OPERATOR SEMIGROUP COMPACTIFICATIONS

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ABSTRACT. A weakly continuous, equicontinuous representation of a semitopological semigroup S on a locally convex topological vector space X gives rise to a family of operator semigroup compactifications of S, one for each invariant subspace of X. We consider those invariant subspaces which are maximal with respect to the associated compactification possessing a given property of semigroup compactifications and show that under suitable hypotheses this maximality is preserved under the formation of projective limits, strict inductive limits and tensor products.

1. INTRODUCTION

Let $U : s \mapsto U_s$ be a weakly continuous, equicontinuous representation of a semitopological semigroup S on a locally convex topological vector space X. If Y is a U_S-invariant subspace of X then the weak* operator closure $\mathcal{U}_Y$ of $U_S|_Y$ in $\mathcal{L}(X, X'')$ is a left topological semigroup compactification of S under a multiplication which may be viewed as the natural extension of operator composition in $\mathcal{L}(X)$. Given a property $\mathcal{P}$ of semigroup compactifications of S one can ask if there exists a maximal invariant subspace Y of X such that the compactification $\mathcal{U}_Y$ has property $\mathcal{P}$. Such a subspace, if it exists, is called a maximal $\mathcal{P}$-subspace of X (or U). For example, if U is weakly almost periodic and $\mathcal{P}$ is the property that a compactification of S is a group, then the maximal $\mathcal{P}$-subspace of X is the space X_{sap} of strongly almost periodic vectors as defined by deLeeuw and Glicksberg [3].

In this paper we prove a general existence and uniqueness theorem for maximal $\mathcal{P}$-spaces. We show that for certain properties $\mathcal{P}$, called regular properties, X has a unique maximal $\mathcal{P}$-subspace, which we denote by $X_{\mathcal{P}}$. We then examine the semigroup theoretic and topological dynamical properties of $\mathcal{U}_{X_{\mathcal{P}}}$ as well as the functional analytic properties of $X_{\mathcal{P}}$. We show that under suitable conditions direct products, projective limits, strict inductive limits and tensor products of maximal $\mathcal{P}$-spaces are again maximal $\mathcal{P}$-spaces (relative to the appropriate derived representations) and we characterize the resulting operator semigroup compactifications in terms of those of the given representations. As an illustration of the sort of result that one can expect, consider an infinite family $\{X_i : i \in I\}$ of unital abelian C^*-algebras and let $X := \bigotimes_{i \in I} X_i$ be the C^*-algebra tensor product of the X_i (see Section 5). For each $i \in I$ let U^i be a weakly almost periodic representation of a semitopological semigroup S_i by unital *-homomorphisms on X_i.

Let $s \mapsto U_s : \prod_i S_i \mapsto \mathcal{L}(X)$ be the tensor product of the representations U^i. It follows from general results in Section 5 that $\mathcal{U}_X$ is an inverse limit of finite tensor products of the $\mathcal{U}^i_{X_i}$. Moreover, if $WAP(S_i)$ is left amenable for every i then $X_{sap} = \bigotimes_i X_{i\,sap}$ and X splits into a direct sum of X_{sap} and a subspace of flight vectors (see [3]).

2. Terminology and Notation

In this section we set down the basic notions of semigroup representations and compactifications needed in the sequel. For details the reader is referred to [1,2].

A *semitopological semigroup* (stsg) is a semigroup S with a topology relative to which multiplication is separately continuous. If multiplication is jointly continuous then S is a *topological semigroup* (tsg). If $U : s \mapsto U_s$ is a representation of a stsg S on a (separated) locally convex topological vector space (lctvs) X we define the subspaces

$$X_{wc} := \{x \in X : s \mapsto U_s x \text{ is weakly continuous}\},$$

$$X_{ap} := \{x \in X : U_S x \text{ is relatively compact}\},$$

$$X_{wap} := \{x \in X : U_S x \text{ is relatively weakly compact}\}.$$

If U is equicontinuous (i.e., U_S is an equicontinuous family) then X_{wc} and X_{ap} are closed. If, additionally, X is quasicomplete, then X_{wap} is closed [2, 6.1.14]. U is *weakly continuous* (*weakly almost periodic, almost periodic*) if $X = X_{wc}$ ($X = X_{wap}$, $X = X_{ap}$). For the right translation representation $s \mapsto R_s$ on $C(S)$, the C^*-algebra of all bounded, continuous, complex-valued functions on S, we use the standard notation $WRC(S) = C(S)_{wc}$, $WAP(S) = C(S)_{wap}$ and $AP(S) = C(S)_{ap}$.

If X and Y are lctvs then $\mathcal{L}(X, Y)$ denotes the space of continuous linear transformations $u : X \mapsto Y$. The adjoint of u is denoted by u'. Note that $\mathcal{L}(X)$ ($= \mathcal{L}(X, X)$) is a stsg under composition and the weak operator topology. If U and V are representations of a stsg S on X and Y, respectively, and if $u \in \mathcal{L}(X, Y)$ satisfies $u \circ U_s = V_s \circ u$ for all $s \in S$, then $u(X_{wc}) \subset Y_{wc}$, $u(X_{wap}) \subset Y_{wap}$ and $u(X_{ap}) \subset Y_{ap}$, as is easily verified.

A semigroup S with a topology is *left* (*right*) *topological* if the mappings $\lambda_t : s \mapsto ts$ ($\rho_t : s \mapsto st$), $t \in S$, are continuous. A compact, Hausdorff, left (or right) topological semigroup S has a smallest ideal $K(S)$, called the *minimal ideal* of S. We use freely the properties of $K(S)$ (see, for example, Sections 1.2 and 1.3 of [2]).

A *left topological compactification* of a stsg S is a pair (φ, S'), where S' is a compact, Hausdorff, left topological semigroup and $\varphi : S \mapsto S'$ is a continuous homomorphism with dense range such that the mappings $\rho_{\varphi(s)} : S' \mapsto S'$, $s \in S$, are continuous. Right topological compactifications are defined analogously. A compactification (φ, S') is said to be (*semi*)*topological* if S' is a (semi)topological semigroup. We sometimes omit reference to the mapping φ and call S' a compactification of S. **Compactifications in this paper are assumed to be left topological unless otherwise stipulated.**

A *homomorphism* from a compactification (φ, S') of a stsg S to a compactification (ψ, S'') of S is a continuous function $\theta : S' \mapsto S''$ such that $\theta \circ \varphi = \psi$. The compactification S'' is then called a *factor* of S' and S' an *extension* of S'' and we write $(\varphi, S') \geq (\psi, S'')$. Note that θ is necessarily unique, surjective and

a semigroup homomorphism. If θ is 1-1 then θ^{-1} is a compactification homomorphism. In this case θ is called an *isomorphism* and we write $(\varphi, S') \cong (\psi, S'')$. If $(\varphi, S') \geq (\psi, S'')$ and $(\varphi, S') \not\cong (\psi, S'')$ we write $(\varphi, S') > (\psi, S'')$. The relation $\geq$ partially orders (equivalence classes of) compactifications of S.

Let F be a translation invariant subspace of $C(S)$. For $\mu \in F'$, the dual space of F, define the *right introversion operator* μ_r on F by $(\mu_r f)(s) := \mu(R_s f)$. F is *right introverted* if $\mu_r F \subset F$ for every $\mu \in F'$. If F is a C^*-subalgebra of $C(S)$ and $\mu_r F \subset F$ for every $\mu \in S^F$, the spectrum of F, then F is said to be *right m-introverted*. *Left introversion* and *left m-introversion* are defined analogously. The space $WRC(S)$ is right introverted [2, 4.5.2], and any closed, translation invariant subspace of $WAP(S)$ is both right and left introverted [2, 4.2.7].

