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A NOTE ON BERNOULLI NUMBERS AND SHINTANI GENERALIZED BERNOULLI POLYNOMIALS

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ABSTRACT. Generalized Bernoulli polynomials were introduced by Shintani in 1976 in order to express the special values at non-positive integers of Dedekind zeta functions for totally real numbers. The coefficients of such polynomials are finite combinations of products of Bernoulli numbers which are difficult to get hold of. On the other hand, Zagier was able to get the explicit formula for the special values in cases of real quadratic number fields.

In this paper, we shall improve Shintani's formula by proving that the special values can be determined by a finite set of polynomials. This provides a convenient way to evaluate the special values of various types of Dedekind functions. Indeed, a much broader class of zeta functions considered by the author [4] admits a similar formula for its special values. As a consequence, we are able to find infinitely many identities among Bernoulli numbers through identities among zeta functions. All these identities are difficult to prove otherwise.

1. IDENTITIES AMONG BERNOULLI NUMBERS

The Bernoulli numbers B_n $(n = 0, 1, 2, \cdots)$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}, \qquad |t| < 2\pi.$$

From the definition, one has $B_0 = 1$, $B_1 = -1/2$. Also $B_{2k+1} = 0$ for $k \geq 1$ since the function

$$\frac{t}{e^t - 1} + \frac{t}{2}$$

is an even function of t by direct verification. Bernoulli numbers are used to express the special values of Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

namely,

$$\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{m-1}B_{2m}}{2(2m)!}, \qquad m \geq 1.$$

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Given the functional equation of $\zeta(s)$, this is equivalent to

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}.$$

In [10], Zagier considered the zeta function associated with a quadratic form $Q = ap^2 + bpq + cq^2$ with a, b, $c > 0$, $D = b^2 - 4ac > 0$. The zeta function is defined as

$$Z_Q(s) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(ap^2 + bpq + cq^2)^{-s} + \frac{1}{2}(a^{-s} + c^{-s})\zeta(2s).$$

He proved that the special value of $Z_Q(s)$ at integer $-n$, $n \geq 0$, is given by

$$\begin{aligned} Z_Q(-n) = (-\frac{B_{2n+2}}{2n+2})[\int_0^{-b/2a}(ax^2+bx+c)^n dx + \int_0^{-b/2c}(a+by+cy^2)^n dy] \\ + \sum_{r=0}^{2n}\frac{B_{r+1}B_{2n-r+1}}{(r+1)(2n-r+1)}d_{r,n} + \begin{cases} -1/2 & \text{if } n=0, \\ 0 & \text{if } n \geq 1. \end{cases} \end{aligned}$$

where $d_{r,n}$ is the coefficient of $x^r y^{n-r}$ in $(ax^2 + bxy + cy^2)^n$.

Remark. Zagier's formula was reproduced in van der Geer's book [5] with a slight mistake. The formula here is a corrected version.

In [4], the author considered the zeta function $Z(P, \beta; s)$ associated with a product of linear forms

$$P(x) = \prod_{j=1}^{n}(a_{1j}x_1 + \cdots + a_{rj}x_r + \delta_j), \qquad \text{Re } a_{ij} > 0,\ \text{Re } \delta_j > 0,$$

defined as

$$Z(P, \beta; s) = \sum_{n_1=1}^{\infty}\cdots\sum_{n_r=1}^{\infty} n_1^{\beta_1}\cdots n_r^{\beta_r}P(n_1, \cdots, n_r)^{-s}, \qquad \text{Re } s > \frac{|\beta| + r}{n}.$$

Here $\beta = (\beta_1, \cdots, \beta_r)$ is a r-tuple of non-negative integers. The special value of $Z(P, \beta; s)$ at integer $-m$, $m \geq 0$, can be determined by $n \cdot 2^r$ polynomials which are related to $x^\beta P^m(x)$. Indeed, one has

$$Z(P, \beta; -m) = \frac{1}{n}\sum_{1 \leq j_1 < \cdots < j_q \leq r} J^{r-q}[\int_{\Delta_j(x_{j_1}, \cdots, x_{j_q})} x^\beta P^m(x) dx_{j_1}\cdots dx_{j_q}]\ .$$

When a_{ij} and δ_j are real numbers, the domain $\Delta_j(x_{j_1}, \cdots, x_{j_q})$ is a simplex in $\mathbb{R}^q$ defined by

$$\begin{cases} x_{j_1} \leq 0, \cdots, x_{j_q} \leq 0, \\ a_{1j}x_1 + \cdots + a_{rj}x_r + \delta_j \geq 0. \end{cases}$$

Also for any polynomial $f(x) = \sum a_\alpha x_1^{\alpha_1}\cdots x_p^{\alpha_p}$ of p variables,

$$\begin{aligned} J^p[f(x)] &= \sum a_\alpha \zeta(-\alpha_1)\cdots\zeta(-\alpha_p) \\ &= \sum a_\alpha (-1)^{|\alpha|}\frac{B_{\alpha_1+1}\cdots B_{\alpha_p+1}}{(\alpha_1+1)\cdots(\alpha_p+1)}. \end{aligned}$$

With the formula we shall derive below for special values of $Z(P, \beta; s)$, we are able to transform identities among zeta functions into identities among Bernoulli numbers. The procedure is demonstrated in the following proposition:

Proposition 1. *For $n \geq 4$, one has*

$$\sum_{p=2}^{n-2} \frac{(2n-2)!}{(2p-2)!(2n-2p-2)!} \cdot \frac{B_{2p}}{2p} \cdot \frac{B_{2n-2p}}{2n-2p} = (-\frac{B_{2n}}{2n}) \frac{(2n+1)(2n-6)}{6(2n-2)(2n-3)} \ .$$

Proof. Consider the zeta function

$$Z(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} n_1 n_2 (n_1+n_2)^{-s}, \qquad \text{Re } s > 4.$$

Let $r = n_1 + n_2$ be a new variable in place of n_2. Then for Re $s > 4$,

$$\begin{aligned} Z(s) &= \sum_{r=2}^{\infty} r^{-s} \cdot \sum_{n_1=1}^{r-1} (r-n_1) n_1 \\ &= \frac{1}{6} \sum_{r=2}^{\infty} r^{-s} (r^3 - r) \\ &= \frac{1}{6} [\zeta(s-3) - \zeta(s-1)]. \end{aligned}$$

One can now make an analytic continuation. In particular, one has for $n \geq 4$

$$\begin{aligned} Z(4-2n) &= \frac{1}{6} [\zeta(1-2n) - \zeta(3-2n)] \\ &= \frac{1}{6} [-\frac{B_{2n}}{2n} + \frac{B_{2n-2}}{2n-2}]. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} Z(4-2n) &= J^2[xy(x+y)^{2n-4}] + J[\int_0^{-x} xy(x+y)^{2n-4} dy + \int_0^{-y} xy(x+y)^{2n-4} dx] \\ &= \sum_{p=0}^{2n-4} \frac{(2n-4)!}{p!(2n-p-4)!} \frac{B_{p+2}}{p+2} \frac{B_{2n-p-2}}{2n-p-2} + (-\frac{B_{2n}}{2n}) \frac{2}{(2n-2)(2n-3)} \\ &= \sum_{p=2}^{n-2} \frac{(2n-4)!}{(2p-2)!(2n-2p-2)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p} + \frac{1}{6} \frac{B_{2n-2}}{2n-2} \\ &\quad + (-\frac{B_{2n}}{2n}) \frac{2}{(2n-2)(2n-3)}. \end{aligned}$$

