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ON THE POLES OF RANKIN-SELBERG CONVOLUTIONS OF MODULAR FORMS

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Abstract. The Rankin-Selberg convolution is usually normalized by the multiplication of a zeta factor. One naturally expects that the non-normalized convolution will have poles where the zeta factor has zeros, and that these poles will have the same order as the zeros of the zeta factor. However, this will only happen if the normalized convolution does not vanish at the zeros of the zeta factor. In this paper, we prove that given any point inside the critical strip, which is not equal to $\frac{1}{2}$ and is not a zero of the Riemann zeta function, there exist infinitely many cusp forms whose normalized convolutions do not vanish at that point.

Introduction

Assume that k is a positive even integer. Let Γ be the modular group $SL_2(\mathbb{Z})$. Denote by θ_k the dimension of the space $S_k(\Gamma)$ of cusp forms. Assume that functions

$$f_{jk}(z) = \sum_{n=1}^{\infty} \psi_{jk}(n) e(nz), \qquad j = 1, 2, \cdots, \theta_k,$$

form an orthogonal base of Hecke eigenforms of $S_k(\Gamma)$ such that the norm of $f_{jk}(z)$ is one under the Petersson inner product of the space. Put

$$R_{jk}(s) = \sum_{n=1}^{\infty} |\psi_{jk}(n)|^2 n^{-s-k+1}.$$

R. A. Rankin expressed in 1939 [7] that $R_{jk}(s)$ may have poles at the complex zeros of $\zeta(2s)$.

In this paper, we prove the following theorem:

Main Theorem. *If $\rho \neq \frac{1}{2}$ is a complex number with $0 < Re\,\rho < 1$, which is not a zero of the Riemann zeta function $\zeta(s)$, then infinitely many cusp forms $f_{jk}(z)$ exist such that $\zeta(2s)R_{jk}(s)$ do not vanish at the point $s = \rho$.*

This paper is divided into three sections. In section 1, we recall some well-known results. The main theorem is proved in section 2. An estimation of the term Π left from section 2 is given in section 3. All notations are defined as they first appear.

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The proof of the main theorem goes briefly as follows: Let η be a zero of the Riemann zeta function on the critical line. Assume that $\rho \neq \frac{1}{2}$ is a complex number with $0 < Re\,\rho \leq \frac{1}{2}$, which is not a zero of the Riemann zeta function. Put

$$R_K(s) = \sum b_k \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{1\leq j\leq \theta_k} R_{jk}(s)$$

where b_k are integers satisfying a number of constraints and where the summation is taken over all positive even integers k. Let $\delta = 1+\nu$ with ν a sufficiently small positive number. Consider the integral

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{(z-1)\pi^{-z}\Gamma(z)\zeta(2z)}{(z-\eta)(z-\rho)} R_K(z) N^z\,dz. \tag{0.1}$$

Argue by contradiction, assuming that $\zeta(2s)R_K(s)$ vanishes at the point ρ. Then the integrand in (0.1) is analytic in the region $-\nu < Re\,z < \delta$ except a simple pole at the origin. Moving the line of integration to $Re\,z = -\nu$, we find that the integral (0.1) is $\ll K^{A+1}$ (Theorem 2.1). On the other hand, it follows from the Petersson trace formula that the integral (0.1) is equal to

$$\begin{aligned}(\frac{K}{2})^A \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{(z-1)\pi^{-z}\Gamma(z)\zeta(2z)}{(z-\eta)(z-\rho)}\zeta(z)N^z dz\\ + \sum_{c=1}^{\infty} \frac{S(n,n;c)}{c} E(\frac{4n\pi}{c}) \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{(z-1)\pi^{-z}\Gamma(z)\zeta(2z)}{(z-\eta)(z-\rho)}(\frac{N}{n})^z dz\end{aligned} \tag{0.2}$$

where $S(n,n;c)$ are Kloosterman's sums and where $E(y) = \sum 2\pi i^k b_k J_{k-1}(y)$. By computation we find that the first term in (0.2) is equal to (Theorem 2.2)

$$\frac{\zeta(\frac{1}{2})2^{-A}}{(1-2\eta)(2\rho-1)} K^{A+1+\frac{\nu}{2}} + \frac{\rho-1}{\rho-\eta}\pi^{-\rho}2^{-A}\Gamma(\rho)\zeta(\rho)\zeta(2\rho)K^{A+2\rho+\rho\nu} + O(K^A). \tag{0.3}$$

After a long tedious computation, we find that the second term in (0.2) is equal to (Theorem 2.6)

$$\begin{aligned}\frac{\zeta(\frac{1}{2})2^{-A}}{(1-2\eta)(1-2\rho)} K^{A+1+\frac{\nu}{2}} + \{\frac{i}{4}\frac{\rho-1}{\rho-\eta}2^{-A}(AL)^{1-2\rho}8^{\rho}\pi^{-\frac{1}{2}}\Gamma(\rho)\Gamma(\rho-\frac{1}{2})\\ \times \zeta(\rho)\zeta(2\rho-1)[e(\frac{\rho-1}{4}) - e(-\frac{\rho-1}{4})] + o_K(1)\}K^{A+1+\rho\nu}\end{aligned} \tag{0.4}$$

when $K \to \infty$. It has been shown that the sum of (0.3) and (0.4) should be $\ll K^{A+1}$. This derives a contradiction. The result of the main theorem then follows. When $\frac{1}{2} < Re\,\rho < 1$, the result of the main theorem follows from the functional identity.

A significant improvement of the main theorem is made following a suggestion of the referee. The author wishes to thank the referee for his/her valuable suggestions, and he also wishes to thank Freydoon Shahidi for helpful comments. For related results, see Deshouillers and Iwaniec [3], Luo [5] and Phillips and Sarnak [6].

1. Preliminary results

Lemma 1.1. [1] *The inequality*

$$|\psi_{jk}(n)| \leq |\psi_{jk}(1)| d(n) n^{\frac{k-1}{2}}$$

holds for all positive integers n, where $d(n)$ is the number of divisors of n.

Lemma 1.2. [7] *Let*

$$r(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\zeta(2s-2k+2)R_{jk}(s-k+1).$$

Then the identity

$$r(s+k-1) = r(k-s)$$

holds for all complex numbers s. Furthermore, $R_{jk}(s)$ has a simple pole at $s = 1$ with a residue $12(4\pi)^{k-1}/(k-1)!$.

Lemma 1.3. [8] *The function $\zeta(2s)R_{jk}(s)/\zeta(s)$ is an entire function.*

The Kloosterman sum $S(m, n; c)$ is defined by

$$S(m, n; c) = \sum_{d \pmod c}^{*} e\left(\frac{md + n\bar{d}}{c}\right),$$

where the sum is taken over a reduced set of residues modulo c and where $\bar{d}$ denotes the inverse of d modulo c.

Lemma 1.4. [4] [11] *The inequality*

$$|S(m, n; c)| \leq (m, n, c)^{\frac{1}{2}} \sqrt{c}\, d(c)$$

holds for all positive integers m, n and c.

The Bessel function of order $k-1$ is defined by

$$J_{k-1}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{z}{2})^{2m+k-1}}{m!(m+k-1)!}, \ -\pi < \arg z \leq \pi.$$

Lemma 1.5. *The inequality*

$$|J_n(x)| \leq \frac{1}{n!}\left(\frac{|x|}{2}\right)^n$$

holds for all real numbers x and for all positive integers n.

Proof. Since the identity

$$J_n(z) = \frac{(\frac{z}{2})^n}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{-itz}(1-t^2)^{n-\frac{1}{2}}\, dt$$

holds for all positive integers n, the stated inequality follows. □

The Bessel function of order $k-1$ can be written as

$$J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-1)\theta + ix\sin\theta}\, d\theta.$$

Lemma 1.6. [9] *We have*

$$\theta_k = \begin{cases} 0, & \text{for } k = 2; \\ [\frac{k}{12}] - 1, & \text{for } k > 2, k \equiv 2 \pmod{12}; \\ [\frac{k}{12}], & \text{for } k \not\equiv 2 \pmod{12}. \end{cases}$$

Lemma 1.7. [2, (4.4)] *The identity*

$$\frac{(k-2)!}{(4\pi)^{k-1}} \sum_{1\le j\le \theta_k} \frac{|\psi_{jk}(n)|^2}{n^{k-1}} = 1 + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(n,n;c)}{c} J_{k-1}(\frac{4n\pi}{c})$$

holds for all positive even integers k and for all positive integers n.