Every left topological compactification (φ, S') of a stsg S is isomorphic to the *canonical F-compactification* (ϵ, S^F) of S. Here, F is the right m-introverted C^*-subalgebra $\varphi^*(C(S'))$ (the *function space* of the compactification), $\epsilon : S \mapsto S^F)$ is the evaluation mapping $\epsilon(s) : f \mapsto f(s)$, and multiplication in S^F is given by $\mu * \nu = \mu_r'(\nu)$. Every translation invariant, right m-introverted C^*-subalgebra F of $C(S)$ which contains the constant functions is the function space of a left topological compactification of S. Similar remarks apply to right topological compactifications.

If S and T are stsgs, $\phi : S \mapsto T$ is a continuous homomorphism and (ψ, T') is a compactification of T, then the induced compactification $(\psi \circ \phi, \overline{\psi \circ \phi(S)})$ of S is called a *subcompactification* of (ψ, T'). If $\{S_i : i \in I\}$ is a family of stsgs and (ψ_i, S_i') is a compactification of S_i, then the compactification $(\prod_i \psi_i, \prod_i S_i')$ of the stsg $\prod_i S_i$ is called the *direct product* of the compactifications (ψ_i, S_i'). If $S_i = S$ for all i and $\psi(s) := (\psi_i(s))_i$, $s \in S$, then the subcompactification $(\psi, \overline{\psi(S)})$ of $(\prod_i \psi_i, \prod_i S_i')$ is called the *subdirect product* of the compactifications (ψ_i, S_i').

A compactification with a given property $\mathcal{P}$ which is invariant under compactification isomorphisms is called a $\mathcal{P}$-*compactification*. A *universal* or *maximal $\mathcal{P}$-compactification* of a stsg S is a $\mathcal{P}$-compactification which is an extension of every $\mathcal{P}$-compactification of S. *Minimal $\mathcal{P}$-compactifications* are defined analogously. Maximal and minimal $\mathcal{P}$-compactifications of S, if they exist, are unique up to isomorphism. We denote the maximal and minimal $\mathcal{P}$-compactifications of S by $S^\mathcal{P}$ and $S_\mathcal{P}$, respectively.

A property $\mathcal{P}$ of compactifications (of a specified or understood family of stsgs) is said to be *inherited by factors* (*extensions, direct products, subcompactifications, subdirect products*) if factors (extensions, direct products, subcompactifications, subdirect products) of $\mathcal{P}$-compactifications are again $\mathcal{P}$-compactifications. For example, let $\mathcal{P}$ be the property of compactifications (φ, S') of a locally compact topological group S that $\lim_{s \to \infty} \varphi(s)$ exists, and let $\mathcal{Q}$ be the property that φ is an isomorphism and $S' \backslash \varphi(S)$ is an ideal of S'. Then $\mathcal{P}$ is inherited by factors, $\mathcal{Q}$ is inherited by extensions and $S^\mathcal{P} = S_\mathcal{Q} = S^\infty$, the one-point compactification of S (with infinity acting as a zero). If $\mathcal{P}$ is, more generally, a property of compact left topological semigroups we say that $\mathcal{P}$ is *inherited by continuous homomorphic images* if any continuous homomorphic image of a compact left topological semigroup with property $\mathcal{P}$ also has property $\mathcal{P}$. Similarly, $\mathcal{P}$ is *inherited by direct products* if the direct product of a family of compact left topological semigroups with property $\mathcal{P}$ also has property $\mathcal{P}$.

We shall call a property $\mathcal{P}$ of compactifications (resp., compact left topological semigroups) *regular* if universal $\mathcal{P}$-compactifications exist and if $\mathcal{P}$ is inherited by

factors (resp., continuous homomorphic images). Note that the conjunction of regular properties is regular. Some examples of regular properties are: the property of being a topological or semitopological semigroup or group, the property of being a semilattice, the property of being a left simple, right simple or simple semigroup, the property of having a left, right or two-sided zero, and the property that the minimal ideal is left simple or right simple. These are also examples of properties which are inherited by direct products.

3. Maximal $\mathcal{P}$-Subspaces of a Representation

Let X be a lctvs with strong dual $X' = X'_\beta$ and let $X'' = (X'_\beta)'$ be the bidual of X with the (natural) topology of uniform convergence on equicontinuous subsets of X'. We identify X with its image in the bidual under the canonical imbedding $x \mapsto \hat{x}$. The topology of X'' is generated by the seminorms

$$q_{A'}(x'') := \sup\{|\langle x', x''\rangle| : x' \in A'\},$$

where A' is an equicontinuous subset of X'. The topology of X is generated by the seminorms $p_{A'} := q_{A'}|_X$.

Following Witz [12] we define the product uv of $u, v \in \mathcal{L}(X, X'')$ by $uv := \tilde{u}' \circ v$, where $\tilde{u} := u'|_{X'} \in \mathcal{L}(X')$. Thus

$$\langle x', uv(x)\rangle = \langle u'(x'), v(x)\rangle \qquad (x \in X,\ x' \in X').$$

Under this multiplication, $\mathcal{L}(X, X'')$ is a left topological semigroup in the weak* operator topology, and the mapping $u \mapsto \tilde{u} : \mathcal{L}(X, X'') \mapsto \mathcal{L}(X')$ is a homeomorphism and a semigroup anti-isomorphism. If $v(x) \in X$ then $(uv)(x) = u(v(x))$, hence multiplication restricted to $\mathcal{L}(X, X'') \times \mathcal{L}(X)$ is separately continuous. In particular, $\mathcal{L}(X)$ is a semitopological subsemigroup of $\mathcal{L}(X, X'')$.

Now let U be an equicontinuous representation of a stsg S on X. For a U-invariant subspace Y of X let $\mathcal{U}_Y$ denote the closure of $U_S|_Y$ in the weak* operator topology of $\mathcal{L}(Y, Y'')$. Then $\mathcal{U}_Y$ is equicontinuous and is a weak* operator compact subsemigroup of $\mathcal{L}(Y, Y'')$. Moreover, $\mathcal{U}_Y \subset \mathcal{L}(Y)$ if and only if $Y \subset X_{wap}$, in which case $\mathcal{U}_Y$ is a stsg and is the weak operator closure of $U_S|_Y$ in $\mathcal{L}(Y)$. If $Y \subset X_{ap}$ then $\mathcal{U}_Y$ is a topological semigroup, and if $Y \subset X_{wc}$ then $\mathcal{U}_Y$ is a left topological compactification of S (with compactification map $s \mapsto U_s|_Y$). We write $\mathcal{U}$ for $\mathcal{U}_X$. Compactifications of semigroups of operators on Banach spaces were studied by deLeeuw and Glicksberg [3,4] in the weakly almost periodic case and by K. Witz in the general case in connection with various ergodic problems [12].

The following simple result will have important consequences in what follows.

Lemma 1. *Let U be a weakly continuous, equicontinuous representation of a stsg S on a lctvs X. If Y is a U-invariant subspace of X then $\mathcal{U} \geq \mathcal{U}_Y$. The compactification homomorphism $\pi_{YX} : \mathcal{U} \mapsto \mathcal{U}_Y$ satisfies*

(1) $$\langle x'|_Y, \pi_{YX}(u)(y)\rangle = \langle x', u(y)\rangle \qquad (u \in \mathcal{U},\ y \in Y,\ x' \in X').$$

Proof. The mapping π_{YX} defined by (1) is well defined, continuous, and maps U_s to $U_s|_Y$. Therefore π_{YX} is a compactification homomorphism, hence $\mathcal{U} \geq \mathcal{U}_Y$.

Note that $\mathcal{U}_Y$ is isomorphic to $\mathcal{U}|_Y := \{u|_Y : u \in \mathcal{U}\} \subset \mathcal{L}(Y, X'')$, where multiplication in $\mathcal{U}|_Y$ is defined by $u|_Y v|_Y = (uv)|_Y$. Indeed, the mapping $\pi_{YX}(u) \mapsto u|_Y$ is easily seen to be a compactification isomorphism.