Comparing the two expressions for $Z(4-2n)$, we get our assertion. □

Remark. The identity of Proposition 1 appears in [7] as a consequence of an identity among Eisenstein series of different weights:

$$\frac{1}{6}(2n+1)(2n-1)(2n-6) G_{2n}(z) = \sum_{p=2}^{n-2} (2p-1)(2n-2p-1) G_{2p}(z) G_{2n-2p}(z),$$

where $G_{2k}(z)$ is the Eisenstein series defined by

$$G_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} (m+nz)^{-2k}.$$

The above procedure can be applied to other zeta functions to get a lot of identities of a similar kind. Here we mention some of them.

(1) Consider the identity of zeta functions:

$$Z_1(s) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(n_1+n_2)^{-s} = \zeta(s-1)-\zeta(s)\ .$$

With $s = 2-2n$, $n \geq 2$, one gets the classical identity due to Euler [1],

$$\sum_{k=1}^{n-1}\frac{(2n)!}{(2k)!(2n-2k)!}B_{2k}B_{2n-2k} = -(2n+1)B_{2n}.$$

(2) Consider the identity of zeta functions:

$$Z_2(s) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}(n_1+n_2+n_3)^{-s} = \frac{1}{2}[\zeta(s-2)-3\zeta(s-1)+2\zeta(s)].$$

By letting $s = 3-2n$ with $n \geq 3$ and using the identity in (1), we get

$$\sum_{\substack{p+q+r=n\\ p,q,r\geq 1}}\frac{(2n)!}{(2p)!(2q)!(2r)!}B_{2p}B_{2q}B_{2r} = \frac{(2n+1)(2n+2)}{2}B_{2n}+\frac{n(2n-1)}{2}B_{2n-2}.$$

(3) Consider the identity of zeta functions:

$$\begin{aligned} Z_3(s) &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\sum_{n_4=1}^{\infty}(n_1+n_2+n_3+n_4)^{-s} \\ &= \frac{1}{6}[\zeta(s-3)-6\zeta(s-2)+11\zeta(s-1)-6\zeta(s)]. \end{aligned}$$

By letting $s = 4-2n$ with $n \geq 4$ and using the identities in (1) and (2), we get

$$\begin{aligned} &\sum_{\substack{p+q+r+s=n\\ p,q,r,s\geq 1}}\frac{(2n)!}{(2p)!(2q)!(2r)!(2s)!}B_{2p}B_{2q}B_{2r}B_{2s} \\ &\qquad = -[\frac{(2n+1)(2n+2)(2n+3)}{6}B_{2n}+\frac{4n^2(2n-1)}{3}B_{2n-2}]. \end{aligned}$$

(4) Consider the identity of zeta functions:

$$Z_4(s) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}n_1^2n_2^2(n_1+n_2)^{-s} = \frac{1}{30}[\zeta(s-5)-\zeta(s-1)]\ .$$

By setting $s = 6 - 2n$, $n \geq 4$, we get

$$\sum_{p=3}^{n-3} \frac{(2n-6)!}{(2p-3)!(2n-2p-3)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p}$$
$$= \frac{(2n+1)(4n^2-26n+60)}{30(2n-3)(2n-4)(2n-5)}(-\frac{B_{2n}}{2n}) + \frac{1}{60}B_{2n-4}.$$

(5) Consider the identity of zeta functions:

$$Z_5(s) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} n_1n_2n_3(n_1+n_2+n_3)^{-s}$$
$$= \frac{1}{120}\zeta(s-5) - \frac{1}{24}\zeta(s-3) + \frac{1}{30}\zeta(s-1).$$

By setting $s = 6 - 2n$ with $n \geq 6$, we get

$$\sum_{\substack{p+q+r=n \\ p,q,r\geq 2}} \frac{(2n-6)!}{(2p-2)!(2q-2)!(2r-2)!} \frac{B_{2p}B_{2q}B_{2r}}{8pqr}$$
$$= (-\frac{B_{2n}}{2n})[\frac{1}{120} - \frac{2n^2-5n}{(2n-2)(2n-3)(2n-4)(2n-5)}] + \frac{1}{80}(\frac{B_{2n-4}}{2n-4}).$$

2. Shintani's generalized Bernoulli polynomials

The classical Bernoulli polynomials $B_n(x)$ $(n = 0, 1, 2, \cdots)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k.$$

Here $\binom{n}{k}$ is the binomial coefficient $\frac{n!}{(n-k)!k!}$. Equivalently, one has

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}, \qquad |t| < 2\pi.$$

Bernoulli polynomials can be used to express the special values at negative integers of Hurwitz's zeta function $\zeta(s;x)$, defined by

$$\zeta(s;x) = \sum_{n=0}^{\infty}(n+x)^{-s}, \qquad x > 0, \mathrm{Re}\ s > 1.$$

In fact, one has for $m \geq 1$,

$$\zeta(1-m;x) = -\frac{B_m(x)}{m}.$$

For any $r \times n$ matrix $A = [a_{jk}]$ with positive entries and any r-tuple of complex numbers $x = (x_1, \cdots, x_r)$, Shintani considered in [6] the zeta function

$$\zeta(s, A, x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \prod_{k=1}^{n} [a_{1k}(n_1 + x_1) + \cdots + a_{rk}(n_r + x_r)]^{-s}.$$

Set

$$\begin{cases} L_j(t) = a_{j1}t_1 + \cdots + a_{jn}t_n, & 1 \le j \le r; \\ L_k^*(u) = a_{1k}u_1 + \cdots + a_{rk}u_r, & 1 \le k \le n. \end{cases}$$

Shintani's generalized Bernoulli polynomial $B_m(A, x)$ is defined as the special value of $\zeta(s, A, x)$ at $s = 1 - m$ up to a constant factor; namely,

$$\zeta(1 - m, A, x) = (-1)^r m^{-n} B_m(A, x).$$

In the same paper, Shintani was able to express $B_m(A, x)$ in terms of a combination of products of Bernoulli polynomials as follows:

$$\frac{B_m(A, x)}{(m!)^n} = \sum_{\substack{|p|=n(m-1)+r \\ p_1, \cdots, p_r \ge 1}} \frac{B_{p_1}(x_1) \cdots B_{p_r}(x_r)}{p_1! \cdots p_r!} C(A, p)$$
$$+ \frac{1}{n} \sum_{S} \sum_{\substack{|q|=n(m-1)+r \\ q_j \ge 1}} \{\prod_{j \in S} \frac{B_{q_1}(x_j)}{q_j!}\} \sum_{k=1}^{n} C(S, q, A)^{(k)}.$$