Lemma 1.8. [10] *Let $|t| \ge t_0$ for some positive number t_0. The inequality*

$$\zeta(\sigma + it) \ll |t|^{\frac{1-\sigma}{2}} \ln(1 + |t|)$$

holds uniformly for $0 \le \sigma \le 1$, and $\zeta(\frac{1}{2} + it) \ll |t|^{\frac{27}{164}}$. For $\sigma \ge -\epsilon$, we have $\zeta(\sigma + it) \ll |t|^{\frac{3}{2}+\epsilon}$.

The following lemma is a variant of Lemma 4.2 in [10].

Lemma 1.9. *Let $F(x)$ and $G(x)$ be real valued functions, which have continuous derivatives. Assume that $|F'(x)| \ge m$ for some positive number m when x belongs to (a, b). If $F'(x)$ is monotone on the interval, and if $G(x)$ is a finite product of monotone functions, then the inequality*

$$\int_a^b G(x)e^{iF(x)}dx \ll \frac{1}{m} \prod_{i=1}^{n} \sup_{x\in(a,b)} |G_i(x)|$$

holds, where $G(x) = G_1(x)\cdots G_n(x)$ with $G_i(x)$ being monotone functions.

Throughout this paper, the following notations are used. The constants implied by $\ll$ and O are absolute constants. A is a large positive even integer such that $A \ge \frac{2}{\nu}$, and L is a large positive integer depending on A. K is a sufficiently large positive even integer. Put $N = K^{2+\nu}$. Denote by b_k the number of solutions $(k_1, \cdots, k_A)$ of the equation

$$k = k_1 + \cdots + k_A,\ LK \le k_j < (L+1)K,\ k_j \text{ even integers.}$$

The number b_k is nonzero only if k is a positive even integer such that $ALK \le k < A(L+1)K$. The identity

$$\sum b_k = (\frac{K}{2})^A$$

holds, where the summation is taken over all positive even integers k.

Lemma 1.10. *The inequality*

$$E(y) \ll K^{A-3+3\nu}$$

holds when $0 < y \leq 4\pi K^{2-4\nu}$, *and the identity*

$$E(y) = \sqrt{2\pi i} y^{-\frac{1}{2}} \sum b_k \beta_k(y) + O(yK^{A-5+5\nu} + K^{A-3+\nu} + K^A/y)$$

holds when $y \geq 4\pi K^{2-14\nu}$, *where*

$$\beta_k(y) = \exp(-\frac{\pi i}{4})\alpha_k(y) - \exp(\frac{\pi i}{4})\alpha_k(-y)$$

with $\alpha_k(y) = \exp(iy + \frac{i(k-1)^2}{2y})$.

Proof. Write

$$E(y) = 2Im\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum b_k e^{i(k\theta - \theta - y\cos\theta)} d\theta\}, \tag{1.1}$$

where the summation is taken over all positive even integers k. The identity

$$\sum b_k e^{ik\theta} = e^{-iA\theta}(e^{iLK\theta} - e^{i(L+1)K\theta})^A (2i\sin\theta)^{-A}$$

holds. It follows that

$$\sum b_k e^{i(k\theta-\theta)} \ll \min\{K^A, \theta^{-A}\} \tag{1.2}$$

for $|\theta| \leq \frac{\pi}{2}$. The inequality

$$e^{-iy\cos\theta} - e^{-iy+\frac{1}{2}iy\theta^2} \ll \min\{1, y\theta^4\} \tag{1.3}$$

holds for all positive numbers y when $|\theta| \leq \frac{\pi}{2}$. By using (1.1)–(1.3), we find

$$E(y) - 2\,\mathrm{Im}\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum b_k e^{i(k\theta-\theta)-iy+\frac{1}{2}iy\theta^2} d\theta\} \ll yK^{A-5+5\nu} + K^{A-3+\nu} \tag{1.4}$$

for positive numbers y when the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of integration is divided into three subintervals with second one being $[-K^{-1+\nu}, K^{-1+\nu}]$. Since

$$\sum b_k \exp(ik\theta - i\theta) = e^{-i(A+1)\theta}(e^{iLK\theta} - e^{i(L+1)K\theta})^A (2i\theta)^{-A} + O(K^A\theta^2)$$

for $|\theta| \leq \frac{\pi}{2}$, and since

$$e^{\frac{1}{2}iy\theta^2} - \sum_{j=0}^{\frac{A}{2}-1} \frac{1}{j!}(\frac{1}{2}iy\theta^2)^j \ll K^{-2}$$

for $0 < y \le 4\pi K^{2-4\nu}$ and for $0 \le |\theta| \le K^{-1+\nu}$, we obtain

$$\begin{aligned}&\int_{-K^{-1+\nu}}^{K^{-1+\nu}} \sum b_k \exp(ik\theta - i\theta + \frac{1}{2}iy\theta^2)d\theta + O(K^{A-3+3\nu}) \\ &= \sum_{j=0}^{\frac{A}{2}-1} \frac{1}{j!}(\frac{1}{2}iy)^j \int_{-K^{-1+\nu}}^{K^{-1+\nu}} \frac{e^{-i(A+1)\theta}(e^{iLK\theta} - e^{i(L+1)K\theta})^A}{(2i)^A\theta^{A-2j}} d\theta.\end{aligned} \tag{1.5}$$

Consider θ as a complex variable. By using the residue theorem around the contour $C_R = \{\theta : -R \le \theta \le R\} \cup \{\theta : |\theta| = R, \text{ Im } \theta > 0\}$, we find that the right side of (1.5) is $\ll K^{A-3+3\nu}$ for $0 < y \le 4\pi K^{2-4\nu}$ when $R \to \infty$. By (1.2), the inequality

$$\int_{K^{-1+\nu}<|\theta|\le\frac{\pi}{2}} \sum b_k e^{i(k-1)\theta+\frac{1}{2}iy\theta^2} d\theta \ll K^{A-3+\nu}$$

holds. It follows from (1.4) that the inequality

$$E(y) \ll K^{A-3+3\nu}$$

holds for $0 < y \le 4\pi K^{2-4\nu}$.

The second assertion of the lemma follows from (1.4) and the identity

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(ik\theta - i\theta - iy + \frac{i}{2}y\theta^2)d\theta = \sqrt{2\pi}\alpha_k(-y)e^{\frac{\pi i}{4}}y^{-\frac{1}{2}} + O(\frac{1}{y})$$

for $y \ge 4\pi K^{2-14\nu}$. □

2. Proof of the Main Theorem

From now on, k is always meant to be an even integer between ALK and $A(L+1)K$. Define

$$\lambda(z) = \frac{(z-1)\pi^{-z}\Gamma(z)\zeta(2z)}{(z-\eta)(z-\rho)}. \tag{2.1}$$

Theorem 2.1. *If $\zeta(2s)R_K(s)$ vanishes at ρ, then*

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \lambda(z)R_K(z)N^z\, dz \ll K^{A+1}.$$

Proof. Since $\zeta(2z)R_{jk}(z)/\zeta(z)$ is an entire function, the left side of the stated inequality is equal to

$$\frac{1}{2\pi i}\int_{-\nu-i\infty}^{-\nu+i\infty} \lambda(z)R_K(z)N^z dz + \sum b_k \frac{\theta_k}{2\eta\rho} \tag{2.2}$$

by the assumption and Lemma 1.2. It follows from Lemma 1.2 that the first term of (2.2) is equal to

$$\sum b_k \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{z2^{2-4z}\pi^{1-3z}\Gamma(z+k-1)\Gamma(z)\zeta(2z)}{(1-z-\eta)(z-1+\rho)\Gamma(k-z)} R_k(z)N^{1-z}dz \tag{2.3}$$

where

$$R_k(z) = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{1 \le j \le \theta_k} R_{jk}(z).$$

By using Stirling's formula, we obtain

$$\Gamma(z+k-1)/\Gamma(k-z) \ll (k|z|)^{1+2\nu}.$$

By Lemma 1.1 and the inequality

$$\Gamma(\delta+it) \ll |t|^{\delta-\frac{1}{2}} \exp(-\frac{\pi}{2}|t|),$$

we find that (2.3) is

$$\ll N^{-\nu} \sum b_k k^{1+2\nu} \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{1 \le j \le \theta_k} |\psi_{jk}(1)|^2. \tag{2.4}$$

We have by Lemma 1.7

$$\frac{(k-2)!}{(4\pi)^{k-1}} \sum_{1 \le j \le \theta_k} |\psi_{jk}(1)|^2 = 1 + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(1,1;c)}{c} J_{k-1}(\frac{4\pi}{c}).$$

