{R}}(w)$. To find $e^{At}u_0$, we solve

$$\tilde{u}_t = u_{\xi\xi}, \quad \tilde{u}(0) = \tilde{u}_0.$$

The restriction of $\tilde{u}(t)$ to $\xi \geq 0$ is $e^{At}u_0$. Using the results from (i), we find that $|t^{1/2}Ae^{At}u_0|_{E_{\mathbb{R}^+}(w)} \leq |t^{1/2}D_\xi^2 \tilde{u}(t)|_{E_{\mathbb{R}}(w)}$. The latter approaches 0 as $t \to 0^+$. Thus $u_0 \in D_A(\frac{1}{2})$.

(iii) The other cases can be treated similarly. $\qquad\square$

Proof of Lemma 2.9. (a) Let $\lambda \in \mathbb{C}$ be such that $Re\lambda > -\sigma_0$. Let $g \in X$ and consider $(L - \lambda)u = g$. The equivalent system is

$$(7.14) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= (\lambda I - M)u - Vv + g. \end{aligned}$$

Let $J = \begin{pmatrix} 0 & I \\ \lambda I - M & -VI \end{pmatrix}$. Since $Re\sigma(\lambda I - M) > 0$, it can be verified that J is hyperbolic with n-dimensional stable and unstable subspaces. Assume that $\eta + i\omega \in \sigma(\lambda I - M)$, and $\mu \in \sigma J$. Then $\mu = [-V \pm \sqrt{V^2 + 4\eta + 4i\omega}]/2$. One can verify that $Re\sqrt{V^2 + 4\eta + 4i\omega} > Re\sqrt{V^2 + 4\eta}$. Thus, $|Re\mu| > \sqrt{V^2 + 4\eta} - |V|$. If $Re\lambda + \sigma_0 > \eta_0$ for some $\eta_0 > 0$, then let $\gamma_0 = \sqrt{V^2 + 4\eta_0} - |V|$, we have

$$\begin{aligned} |e^{J\xi}P_s| &< Ke^{-\gamma_0\xi}, \quad \xi \geq 0, \\ |e^{J\xi}P_u| &< Ke^{\gamma_0\xi}, \quad \xi \leq 0, \end{aligned}$$

for some $K > 0$, where P_s and P_u are the projections to stable and unstable subspaces of $\mathbb{R}^{2n}$. Therefore, if γ is the constant used to define w, with $|\gamma| < \gamma_0$,

$$(7.15)$$
$$\begin{pmatrix} u \\ v \end{pmatrix}(\xi) = \int_{-\infty}^\xi e^{J(\xi-\eta)}P_s \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} d\eta + \int_\infty^\xi e^{J(\xi-\eta)}P_u \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} d\eta$$

is the unique solution to (7.14) that is in $E_{\mathbb{R}}^2(w)$. It is easy to show that $|u|_{E^2(w)} \leq C|g|_{E(w)}$. Thus, $\lambda \in \rho(L)$. This proves the case $X = E_{\mathbb{R}}(w)$.

Now consider $X = E_{\mathbb{R}^\pm}(w)$ and $g \in X$. Replace g by its even extension $\tilde{g}$ and consider (7.14). The solution $\begin{pmatrix} u \\ v \end{pmatrix}$ can be expressed by (7.15) and is even in $\xi \in \mathbb{R}$. Consider its restriction on $\xi \in \mathbb{R}^\pm$, then $u \in E_{\mathbb{R}^\pm}^2(w) \cap BC$.

(b) From part (a), if $Re\lambda > -\sigma_0 + \eta_0$, equation $u_{\xi\xi} + Vu_\xi + Df(p^i)u - \lambda u = 0$ has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. Therefore

$$(7.16) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= [\lambda I - Df(q)]u - Vv \end{aligned}$$

has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ with projections $P_s(t) + P_u(t) = I$, $t \in \mathbb{R}^-$ and $\mathbb{R}^+$ respectively. If λ is such that $\mathbb{R}P_u(0^-) \cap \mathbb{R}P_s(0^+) = \{0\}$, then (7.16) has exponential dichotomy on $\mathbb{R}$. Similar to part (a), we can show $\lambda \in \rho(L_q)$. If λ is such that $\mathcal{R}P_u(0^-) \cap \mathcal{R}P_s(0^+) \neq \{0\}$, then λ is an eigenvalue of finite multiplicity. $\qquad\square$

Proof of Lemma 4.1. (i) P1 comes from H1 by the implicit function theorem and the continuous dependence of eigenvalues on the parameter x.