Here $C(A, p)$ is the coefficient of $(t_1 \cdots t_n)^{m-1}$ in the polynomial $\prod_{j=1}^{r} L_j(t)^{p_j - 1}$, S ranges over all non-empty subsets of $I = \{1, 2, \cdots, r\}$, and $C(S, q, A)^{(k)}$ is the coefficient of $(t_1 \cdots t_{k-1} t_{k+1} \cdots t_n)^{m-1}$ in the Taylor expansion of the function

$$\prod_{j \in S} L_j(t)^{q_j - 1} / \prod_{j \notin S} L_j(t)\big|_{t_k = 1}.$$

Such a formula is useful in calculating the special values at non-positive integers of Dedekind zeta functions of totally real number fields. However, it is typically quite painful and laborious to compute $C(A, p)$ as well as $C(S, q, A)^{(k)}$.

Here we shall prove that $\zeta(1 - m, A, x)$ can be determined by $n \cdot 2^r$ polynomials which can be obtained from $[P(u)]^{m-1} = \prod_{k=1}^{n} L_k^*(u)^{m-1}$ by integrating over certain simplexes. We need a correspondence from polynomials to products of Bernoulli polynomials.

Given any subset S of $I = \{1, 2, \cdots, r\}$ and polynomial $g(u)$ with variables in $\{u_i | i \in S\}$. If

$$g(u) = \sum_{|\alpha|=0}^{p} b_\alpha \cdot \prod_{i \in S} u_i^{\alpha_i},$$

we let

$$J_S[g] = \sum_{|\alpha|=0}^{p} b_\alpha \cdot \prod_{i \in S} \frac{-B_{\alpha_i+1}(x_i)}{\alpha_i + 1}.$$

When $S = \varnothing$, we let $J_\phi[c] = c$ for any constant c.

Then Shintani's formula for $\zeta(1 - m, A, x)$ can be reformulated as follows:

Theorem. *In the above notation, for any positive integer m,*

$$\zeta(1-m,A,x)=\frac{1}{n}\sum_{S}J_S\left[\sum_{k=1}^{n}\int_{\Delta_{k,S}(u)}P^{m-1}(u)\prod_{j\notin S}du_j\right].$$

Here S ranges over all non-empty subsets of $I=\{1,2,\cdots,r\}$ and $\Delta_{k,S}(u)$ is the simplex defined by

$$\Delta_{k,S}(u):\ (u_j)_{j\notin S},\ u_j\le 0,\ L_k^*(u)=a_{1k}u_1+\cdots+a_{rk}u_r\ge 0.$$

To prove the theorem, we need to evaluate $C(A,p)$ and $C(S,q,A)^{(k)}$ explicitly. Here we transform them into integrals so that we can apply the results obtained in [4].

Let E^n be the standard simplex in $\mathbb{R}^n$ defined by

$$E^n:\ (u_1,\cdots,u_n)\in\mathbb{R}^n,\ u_i\ge 0,\ \sum_{i=1}^{n}u_i=1.$$

Let $dU=du_1\cdots du_{n-1}$ be the Euclidean measure on E^n.

Lemma 1. *For any homogeneous polynomial $P(u)$ of degree $n(m-1)$, the integral*

$$F(s)=\frac{1}{\Gamma(s)^{n-1}}\int_{E^n}(u_1\cdots u_n)^{s-1}P(u)dU,\qquad \mathrm{Re}\ s>0,$$

as a function of s, has an analytic continuation to the whole s-plane. Furthermore,

$$F(1-m)=(-1)^{(n-1)(m-1)}n[(m-1)!]^{n-1}\times[\textit{coefficient of }(u_1\cdots u_n)^{m-1}\textit{in }P(u)].$$

Proof. Let

$$P(u)=\sum_{|\beta|=n(m-1)}C_\beta u_1^{\beta_1}\cdots u_n^{\beta_n}.$$

Then, for Re $s>0$, a term by term integration yields

$$F(s)=\sum_{|\beta|=n(m-1)}C_\beta\frac{\Gamma(s+\beta_1)\cdots\Gamma(s+\beta_n)}{\Gamma(s)^{n-1}\Gamma(ns+|\beta|)}.$$

For fixed $\beta=(\beta_1,\cdots,\beta_n)$ with $|\beta|=n(m-1)$, let

$$A_\beta(s)=\frac{\Gamma(s+\beta_1)\cdots\Gamma(s+\beta_n)}{\Gamma(s)^{n-1}\Gamma(ns+|\beta|)}.$$

Note that any possible poles of $\Gamma(s+\beta_1)\cdots\Gamma(s+\beta_n)$ are cancelled by the poles arising from $\Gamma(s)^{n-1}\Gamma(ns+|\beta|)$. Thus $A_\beta(s)$ is a regular function of s. Also $A_\beta(1-m)=0$ unless $\beta_1=\beta_2=\cdots=\beta_n=m-1$. For such an exceptional case, we have

$$A_\beta(1-m)=(-1)^{(n-1)(m-1)}n[(m-1)!]^{n-1}.$$

This proves our assertion. □

With Lemma 1, we are able to express the coefficient $C(A,p)$ in Shintani's formula as follows.

Corollary. *For any r-tuple of positive integers $p = (p_1, \ldots, p_r)$ with $|p| = n(m-1)+r$, one has*

$$nC(A,p) = (-1)^{(n-1)(m-1)}[(m-1)!]^{n-1}F(1-m),$$

where for $\operatorname{Re} s > 0$,

$$F(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \ldots u_n)^{s-1} \prod_{j=1}^{r} L_j(u)^{p_j-1} dU. \square$$

We now proceed to consider the same type of integral as $F(s)$ but with $p_j = 0$ for some j. First we need a lemma to evaluate the integral around a particular vertex of E^n. For any positive number $h < 1$, we let $E_j^n(h)$ be the subset of E^n defined by

$$\begin{aligned} E_j^n(h) &= \{(u_1, \ldots, u_n) \in E^n \mid 0 \le u_1 + \ldots + u_{j-1} + u_{j+1} + \ldots + u_n \le h\} \\ &= \{(u_1, \ldots, u_n) \in E^n \mid 1-h \le u_j \le 1\}. \end{aligned}$$

Lemma 2. *For* $\operatorname{Re} s > 0$ *and nonegative integers* $\alpha_1, \ldots, \alpha_n$, *one has*

$$\begin{aligned} &\int_{E_j^n(h)} \prod_{i=1}^{n} (u_i)^{s+\alpha_i-1} dU \\ &\qquad = \prod_{i \ne j} \Gamma(s+\alpha_i) \cdot \frac{1}{\Gamma((n-1)s + |\alpha|_j)} \int_0^h t^{(n-1)s+|\alpha|_j-1}(1-t)^{s+\alpha_j-1} dt \end{aligned}$$

where $|\alpha|_j = |\alpha| - \alpha_j$.