Since $S(1,1;c)$ is bounded by $\sqrt{c}d(c)$, and since $J_{k-1}(\frac{4\pi}{c}) \ll \frac{1}{c}$ by Lemma 1.5 when $k > 1$, we have

$$\sum_{c=1}^{\infty} \frac{S(1,1;c)}{c} J_{k-1}(\frac{4\pi}{c}) \ll 1.$$

It follows that (2.4) is $\ll K^{A+1}$. By Lemma 1.6 and (2.2) the stated inequality follows. □

Theorem 2.2. *Let $\Phi(x)$ be given by*

$$(\rho-\eta)\Phi(x) = \int_1^{\infty} \{(1-\eta)t^{\eta-1} + (\rho-1)t^{\rho-1)}\}\psi(\frac{t}{x})dt$$

where $\psi(x) = \sum_{n=1}^{\infty} \exp(-\pi n^2 x)$. If $\zeta(2s)R_K(s)$ vanishes at the point ρ, then

$$\begin{aligned} &\sum_{n=1}^{\infty} \Phi(\frac{N}{n}) \sum_{c=1}^{\infty} \frac{S(n,n;c)}{c} E(\frac{4n\pi}{c}) \\ &= \frac{\zeta(\frac{1}{2})(\frac{K}{2})^A}{(1-2\eta)(1-2\rho)} \sqrt{N} + \frac{1-\rho}{\rho-\eta} \pi^{-\rho} \Gamma(\rho)\zeta(\rho)\zeta(2\rho)(\frac{K}{2})^A N^{\rho} + O(K^{A+1}). \end{aligned} \tag{2.5}$$

Proof. By Theorem 2.1 and Lemma 1.7, the identity

$$\sum_{n=1}^{\infty} \{(\frac{K}{2})^A + \sum_{c=1}^{\infty} \frac{S(n,n;c)}{c} E(\frac{4n\pi}{c})\} \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \lambda(z)(\frac{N}{n})^z = O(K^{A+1})$$

holds. By the identity

$$\frac{(\rho-\eta)(z-1)}{(z-\eta)(z-\rho)} = \int_1^\infty \{(1-\eta)t^{\eta-1} + (\rho-1)t^{\rho-1}\}t^{-z}dt$$

for $Re\, z = \delta$, we find

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \lambda(z)x^z dz = \Phi(x).$$

It follows that

$$\sum_{n=1}^{\infty} \Phi(\frac{N}{n}) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \lambda(z)\zeta(z)N^z dz. \tag{2.6}$$

The right side of (2.6) is equal to

$$\frac{\zeta(\frac{1}{2})\sqrt{N}}{(1-2\eta)(2\rho-1)} + \frac{\rho-1}{\rho-\eta}\pi^{-\rho}\Gamma(\rho)\zeta(\rho)\zeta(2\rho)N^\rho - \frac{1}{4\eta\rho} + O(N^{-\nu}),$$

and then the stated identity follows. □

The second term on the left side of (2.5) will now be estimated.

Lemma 2.3. *Let*

$$\Omega(n,c,k) = \Phi(\frac{N}{n})J_{k-1}(\frac{4n\pi}{c})\frac{S(n,n;c)}{c}.$$

Then the inequalities

$$\sum_{n\ge K^{2+2\nu}} \sum_{c=1}^{\infty} \Omega(n,c,k) \ll K^{-3}; \tag{2.7}$$

$$\sum_{n<K^{2+2\nu}} \sum_{c\ge K^{1+2\nu}} \Omega(n,c,k) \ll K^{-A}, \tag{2.8}$$

and

$$\sum_{n\le K} \sum_{c<K^{1+2\nu}} \Omega(n,c,k) \ll K^{-A} \tag{2.9}$$

hold.

Proof. By using the identity

$$2\pi(k-1)J_{k-1}(x) = x\int_{-\pi}^{\pi} \cos\theta e^{-i(k-1)\theta + ix\sin\theta}d\theta,$$

we find that $J_{k-1}(\frac{4n\pi}{c}) \ll \frac{n}{c}$. Since the inequality

$$\psi(\frac{n}{N}t) \le \exp(-\frac{n}{N})\psi(\frac{t}{2})$$

holds for $n \ge N$ and for $t \ge 1$, $\Phi(\frac{N}{n}) \ll \exp(-\frac{n}{N})$ when $n \ge N$. It follows from Lemma 1.4 that the left side of (2.7) is

$$\ll \sum_{n\ge K^{2+2\nu}} n^{\frac{3}{2}}e^{-\frac{n}{N}} \sum_{c=1}^{\infty} \frac{\sqrt{c}d(c)}{c^2} \ll K^{-3}.$$

We have $\Phi(\frac{N}{n}) \ll \sqrt{N}\ln K$ for all positive integers n. It follows from Lemma 1.5 and Stirling's formula that $J_{k-1}(\frac{4n\pi}{c}) \ll \frac{n}{c}K^{-2A}$ for $n < K^{2+2\nu}$ and for $c \ge K^{1+2\nu}$. The left of (2.8) is then $\ll K^{-A}$.

Similarly, we can show that the left side of (2.9) is $\ll K^{-2A}$. □

Lemma 2.4. *Let*

$$\Omega(n,c)=\Phi(\frac{N}{n})E(\frac{4n\pi}{c})\frac{S(n,n;c)}{c}.$$

Then the inequalities

$$\sum_{K<n\le K^{2-4\nu}}\ \sum_{c<K^{1+2\nu}}\Omega(n,c)\ll K^{A+1} \tag{2.10}$$

and

$$\sum_{K^{2-4\nu}<n<K^{2+2\nu}}\ \sum_{K^{10\nu}\le c<K^{1+2\nu}}\Omega(n,c)\ll K^{A+1} \tag{2.11}$$

hold.

Proof. Since $\Phi(\frac{N}{n})\ll K^{\frac{1}{2}+\nu}$ for $n>K$, it follows from Lemma 1.10 and Lemma 1.4 that the left side of (2.10) is

$$\ll\sum_{K<n\le K^{2-4\nu}}K^{\frac{1}{2}+\nu}\sum_{c<K^{1+2\nu}}d(c)K^{A-3+3\nu}\ll K^{A+\frac{1}{2}+3\nu}.$$

For the second inequality (2.11), $\Phi(\frac{N}{n})\ll K^{3\nu}$ when $n>K^{2-4\nu}$. It follows from Lemma 1.10 that the left side of (2.11) is $\ll K^{A+11\nu}$. □

Theorem 2.5. *Let*

$$\Pi=\frac{i}{\sqrt{2}}\sum b_k\sum_{K^{2-4\nu}<n<K^{2+2\nu}}\frac{1}{\sqrt{n}}\Phi(\frac{N}{n})\sum_{c<K^{10\nu}}\frac{S(n,n;c)}{\sqrt{c}}\beta_k(\frac{4n\pi}{c}).$$

Then the identity

$$\sum_{n=1}^{\infty}\Phi(\frac{N}{n})\sum_{c=1}^{\infty}\frac{S(n,n;c)}{c}E(\frac{4n\pi}{c})=\Pi+O(K^{A+1})$$

holds.

Proof. By Lemma 1.10, the identity

$$\sqrt{2}E(\frac{4n\pi}{c})=i(\frac{c}{n})^{\frac{1}{2}}\sum b_k\beta_k(\frac{4n\pi}{c})+O(K^{A-2+14\nu})$$

holds for $c<K^{10\nu}$ when $K^{2-4\nu}<n<K^{2+2\nu}$. The theorem then follows from Lemma 2.3 and Lemma 2.4. □

Proof of the Main Theorem. If $\zeta(2s)R_K(s)$ vanishes at the point ρ, it follows from Theorem 2.2, Theorem 2.5 and the following Theorem 2.6 that

$$\frac{1-\rho}{\rho-\eta}\pi^{-\rho}\Gamma(\rho)\zeta(\rho)\zeta(2\rho)(\frac{K}{2})^{A}N^{\rho}+O(K^{A+1})=\{\frac{i}{4}\frac{\rho-1}{\rho-\eta}2^{-A}(AL)^{1-2\rho}8^{\rho}\pi^{-\frac{1}{2}}$$
$$\times\Gamma(\rho)\Gamma(\rho-\frac{1}{2})\zeta(\rho)\zeta(2\rho-1)\varphi(\rho-\frac{1}{2})+o_K(1)\}K^{A+1+\rho\nu}.$$

This identity derives a contradiction. Hence a cusp form $f_{jk}(z)$ exists for some k between ALK and $A(L+1)K$ such that $\zeta(2s)R_{jk}(s)$ does not vanish at the point ρ. By letting $K\to\infty$, we obtain infinitely many such cusp forms.