(ii) Equation (4.4) has a heteroclinic solution $q(\xi)$ when $x = x^i$ and $V^i(x^i) = 0$. Melnikov's method is used to determine the existence of heteroclinic solutions (u, v) near (q, q_ξ) for the following system

$$(7.17) \qquad \begin{aligned} u_\xi &= v, \\ v_\xi &= -f_0(u, x) - Vv. \end{aligned}$$

Here $V \in \mathbb{R}$ and x are parameters. Based on H1, H2 and Lemma 2.2, the linear variational equations for (7.17) around $(u, v) = (q, q_\xi)$ has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ with $\mathcal{R}P_s(0^+) \cap \mathcal{R}P_u(0^-)$ spanned by $\{(q_\xi(0), q_{\xi\xi}(0))\}$, when $V = 0$ and $x = x^i$. Let $G(x, V)$ denote the function that measures the distance between $W^u(p^1)$ and $W^s(p^2)$ along the direction $(\psi_{i1}(0), , \psi_{2i}(0))$ based on the point $(q(0), q_\xi(0))$. cf. (7.6). Notice that $(\psi_{i1}(0), \psi_{2i}(0))$ is transverse to $T_{q(0)}W^u(p') + T_{q(0)}W^s(p^2)$, thus, $G(x, V)$ is well defined. From [26],

$$
(7.18) \quad
\begin{aligned}
\frac{\partial G(x^i, 0)}{\partial x} &= \int_{-\infty}^{\infty} \psi_{i2}^\tau(\xi) f_{0x}(q^i(\xi), x^i) d\xi, \\
\frac{\partial G(x^i, 0)}{\partial V} &= \int_{-\infty}^{\infty} \psi_{i2}^\tau(\xi) q_\xi^i(\xi) d\xi.
\end{aligned}
$$

From (4.1), $\frac{\partial G(x^i, 0)}{\partial V} \neq 0$. Therefore there exist $V^i(x)$ such that $G(x, V^i(x)) = 0$ for x in a neighborhood of x^i. The method in [26] also insures that the heteroclinic solution $q^i(\xi, x)$ depends smoothly on x.

(iii) Define $L_x^i \varphi$ by the left hand side of (4.5). When $x = x^i$, $L_x^i : C_{bu}^2 \to C_{bu}^0$ is Fredholm with index equal to zero, see [26]. From [30, page 115], if x is in a small neighborhood of x^i, L_x^i is Fredholm with index zero. Also $\dim \mathcal{K}L_x^i \leq \dim \mathcal{K}L_{x^i}^i$. Since $q_\xi \in \mathcal{K}L_x^i$, we have $\dim \mathcal{K}L_x^i = 1$ for all $x \in O^i$. Therefore (4.6) has a unique bounded solution $\psi_i(\xi, x)$ up to constant multiples.

It remains to show that by choosing $|\psi_i(\xi, x)| = 1$, ψ_i is a smooth function of x. Let $\{U_1 = (q_\xi(0), q_{\xi\xi}(0)), U_2, \ldots, U_n\}$ be an orthogonal basis for $\mathcal{R}P_u(0^-)$ and $\{\overline{U}_1 = U_1, \overline{U}_2, \ldots, \overline{U}_n\}$ be an orthogonal basis for $\mathcal{R}P_s(0^+)$ of the system

$$
(7.19) \quad
\begin{aligned}
u_\xi &= v, \\
v_\xi &= f_{0u}(q^i(\xi, x), x)u - V^i(x)v,
\end{aligned}
$$

when $x = x^i$ and $V^i(x) = 0$. System (7.19) has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ when $x \in O^i$. Let $P_u(x, t)$ and $P_s(x, t)$ denote the projections to the unstable and stable spaces. $P_u(x^i, t) = P_u(t)$ and $P_s(x^i, t) = P_s(t)$.

Assume that x is near x^i so that $V^i(x)$ is near zero. For each U_i, $2 \leq i \leq n$, there exists a unique $\Delta U_i \in \mathcal{R}P_s(0^-)$ such that $U_i + \Delta U_i \in \mathcal{R}P_u(x, 0^-)$. Also for each $\overline{U}_i$, $2 \leq i \leq n$, there exists a unique $\Delta \overline{U}_i \in \mathcal{R}P_u(0^+)$ such that $\overline{U}_i + \Delta \overline{U}_i \in \mathcal{R}P_s(x, 0^+)$. The functions ΔU_i and $\Delta \overline{U}_i$ are smooth functions of x. See [18] for details. For $i = 1$ let $U_1 + \Delta U_1 = (q_\xi^i(0, x), q_{\xi\xi}^i(0, x))$. ΔU_1 is also smooth in x. In particular, $|\Delta U_i| + |\Delta \overline{U}_i| = O(|x - x^i|)$ for all i. This proves that if $x - x^i$ is small,

$$
\mathcal{R}P_u(x, 0^-) + \mathcal{R}P_s(x, 0^+) = \mathrm{span}(U_1 + \Delta U_1 \ldots U_n + \Delta U_n \overline{U}_2 + \Delta \overline{U}_2 \ldots \overline{U}_n + \Delta \overline{U}_n),
$$

depends smoothly on x. The adjoint equation of (7.19) can be found in (7.6) where $V = V^i$, $q = q^i$. Let $(\psi_{i1}(x), \psi_{i2}(x))$ be the unique bounded solution to the adjoint equation with $|\psi_{i2}(0)| = 1$. Let $U_0 = (\psi_{i1}(0), \psi_{i2}(0))$ that is orthogonal to $\mathcal{R}P_u(0^-) + \mathcal{R}P_s(0^+)$. By a standard projection method, there exists a unique $\Delta U_0 \in \mathcal{R}P_u(x, 0^-) + \mathcal{R}P_s(x, 0^+)$ such that $U_0 + \Delta U_0$ is orthogonal to $\mathcal{R}P_u(x, 0^-) + \mathcal{R}P_s(x, 0^+)$. It can be shown that (7.6) has a bounded solution with the initial data $U_0 + \Delta U_0$, cf. [18]. A normalized solution $(\psi_{i1}(0, x), \psi_{i2}(0, x))$ with the initial data $(\psi_{i1}(0, x), \psi_{i2}(0, x))$, $|\psi_{i2}(0, x)| = 1$, can obtained by rescaling. Let $\psi_i(0, x) = \psi_{i2}(0, x)$. This proves the smooth dependence of $\psi_i(0, x)$ on x.