Proof. Set

$$\begin{cases} t = u_1 + \ldots + u_{j-1} + u_{j+1} + \ldots + u_n, \\ u_i' = \dfrac{u_i}{t} \qquad (i \ne j). \end{cases}$$

Then $u' = (u_1', \ldots, u_{j-1}', u_{j+1}', \ldots, u_n') \in E^{n-1}$ and

$$\frac{\partial(t, u_1', \ldots, u_n')}{\partial(u_1, u_2, \ldots, u_{n-1})} = t^{2-n}.$$

So if we take t and u' as new variables in place of $u_1, u_2, \ldots, u_n$, the integral is transformed into

$$\int_{E^{n-1}} \prod_{i \ne j} (u_i')^{s+\alpha_i-1} dU' \int_0^h t^{(n-1)s+|\alpha|_j-1}(1-t)^{s+\alpha_j-1} dt.$$

The first integral in the above is the standard β-function of several variables; it is equal to

$$\prod_{i \ne j} \Gamma(s+\alpha_i) \cdot \frac{1}{\Gamma((n-1)s + |\alpha|_j)}.$$

Our assertion thus follows. $\square$

We are now ready to evaluate the integral $F(s)$ at $s = 1 - m$ when $p_j = 0$ for some j. With a permutation in the indices, we can suppose that

$$p_1 = \ldots = p_q = 0, \quad p_{q+1} \geq 1, \ldots, p_r \geq 1, \qquad 1 \leq q \leq r.$$

For Re $s > 0$, set

$$G(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \ldots u_n)^{s-1} \prod_{j=q+1}^{r} L_j(u)^{p_j - 1} \prod_{i=1}^{q} L_i(u)^{-1} dU$$

with $p_{q+1} + \ldots + p_r = n(m-1) + r$. Note that $\prod_{j=q+1}^{r} L_j(u)^{p_j-1}$ is a homogeneous polynomial in $u_1, \ldots, u_n$ of degree $n(m-1) + q$. Put

$$\prod_{j=q+1}^{r} L_j(u)^{p_j-1} = \sum_{|\alpha| = n(m-1)+q} b_\alpha u_1^{\alpha_1} \ldots u_n^{\alpha_n}.$$

Then

$$G(s) = \sum_{|\alpha| = n(m-1)+q} b_\alpha G_\alpha(s)$$

with

$$G_\alpha(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} \prod_{i=1}^{n} u_i^{s+\alpha_i - 1} \prod_{i=1}^{q} L_i(u)^{-1} dU.$$

For Re $s > 0$, $G_\alpha(s)$ is an analytic function of s, and it has analytic continuation in the whole complex plane as we shall see. We are interested in the special value $G_\alpha(1-m)$.

Lemma 3. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers with $|\alpha| = n(m-1) + q$, $q \geq 1$. If $\alpha_j, \alpha_k \geq m$ for $j \neq k$, then $G_\alpha(1-m) = 0$.*

Proof. Since the singularities of the integral $G_\alpha(s)$ arise from integration around vertices of E^n, to show $G_\alpha(1-m) = 0$, it suffices to prove that the partial integration

$$G_\alpha^p(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E_p^n(h)} \prod_{i=1}^{n} u_i^{s+\alpha_i-1} \prod_{i=1}^{q} L_i(u)^{-1} dU$$

vanishes at $s = 1 - m$. But this follows from the fact that the difference

$$G_\alpha(s) - \sum_{p=1}^{n} G_\alpha^p(s)$$

can be eliminated via $\Gamma(s)^{1-n}$ when $s = 1 - m$. In particular, one has

$$G_\alpha(1-m) = \sum_{p=1}^{n} G_\alpha^p(1-m).$$

Around the vertex $u_p = 1$, suppose that

$$\prod_{i=1}^{q} L_i(u)^{-1} = \sum_{|\beta|=0}^{\infty} C_\beta u_1^{\beta_1} \dots u_{p-1}^{\beta_{p-1}} u_{p+1}^{\beta_{p+1}} \dots u_n^{\beta_n}.$$

By Lemma 2 and a term by term integration, we get

$$G_\alpha^p(s) = \sum_{|\beta|=0}^{\infty} C_\beta \prod_{i \neq p} \Gamma(s + \alpha_i + \beta_i) \cdot \Gamma(s)^{1-n} \cdot I_{\alpha,\beta,p}(s)$$

where

$$I_{\alpha,\beta,p}(s) = \frac{1}{\Gamma(n-1)s + |\alpha|_p + |\beta|)} \int_0^h t^{(n-1)s+|\alpha|_p+|\beta|-1}(1-t)^{s+\alpha_p-1} dt.$$

$I_{\alpha,\beta,p}(s)$ is an analytic function of s for Re $s > 0$ and it has analytic continuation in the whole complex plane. Note that $\Gamma(s)^{1-n}$ has a zero of order $n-1$ at $s = 1-m$ while the order of the pole at $s = 1-m$ of $\prod_{i \neq p} \Gamma(s + \alpha_i + \beta_i)$ is no greater than $n-2$ since $\alpha_i \geq m$ for at least one i. So $G_\alpha^p(1-m) = 0$ for all p. This proves our assertion. □

By the above lemma, we know that $G_\alpha(1-m) = 0$ unless $\alpha_j \geq m$ for exactly one j. Now we proceed to consider these exceptional cases.

Lemma 4. *Suppose that $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of non-negative integers such that $|\alpha| = n(m-1) + q, q \geq 1, \alpha_1, \dots, \alpha_{n-1} \leq m-1$ and $\alpha_n \geq m$. Then*

$$\begin{aligned} G_\alpha(1-m) = {} & (-1)^{(n-1)(m-1)}[(m-1)!]^{n-1} \\ & \times \textit{the coefficient of } \prod_{i=1}^{n-1} u_i^{m-1-\alpha_i} \textit{ in the Taylor expansion} \\ & \textit{of the function } (1 - u_1 - \dots - u_{n-1})^{\alpha_n - m} \prod_{i=1}^{q} L_i(u)^{-1}. \end{aligned}$$

Proof. Suppose that around $u_j = 1$, the Taylor expansion of $\prod_{i=1}^{q} L_i(u)^{-1}$ is given by

$$\sum_{|\beta|=0}^{\infty} C_\beta u_1^{\beta_1} \dots u_{j-1}^{\beta_{j-1}} u_{j+1}^{\beta_{j+1}} \dots u_n^{\beta_n}.$$