When $\frac{1}{2}<Re\,\rho<1$, it follows from Lemma 1.2 and the above argument that the main theorem is true. □

Theorem 2.6. *Let* $\varphi(s) = e(\frac{s}{4} - \frac{1}{8}) - e(-\frac{s}{4} + \frac{1}{8})$. *Then*

$$\Pi = \frac{\zeta(\frac{1}{2})(\frac{K}{2})^A}{(1-2\eta)(1-2\rho)}\sqrt{N} + \{c + o_K(1)\}K^{A+1+\rho\nu}$$

when $K \to \infty$, *where*

$$c = \frac{i}{4}\frac{\rho-1}{\rho-\eta}2^{-A}(AL)^{1-2\rho}8^{\rho}\pi^{-\frac{1}{2}}\Gamma(\rho)\Gamma(\rho-\frac{1}{2})\zeta(\rho)\zeta(2\rho-1)\varphi(\rho-\frac{1}{2}).$$

The next section is devoted to the proof of Theorem 2.6.

3. Proof of Theorem 2.6

Put $a = K^{2-4\nu}$ and $b = K^{2+2\nu}$. By ϵ we always mean a sufficiently small positive number. Let $u(x)$ be a smooth function, which is a product of two monotone functions and which equals zero outside (a, b) and equals one inside $(a+\epsilon, b-\epsilon)$.

Theorem 3.1. *Let*

$$\gamma_k(c) = \int_a^b \Phi(\frac{N}{x})x^{-\frac{1}{2}}e(-\frac{1}{8} + \frac{c(k-1)^2}{16\pi^2 x})dx.$$

Then the identity

$$\Pi = \frac{i}{\sqrt{2}}\sum b_k \sum_{c<K^{10\nu}} \frac{\tau(c)}{\sqrt{c}}\{\gamma_k(c) - \bar{\gamma}_k(c)\} + O(K^A)$$

holds, where $\tau(c) = l$ *if* $c = l^2 p_1 \dots p_s$ *with* $p_1, \dots, p_s$ *distinct prime numbers.*

Proof. By the definition of Kloosterman's sums, we can write

$$(\rho-\eta)\Pi = \frac{i}{\sqrt{2}}\sum b_k \sum_{c<K^{10\nu}} \frac{1}{\sqrt{c}}\int_1^\infty \{(1-\eta)t^{\eta-1} + (\rho-1)t^{\rho-1}\}dt$$
$$\times \{\sum_{d\,(\mathrm{mod}\,c)}^{*} \sum_{0\le r<c} e(\frac{d+\bar{d}}{c}r) \sum_{n\equiv r\,(\mathrm{mod}\,c)} n^{-\frac{1}{2}}u(n)\psi(\frac{n}{N}t)\beta_k(\frac{4n\pi}{c})\}.$$

Using the Poisson summation formula, we obtain

$$\begin{aligned}(3.1)\qquad &\sum_{n\equiv r\,(\mathrm{mod}\,c)} n^{-\frac{1}{2}}u(n)\psi(\frac{n}{N}t)\beta_k(\frac{4n\pi}{c})\\ &= \frac{1}{\sqrt{c}}\sum_{h=-\infty}^{+\infty} e(-\frac{hr}{c})\int_{-\infty}^{+\infty} x^{-\frac{1}{2}}u(cx)\psi(\frac{tcx}{N})\beta_k(4\pi x)e(hx)dx.\end{aligned}$$

Let $G(x) = x^{-\frac{1}{2}}u(cx)\psi(\frac{c}{N}tx)$. Then $G(x)$ is a product of four monotone functions. If $F(x) = hx + 2x + \frac{(k-1)^2}{16\pi^2 x}$, then $F''(x) > 0$ and $F'(x)$ is monotone for positive numbers x. The integral on the right side of (3.1) is equal to

$$(3.2)\qquad e(-\frac{1}{8})\int_{-\infty}^{+\infty} G(x)e(F(x))dx - e(\frac{1}{8})\int_{-\infty}^{+\infty} G(x)e(2hx - F(x))dx.$$

When x belongs to $[\frac{a}{c}, \frac{b}{c}]$, we have

$$|F'(x)| \geq \min\{|h+\frac{3}{2}|, |h+2|\} = m_h$$

when $h \neq -2$. Write

$$(G(x)/F'(x))' = G_1(x) + cu'(cx)\psi(\frac{c}{N}tx)x^{-\frac{1}{2}}/F'(x).$$

Then $G_1(x)$ is a sum of three functions which are finite products of monotone functions. When these monotone functions are replaced by their maximums on $[\frac{a}{c}, \frac{b}{c}]$ in $G_1(x)$, the resulting sum corresponding to $G_1(x)$ is

$$\ll \frac{K^{-1}}{m_h}\{\psi(tK^{-5\nu}) - t\psi'(tK^{-5\nu})\}.$$

It follows from Lemma 1.9 that

$$\int_{-\infty}^{+\infty} G_1(x)e(F(x))dx \ll \frac{K^{-1}}{m_h^2}\{\psi(tK^{-5\nu}) - t\psi'(tK^{-5\nu})\}.$$

By partial integration, we find

$$\int_{-\infty}^{+\infty} \frac{cu'(cx)x^{-\frac{1}{2}}\psi(ctx/N)}{F'(x)}e(F(x))dx \ll \frac{1}{m_h^2}K^{-1+27\nu}\{\psi(tK^{-5\nu}) - t\psi'(tK^{-5\nu})\}.$$

It follows that the inequality

$$\int_{-\infty}^{+\infty} G(x)e(F(x))dx \ll \frac{1}{m_h^2}K^{-1+27\nu}\{\psi(tK^{-5\nu}) - t\psi'(tK^{-5\nu})\} \tag{3.3}$$

holds when $h \neq -2$.

Let $m_h' = \min\{|h-\frac{3}{2}|, |h-2|\}$. Similarly, we can show that the inequality

$$\int_{-\infty}^{+\infty} G(x)e(2h-F(x))dx \ll \frac{1}{(m_h')^2}K^{-1+27\nu}\{\psi(tK^{-5\nu}) - t\psi'(tK^{-5\nu})\} \tag{3.4}$$

holds when $h \neq 2$.

It follows from (3.1)–(3.4) that

$$\begin{aligned}\Pi = \frac{i}{\sqrt{2}}\sum b_k \sum_{c<K^{10\nu}} \frac{1}{c} \sideset{}{^*}\sum_{d\,(\mathrm{mod}\,c)} \int_{-\infty}^{+\infty} \Phi(\frac{N}{cx})u(cx)x^{-\frac{1}{2}} \sum_{0\leq r<c} e(\frac{d+\bar{d}}{c}r) \\ \times \{e(\frac{2r}{c}-\frac{1}{8})\alpha_k(4\pi x)e(-2x) - e(-\frac{2r}{c}+\frac{1}{8})\alpha_k(-4\pi x)e(2x)\}dx + O(K^A).\end{aligned}$$

The identities

$$\sum_{0\leq r<c} e(\frac{d+\bar{d}+2}{c}r) = \begin{cases} 0, & \text{if } d+\bar{d} \not\equiv -2\,(\mathrm{mod}\,c); \\ c, & \text{if } d+\bar{d} \equiv -2\,(\mathrm{mod}\,c)\end{cases}$$

and

$$\sum_{0\le r<c} e(\frac{d+\bar{d}-2}{c}r) = \begin{cases} 0, & \text{if } d+\bar{d}\not\equiv 2 \,(\mathrm{mod}\, c); \\ c, & \text{if } d+\bar{d}\equiv 2 \,(\mathrm{mod}\, c) \end{cases}$$

hold. The following two equations

$$\begin{cases} d\bar{d}\equiv 1 \,(\mathrm{mod}\, c) \\ d+\bar{d}\equiv -2 \,(\mathrm{mod}\, c) \\ 1\le d,\bar{d}\le c, (c,d)=1 \end{cases} \quad \text{and} \quad \begin{cases} d\bar{d}\equiv 1 \,(\mathrm{mod}\, c) \\ d+\bar{d}\equiv 2 \,(\mathrm{mod}\, c) \\ 1\le d,\bar{d}\le c, (c,d)=1 \end{cases}$$

have the same number of solutions, which is denoted by $\tau(c)$. By computation, we find that $\tau(c) = l$ if $c = l^2 p_1 \cdots p_s$. Every positive integer c can be written uniquely in this form. It follows that

$$\begin{aligned} \Pi =& \frac{i}{\sqrt{2}} \sum b_k \sum_{c<K^{10\nu}} \frac{\tau(c)}{\sqrt{c}} \int_a^b \Phi(\frac{N}{y}) u(y) y^{-\frac{1}{2}} \\ & \times \{e(-\frac{1}{8}+\frac{c(k-1)^2}{16\pi^2 y}) - e(\frac{1}{8}-\frac{c(k-1)^2}{16\pi^2 y})\} dy + O(K^A). \end{aligned} \tag{3.5}$$