Example 1. Consider the right translation representation $s \mapsto R_s$ on the Banach space $X := WRC(S)$. Let Y be a right translation invariant subspace of X. Denote by Y^r the right introverted subspace generated by Y and let Y^a denote the (necessarily right m-introverted) C^*-algebra generated by Y^r and the constant functions . We use the compactification homomorphism of Lemma 1 to show that

(a) $\mathcal{R}_Y \cong \mathcal{R}_{Y^r}$;

(b) $\mathcal{R}_Y \cong S^{Y^a}$ if S has a right identity and $Y^a \subset WRC(S)$.

(The inclusion $Y^a \subset WRC(S)$ in (b) holds if S is a k-space or a complete metric semitopological group, since in each case $WRC(S)$ is an algebra [2, p. 173].)

To verify (a) note first that for $\mu, \nu \in Y^{r\prime}$, $s \in S$ and $f \in Y^r$ we have $\langle \nu, R_s(\mu_r f) \rangle = \langle R_s f, \mu'_r(\nu) \rangle$, so letting R_s tend to u in $\mathcal{R}_{Y^r}$ we obtain

$$(2) \qquad \langle \nu, u(\mu_r f) \rangle = \langle \mu'_r(\nu), u(f) \rangle \qquad (\mu, \nu \in Y^{r\prime}, \ u \in \mathcal{R}_{Y^r}, f \in Y^r).$$

If $f \in Y$, then the right side of (2) is $\langle \mu'_r(\nu)|_Y, \pi_{Y Y^r}(u)(f) \rangle$, which shows that $\pi_{Y Y^r}$ is 1-1 and therefore a compactification isomorphism.

To verify (b) define $\theta : \mathcal{R}_{Y^r} \mapsto S^{Y^a}$ by

$$\theta(\pi_{Y^r Y^a}(u))(f) = \langle \epsilon(1_r), u(f) \rangle \qquad (u \in \mathcal{R}_{Y^a}, f \in Y^a),$$

where 1_r is a right identity of S. θ is well-defined, for if $\pi_{Y^r Y^a}(u) = \pi_{Y^r Y^a}(v)$, $u, v \in \mathcal{R}_{Y^a}$, then, since u and v are weak* operator limits of homomorphisms, $\langle \nu, u(fg) \rangle = \langle \nu, v(fg) \rangle$ for all $f, g \in Y^r$ and all $\nu \in S^{Y^a}$, hence $\langle \epsilon(1_r), u(h) \rangle = \langle \epsilon(1_r), v(h) \rangle$ for all $h \in Y^a$. Since θ is continuous and $\theta(R_s) = \epsilon(s)$, θ is a compactification homomorphism. Moreover, for $f \in Y^r$ and $v \in \mathcal{R}_{Y^r}$, $\theta(v)(f) = \langle \epsilon(1_r)|_{Y^r}, v(f) \rangle$ from (1), hence, from (2), $\theta(v)(\mu_r f) = \langle \mu * (\epsilon(1_r)|_{Y^r}), v(f) \rangle = \langle \mu, v(f) \rangle$ for all $\mu \in Y^{r\prime}$. Therefore θ is 1-1 and hence is an isomorphism.

Definition 1. Let U be an equicontinuous representation of a stsg S on a lctvs X and let $\mathcal{P}$ be a property of compactifications of S. A U-invariant subspace Y of X is called a $\mathcal{P}$-*subspace* of X (or of the representation U) if the compactification $\mathcal{U}_Y$ has property $\mathcal{P}$. A $\mathcal{P}$-subspace of X which is not properly contained in another $\mathcal{P}$-subspace is called a *maximal $\mathcal{P}$-subspace of X*. If X is a $\mathcal{P}$-space then U is called a $\mathcal{P}$-*representation* of S.

Note that the closure of a $\mathcal{P}$-subspace is a $\mathcal{P}$-subspace ($\pi_{Y \overline{Y}}$ is a compactification isomorphism).

The following example shows that a representation need not have a maximal $\mathcal{P}$-space.

Example 2. Let $S = (\mathbb{R}, +)$, $X = WRC(S)$, $U_s = R_s$, and let $\mathcal{P}$ be the property of compactifications (φ, S') of S that φ is not 1-1. For $p > 0$ let Y_p be the space of all periodic functions in X with p as a period. An invariant subspace Y of X is a $\mathcal{P}$-subspace if and only if $Y \subset Y_p$ for some $p > 0$. Since Y_p is a proper subset of Y_{2p}, a maximal $\mathcal{P}$-subspace cannot exist. Note that if $\mathcal{Q}$ is the negation of $\mathcal{P}$ (namely, the property that φ is 1-1), then X is the unique maximal $\mathcal{Q}$-subspace, and each nonperiodic function in X generates a *minimal $\mathcal{Q}$-subspace* (analogously defined).

The next example shows that maximal $\mathcal{P}$-spaces need not be unique.

Example 3. Let $\mathcal{P}$ be the property of compactifications (φ, S') of a stsg S that φ is constant on a nontrivial subgroup of S. If $X = WRC(S)$ and $U_s = R_s$ then an invariant subspace Y of X is a $\mathcal{P}$-subspace if and only if there exists a nontrival subgroup T of S such that $Y \subset Y_T := \{f \in X : f(st) = f(st') \text{ for all } t, t' \in T, \ s \in S\}$. Let S be a discrete, commutative semigroup which is the disjoint union of nontrivial simple subgroups S_1 and S_2 such that $s_1 s_2 = s_2$, $s_i \in S_i$. Then S_1 and S_2 are the only nontrivial subgroups of S, and since $Y_{S_j} = \{f : f|_{S_j} \text{ is constant}\}$ it follows that Y_{S_1} and Y_{S_2} are distinct maximal $\mathcal{P}$-subspaces of X.

For regular properties, the pathologies exhibited by Examples 2 and 3 cannot occur. Indeed, if $\mathcal{P}$ is regular then the union of all $\mathcal{P}$-subspaces of X is a $\mathcal{P}$-space. To prove this we use the first part of the following lemma.

Lemma 2. *Let I be a directed set and let $\{(\psi_i, S_i') : i \in I\}$ be a family of compactifications of a stsg S such that $S_i' \leq S_j'$ whenever $i \leq j$. Let*

$$S' := \varprojlim S_i' = \{(u_i)_i \in \prod_{i \in I} S_i' : \theta_{ij}(u_j) = u_i, \ i \leq j\}$$

be the inverse limit of the semigroups S_i' relative to the compactification homomorphisms $\theta_{ij} : S_j' \mapsto S_i'$, $i \leq j$, and set $\psi(s) := (\psi_i(s))_i$, $s \in S$. Then (ψ, S') is the compactification of S which is minimal with respect to the property that $(\psi, S') \geq (\psi_i, S_i')$ for all i. Moreover, $K(S') = \varprojlim K(S_i')$ (relative to the mappings $\theta_{ij}|_{K(S_j')}$).

Proof. Let (ϕ, S'') be an extension of every (ψ_i, S_i') and let $\theta_i : S'' \mapsto S_i'$ be the corresponding compactification homomorphism. Define $\theta : S'' \mapsto \prod_i S_i'$ by $\theta(u) = (\theta_i(u))_i$. We claim that $\theta(S'') = S'$. Indeed, if $i \leq j$ then $\theta_{ij} \circ \theta_j = \theta_i$, hence $\theta(S'') \subset S'$. For the reverse inclusion let $(u_i)_i \in S'$ and note that since I is directed and θ_i is surjective, the sets $\theta_i^{-1}(u_i)$, $i \in I$, have the finite intersection property and hence must contain a common point $u \in \theta^{-1}((u_i)_i)$. Therefore θ maps S'' continuously onto S'. Since $\theta \circ \phi = \psi$ it follows that (ψ, S') is a compactification of S and a factor of (ϕ, S'').