As in the previous lemma, we have for Re $s > 0$,

$$\begin{aligned} G_\alpha^j &= \frac{1}{\Gamma(s)^{n-1}} \int_{E_j^n(h)} \prod_{i=1}^{n} u_i^{s+\alpha_i-1} \prod_{i=1}^{q} L_i(u)^{-1} dU \\ &= \sum_{|\beta|=0}^{\infty} C_\beta \prod_{i \neq j} \Gamma(s + \alpha_i + \beta_i) \cdot \Gamma(s)^{1-n} \cdot I_{\alpha,\beta,j}(s), \end{aligned}$$

where

$$I_{\alpha,\beta,j}(s) = \frac{1}{\Gamma((n-1)s + |\alpha|_j + |\beta|)} \int_0^h t^{(n-1)s+|\alpha|_j+|\beta|-1}(1-t)^{s+\alpha_j-1}dt.$$

If $j \neq n$, then the product of gamma functions $\gamma(s) = \prod_{i\neq j} \Gamma(s+\alpha_i+\beta_i) \cdot \Gamma(s)^{1-n}$ will vanish at $s = 1-m$. So only $G_\alpha^n(s)$ has a nontrivial contribution to $G_\alpha(s)$ at $s = 1-m$. In the case $j = n$, the factor $(1-t)^{s+\alpha_n-1}$ in $I_{\alpha,\beta,n}(s)$ has no further pole at $t = 1$, since $\alpha_n \geq m$. So we can replace $I_{\alpha,\beta,n}(s)$ by

$$\begin{aligned}\frac{1}{\Gamma((n-1)s + |\alpha|_n + |\beta|)} &\int_0^1 t^{(n-1)s+|\alpha|_n+|\beta|-1}(1-t)^{s+\alpha_n-1}dt \\ &= \frac{\Gamma(s+\alpha_n)}{\Gamma(ns+|\alpha|+|\beta|)}.\end{aligned}$$

In fact, the difference between the two integrals can be eliminated by the zero at $s = 1-m$ of $\Gamma(s)^{1-n}$. Consequently, we have

$$G_\alpha^n(1-m) = \sum_{|\beta|=0}^{\infty} C_\beta A_{\alpha,\beta}(1-m)$$

where

$$A_{\alpha,\beta}(s) = \frac{\Gamma(s+\alpha_n)\prod_{i=1}^{n-1}\Gamma(s+\alpha_i+\beta_i)}{\Gamma(s)^{n-1}\Gamma(ns+|\alpha|+|\beta|)}.$$

Note that $A_{\alpha,\beta}(s)$ is an analytic function of s for Re $s > 0$ and it has analytic continuation in the whole complex s-plane. Also $A_{\alpha,\beta}(1-m) = 0$ unless $\alpha_i + \beta_i \leq m-1$ for $1 \leq i \leq n-1$. For such exceptional cases, one has

$$A_{\alpha,\beta}(1-m) = (-1)^{|\alpha|_n+|\beta|}\frac{(\alpha_n-m)!}{(\beta+q-1)!}\prod_{i=1}^{n-1}\frac{(m-1)!}{(m-1-\alpha_i-\beta_i)!}.$$

It follows that

$$\begin{aligned}G_\alpha(1-m) &= G_\alpha^n(1-m) \\ &= \sum_{0\leq\beta_i\leq m-1-\alpha_i}(-1)^{|\alpha|_n+|\beta|}C_\beta\frac{(\alpha_n-m)!}{(\beta+q-1)!}\prod_{i=1}^{n-1}\frac{(m-1)!}{(m-1-\alpha_i-\beta_i)!} \\ &= (-1)^{(n-1)(m-1)}[(m-1)!]^{n-1} \times \text{ the coefficient of } \prod_{i=1}^{n-1} u_i^{m-1-\alpha_i} \text{ in the} \\ &\quad \text{Taylor expansion of the function } (1-u_1-\ldots-u_{n-1})^{\alpha_n-m}\prod_{j=1}^{q} L_j(u)^{-1}.\end{aligned}$$

With Lemma 3 and 4, we are able to find the special value $G(1-m)$ in terms of coefficients of functions

$$u_j^{-m}\prod_{i=q+1}^{r} L_i(u)^{p_i-1}/\prod_{i=1}^{q} L_i(u) \qquad (j = 1, \ldots, n).$$

This is equivalent to compute Shintani's coefficients $C(s,q,A)$ $(k = 1,\dots,n)$, which are defined in terms of coefficients of the function

$$\prod_{i=q+1}^{r} L_i(u)^{p_i-1} / \prod_{i=1}^{q} L_i(u) \mid_{u_j=1} .$$

Precisely, we have the following.

Lemma 5. *Let*

$$f_p(u) = \prod_{i=q+1}^{r} L_i(u)^{p_i-1} / \prod_{i=1}^{q} L_i(u)$$

with $|p| = n(m-1)+r$. *Then the coefficient of* $(u_1 \dots u_{n-1})^{m-1}$ *in the Taylor expansion of* $f_p(u_1,\dots,u_{n-1},1)$ *at* $u_1 = \dots = u_{n-1} = 0$ *is equal to the coefficient of* $(u_1,\dots,u_{n-1})^{m-1}$ *in the Taylor expansion of*

$$g(u_1,\dots,u_{n-1}) = (1-u_1-\dots-u_{n-1})^{-m} f(u_1,\dots,u_{n-1},1-u_1-\dots-u_{n-1})$$

at $u_1 = \dots = u_{n-1} = 0$.

Proof. For Re $s > 0$, consider the integral

$$G(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \dots u_n)^{s-1} f_p(u) dU.$$

For $1 \le j \le n$, we let F_j be the subset of E^n defined by

$$F_j = \{u = (u_1,\dots,u_n) \in E^n \mid u_i \le u_j \text{ for all } i \ne j\}.$$

Then $F_1,\dots,F_n$ are pairwise non-overlapping and $E^n = \bigcup_{j=1}^n F_j$. Let

$$G_j(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{F_j} (u_1 \dots u_n)^{s-1} f_p(u) dU.$$

Now we focus our attention on the case $j = n$. F_n contains the vertex $V_n = (0,\dots,0,1)$ of E_n, so by Lemma 4, we have

$$\begin{aligned} G_n(1-m) = {} & (-1)^{(n-1)(m-1)} [(m-1)!]^{n-1} \\ & \times \text{the coefficient of } (u_1 \dots u_{n-1})^{m-1} \text{ in the Taylor expansion of} \\ & g(u_1,\dots,u_{n-1}) \text{ at } \ u_1 = \dots = u_{n-1} = 0. \end{aligned}$$

We want to prove $G_n(1-m)$ can also be given by the coefficient of $f_p(u_1,\dots,u_{n-1},1)$. With the change of variables: $(u_1,\dots,u_n) = u(v_1,\dots,v_{n-1},1)$, we then have

$$G_n(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{1/n}^{1} u^{ns+n(m-1)-2} du \int_{C(u)} (v_1 \dots v_{n-1})^{s-1} f_p(v_1,\dots,v_{n-1}) dV$$

where $C(u)$ is the simplex in $\mathbb{R}^{n-1}$ defined by

$$C(u) = \{(v_1,\dots,v_{n-1}) \mid 0 \le v_i \le 1, v_1 + \dots + v_{n-1} = (1-u)/u\}.$$