Since $u(y)$ differs at most one from the constant function 1 on the union of $(a, a+\epsilon)$ and $(b-\epsilon, b)$, and since the sum of terms in (3.5) involving the integration over the union is $\ll K^A$, the stated identity follows. $\square$

Let $v(x)$ be a monotone function such that $v^{(i)}(x) \ll 1$ for $i = 0, 1, 2$, which equals zero when $x \ge K^{10\nu}$ and equals one when $x \le K^{10\nu} - \epsilon$. Put

$$T_k(s) = \int_0^\infty \{\gamma_k(x) - \bar{\gamma}_k(x)\} v(x) x^{s-1} dx$$

for $s = \tau + it$ with $\tau > 0$. Then $T_k(s)$ has an analytic continuation to the half-plane $Re\, s > -1$ except at the point $s = 0$ where it has a possible simple pole. By the inversion formula of Mellin's transform, we have

$$v(x)\{\gamma_k(x) - \bar{\gamma}_k(x)\} = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} T_k(s) x^{-s} ds.$$

Put

$$Z(s) = \frac{\zeta(2s)\zeta(s+\frac{1}{2})}{\zeta(2s+1)}.$$

It follows from Theorem 3.1 that

$$\Pi = \frac{i}{\sqrt{2}} \sum b_k \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} T_k(s) Z(s) ds + O(K^A) \tag{3.6}$$

for $\tau > \frac{1}{2}$.

Let

$$\phi(x) = e(-\frac{1}{8}+x) - e(\frac{1}{8}-x).$$

By changing the routes of the integrations, we find

$$\int_0^\infty x^{s-1}\phi(x) dx = (2\pi)^{-s}\Gamma(s)\varphi(s) \tag{3.7}$$

for $0 < Re\ s < 1$.

For the convenience, denote $16\pi^2(k-1)^{-2}$ by β.

Lemma 3.2. *We have $T_k(\frac{1}{2}) \ll K$ and*

$$T_k'(\frac{1}{2}) = \frac{4\pi^2 i}{\sqrt{2}(k-1)}\int_a^b \Phi(\frac{N}{y})dy + O(K).$$

Proof. Write

$$T_k(s) = \beta^s \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy \int_0^\infty v(\beta yx)x^{s-1}\phi(x)dx.$$

By (3.7) the identity

$$T_k(\frac{1}{2}) = \sqrt{\beta}\int_a^b \Phi(\frac{N}{y})dy \int_{\frac{K^{10\nu-\epsilon}}{\beta y}}^\infty x^{-\frac{1}{2}}\{v(\beta yx)-1\}\phi(x)dx \tag{3.8}$$

holds. It follows from Lemma 1.9 that the innermost integral on the right side of (3.8) is $\ll K^{-1-5\nu}\sqrt{y}$. By considering the integrations on $[a, K^2]$, $[K^2, N]$ and $[N, b]$ respectively, we find that $T_k(\frac{1}{2}) \ll K$.

A similar argument shows that

$$T_k'(\frac{1}{2}) = \sqrt{\beta}\int_a^b \Phi(\frac{N}{y})dy \int_0^\infty v(\beta yx)x^{-\frac{1}{2}}\phi(x)\ln x dx + O(K).$$

Since $x^{-\frac{1}{2}}\ln x$ is monotone on $(9, +\infty)$, the inequality

$$\int_{\frac{K^{10\nu-\epsilon}}{\beta y}}^\infty x^{-\frac{1}{2}}\ln x\{v(\beta yx)-1\}\phi(x)dx \ll K^{-1-5\nu}\sqrt{y}\ln K$$

holds. The second assertion then follows from (3.7). □

Theorem 3.3. *The identity*

$$\Pi = -6\int_a^b \Phi(\frac{N}{y})dy \sum \frac{b_k}{k-1} + \frac{i}{\sqrt{2}}\sum b_k \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} T_k(s)Z(s)ds + O(K^{A+1})$$

holds.

Proof. By Lemma 1.8 the inequality

$$Z(s) \ll |s|^{\frac{3}{4}(1-2Re\, s)}\ln^3|s|$$

holds for $|s| \geq t_0 > 1$ when $0 \leq Re\, s \leq \frac{1}{2}$. Since $T_k(s) \ll_K \frac{1}{|s|}$ when the real part of s is bounded, the line of integration for the integral

$$\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} T_k(s)Z(s)ds \tag{3.9}$$

with $\tau > \frac{1}{2}$ can be moved to the imaginary axis. Write

$$\zeta(s) = f(s) + \frac{1}{s-1},$$

where $f(s)$ is analytic in the half-plane $Re\, s > -1$. It follows that the integral (3.9) is equal to

$$\frac{3f(1)}{2\zeta(2)}T_k(\frac{1}{2}) - \frac{\zeta'(2)}{\zeta(2)^2}T_k(\frac{1}{2}) + \frac{T_k'(\frac{1}{2})}{2\zeta(2)} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} T_k(s)Z(s)ds.$$

The stated identity then follows from the identity (3.6) and Lemma 3.2. □

Lemma 3.4. *The identity*

$$\int_{-iK^{8\nu}}^{iK^{8\nu}} T_k(s)Z(s)ds = \int_{-iK^{8\nu}}^{iK^{8\nu}} \Gamma(s)(\frac{\beta}{2\pi})^s\varphi(s)Z(s)ds\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy + O(K)$$

holds.

Proof. Assume that s is on the imaginary axis and bounded by $K^{8\nu}$. It follows from (3.7) and Lemma 1.9 that $T_k(s)$ can be written as

$$\{(\frac{\beta}{2\pi})^s\Gamma(s)\varphi(s) + O(K^{-8\nu})\}\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy + \beta^s\int_a^b \Phi(\frac{N}{y})B(y,s)y^{s-\frac{1}{2}}dy$$

where

$$B(y,s) = \int_{K^{8\nu}}^{\infty} x^{s-1}v(\beta yx)\phi(x)dx.$$

It follows from Lemma 1.9 that $B(y,s) \ll K^{-8\nu}$. By partial integration, we have

$$(3.10)\qquad \begin{aligned}(s+\frac{1}{2})\int_a^b \Phi(\frac{N}{y})B(y,s)y^{s-\frac{1}{2}}dy &= -\int_a^b \frac{d}{dy}\Phi(\frac{N}{y})B(y,s)y^{s+\frac{1}{2}}dy\\ &- \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy\int_{K^{8\nu}}^{\infty}\frac{d}{dx}v(\beta yx)x^s\phi(x)dx + O(K^{1-7\nu}).\end{aligned}$$

Dividing $[a,b]$ into several small intervals and considering the integration on each small interval respectively, we find that the first term on the right side of (3.10) is $\ll K^{1-7\nu}$. The second term on the right side of (3.10) equals

$$(3.11)\qquad \beta^{-s}\int_0^{\infty} v'(x)x^s dx\int_{b^{-1}}^{a^{-1}} \Phi(yN)y^{-\frac{3}{2}}\phi(\frac{yx}{\beta})dy.$$

Divide $[\frac{1}{b},\frac{1}{a}]$ into several small intervals, and divide the integral

$$\int_{\frac{1}{b}}^{\frac{1}{a}} y^{-\frac{3}{2}}\psi(\frac{t}{yN})\phi(\frac{xy}{\beta})dy$$

into several integrals correspondingly. By using Lemma 1.9, we find that (3.11) is $\ll K^{1-\frac{25}{4}\nu}$.

We have

$$(s+\frac{1}{2})\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy \ll K^{1+\nu}.$$

It follows that the identity

$$(3.12)\qquad T_k(s) = \beta^s(2\pi)^{-s}\Gamma(s)\varphi(s)\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy + O(\frac{K^{1-\frac{25}{4}\nu}}{1+|s|})$$

holds. By Lemma 1.8, $Z(s) \ll (1+|s|)^{\frac{2}{3}}$ when s is on the imaginary axis. Then the stated identity follows from (3.12). $\square$

Lemma 3.5. *The inequality*

$$\int T_k(s)Z(s)ds \ll K$$

holds, where the integration is taken over the set of all complex numbers s on the imaginary axis whose absolute values are greater than $K^{45\nu}$.