To prove the last assertion of the lemma, observe that the inverse limit $J := \varprojlim K(S_i')$ is defined since $\theta_{ij}(K(S_j')) = K(S_i')$, $i \leq j$. Let $p_i : S' \mapsto S_i'$ be the projection homomorphism. Since $p_i(S') = S_i'$, $p_i(K(S')) = K(S_i')$. Therefore J contains the idempotents of $K(S')$ so J is nonempty and hence is an ideal containing $K(S')$. If $s' = (s_i')_i \in J$ then for each i there exists a unique idempotent $e_i \in K(S_i')$ such that $s_i' \in e_i S_i' e_i$. If $i \leq j$ then $d_i := \theta_{ij}(e_j)$ is an idempotent in $K(S_i')$ and $s_i' = \theta_{ij}(s_j') = \theta_{ij}(e_j s_j' e_j) = d_i s_i' d_i$ so $e_i = d_i$. Therefore $e = (e_i)_i \in S'$ and $es'e = s'$. Thus, to show that $s' \in K(S')$ it suffices to show that $e \in K(S')$. This follows easily from the characterization of idempotents in the minimal ideal of a semigroup as minimal relative to the natural partial ordering of idempotents (see, for example, [2, 1.2.29]).

The compactification (ψ, S') of Lemma 2 is called the *inverse limit* of the compactifications (ψ_i, S_i') and is denoted by $\varprojlim(\psi_i, S_i')$ (or simply by $\varprojlim S_i'$).

Corollary 1. *If $\mathcal{P}$ is a regular property of compactifications and if each (ψ_i, S_i') is a $\mathcal{P}$-compactification of S, then $\varprojlim(\psi_i, S_i')$ is a $\mathcal{P}$-compactification of S.*

Corollary 2. *Let U be a weakly continuous, equicontinuous representation of a stsg S on a lctvs X, let I be a directed set and let $\{Y_i : i \in I\}$ be a family of U-invariant subspaces of X such that $Y_i \subset Y_j$ whenever $i \leq j$. Then $Y := \bigcup_{i \in I} Y_i$ is a U-invariant subspace of X and $\mathcal{U}_Y \cong \varprojlim \mathcal{U}_{Y_i}$. Hence, if $\mathcal{P}$ is a regular property of compactifications and each Y_i is a $\mathcal{P}$-space, then Y is a $\mathcal{P}$-space.*

Proof. Y is clearly a U-invariant subspace of X and $\mathcal{U}_Y \geq \mathcal{U}_{Y_i}$ for all i. By Lemma 2, $\mathcal{U}_Y \geq \varprojlim \mathcal{U}_{Y_i}$ under the compactification homomorphism $\Psi : \mathcal{U}_Y \mapsto \prod_{i \in I} \mathcal{U}_{Y_i}$, $u \mapsto (\pi_{Y_i Y}(u))_i$. It follows from (1) that Ψ is 1-1 and hence is an isomorphism. The last assertion of the corollary is a consequence of Corollary 1.

Theorem 1. *Let U be a weakly continuous, equicontinuous representation of a stsg S on a lctvs X. If $\mathcal{P}$ is a regular property of compactifications then X has a unique maximal $\mathcal{P}$-subspace $X_{\mathcal{P}}$. Moreover, $X_{\mathcal{P}}$ is closed and the $\mathcal{P}$-subspaces of X are precisely the invariant subspaces of $X_{\mathcal{P}}$.*

Proof. We show first that if Y and Z are $\mathcal{P}$-subspaces then so is $W := Y + Z$. Let S' be the universal left topological compactification of S and consider the compactification homomorphisms $\phi : S' \mapsto S^{\mathcal{P}}$, $\phi_w : S' \mapsto \mathcal{U}_W$, $\theta_Y : S^{\mathcal{P}} \mapsto \mathcal{U}_Y$ and $\theta_Z : S^{\mathcal{P}} \mapsto \mathcal{U}_Z$. Define $\theta_w : S^{\mathcal{P}} \mapsto \mathcal{U}_W$ by $\theta_w(\phi(t)) = \phi_w(t)$, $t \in S'$. θ_w is well defined, since if $w' \in W'$, $y \in Y$ and $z \in Z$ then from (1)

$$\langle w', \phi_w(t)(y+z) \rangle = \langle w', \phi_w(t)(y) \rangle + \langle w', \phi_w(t)(z) \rangle$$
$$= \langle w'|_Y, (\pi_{YW} \circ \phi_w)(t)(y) \rangle + \langle w'|_Z, (\pi_{ZW} \circ \phi_w)(t)(z) \rangle$$
$$= \langle w'|_Y, (\theta_Y \circ \phi)(t)(y) \rangle + \langle w'|_Z, (\theta_Z \circ \phi)(t)(z) \rangle.$$

Since θ_w is obviously a compactification homomorphism, the regularity of $\mathcal{P}$ implies that W is a $\mathcal{P}$-subspace of X.

Now let $\mathcal{Y}$ denote the family of all $\mathcal{P}$-subspaces of X. Since the trivial semigroup is a $\mathcal{P}$-compactification of S (it is a homomorphic image of $S^{\mathcal{P}}$), $\{0\}$ is a $\mathcal{P}$-subspace of X, so $\mathcal{Y} \neq \emptyset$. By the first part of the proof, $\mathcal{Y}$ is directed upward by inclusion. Therefore, by Corollary 2 of Lemma 2, $X_{\mathcal{P}} := \bigcup \mathcal{Y}$ is a $\mathcal{P}$-subspace of X. Since $\overline{X}_{\mathcal{P}}$ is also a $\mathcal{P}$-subspace, $X_{\mathcal{P}}$ is closed. That any invariant subspace of $X_{\mathcal{P}}$ is a $\mathcal{P}$-space is consequence of Lemma 1 and the regularity of $\mathcal{P}$.

Theorem 2. *Let U be a weakly continuous representation of a stsg S on a commutative C^*-algebra X. If each U_s is a *-homomorphism and if either U_S is weakly almost periodic or X has separable dual, then $X_{\mathcal{P}}$ is a C^*-subalgebra of X. If X and the operators U_s are unital then $X_{\mathcal{P}}$ is unital.*

Proof. Let $\Omega(X)$ denote the spectrum of X with the weak* topology. Since $u \in \mathcal{U}$ is the weak* operator limit of *-homomorphisms we have

$$\text{(3)} \qquad \langle \tau, u(xy) \rangle = \langle \tau, u(x) \rangle \langle \tau, u(y) \rangle \qquad (x, y \in X, \ \tau \in \Omega(X))$$

and

$$\text{(4)} \qquad \langle \tau, u(x^*) \rangle = \overline{\langle \tau^*, u(x) \rangle} \qquad (x \in X, \tau \in X'),$$

where $\tau^*(x) := \overline{\tau(x^*)}$. Let $Y = X_{\mathcal{P}}$ and let Z be the (necessarily U_S-invariant) C^*-algebra generated by Y. To show that $Y = Z$ it suffices to show that the

compactification homomorphism $\pi_{YZ} : \mathcal{U}_Z \mapsto \mathcal{U}_Y$ is 1-1. A typical member of Z is a limit of sums

$$z = \sum_i x_{i1} x_{i2} \cdots x_{in_i} y_{i1}^* y_{i2}^* \cdots y_{im_i}^* \qquad (x_{ik}, y_{ik} \in Y).$$

For any $\tau \in \Omega(Z)$ and $u \in \mathcal{U}_Z$ we have, by (1), (3) and (4),

$$\langle \tau, u(z) \rangle = \sum_i \prod_j \langle \tau, u(x_{ij}) \rangle \prod_k \overline{\langle \tau^*, u(y_{ik}) \rangle}$$

(5)
$$= \sum_i \prod_j \langle \tau|_Y, \pi_{YZ}(u)(x_{ij}) \rangle \prod_k \overline{\langle \tau^*|_Y, \pi_{YZ}(u)(y_{ik}) \rangle}.$$

Now let $u, v \in \mathcal{U}_Z$ with $\pi_{YZ}(u) = \pi_{YZ}(v)$. From (5),

(6)
$$\langle \tau, u(z) \rangle = \langle \tau, v(z) \rangle \qquad (\tau \in \Omega(Z), \ z \in Z).$$

We claim that $u = v$. If U_S is weakly almost periodic this follows from (6) and the Gelfand representation. Now suppose that X has separable dual and let $z \in Z$. Then $\mathcal{U}(z)$ is weak* metrizable hence there exist *sequences* $U_{s_n} z \overset{w^*}{\to} u(z)$ and $U_{t_n} z \overset{w^*}{\to} v(z)$. From (6) we have $\langle (U_{s_n} - U_{t_n})(z), \tau \rangle \to 0$, $\tau \in \Omega(Z)$. Since an arbitrary member z' of Z' can be represented by a Borel measure on $\Omega(Z)$, the dominated convergence theorem implies that $\langle z', u(z) - v(z) \rangle = 0$ so $u = v$ in the separable case as well. Therefore $Z = X_{\mathcal{P}}$.