Observe that the singularities of $G_n(s)$ arise from the integration around $u = 1$. In order to get $G_n(1-m)$, we consider

$$G_{n,q}(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{1-\delta}^{1} u^{ns+n(m-1)-2} du \\ \times \int_{C(u)} (v_1 \dots v_{n-1})^{s-1} f_p(v_1, \dots, v_{n-1}, 1) dV$$

instead for sufficiently small $\delta > 0$. For $v_1 + \ldots + v_{n-1} < \frac{\delta}{1-\delta}$, suppose that

$$f_p(v_1, \dots, v_{n-1}, 1) = \sum_{|\alpha|=0}^{\infty} b_\alpha v_1^{\alpha_1} \dots v_{n-1}^{\alpha_{n-1}}.$$

With the change of variable $v_i = \dfrac{1-u}{u} v_i'$ $(i = 1, \dots, n-1)$ and a term by term integration, it yields for Re $s > 0$ that

$$G_{n,\delta}(s) = \sum_{|\alpha|=0}^{\infty} b_\alpha \frac{\Gamma(s+\alpha_1) \cdots \Gamma(s+\alpha_{n-1})}{\Gamma((n-1)s + |\alpha|)\Gamma(s)^{n-1}} \\ \times \int_{1-\delta}^{1} (1-u)^{(n-1)s+|\alpha|-1} u^{s+n(m-1)-|\alpha|-1} du.$$

If $|\alpha| > (n-1)(m-1)$, the integrals appear in $G_{n,\delta}(s)$ are convergent for $s = 1-m$ while the product of gamma functions vanishes at $s = 1-m$. On the other hand, if $|\alpha| < (n-1)(m-1)$, then $u^{s+n(m-1)-|\alpha|-1}$ is a regular function of u at $s = 1-m$ and $u = 0$, so one can replace the integral in $G_{n,\delta}(s)$ by

$$\int_0^1 (1-u)^{(n-1)s+|\alpha|-1} u^{s+n(m-1)-|\alpha|-1} du \\ = \frac{\Gamma((n-1)s + |\alpha|)\Gamma(s + n(m-1) - |\alpha|)}{\Gamma(ns + n(m-1))}.$$

Hence the contribution to $G_{n,\delta}(1-m)$ from those terms with $|\alpha| < (n-1)(m-1)$ is given by

$$\sum_{|\alpha|<(n-1)(m-1)} b_\alpha a_\alpha(1-m),$$

where

$$a_\alpha(s) = \frac{\Gamma(s+\alpha_1) \dots \Gamma(s+\alpha_{n-1})\Gamma(s + n(m-1) - |\alpha|)}{\Gamma(s)^{n-1}\Gamma(ns + n(m-1))}.$$

However $a_\alpha(1-m) = 0$. It remains to consider the cases $|\alpha| = (n-1)(m-1)$. In these case $\dfrac{\Gamma(s+\alpha_1) \dots \Gamma(s+\alpha_{n-1})}{\Gamma(s)^{n-1}}$ vanishes at $s = 1-m$ unless $\alpha_1 = \cdots = \alpha_{n-1} = m-1$. Also the special value at $s = 1-m$ of the integral

$$\frac{1}{\Gamma((n-1)s + (n-1)(m-1))} \int_{1-\delta}^{1} (1-u)^{(n-1)(s+m-1)-1} u^{s+m-2} du$$

is 1 by the residue theorem. Consequently we have

$$G_{n,\delta}(1-m) = (-1)^{(n-1)(m-1)}[(m-1)!]^{n-1} b_{m-1,\dots,m-1}.$$

This proves our assertion. □

Integrals of types $F(s)$ and $G(s)$ are considered in [4] with the notation $H_\beta(s)$:

$$H_\beta(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \dots u_n)^{s-1} \prod_{j=1}^{r} [L_j(u)]^{\beta_j - 1} dU,$$

where $\beta = (\beta_1, \dots, \beta_r)$ is an r-tuple of non-negative integers with $|\beta| = n(m-1)+r$. Also the special value $H_\beta(1-m)$ is given explicitly there. Now we are ready to prove the theorem.

Proof of the theorem. For Re $s > 0$, one has

$$\Gamma(s)^n \prod_{k=1}^{n} [a_{1k}(n_1 + x_1) + \dots + a_{rk}(n_r + x_r)]^{-s} = \int_0^\infty \cdots \int_0^\infty (y_1 \dots y_n)^{s-1}$$
$$\times \exp[-\sum_{k=1}^{n} [a_{1k}(n_1 + x_1) + \dots + a_{rk}(n_r + x_r)] y_k] dy_1 \dots dy_n.$$

Therefore for Re $s > r/n$, one has

$$\zeta(s, A, x)\Gamma(s)^n = \int_0^\infty \dots \int_0^\infty (y_1 \dots y_n)^{s-1} \prod_{j=1}^{r} \frac{\exp[(1-x_j)L_j(y)]}{\exp[L_j(y)] - 1} dy_1 \dots dy_n.$$

With the change of variables

$$(y_1, \dots, y_n) = t(u_1, \dots, u_n), \qquad t > 0, (u_1, \dots, u_n) \in E^n,$$

the integral becomes

$$\zeta(s, A, x)\Gamma(s)^n = \int_0^\infty t^{ns-1} dt \int_{E^n} (u_1, \dots, u_n)^{s-1} \prod_{j=1}^{r} \frac{\exp[t(1-x_j)L_j(u)]}{\exp[tL_j(u)] - 1} dU.$$

Rewrite the above formula as

$$\zeta(s, A, x)\Gamma(s) = \int_0^\infty t^{ns-r-1} I_n(s, t, x) dt,$$

where

$$I_n(s, t, x) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \dots u_n)^{s-1} \prod_{j=1}^{r} \frac{t \exp[t(1-x_j)L_j(u)]}{\exp[tL_j(u)] - 1} dU.$$

$I_n(s, t, x)$ is an analytic function of s for Re $s > 0$ and has an analytic continuation since it is an integration of a continuous function over E_n with respect to the measure

$$\frac{1}{\Gamma(s)^{n-1}} (u_1 \dots u_n)^{s-1} dU.$$

On the other hand, $I_n(s,t,x)$ is a rapidly decreasing function in t, and for sufficiently small t, it has the power series expansion

$$\sum_{|\beta|=0}^{\infty} t^{|\beta|} \frac{B_{\beta_1}(1-x_1)\dots B_{\beta_r}(1-x_r)}{\beta_1!\dots\beta_r!} H_\beta(s).$$

Now the special value of $\zeta(s,A,x)$ at $s=1-m$ can be determined by comparing the residues at $s=1-m$ in both sides of equation

$$\zeta(s,A,x)\Gamma(s) = \int_0^\infty t^{ns-r-1} I_n(s,t,x)dt.$$

The result is

$$\zeta(1-m,A,x) = \frac{(-1)^{m-1}(m-1)!}{n} \sum_{|\beta|=n(m-1)+r} \frac{(-1)^\beta B_{\beta_1}(x_1)\cdots B_{\beta_r}(x_r)}{\beta_1!\cdots\beta_r!} H_\beta(1-m).$$