(iv) From Lemma 2.9, (b), $\sigma\{L_x^i\} \cap \{Re\lambda > -\sigma_0 + \eta_0\}$ consists of isolated eigenvalues of finite order. When $x = x^i$, $V^i(x) = 0$, from H5, $\lambda = 0$ is a simple

eigenvalue, all the other eigenvalues satisfy $Re\lambda < -\alpha_0$. Since eigenvalues depend continuously on x. Thus, if $0 < \overline{\alpha}_0 = \min\{\alpha_0, \sigma_0 - \eta_0\}$, there exists $\epsilon > 0$ such that if $|x - x^i| < \epsilon$ then $\lambda = 0$ is the only eigenvalue in $\{Re\lambda > -\overline{\alpha}_0\}$ and is simple.

(v) From H6, (4.7) is valid if $x = x^i$. Since both integrals in (4.7) depend continuously on x, thus (4.7) is valid for $|x - x^i| < \epsilon$ if $\epsilon > 0$ is small. The formula for $\frac{\partial V^i(x^i)}{\partial x}$ follows from (7.18). □

Proof of Lemma 4.2. L^i_x is a Fredholm operator in $E^m_{\mathbb{R}}(w)$ with index zero. From Lemma 4.1, (iii), $Ker\{L^i_x\} = \text{span}\{q^i_\xi(\cdot, x)\}$ is one-dimensional. Therefore

$$\text{Range}\{L^i_x\} = \{\psi_i(\cdot, x)\}$$

is of codimension one. Consider the mapping $\mathcal{F}: (x, u, V_1) \to (g, h)$ as follows

(7.20)
$$\begin{aligned} L^i_x u - V_1 q^i_\xi(\cdot, x) &= g, \\ \langle u, q^i_\xi(\cdot, x)\rangle &= h. \end{aligned}$$

$\mathcal{F}: O^i \times E^{m+2}_{\mathbb{R}}(w) \times \mathbb{R} \to E^m_{\mathbb{R}}(w) \times \mathbb{R}$ is C^∞, in fact, linear with respect to $u \in E^{m+2}_{\mathbb{R}}(w)$ and $V_1 \in \mathbb{R}$. It can be verified that $\partial\mathcal{F}/\partial(u, V_1)$ is a linear homeomorphism in the indicated norms. We only need to show that (u, V_1) is uniquely solvable from (7.20) for any $(g, h) \in E^m_{\mathbb{R}}(w) \times \mathbb{R}$. If we choose $V_1 = \langle\psi_i(\cdot, x), g\rangle \cdot \langle\psi_i(\cdot, x), q^i_\xi(\cdot, x)\rangle^{-1}$, then $g - V_1 q^i_\xi \in \text{Range}\{L^i_x\}$. Any two solutions of the first of (7.20) differ by a multiple of $q^i_\xi \in Ker\{L^i_x\}$, that can be determined by the second of (7.20). Let $h = 0$ and denote the solutions by $V_1 = V^i_*(x, g)$ and $u = u(\cdot, x, g)$.

The smoothness of $V^i_*(x, g)$ and $u(\cdot, x, g)$ on (x, g) also follows from the Implicit Function Theorem applied on the function $\mathcal{F}$. □

Proof of Lemma 5.1. According to Lemma 2.9, (b), $\mathbb{C}_1 = \sigma\{L_{q^i}\} \cap \{Re\lambda > -\partial_0 + \eta_0\}$ consists of only eigenvalues. When $i = 0, r$, from H5, L_{q^i} has no eigenvalues in $\mathbb{C}_1$ in the space $C_{bu}(\mathbb{R}^+, \mathbb{R}^n)$. Thus, it also has no eigenvalues in $\mathbb{C}_1$ in the space $B_{\mathbb{R}^\pm}(w)$. This proves the case $i = 0, r$. When $1 \le i \le r - 1$, from H5 again, in $C_{bu}(\mathbb{R}, \mathbb{R}^n)$, the only eigenvalue of L_{q^i} in $\mathbb{C}_1$ is $\lambda = 0$, simple. Thus the only eigenvalue of L_{q^i} in $B_{\mathbb{R}}(w)$ is also $\lambda = 0$, simple. From Lemma 2.8, $q^i(\cdot, \overline{\eta}^i)$ is asymptotically stable modulo spatial shifts. □

Proof of Lemma 5.2. The proof is exactly like that of Lemma 5.1. □

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