If X and the operators U_s are unital and if W denotes the C^*-algebra generated by the unit and $X_{\mathcal{P}}$, then $\pi_{X_{\mathcal{P}}W}$ is an isomorphism hence $W = X_{\mathcal{P}}$.

If U is a weakly continuous, equicontinuous representation of a stsg S on a lctvs X then by Theorem 1 there exist maximal subspaces X_{stsg} and X_{tsg} of X such that $\mathcal{U}_{X_{stsg}}$ is a stsg and $\mathcal{U}_{X_{tsg}}$ is a tsg. By maximality,

(7)
$$X_{wap} \subset X_{stsg} \quad \text{and} \quad X_{ap} \subset X_{tsg}.$$

We make use of these spaces in the following examples.

Example 4. Consider the right translation representation R on the space $X = WRC(S)$. If S has a topological left (or right) identity then equality holds in (7). Indeed, for sequences $\{R_{s_n}\}$ and $\{R_{t_m}\}$ and for $f \in WRC(S)_{stsg}$ and $\tau \in WRC(S)'$, the separate continuity of multiplication in $\mathcal{R}_{WRC(S)_{stsg}}$ implies that $\lim_n \lim_m \langle R_{s_n} R_{t_m} f, \tau \rangle = \lim_m \lim_n \langle R_{s_n} R_{t_m} f, \tau \rangle$ whenever all limits exist. Taking τ to be a weak* limit point of the net of evaluations at the topological left identity of S gives $\lim_n \lim_m f(s_n t_m) = \lim_m \lim_n f(s_n t_m)$, which is the double limit criterion for weak almost periodicity of f. Thus $WAP(S) = WRC(S)_{stsg}$. To see that $AP(S) = WRC(S)_{tsg}$, let $f \in WRC(S)_{tsg}$ and let $R_{s_\alpha} \to u$ in $\mathcal{R}_{WRC(S)_{tsg}}$. The joint continuity of multiplication in $\mathcal{R}_{WRC(S)_{tsg}}$ implies that $R_{s_\alpha} f(t) = \langle \tau, R_t R_{s_\alpha} f \rangle$ converges uniformly in t to $\langle \tau, R_t u f \rangle$, hence $f \in AP(S)$.

The inclusions in (7) can be strict. For example, let $S = [0, \infty)^3$ with the usual topology and with multiplication $(a, b, c)(a', b', c') = (0, ac', 0)$. For any

$f \in WRC(S)$ and $s, t \in S$, $R_s R_t f$ is the constant function $f(0, 0, 0)$ hence $(uv)(f) = f(0, 0, 0)$ for all $u, v \in \mathcal{R}_{WRC(S)}$. Therefore, trivially, $WRC(S)_{stsg} = WRC(S)_{tsg} = WRC(S)$. But it is easy to construct functions f in $WRC(S) \backslash WAP(S)$. For example, if $g(a) = \arctan(\ln a)$, $a > 0$, and $g(0) = -\pi/2$, then $f(a, b, c) = g(b)$ is such a function.

In the next examples we consider maximal $\mathcal{P}$-spaces which arise from certain topological dynamics considerations. These are based on the observation that for each $x \in X$ the semigroup S acts on $\mathcal{U}(x)$ by

$$(8) \qquad\qquad (s, u(x)) \mapsto (U_s u)(x),$$

the action being jointly continuous if $x \in X_{tsg}$ and separately continuous if S is commutative or if $x \in X_{stsg}$. In the latter case the map

$$\theta_x : \mathcal{U} \mapsto \Sigma_x, \quad \theta_x(u)(v(x)) = uv(x)$$

is a compactification homomorphism, where Σ_x is the enveloping semigroup of the action.

Example 5. Call a vector $x \in X$ *distal* (with respect to the representation U) if the action in (8) is distal, i.e., if $\lim_\alpha (U_{s_\alpha} u)(x) = \lim_\alpha (U_{s_\alpha} v)(x)$ implies $u(x) = v(x)$. Denote the set of all distal vectors by X_d. We show that if $x \in X_{stsg}$ then the following statements are equivalent:
  (a) $x \in X_d$.
  (b) $u, u_1, u_2 \in \mathcal{U}$ and $uu_1(x) = uu_2(x) \Rightarrow u_1(x) = u_2(x)$.
  (c) $ev(x) = v(x)$ for all (minimal) idempotents $e \in \mathcal{U}$ and all $v \in \mathcal{U}$.
   The equivalence of (a) and (b) is an obvious consequence of the separate continuity of multiplication in $\mathcal{U}_{X_{stsg}}$. That (b) $\Rightarrow$ (c) is seen by taking $u = e$, $u_1 = ev$ and $u_2 = v$ in (b), where e is an arbitrary idempotent in $\mathcal{U}$. For (c) $\Rightarrow$ (a) let $e \in \mathcal{U}$ be a minimal idempotent (i.e., an idempotent in $K(\mathcal{U})$) such that $ev(x) = v(x)$ for all $v \in \mathcal{U}$. Then $\theta_x(e)$ is the identity of Σ_x, hence $\Sigma_x = \theta_x(\mathcal{U}) = \theta_x(e\mathcal{U}e)$. Since $e\mathcal{U}e$ is a group, it follows from a well known theorem of Ellis [5,6] that $x \in X_d$.

If S is commutative then similar arguments show that for *arbitrary* $x \in X$, $x \in X_d$ iff $ev(x) = ve(x) = v(x)$ for all $e^2 = e, v \in \mathcal{U}_{X_d}$.

It follows from (c) that $X_{stsg} \cap X_d$ is a closed, invariant subspace of X and is maximal with respect to the property that $\mathcal{U}_{X_{stsg} \cap X_d}$ is a right simple stsg compactification of S. Similarly, if S is commutative then X_d is a closed, invariant subspace of X and is maximal with respect to the property that $\mathcal{U}_{X_d}$ is a left topological group.

Example 6. Call a vector $x \in X$ *minimal* (with respect to U) if for each $v \in \mathcal{U}$ there exists $u \in \mathcal{U}$ such that $uv(x) = x$. Denote the set of all minimal vectors by X_m and set $X_{md} := X_m \cap X_d$. We show that for $x \in X_{stsg}$,
  (a) $x \in X_m \iff$ there exists a minimal idempotent $e \in \mathcal{U}$ such that $e(x) = x$;
  (b) $x \in X_{md} \iff ev(x) = v(x)$ and $e(x) = x$ for all $v \in \mathcal{U}$ and all (minimal) idempotents $e \in \mathcal{U}$.
(Part (a) is proved in [3] for weakly almost periodic semigroups of operators on Banach spaces.) Since the homomorphism $\pi_{X_{stsg}X}$ maps the minimal idempotents of $\mathcal{U}$ onto the minimal idempotents of $\mathcal{U}_{X_{stsg}}$, we may suppose that $X = X_{stsg}$.