For $\beta_1 \geq 1,\cdots,\beta_r \geq 1$, $\prod_{j=1}^r [L_j(u)]^{\beta_j-1}$ is a homogeneous polynomial in $u_1,\cdots,u_n$ of degree $n(m-1)$. By Proposition 4 of [4], we have

$$\frac{(-1)^{m-1}(m-1)!}{n} \sum_{\substack{|\beta|=n(m-1)+r \\ \beta_1\geq 1,\cdots,\beta_r\geq 1}} \frac{(-1)^\beta B_{\beta_1}(x_1)\cdots B_{\beta_r}(x_r)}{\beta_1!\cdots\beta_r!} H_\beta(1-m) = J_I[P^{m-1}(u)],$$

where $I=\{1,2,\cdots,r\}$ and $P(u)=\prod_{j=1}^n (a_{1j}u_1+\cdots+a_{rj}u_r)$.

If $\beta_1=\cdots=\beta_q=0$, $\beta_{q+1}\geq 1,\cdots,\beta_r\geq 1$, we let $S=\{q+1,\cdots,r\}$. By Propositions 5 and 6 of [4], we get

$$\frac{(-1)^{m-1}(m-1)!}{n} \sum_{\substack{|\beta|=n(m-1)+r \\ \beta_1=\cdots=\beta_q=0,\beta_{q+1}\geq 1,\cdots,\beta_r\geq 1}} \frac{(-1)^\beta B_{\beta_{q+1}}(x_{q+1})\cdots B_{\beta_r}(x_r)}{\beta_{q+1}!\cdots\beta_r!} H_\beta(1-m)$$
$$= \frac{1}{n} J_S[\sum_{j=1}^n \int_{\Delta_{j,S}(u)} P^{m-1}(u)du_1\cdots du_q],$$

where $\Delta_{j,S}(u)$ is the simplex in $\mathbb{R}^q$ defined by

$$\Delta_{j,S}(u): \begin{cases} (u_1,\cdots,u_q)\in\mathbb{R}^q, \\ u_1\leq 0,\cdots,u_q\leq 0, \quad L_j(u)=a_{1j}u_1+\cdots+a_{rj}u_r\geq 0. \end{cases}$$

We now compare our formula with the formula obtained by Shintani in [6]. First we consider the coefficient $C(A,p)$ that appears in the first part of Shintani's formula for $\zeta(1-m,A,x)$. According to his result, it is the coefficient of $(t_1\dots t_n)^{m-1}$ in the polynomial function $\prod_{j=1}^r L_j(t)^{p_j-1}$. As we obtained in Lemma 1, it is the special value of the integral

$$F(s) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1\dots u_n)^{s-1} \prod_{j=1}^r L_j(u)^{p_j-1} dU$$

up to a constant factor. Indeed, we have

$$nC(A,p) = (-1)^{(n-1)(m-1)}[(m-1)!]^{n-1}F(1-m).$$

However we have the further result that $C(A,p)$ is nothing but the coefficient of

$$\frac{(-1)^{(n(m-1))}}{(p_1-1)!\dots(p_r-1)!}u_1^{p_1-1}\dots u_r^{p_r-1}$$

in the polynomial function

$$\prod_{k=1}^{n}[L_k^*(u)]^{m-1} = \prod_{k=1}^{n}[a_{1k}u_1+\dots+a_{rk}u_r]^{m-1}.$$

In fact, if

$$\prod_{k=1}^{n}[L_k^*(u)]^{m-1} = \sum_{|\alpha|=n(m-1)} C_\alpha u_1^{\alpha_1}\dots u_r^{\alpha_r},$$

then

$$\sum_{|p|=n(m-1)+r}\frac{B_{p_1}(x_1)\dots B_{p_r}(x_r)}{p_1!\dots p_r!}C(A,p)$$
$$= \sum_{|\alpha|=n(m-1)} C_\alpha(-1)^{n(m-1)}\frac{B_{\alpha_1+1}(x_1)\dots B_{\alpha_r+1}(x_r)}{(\alpha_1+1)\dots(\alpha_r+1)}.$$

Secondly, consider the coefficient $C(S,q,A)^{(k)}(k=1,\dots,n)$. It is the coefficient of $(t_1\dots t_{k-1}t_{k+1}\dots t_n)^{m-1}$ in the Taylor expansion of the function

$$\prod_{j\in S}L_j(t)^{q(j)-1}/\prod_{j\notin S}L_j(t)\mid_{t_k=1}.$$

Again we rewrite such a coefficient as the special value of the integral

$$G(s) = \frac{1}{\Gamma(s)}\int_{E^n}(u_1\dots u_n)^{s-1}\prod_{j\in S}L_j(u)^{q(j)-1}/\prod_{j\notin S}L_j(u)dU.$$

The special value of $G(s)$ at $1-m$ can be obtained by computing the coefficient of certain functions expanded at vertices of E^n. The difficulty in such an algorithm is the discovery that these coefficients are also coefficients of polynomials which can be obtained from $\prod_{k=1}^{n}[L_k^*(u)]^{m-1}$ in an elementary way. For example, for the case $S=\{q+1,\dots,r\}$, the polynomial function is given by

$$\frac{1}{n}\sum_{j=1}^{n}\int_{\Delta_j(u)}\prod_{k=1}^{n}[L_k^*(u)]^{m-1}du_1\dots du_q,$$

where $\Delta_j(u)$ is the simplex in $\mathbb{R}^q$ defined by

$$\Delta_j(u): u_1\le 0,\dots,u_q\le 0,\qquad L_j^*(u)\ge 0.$$

3. Identities among Bernoulli polynomials

Here we discuss two kinds of identities among Bernoulli polynomials.

Proposition 2. *For a positive integer $n \geq 2$, one has*

$$\sum_{k=1}^{2n-1} \frac{(2n)!}{k!(2n-k)!} B_k(\alpha) B_{2n-k}(\beta)$$
$$= -[(2n-1)B_{2n}(\alpha+\beta) + 2n(1-\alpha-\beta)B_{2n-1}(\alpha+\beta) + B_{2n}(\alpha) + B_{2n}(\beta)].$$

Proof. For the time being, we assume that α and β are positive numbers. For $\operatorname{Re} s > 1$, consider the identity of zeta functions

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [(n_1+\alpha)+(n_2+\beta)]^{-s} = \sum_{k=0}^{\infty} \frac{k+1}{k+\alpha+\beta}$$
$$= \zeta(s-1;\alpha+\beta) + (1-\alpha-\beta)\zeta(s;\alpha+\beta).$$