Proof. Assume that s is on the imaginary axis with absolute value greater than $K^{45\nu}$. Write

$$T_k(s) = \beta^s \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}} \int_0^\infty \frac{d^2}{dx^2}\{v(\beta yx)\phi(x)\}\frac{x^{s+1}}{s(s+1)}dx,$$

which is equal to

$$\frac{\beta^s}{s(s+1)}\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}} \int_0^{K^{15\nu}} v(\beta yx)\phi''(x)x^{s+1}dx + O(K^{1+15\nu}|s|^{-2}). \tag{3.13}$$

By Lemma 1.9 the innermost integral in the first term of (3.13) is $\ll K^{1-15\nu}$. It follows that the first term of (3.13) is $\ll K^{1-14\nu}|s|^{-2}$. Since $Z(s) \ll |s|^{\frac{2}{3}}$, the stated inequality follows. □

Lemma 3.6. *Let I be the set of all real numbers whose absolute values are between $K^{8\nu}$ and $K^{45\nu}$. Then the identity*

$$\int_{iI} T_k(s)Z(s)ds = \int Z(s)\beta^s ds \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy \int_{K^{8\nu}/8}^\infty v(\beta yx)\phi(x)x^{s-1}dx + O(K)$$

holds, where the integration is taken over the set $\frac{1}{2}+\epsilon+iI$.

Proof. First assume that s belongs to iI. Write

$$T_k(s) = \beta^s \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}\{\int_{K^{8\nu}/8}^\infty v(\beta yx)\phi(x)x^{s-1}dx + \int_0^{K^{8\nu}/8} \phi(x)x^{s-1}dx\}dy.$$

The inequality

$$(s+\frac{1}{2})\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy \ll K^{1+\nu}$$

holds. By partial integration and Lemma 1.9, we have

$$\int_0^{K^{8\nu}/8} \phi(x)x^{s-1}dx \ll |s|^{-1}.$$

It follows that

$$T_k(s) = \beta^s \int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}\{\int_{K^{8\nu}/8}^\infty v(\beta yx)\phi(x)x^{s-1}dx\}dy + O(K^{1+\nu}|s|^{-2}).$$

By Lemma 1.8 the left side of the stated identity is equal to

$$\int_{iI} Z(s)\beta^s\{\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}J(y,s)dy\}ds + O(K), \tag{3.14}$$

where

$$J(y,s)=\int_{K^{8\nu}/8}^{\infty}v(\beta yx)\phi(x)x^{s-1}dx. \tag{3.15}$$

Now assume that $s=\tau+it$ with $0\le\tau\le\frac{1}{2}+\epsilon$ and $|t|=K^{8\nu}$ or $K^{45\nu}$. Divide the interval of the integration (3.15) into three subintervals with the second one given by $[\frac{|t|}{2\pi}-\epsilon|t|^{\frac{3}{4}},\frac{|t|}{2\pi}+\epsilon|t|^{\frac{3}{4}}]$. By using Lemma 1.9 for the integrations on the first and the third subintervals, we find that

$$J(y,s)\ll|t|^{\frac{1}{4}}(|t|^{\tau-\frac{1}{2}}+K^{8(\tau-1)\nu}). \tag{3.16}$$

By partial integration and the inequality (3.16), the identity

$$\begin{aligned}\beta^s\int_a^b\Phi(\frac{N}{y})y^{s-\frac{1}{2}}J(y,s)dy&=\frac{-\beta^s}{s+\frac{1}{2}}\int_a^b\Phi(\frac{N}{y})y^{s+\frac{1}{2}}\frac{d}{dy}J(y,s)dy\\&\quad+O(K^{1+(2\tau+1)\nu}|t|^{-\frac{3}{4}}[|t|^{\tau-\frac{1}{2}}+K^{8(\tau-1)\nu}])\end{aligned} \tag{3.17}$$

holds. We have

$$\beta^s\int_a^b\Phi(\frac{N}{y})y^{s+\frac{1}{2}}\frac{d}{dy}J(y,s)dy=\int_a^b\Phi(\frac{N}{y})y^{-\frac{1}{2}}\{\int_0^\infty v'(x)x^s\phi(\frac{x}{\beta y})dx\}dy.$$

Dividing $[a,b]$ into several small subintervals and considering the integrations on each small subintervals, we find that the first term on the right side of (3.17) is $\ll\frac{1}{|t|}K^{1+(10\tau+\frac{3}{4})\nu}$. By Lemma 1.8, we have $Z(s)\ll|t|^{\max\{0,(2-4\tau)/3\}}\ln^3|t|$. It follows that the integrand of the integration with respect to s in (3.14) is

$$\ll K|t|^{\max\{0,\,\frac{2-4\tau}{3}\}-1}\ln^3|t|\{K^{(10\tau+\frac{3}{4})\nu}+K^{(2\tau+1)\nu}|t|^{\frac{1}{4}}[|t|^{\tau-\frac{1}{2}}+K^{8(\tau-1)\nu}]\},$$

which is $\ll K$. Then the stated identity follows. □

Lemma 3.7. *The inequality*

$$\sum_{c\le x}\tau(c)\ll x\ln x$$

holds when $x\to\infty$.

Proof. Let $\tau>\frac{3}{2}$, and put

$$h(x)=\int_0^x\{\sum_{c\le t}\tau(c)\}dt.$$

By the inversion formula of Mellin's transform, we have

$$\frac{h(x)}{x}=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\zeta(2s-1)\zeta(s)}{s(s+1)\zeta(2s)}x^s ds.$$

Write $\zeta(s)=f(s)+\frac{1}{s-1}$. Then

$$\frac{h(x)}{x}=\frac{x\ln x}{4\zeta(2)}+(\frac{3f(1)}{4\zeta(2)}-\frac{3}{8\zeta(2)}-\frac{\zeta'(2)}{2\zeta(2)^2})x+\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\zeta(2s-1)\zeta(s)}{s(s+1)\zeta(2s)}x^s ds.$$

We know

$$\sum_{c\le x}\tau(c)\le 2\frac{h(2x)}{2x}-\frac{h(x)}{x}.$$

By Lemma 1.8, $\zeta(2s-1)\zeta(s)/\zeta(2s)\ll|t|^{\frac{2}{3}}$ when $Re\,s=\frac{1}{2}$, and the stated inequality follows. □

Theorem 3.8. *The identity*

$$\int Z(s)\beta^s\{\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy\int_{K^{8\nu}/8}^{\infty} v(\beta yx)\phi(x)x^{s-1}dx\}ds = O(K)$$

holds, where the integration is over the set $\frac{1}{2}+\epsilon+iI$.

Proof. Assume that the imaginary part of s belongs to I with $Re\, s = \frac{1}{2}+\epsilon$. Let $J(y,s)$ be given as in the proof of Lemma 3.6. We have

$$\beta^s\int_a^b \Phi(\frac{N}{y})\frac{d}{dy}J(y,s)y^{\frac{1}{2}+s}dy = \int_0^{\infty} v'(x)x^s dx\int_{b^{-1}}^{a^{-1}} \Phi(yN)y^{-\frac{3}{2}}\phi(\frac{x}{\beta}y)dy. \tag{3.18}$$

An argument similar to the estimation of (3.11) shows that the left side of (3.18) is $\ll K^{1-\nu}$. Since $Z(s)$ is bounded, it follows by partial integration that the left side of the stated identity equals

$$-\int \frac{Z(s)\beta^s}{\frac{1}{2}+s}\{a^{\frac{1}{2}+s}\Phi(\frac{N}{a})J(a,s)+\int_a^b \frac{d}{dy}\Phi(\frac{N}{y})y^{\frac{1}{2}+s}J(y,s)dy\}ds + O(K). \tag{3.19}$$

Let c be a positive integer. Denote by $I_1(c)$ the interval

$$[\max\{K^{8\nu}/8,(c-\sqrt{c}K^{-2\nu})/(\beta a)\},(c+\sqrt{c}K^{-2\nu})/(\beta a)]$$

when $c\le K^{11\nu}$, and the empty set when $c>K^{11\nu}$. Put

$$I(c) = [K^{8\nu}/8,\infty) - I_1(c).$$

The first term of (3.19) can be written as

$$\sum_{c=1}^{\infty}\frac{\tau(c)}{c^{1+\epsilon}}\beta^{\frac{1}{2}+\epsilon}a^{1+\epsilon}\Phi(\frac{N}{a})\{\int_{I_1(c)}+\int_{I(c)}\}x^{\epsilon-\frac{1}{2}}v(\beta ax)\phi(x)dx\int_I \frac{(\beta ax/c)^{it}}{1+\epsilon+it}dt. \tag{3.20}$$