To prove (a), let $x \in X_m$. Then, for any minimal idempotent d, the set $\{u \in \mathcal{U}d : u(x) = x\}$ is a nonempty, closed subsemigroup of $\mathcal{U}$ and hence contains an idempotent. Conversely, assume that $e(x) = x$ for some minimal idempotent $e \in \mathcal{U}$. If we choose $w \in \mathcal{U}$ so that $weve = e$, then $x = weve(x) = wev(x)$, so $x \in X_m$.

To prove (b) let $x \in X_{md}$ and let $e^2 = e \in \mathcal{U}$. Choose $u \in \mathcal{U}$ such that $ue(x) = x$. Then $e(x) = eue(x) = ue(x) = x$, the middle equality by (c) of Example 5. The converse follows from (a) of this example and from (c) of Example 5.

As a consequence of (b), $X_{md} \cap X_{stsg}$ is a closed, invariant subspace of X and is maximal with respect to the property that $\mathcal{U}_{X_{md} \cap X_{stsg}}$ is a topological group with identity the identity operator. Hence, if U_S is weakly almost periodic then X_{md} is the space X_{sap} of strongly almost periodic vectors of X [2,3] and $Y = X_d$ has the deLeeuw-Glicksberg *splitting property*:

$$Y = X_{sap} \oplus (Y \cap \ker e) \text{ for all minimal idempotents } e \in \mathcal{U}.$$

In this connection, let X_{umri} denote the subspace of X which is maximal with respect to the property that $\mathcal{U}_{X_{umri}}$ has a unique minimal right ideal. It follows from [3, Theorem 4.10] (or [2, 6.2.14]) that if U_S is weakly almost periodic then X_{umri} is the largest invariant subspace of X with the splitting property.

We note for future use that if U_S is weakly almost periodic and $X = X_{umri}$ then $X_{md} = X_m$. This follows from (a) and (b) and the fact that $eve = ve$ and $de = e$ for all minimal idempotents $d, e \in \mathcal{U}$ and all $v \in \mathcal{U}$ [2, 1.2.17].

Example 7. Call a vector $x \in X$ *proximal* (with respect to U) if the action of (8) is proximal, i.e., if for each pair u, $v \in \mathcal{U}$ there exists a net $\{U_{s_\alpha}\}$ such that $\lim_\alpha (U_{s_\alpha} u)(x) = \lim_\alpha (U_{s_\alpha} v)(x)$. Denote the set of proximal vectors by X_p. We show that if $x \in X_{stsg}$ then

$$x \in X_p \iff wu(x) = w(x) \text{ for all } w \in K(\mathcal{U}) \text{ and } u \in \mathcal{U}.$$

Note first that since $\mathcal{U}_{X_{stsg}}$ is semitopological, $x \in X_p \iff$ for each pair $u, v \in \mathcal{U}$ there exists $w \in \mathcal{U}$ such that $wu(x) = wv(x)$. Hence, if $x \in X_p$ then $wu(x) = wv(x)$ for all $w \in \bigcup \{\mathcal{U}e : e$ a minimal idempotent$\} = K(\mathcal{U})$. Taking $w \in \mathcal{U}e$ and $v = e$ we have $wu(x) = we(x) = w(x)$ for all $u \in \mathcal{U}$. The converse is clear.

As a consequence, $X_{stsg} \cap X_p$ is a closed invariant subspace of X and is maximal with respect to the property that $K(\mathcal{U}_{X_{stsg} \cap X_p})$ is the set of left zeros of $\mathcal{U}_{X_{stsg} \cap X_p}$ (or, equivalently, is a left zero semigroup). Also, $X_{stsg} \cap X_d \cap X_p$ is maximal with respect to the property that $\mathcal{U}_{X_{stsg} \cap X_d \cap X_p}$ consists of a single projection (with range $X_{stsg} \cap X_{md} \cap X_p$), and $X_{stsg} \cap X_{md} \cap X_p$ is maximal with respect to the property that $\mathcal{U}_{X_{stsg} \cap X_{md} \cap X_p}$ consists of the identity operator.

4. Inductive and Projective Limits of $\mathcal{P}$-Subspaces

In this section we prove that the property of being a $\mathcal{P}$-space is preserved under the formation of projective and strict inductive limits. We treat the easy inductive limit case first.

Theorem 3. *Let* $X = \bigcup_n X_n$ *be the strict inductive limit of a sequence of lctvs* X_n. *Let* S *be a stsg and for each* n *let* U^n *be an equicontinuous representation of* S

on X_n such that $U_s^{n+1}|_{X_n} = U_s^n$. For $s \in S$ define U_s on X by $U_s|_{X_n} = U_s^n$. Then $s \mapsto U_s$ is an equicontinuous representation of S on X such that

$$X_{wc} = \bigcup_n X_{n\,wc}, \quad X_{wap} = \bigcup_n X_{n\,wap} \quad and \quad X_{ap} = \bigcup_n X_{n\,ap}.$$

Moreover, if $\mathcal{P}$ is a regular property of compactifications of S and if each U^n is weakly continuous then U is weakly continuous and

$$X_{\mathcal{P}} = \bigcup_n X_{n\,\mathcal{P}}.$$

Proof. U is clearly a representation on X. To see that U is equicontinuous, let p be a continuous seminorm on X. Then $p|_{X_n}$ is a continuous seminorm on X_n, hence there exists a continuous seminorm q_n on X_n such that $p(U_sx) \leq q_n(x)$ for all $x \in X_n$ and $s \in S$. By definition of inductive limit topology, $q(x) = \sup_{s \in S} p(U_sx)$ defines a continuous seminorm on X and $p(U_sx) \leq q(x)$ for all $x \in X$ and $s \in S$.

The assertions regarding the spaces X_{wc}, X_{wap}, and X_{ap} are clear, since the original topology of X_n is the relative topology from X. The last assertion of the theorem follows from the observation that $X_{n\,\mathcal{P}} = X_{\mathcal{P}} \cap X_n$.

Corollary. *If, for each n, X_n is a strictly convex Banach space, U^n is weakly almost periodic and $\|U_s^n\| \leq 1$ for all s, then X has the splitting property (see Example* 6).

Proof. The strict convexity implies that $X_n = X_{n\,umri}$ [4, Corollary 4.12] or [2, 6.2.15]. Hence, by Theorem 3, $X = X_{umri}$, so X has the splitting property.

Theorem 3 may be applied to the following typical setting: Let S be a topological group acting continuously on the right on a locally compact topological space Ω and let $\{\Omega_n\}$ be an increasing sequence of open invariant sets whose union is Ω. Let X_n be the normed linear space (sup norm) of all continuous, complex valued functions on Ω with compact support contained in Ω_n and define U_s^n on X_n by $U_s^n f(\omega) = f(\omega s)$. The inductive limit of the spaces X_n is the space of all continuous functions on Ω with compact support.

For the projective limit case we need the following lemma.

Lemma 3. *Let U and V be weakly continuous, equicontinuous representations of stsgs S and T on lctvs X and Y, respectively. If $\theta : S \mapsto T$ is a continuous, surjective homomorphism and $w \in \mathcal{L}(X,Y)$ satisfies $w \circ U_s = V_{\theta(s)} \circ w$ for all $s \in S$, then there exists a continuous, surjective semigroup homomorphism $\pi_X^w : \mathcal{U} \mapsto \mathcal{V}_{w(X)}$ such that $\pi_X^w(u) \circ w = w'' \circ u$ and $\pi_X^w(U_s) = V_{\theta(s)}|_{w(X)}$. Moreover, if $T = S$ and if θ is the identity map on S then $\pi_Z^w \circ \pi_{ZX} = \pi_X^w$ for any U-invariant subspace Z of X with $w(Z) = Y$.*

Proof. We may assume that w is surjective. For $u \in \mathcal{U}$ define $\pi_X^w(u) \in \mathcal{L}(Y)$ by

$$\pi_X^w(u)(w(x)) = w''(u(x)) \qquad (x \in X).$$

If $U_{s_\alpha} \xrightarrow{w^*} u$ and $V_{\theta(s_\alpha)} \xrightarrow{w^*} v$ then

$$\langle y', w''(u(x)) \rangle = \lim_\alpha \langle U_{s_\alpha}(x), w'(y') \rangle = \lim_\alpha \langle V_{\theta(s_\alpha)}(w(x)), y' \rangle = \langle y', v(w(x)) \rangle,$$

hence $\pi_X^w(u)$ is well defined and equals v. Therefore π_X^w maps $\mathcal{U}$ into $\mathcal{V}$ such that $\pi_X^w(U_s) = V_{\theta(s)}$. Since $\pi_X^w(u)$ is obviously continuous, it is a semigroup homomorphism.