From the special value at $s = 2-2n$, we get

$$\sum_{k=1}^{2n-1} \frac{(2n-2)!}{k!(2n-k)!} B_k(\alpha) B_{2n-k}(\beta) + \frac{1}{(2n-1)2n} B_{2n}(\alpha) + \frac{1}{(2n-1)2n} B_{2n}(\beta)$$
$$= -\frac{1}{2n} B_{2n}(\alpha+\beta) - \frac{1-\alpha-\beta}{2n-1} B_{2n-1}(\alpha+\beta).$$

Multiplying both sides by $2n(2n-1)$, we get our assertion for $\alpha > 0$ and $\beta > 0$. Both sides of the identity are polynomial functions in α and β; they are equal for $\alpha > 0$ and $\beta > 0$. It follows that they are equal for all α and β. □

Proposition 3. *For a positive integer $n > 1$, one has*

$$\sum_{k=1}^{n-1} \frac{(2n)!}{(2k)!(2n-2k)!} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k}$$
$$= -[(2n-1)2^{2n} + 6^{2n} + 4^{2n}]B_{2n} + (16n)6^{2n-2} B_{2n}(\frac{1}{3}).$$

Proof. Consider the zeta function

$$Z(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (4n_1+6n_2)^{-s} + (4^{-s}+6^{-s})\zeta(s)$$

which is equal to the Dirichlet series

$$\sum_{k=1}^{\infty} a(k) k^{-s},$$

where $a(k)$ is the number of non-negative integral solutions of the equation $4x+6y = k$. Obviously, $a(k) = 0$ if k is odd. For even k, we have

$$a(k) = \frac{k+5}{12} + \frac{1}{4}\delta_1(k) + \frac{1}{3}\delta_2(k),$$

with

$$\delta_1(k) = \begin{cases} 1 & \text{if } k \equiv 0 (\text{mod } 4), \\ -1 & \text{if } k \equiv 2 (\text{mod } 4), \end{cases} \qquad \delta_2(k) = \begin{cases} 1 & \text{if } k \equiv 0 (\text{mod } 6), \\ -1 & \text{if } k \equiv 2 (\text{mod } 6), \\ 0 & \text{if } k \equiv 4 (\text{mod } 6). \end{cases}$$

Consequently, from two expressions of $Z(2-2n)$, we get

$$\sum_{k=1}^{n-1} \frac{(2n)!}{(2k)!(2n-2k)!} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k}$$
$$= -[(2n-1)2^{2n} + 6^{2n} + 4^{2n}] B_{2n} + (3n) 4^{2n-1} B_{2n-1}(\frac{1}{2}) + (16n) 6^{2n-2} B_{2n}(\frac{1}{3}).$$

Note that $B_{2n-1}(\frac{1}{2}) = 0$. So we have the desired identity. □

Remark. $a(k)$ is just the dimension of the space of modular forms of weight k of one variable in the upper half plane.

4. Generalized Bernoulli numbers

Let χ be a Dirichlet character with conductor N. The L-series defined by

$$L(s;\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \qquad \text{Re } s > 1,$$

is a holomorphic function of s for Re $s > 1$. Also it has an analytic continuation which is holomorphic in the whole complex plane except for a possible simple pole at $s = 1$ when χ is the trivial character χ_0 defined by

$$\chi_0(n) = 1 \quad \text{if } (n, N) = 1.$$

Especially, the special value of $L(s;\chi)$ at a non-positive integer $-m$ is given by

$$L(-m;\chi) = -\frac{B_{m+1,\chi}}{m+1},$$

where the *generalized Bernoulli number* $B_{m,\chi}$ is defined by

$$(\sum_{j=1}^{N} \chi(j) e^{jt}) t (e^{Nt} - 1)^{-1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_{m,\chi}, \qquad |t| < \frac{2\pi}{N}.$$

See also [9] for the definition. From the definition, we see immediately that

$$B_{m+1,\chi} = N^m \sum_{j=1}^{N} \chi(j) B_{m+1}(\frac{j}{N}).$$

In the following, we compute the special values of an L-series by the result of the author in [3].

Proposition 4. *Suppose that χ is not a trivial character and*

$$P(x) = \prod_{j=1}^{k}(a_j x + \delta_j), \qquad a_j > 0,\ \mathrm{Re}\ \delta_j/a_j > -1,$$

is a polynomial with real coefficients. Define

$$L(s; P, \chi) = \sum_{n=1}^{\infty} \chi(n) P(n)^{-s}, \qquad \mathrm{Re}\ s > 1/k.$$

Then for any integer $m \geq 0$,

$$L(-m; P, \chi) = \sum_{r=0}^{km} a_r (-\frac{B_{r+1,\chi}}{r+1}),$$

where a_r is the coefficient of x^r in $P^m(x)$.

Proof. Replace $P(x)$ by $P^m(x)$ if necessary; we therefore need only consider the cases $m = 0$ and $m = 1$. Rewrite the zeta function as

$$L(s; P, \chi) = \sum_{j=1}^{N} \chi(j) \sum_{n=0}^{\infty} P(j + nN)^{-s}.$$

Set

$$P(x) = \sum_{i=0}^{k} b_i x^i \quad \text{and} \quad Z_{P,j}(s) = \sum_{n=0}^{\infty} P(j + nN)^{-s}.$$

Then by the main theorem of [3], we have

$$Z_{P,j}(-1) = P(j) + J[P(j + Nx)] + \frac{1}{k} \sum_{i=1}^{k} \int_0^{\mu_{ij}} P(j + Nx) dx,$$

where $\mu_{ij} = -(ja_i + \delta_j)/a_i N$ $(i = 1, \cdots, k)$ are zeros of $P(j + Nx)$. By a change of variable in the integral of $Z_{P,j}(-1)$, we get

$$\begin{aligned} Z_{P,j}(-1) &= P(j) + J[P(j + Nx)] + \frac{1}{kN} \sum_{i=1}^{k} \int_j^{-\delta_i/a_i} P(x) dx \\ &= P(j) + J[P(j + Nx)] + \frac{1}{kN} \sum_{i=1}^{k} \int_0^{-\delta_i/a_i} P(x) dx - \frac{1}{N} \int_0^{j} P(x) dx. \end{aligned}$$

A direct verification shows that

$$P(j) + J[P(j + Nx)] - \frac{1}{N} \int_0^{j} P(x) dx = -\sum_{i=0}^{k} b_i \frac{N_i}{i+1} B_{i+1}(\frac{j}{N}).$$

It follows that

$$\begin{aligned} L(-1;P,\chi) &= \sum_{j=1}^{N} \chi(j) Z_{P,j}(-1) \\ &= -\sum_{i=0}^{k} \frac{b_i}{i+1} \sum_{j=1}^{N} \chi(j) N^i B_{i+1}(\frac{j}{N}) + \frac{1}{kN} \sum_{j=1}^{N} \chi(j) \sum_{i=1}^{k} \int_0^{-\delta_j/a_i} P(x)dx \\ &= -\sum_{i=0}^{k} \frac{b_i}{i+1} B_{i+1,\chi}. \end{aligned}$$

In the same way, we can compute the special value $L(0;P,\chi)$. This proves our assertion. □

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