The sum of terms in (3.20) involving the integration over $I_1(c)$ is by Lemma 3.7

$$\ll K^{1-\nu}\sum_{I_1(c)\neq\emptyset}\frac{\tau(c)}{c}\ll K.$$

The inequality

$$\ln^{-1}(\beta ax/c)\ll\sqrt{c}K^{2\nu}$$

holds when x belongs to $I(c)$ with $c\ge\beta aK^{8\nu}/8$. Dividing $I(c)$ into three subsets with the second one being $I(c)\cap[t-\sqrt{|t|},t+\sqrt{|t|}]$, we find by Lemma 1.9 that

$$\int_{I(c)} x^{\epsilon-\frac{1}{2}}v(\beta ax)\frac{(\beta ax/c)^{it}}{\ln(\beta ax/c)}\phi(x)dx\ll\sqrt{|t|}(|t|^{\epsilon-\frac{1}{2}}+K^{(8\epsilon-4)\nu})\sqrt{c}K^{2\nu}.$$

By partial integration we obtain the inequality

$$\int_{I(c)} x^{\epsilon-\frac{1}{2}}v(\beta ax)\phi(x)dx\int_I\frac{(\beta ax/c)^{it}}{1+\epsilon+it}dt\ll\sqrt{c}K^{(8\epsilon-6)\nu}.$$

The sum of terms in (3.20) involving the integration over $I(c)$ with c between $\beta a K^{8\nu}/8$ and $K^{11\nu}$ is by Lemma 3.7

$$\ll K^{1-7\nu} \sum_{c \le K^{11\nu}} \frac{\tau(c)}{c^{\frac{1}{2}+\epsilon}} \ll K.$$

We have $\beta ax/c \ll K^{-\nu}$ for $c > K^{11\nu}$ when x belongs to $[K^{8\nu}/8, K^{10\nu}/(\beta a)]$. Then $\ln^{-1}(\beta ax/c) \ll \ln^{-1} K$. It follows that the sum of terms in (3.20) involving the integration over $I(c)$ with $c > K^{11\nu}$ is

$$\ll K^{1-\nu} \sum_{c > K^{11\nu}} \frac{\tau(c)}{c^{1+\epsilon}} \ll K.$$

Since c is a positive integer, there exists a positive number ϵ_1, which does not depend on c, such that $c < \beta a K^{8\nu}/8 - \epsilon_1$ whenever $c < \beta a K^{8\nu}/8$. Hence,

$$\ln^{-1}(\beta ax/c) \ll K^{4\nu}.$$

It follows that the sum of terms in (3.20) with $c < \beta a K^{8\nu}/8$ is by Lemma 3.7

$$\ll K^{1-\nu} \sum_{c < \beta a K^{8\nu}/8} \frac{\tau(c)}{c^{1+\epsilon}} \ll K.$$

Therefore, the first term (3.20) of (3.19) is $\ll K$.

Assume that c is a positive integer. Let $a \le y \le b$. Denote by $I_1(c, y)$ the interval

$$[\max\{K^{8\nu}/8, (c - \sqrt{c}K^{-\nu})/(\beta y)\}, (c + \sqrt{c}K^{-\nu})/(\beta y)]$$

when $c \le K^{11\nu}$, and empty set when $c > K^{11\nu}$. Put

$$I(c, y) = [K^{8\nu}/8, \infty) - I_1(c, y).$$

The second term of (3.19) can be written as

$$\int_a^b \frac{d}{dy}\Phi(\frac{N}{y}) y^{1+\epsilon} dy \{\sum_{c=1}^{\infty} \frac{\tau(c)}{c^{1+\epsilon}} \beta^{\frac{1}{2}+\epsilon}\} \int x^{\epsilon-\frac{1}{2}} v(\beta y x)\phi(x) dx \int_I \frac{(\beta y x/c)^{it}}{1+\epsilon+it} dt \tag{3.21}$$

where the integration is taken over the union of $I_1(c, y)$ and $I(c, y)$.

The sum of terms in (3.21) involving the integration over $I_1(c, y)$ is

$$\ll K^{-\nu} \ln K \int_a^b |\frac{d}{dy}\Phi(\frac{N}{y})| y^{\frac{1}{2}} \{\sum_{I_1(c,y) \ne \emptyset} \frac{\tau(c)}{c}\} dy \ll K$$

when the interval of integration $[a, b]$ is divided into several small subintervals.

The sum of terms in (3.21) involving the integration on $I(c, y)$ with $c \ge K^{11\nu}$ is

$$\ll K^{(6\epsilon-8)\nu-1} \int_a^b |\frac{d}{dy}\Phi(\frac{N}{y})| y^{1+\epsilon} dy \sum_{c \ge K^{11\nu}} \frac{\tau(c)}{c^{1+\epsilon}} \ll K.$$

The sum of terms in (3.21) involving the integration over $I(c,y)$ with c between $\beta y K^{8\nu}/8$ and $K^{11\nu}$ is by Lemma 3.7

$$\ll K^{(6\epsilon-7)\nu-1}\int_a^b |\frac{d}{dy}\Phi(\frac{N}{y})|y^{1+\epsilon}dy \sum_{c<K^{11\nu}} \frac{\tau(c)}{c^{\frac{1}{2}+\epsilon}} \ll K.$$

Divide $I(c,y)$ into two subsets, $[K^{8\nu}/8, K^{8\nu}/8+K^{\frac{5}{2}\nu}]\cap I(c,y)$ and the complement of this set in $I(c,y)$ when $c<\beta y K^{8\nu}/8$. The sum of terms in (3.21) involving the integration over the first subset is

$$\ll K^{(6\epsilon-4)\nu+\frac{5}{2}\nu-1}\ln K\int_a^b |\frac{d}{dy}\Phi(\frac{N}{y})|y^{1+\epsilon}dy \sum_{c<K^{10\nu}} \frac{\tau(c)}{c^{1+\epsilon}} \ll K.$$

The sum of terms in (3.21) involving the integration over the second subset is

$$\ll K^{(6\epsilon-8)\nu+\frac{11}{2}\nu-1}\int_a^b |\frac{d}{dy}\Phi(\frac{N}{y})|y^{1+\epsilon}dy \sum_{c<K^{10\nu}} \frac{\tau(c)}{c^{1+\epsilon}} \ll K.$$

The stated identity follows. □

Theorem 3.9. *The identity*

$$\int_{-i\infty}^{i\infty} T_k(s)Z(s)ds = \int_{-iK^{8\nu}}^{iK^{8\nu}} (s+\frac{1}{2})\omega(s)ds\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy + O(K)$$

holds, where $\omega(s)=\Gamma(s)\beta^s(2\pi)^{-s}Z(s)\varphi(s)/(s+\frac{1}{2})$.

Proof. The stated identity follows from Lemma 3.4–Lemma 3.7 and Theorem 3.8. □

For the convenience, denote by Δ the interval $[-K^{8\nu}, K^{8\nu}]$.

Lemma 3.10. *We have*

$$\begin{aligned}&2\pi i\int_{i\Delta}\omega(s)\{a^{s+\frac{1}{2}}\Phi(\frac{N}{a})+\int_N^b \frac{d}{dy}\Phi(\frac{N}{y})y^{s+\frac{1}{2}}dy\}ds\\ &=\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}ds\int_{\delta-i\infty}^{\delta+i\infty} z\lambda(z)/(s-z+\frac{1}{2})dz+O(K).\end{aligned}$$

Proof. We have

$$2\pi i\int_N^b \frac{d}{dy}\Phi(\frac{N}{y})y^{s+\frac{1}{2}}dy = \int_{\delta-i\infty}^{\delta+i\infty} z\lambda(z)N^{s+\frac{1}{2}}\frac{K^{(s-z+\frac{1}{2})\nu}-1}{z-s-\frac{1}{2}}dz.$$

The inequality

$$\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}ds\int_{\delta-i\infty}^{\delta+i\infty} z\lambda(z)K^{(s-z+\frac{1}{2})\nu}/(z-s-\frac{1}{2})dz \ll K$$

holds. It is seen that the identity

$$\int_{i\Delta}\omega(s)a^{s+\frac{1}{2}}\Phi(\frac{N}{a})ds = \int_{\frac{1}{2}+\epsilon+i\Delta}\omega(s)a^{s+\frac{1}{2}}\Phi(\frac{N}{a})ds+O(K) \tag{3.22}$$

holds. For $Re\, s=\frac{1}{2}+\epsilon$, the integrand of the integration on the right side of (3.22) is $\ll K^{1-\nu}|s|^{\epsilon-1}$. It follows that the left side of (3.22) is $\ll K$. Then the stated identity follows. □

Lemma 3.11. *We have*

$$\int_{i\Delta}\omega(s)ds\int_a^N \frac{d}{dy}\Phi(\frac{N}{y})y^{s+\frac{1}{2}}dy = O(K)+\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}\{\frac{\eta(1-\eta)\pi^{-\eta}\Gamma(\eta)\zeta(2\eta)}{(\eta-\rho)(s-\eta+\frac{1}{2})}$$
$$+\frac{\rho(\rho-1)\pi^{-\rho}\Gamma(\rho)\zeta(2\rho)}{(\eta-\rho)(s-\rho+\frac{1}{2})}+\frac{1}{2(1-2\eta)(1-2\rho)s}-\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{z\lambda(z)}{s-z+\frac{1}{2}}dz\}ds.$$

Proof. It will first be shown that the inequality

$$\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}K^{-5(s-\eta+\frac{1}{2})\nu}/(s-\eta+\frac{1}{2})ds \ll K \tag{3.23}$$

holds. The integrand of the integration in (3.23) is $\ll K^{1+\frac{\nu}{2}}(1+|s|)^{-\frac{7}{4}}$ when the real part of s is nonnegative. It follows that the left side of (3.23) differs by a term of order K from an integral, which has the same integrand and whose interval of integration is taken over the set $\frac{1}{2}+\epsilon+i\Delta$. Note that the integrand has a simple pole at $s=\frac{1}{2}$ with a residue of order less than K. When $Re\, s=\frac{1}{2}+\epsilon$, the integrand is $\ll K^{1-\frac{3}{2}\nu}|s|^{\epsilon-2}$. It follows that the inequality (3.23) holds.