To prove the last assertion of the lemma, let $u \in \mathcal{U}, x \in X$ and $y' \in Y'$. Then

$$\langle y', \pi_Z^w \circ \pi_{ZX}(u)(w(x)) \rangle = \langle (w|_Z)'(y'), \pi_{ZX}(u)(x) \rangle = \langle w'(y')|_Z, \pi_{ZX}(u)(x) \rangle$$
$$= \langle w'(y'), u(x) \rangle = \langle y', \pi_X^w(u)(w(x)) \rangle.$$

Corollary.

(a) *If $\mathcal{P}$ is a regular property of compact left topological semigroups then $w(X_{\mathcal{P}}) \subset Y_{\mathcal{P}}$.*

(b) *If $\mathcal{P}$ is a regular property of compactifications of S and if $T = S$ and θ is the identity map then $w(X_{\mathcal{P}}) \subset Y_{\mathcal{P}}$.*

For the next result recall that if X is the direct product $\prod_{i \in I} X_i$ of a family of lctvs X_i then X'_β may be identified with the locally convex direct sum $\bigoplus_{i \in I} X'_{i\beta}$ under $x' \leftrightarrow \sum_i x'_i$, where $x'(x) = \sum_i \langle x_i, x'_i \rangle$, $x = (x_i)_i \in X$ [11, p. 192]. Thus $X'' = (X'_\beta)'$ is algebraically isomorphic to $\prod_{i \in I} X''_i$ under $x'' \leftrightarrow (x''_i)_i$, where $x''(\sum_i x'_i) = \sum_i \langle x'_i, x''_i \rangle$ [11, p. 138]. Since the natural topology of X'' is the topology of uniform convergence on subsets of $\bigoplus_{i \in I} X'_i$ of the form $(\prod_{i \in a} N_i \times \prod_{i \notin a} X_i)^\circ = \Gamma_{i \in a} N_i^\circ$, where N_i is a convex, circled neighborhood of 0 in X_i, a is a finite subset of I, and Γ denotes the convex circled hull operation, it follows that the natural topology of X'' is the product of the natural topologies of the X''_i.

Theorem 4. *For each $i \in I$ let U^i be an equicontinuous representation of a stsg S_i on a lctvs X_i. Define the product representation $U = \prod_{i \in I} U^i$ of $T := \prod_{i \in I} S_i$ on $X := \prod_{i \in I} X_i$ by*

$$U_s(x) = (U_{s_i}^i x_i)_i \qquad \big(s = (s_i)_i \in T, \ x = (x_i)_i \in X\big).$$

Then

(a) *U is equicontinuous and $\mathcal{U} = \prod_i \mathcal{U}^i$;*

(b) *$X_{wc} = \prod_{i \in I} X_{i\,wc}$, $X_{wap} = \prod_{i \in I} X_{i\,wap}$ and $X_{ap} = \prod_{i \in I} X_{i\,ap}$;*

(c) *if each U^i is weakly continuous and if $\mathcal{P}$ is a regular property of compact left topological semigroups which is inherited by direct products then $X_{\mathcal{P}} = \prod_{i \in I} X_{i\,\mathcal{P}}$.*

Proof. Part (a) is clear, in light of the above discussion. The first equality in (b) follows from $\sigma(X, X') = \prod_i \sigma(X_i, X'_i)$, and the remaining equalities follow from $U_T x = \prod_i U_{S_i}^i x_i$. To prove (c), let $Y_i := X_{i\,\mathcal{P}}$ and $Y := \prod_{i \in I} Y_i$ and note that since $\mathcal{U}_Y = \prod_i \mathcal{U}_{Y_i}^i$ and $\mathcal{P}$ is inherited by direct products, $Y \subset X_{\mathcal{P}}$. For the reverse inclusion apply the corollary of Lemma 3 to the projection transformation $w_j : \prod_{i \in I} X_i \mapsto X_j$ and the projection homomorphism $\theta_j : T \mapsto S_j$ to obtain $w_j(X_{\mathcal{P}}) \subset Y_j$.

Theorem 5. *Let I be a directed set and for each $i \in I$ let U^i be an equicontinuous representation of a stsg S on a lctvs X_i. Let*

$$Y := \varprojlim X_i = \{(x_i)_i \in \prod_{i \in I} X_i : w_{ij}(x_j) = x_i, i \leq j\}$$

be a projective limit of the spaces X_i, where the $w_{ij} \in \mathcal{L}(X_j, X_i)$ satisfy $w_{ii} = \mathrm{id}_{X_i}$, $w_{ij} \circ w_{jk} = w_{ik}$, and $w_{ij} \circ U_s^j = U_s^i \circ w_{ij}$, $i \leq j \leq k$. Define V_s on Y by $V_s((x_i)_i) = (U_s^i x_i)_i$. Then V is an equicontinuous representation of S and

$$Y_{wc} = \varprojlim X_{i\,wc}, \qquad Y_{wap} = \varprojlim X_{i\,wap}, \quad and \quad Y_{ap} = \varprojlim X_{i\,ap}$$

(relative to the appropriate restrictions of the mappings w_{ij}). If each U^i is weakly continuous and if $\mathcal{P}$ is a regular property of compact left topological semigroups inherited by direct products, then

$$Y_{\mathcal{P}} = \varprojlim X_{i\,\mathcal{P}}.$$

Proof. The projective limits in the statement of the theorem exist since $w_{ij}(X_{j\,wc}) \subset X_{i\,wc}$, $w_{ij}(X_{j\,wap}) \subset X_{i\,wap}$, $w_{ij}(X_{j\,ap}) \subset X_{i\,ap}$ and, by the corollary to Lemma 3, $w_{ij}(X_{j\,\mathcal{P}}) \subset X_{i\,\mathcal{P}}$. Let U, X and T be as in the statement of Theorem 4. The representation V in the statement of the present theorem is the restriction to Y of the representation $s \mapsto W_s := U_{\iota(s)}$ on X, where $\iota : S \mapsto T$ denotes the injection onto the diagonal. Clearly, W is equicontinuous and has properties (b) and (c) of Theorem 4. The theorem now follows from the observation that $Y \cap X_{\mathcal{P}} = Y_{\mathcal{P}}$, $Y \cap X_{wc} = Y_{wc}$, etc.

Here's a simple example to which Theorem 5 may be applied: Let $\{t_i : i \in I\}$ be a net of units of a stsg S with identity and set $s_{ij} = t_j^{-1} t_i$, $i \leq j$. Let F be a translation invariant, linear subspace of $C(S)$ and let $U_s^i = R_s$ on $X_i = F$. Then the left translation operators $w_{ij} = L_{s_{ij}}$ have the required properties.

For another example, let $\mathcal{J}$ be a family of right ideals of a stsg S such that $\bigcup \mathcal{J} = S$ and let $C(S, \mathcal{J})$ denote the set of all continuous, complex-valued functions on S which are bounded on each member of $\mathcal{J}$. The right translation representation of S on $C(S, \mathcal{J})$ is obviously equicontinuous with respect to the locally convex topology generated by the seminorms $p_J : f \mapsto \sup_{s \in J} |f(s)|$, $J \in \mathcal{J}$, and is a uniformly bounded representation on the Banach space $C(J)$. If $\mathcal{J}$ is directed upward by inclusion then $C(S, \mathcal{J})$ is a projective limit of the Banach spaces $C(J)$ relative to the restriction mappings $w_{JK} : C(K) \mapsto C(J)$, $J, K \in \mathcal{J}$, $J \subset K$ (which clearly commute with R_s, $s \in S$). For example, one could take $S = (0, +\infty)$ under addition or $\vee$ and $\mathcal{J} = \{(1/n, +\infty) : n \in \mathbb{N}\}$.