A similar argument shows that

$$\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}K^{-5(s-\rho+\frac{1}{2})\nu}/(s-\rho+\frac{1}{2})ds \ll K.$$

It will now be shown that the inequality

$$\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}K^{-5s\nu}\frac{ds}{s} \ll K \tag{3.24}$$

holds, where the integration around the point $s=0$ is taken along the route $\epsilon e^{i\theta}$ with $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. The integrand of the integration in (3.24) is $\ll K^{1+\frac{\nu}{2}}|s|^{-\frac{7}{4}}$ for $Re\, s\ge 0$. The left side of (3.24) differs by a term of order K from an integral, which has the same integrand and whose interval of integration is taken over the set $\frac{1}{4}+i\Delta$. When $Re\, s=\frac{1}{4}$, the integrand is $\ll K^{1-\frac{\nu}{2}}|s|^{-\frac{7}{4}}$. This implies the inequality (3.24).

It will finally be shown that the inequality

$$\int_{i\Delta}\omega(s)N^{s+\frac{1}{2}}\{\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}z\lambda(z)K^{-5(s-z+\frac{1}{2})\nu}/(s-z+\frac{1}{2})dz\}ds \ll K \tag{3.25}$$

holds. When s is on the imaginary axis, the integrand of the integration with respect to s in (3.25) is $\ll K^{1-2\nu}(1+|s|)^{-\frac{11}{6}}$, and the inequality (3.25) follows.

The identity

$$\int_a^N\frac{d}{dy}\Phi(\frac{N}{y})y^{s+\frac{1}{2}}=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}z\lambda(z)N^{s+\frac{1}{2}}\frac{1-K^{-5(s-z+\frac{1}{2})\nu}}{z-s-\frac{1}{2}}dz \tag{3.26}$$

holds. Moving the line of integration on the right side of (3.26) to the line $Re\ z=-\epsilon$ and using (3.23)–(3.25), we obtain the stated identity. $\square$

Theorem 3.12. *The identity*

$$\Pi = -6\int_a^b \Phi(\frac{N}{y})dy \sum \frac{b}{k-1} + i\sqrt{2}\sum b_k \frac{1}{2\pi i}\int_{-iK^{8\nu}}^{iK^{8\nu}} \Gamma(s)(\frac{\beta}{2\pi})^s \varphi(s)$$
$$\times \frac{\zeta(2s)\zeta(s+\frac{1}{2})\Gamma(s+\frac{1}{2})(2s-1)}{(2s+1-2\eta)(2s+1-2\rho)}(\frac{N}{\pi})^{s+\frac{1}{2}}ds + O(K^{A+1})$$

holds.

Proof. Use partial integration for the integral

$$\int_a^b \Phi(\frac{N}{y})y^{s-\frac{1}{2}}dy,$$

and move the line of integration for the integral

$$\int_{\delta-i\infty}^{\delta+i\infty} \frac{z\lambda(z)}{s-z+\frac{1}{2}}dz$$

to the line $Re\ z = -\epsilon$. Then the stated identity follows from Theorem 3.3, Theorem 3.9, Lemma 3.10 and Lemma 3.11. □

Proof of Theorem 2.6. From Theorem 3.12, we obtain

$$\Pi = -6\sum \frac{b_k}{k-1}\int_a^b \Phi(\frac{N}{y})dy + \frac{\zeta(\frac{1}{2})\sqrt{N}}{(1-2\eta)(1-2\rho)}(K/2)^A$$
$$+ c_\rho N^\rho \sum b_k(k-1)^{1-2\rho} + O(K^{A+1}),$$

where

$$c_\rho = \frac{i}{4}\frac{\rho-1}{\rho-\eta}8^\rho \pi^{-\frac{1}{2}}\Gamma(\rho)\Gamma(\rho-\frac{1}{2})\zeta(\rho)\zeta(2\rho-1)\varphi(\rho-\frac{1}{2}).$$

The identity

$$\int_a^b \Phi(\frac{N}{y})dy = -\int_a^b y\frac{d}{dy}\Phi(\frac{N}{y})dy + O(K^{2-\nu})$$

holds. Write

$$\frac{2\pi i}{N}\int_a^b y\frac{d}{dy}\Phi(\frac{N}{y})dy$$
$$= \int_{\delta-i\infty}^{\delta+i\infty} \frac{z}{z-1}\lambda(z)K^{(1-z)\nu}dz + \int_{\delta-i\infty}^{\delta+i\infty} \frac{z}{1-z}\lambda(z)K^{-5(1-z)\nu}dz = J_1 + J_2.$$

By moving the lines of the integrations, we find that

$$J_1 = \int_{2-i\infty}^{2+i\infty} \frac{z}{z-1}\lambda(z)K^{(1-z)\nu}dz \ll K^{-\nu}$$

and

$$J_2 = \int_{\frac{2}{3}-i\infty}^{\frac{2}{3}+i\infty} \frac{z}{1-z}\lambda(z)K^{-5(1-z)\nu}dz \ll K^{-\nu}.$$

It follows that

$$\Pi = \frac{\zeta(\frac{1}{2})\sqrt{N}}{(1-2\eta)(1-2\rho)}(K/2)^A + c_\rho N^\rho \sum b_k k^{1-2\rho} + O(K^{A+1}).$$

We have

(3.27)
$$\lim_{K\to\infty}\left(\frac{2}{K}\right)^A \sum b_k \left(\frac{k}{K}\right)^{1-2\rho} = \int_L^{L+1}\cdots\int_L^{L+1}(t_1+\cdots+t_A)^{1-2\rho}dt_1\ldots dt_A.$$

The right side of (3.27) equals $(AL)^{1-2\rho}+o_L(1)$ when L is large enough. Therefore, the identity

$$\Pi = \frac{\zeta(\frac{1}{2})\sqrt{N}}{(1-2\eta)(1-2\rho)}(K/2)^A + \{c_\rho 2^{-A}(AL)^{1-2\rho} + o_K(1)\}K^{A+1+\rho\nu}$$

holds. This completes the proof of Theorem 2.6. □

References

1. P. Deligne, *La conjecture de Weil I*, Publ. Math. I.H.E.S. **43**, 273–307 (1974). MR **49:**5013
2. J. -M. Deshouillers and H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. **70**, 219–288 (1982). MR **84m:**10015
3. J.-M. Deshouillers and H. Iwaniec, *The non-vanishing of the Rankin-Selberg zeta-functions at special points*, in "The Selberg Trace Formula and Related Topics," Contemp. Math. 53, Amer. Math. Soc., Providence, R. I., 51–95 (1986). MR **88d:**11047
4. T. Estermann, *On Kloosterman's sums*, Mathematika **8**, 83–86 (1961). MR **23:**A3716
5. W. Luo, *On the non-vanishing of Rankin-Selberg L-functions*, Duke Math. J. **69**, 411–425 (1993). MR **93m:**11040
6. R. Phillips and P. Sarnak, *On cusp form for cofinite subgroups of $PSL(2,\mathbb{R})$*, Invent. Math. **80**, 339–364 (1985). MR **86m:**11037
7. R. A. Rankin, *Contributions of the theory of Ramanujan's functions $\tau(n)$ and similar arithmetical functions*, Proc. Cambridge Phil. Soc. **35**, 357–372 (1939). MR **1:**69
8. G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. **31**, 79–98 (1975). MR **52:**3064
9. G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, 1971. MR **47:**3318
10. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford, 1950. MR **13:**741
11. A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. **34**, 204–207 (1948). MR **10:**234e

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