5. Tensor Products of $\mathcal{P}$-Subspaces

If X and Y are lctvs we denote by $X \otimes Y$ the completion of the algebraic tensor product $X \odot Y$ in the topology of bi-equicontinuous convergence. This is the topology generated by the seminorms

$$p_{A' \otimes B'}(z) := \sup\{|\langle z, z' \rangle| : z' \in A' \otimes B'\},$$

where $A' \subset X'$ and $B' \subset Y'$ are equicontinuous, $A' \otimes B' := \{x' \otimes y' : x' \in A', y' \in B'\}$ and $x' \otimes y'$ is the unique member of $(X \otimes Y)'$ satisfying $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle$. More generally, if $X_1, X_2, \dots, X_n$ are lctvs then $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ denotes the completion of the algebraic tensor product $X_1 \odot X_2 \odot \cdots \odot X_n$ in the topology of n-fold-equicontinuous convergence. Equivalently, $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ may be defined inductively as $(X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1}) \otimes X_n$.

In this section we determine general conditions under which the tensor product of $\mathcal{P}$-spaces is a $\mathcal{P}$-space. We begin with the following key result.

Lemma 4. *Let X and Y be lctvs. Then $(x, y) \mapsto x \otimes y : X \times Y \mapsto X \otimes Y$ extends to a continuous bilinear mapping $(x'', y'') \mapsto x'' \otimes y'' : X'' \times Y'' \mapsto (X \otimes Y)''$ such that*

(a) $x'' \mapsto x'' \otimes y'' : X'' \mapsto (X \otimes Y)''$ *is weak* continuous for each $y'' \in Y''$;*

(b) $y'' \mapsto x \otimes y'' : Y'' \mapsto (X \otimes Y)''$ *is weak* continuous for each $x \in X$;*

(c) $\langle x' \otimes y', x'' \otimes y'' \rangle = \langle x', x'' \rangle \langle y', y'' \rangle$ *for all $x' \in X'$ and $y' \in Y'$;*

(d) *if X and Y are normed spaces then $||x'' \otimes y''|| \le ||x''|| \, ||y''||$.*

Proof. For $x \in X$ define $\ell_x \in \mathcal{L}(Y, X \otimes Y)$ by $\ell_x(y) = x \otimes y$ and set $x \otimes y'' := \ell_x''(y'')$. The mapping $(x, y'') \mapsto x \otimes y'' : X \times Y'' \to (X \otimes Y)''$ is clearly bilinear, and for $z' \in (X \otimes Y)'$ and $y \in Y$ we have $\langle x \otimes y, z' \rangle = \langle y, \ell_x'(z') \rangle = \langle z', \ell_x''(\hat{y}) \rangle = \langle z', x \otimes \hat{y} \rangle$, so $y'' \mapsto x \otimes y''$ is an extension of $y \mapsto x \otimes y$. Moreover, (b) holds since $\langle z', x \otimes y'' \rangle = \langle \ell_x'(z'), y'' \rangle$.

Let C' be an equicontinuous subset of $(X \otimes Y)'$ and choose equicontinuous sets $A' \subset X'$ and $B' \subset Y'$ such that $p_{C'} \le p_{A' \otimes B'}$. Then

$$|\langle y, \ell_x'(z') \rangle| = |\langle x \otimes y, z' \rangle| \le p_{A'}(x) p_{B'}(y) \qquad (x \in X, y \in Y, z' \in C');$$

hence $B_1' := \{ \ell_x'(z') : p_{A'}(x) \le 1, z' \in C' \}$ is an equicontinuous subset of Y' and $q_{C'}(x \otimes y'') \le q_{B_1'}(y'')$ for all x with $p_{A'}(x) \le 1$. Therefore

$$(9) \qquad q_{C'}(x \otimes y'') \le p_{A'}(x) q_{B_1'}(y'') \qquad (x \in X, y'' \in Y'').$$

Fix $y'' \in Y''$ and define $r_{y''} : (X \otimes Y)' \mapsto X'$ by

$$\langle x, r_{y''}(z') \rangle = \langle z', x \otimes y'' \rangle \qquad (x \in X, z' \in (X \otimes Y)').$$

Let A be any bounded subset of X and let $B = \{ y_\alpha \}$ be a bounded net in Y such that $y_\alpha \xrightarrow{w^*} y''$. Then $A \otimes B := \{ x \otimes y : x \in A, y \in B \}$ is bounded in $X \otimes Y$ and

$$|\langle x, r_{y''}(z') \rangle| = \lim_\alpha |\langle x \otimes y_\alpha, z' \rangle| \le \sup\{ |\langle z, z' \rangle| : z \in A \otimes B \} \quad (x \in A, z' \in (X \otimes Y)').$$

Thus $r_{y''} \in \mathcal{L}((X \otimes Y)'_\beta, X'_\beta)$ so $r_{y''}' : X'' \mapsto (X \otimes Y)''$. Set $x'' \otimes y'' := r_{y''}'(x'')$. Then $(x'', y'') \mapsto x'' \otimes y''$ is bilinear and for any $z' \in (X \otimes Y)'$ and $x \in X$ we have $\langle z', \hat{x} \otimes y'' \rangle = \langle z', r_{y''}'(\hat{x}) \rangle = \langle x, r_{y''}(z') \rangle = \langle z', x \otimes y'' \rangle$, so $x'' \mapsto x'' \otimes y''$ is an extension of $x \mapsto x \otimes y''$. Moreover, (a) holds since $\langle z', x'' \otimes y'' \rangle = \langle r_{y''}'(z'), x'' \rangle$.

To establish the continuity of $(x'', y'') \mapsto x'' \otimes y''$, let C' be an equicontinuous subset of $(X \otimes Y)'$ and choose equicontinuous sets $A' \subset X'$ and $B_1' \subset Y'$ as in (9). Then

$$|\langle x, r_{y''}(z') \rangle| = |\langle z', x \otimes y'' \rangle| \le p_{A'}(x) q_{B_1'}(y'') \qquad (x \in X, y'' \in Y'', z' \in C'),$$

hence $A_1' := \{ r_{y''}(z') : q_{B_1'}(y'') \le 1, z' \in C' \}$ is an equicontinuous subset of X' and $q_{C'}(x'' \otimes y'') \le q_{A_1'}(x'')$ for all y'' with $q_{B_1'}(y'') \le 1$. Thus

$$(10) \qquad q_{C'}(x'' \otimes y'') \le q_{A_1'}(x'') q_{B_1'}(y'') \qquad (x'' \in X'', y'' \in Y''),$$

which shows that $(x'', y'') \mapsto x'' \otimes y''$ is continuous.

Part (c) is a consequence of (a) and (b) and the weak* density of a locally convex space in its bidual. Part (d) follows easily from (10).

Note that the bilinear extension $(x'', y'') \mapsto x'' \otimes y''$ of $(x, y) \mapsto x \otimes y$ is unique with respect to properties (a) and (b) of Lemma 4. We shall refer to this extension as the *right continuous tensor product*. Obviously, a unique left continuous tensor product also exists. For finitely many lctvs $X_1, X_2, \dots, X_n$ we define the (right continuous) tensor product $x_1'' \otimes x_2'' \otimes \cdots \otimes x_n''$ inductively by

$$x_1'' \otimes x_2'' \otimes \cdots \otimes x_n'' = (x_1'' \otimes x_2'' \otimes \cdots \otimes x_{n-1}'') \otimes x_n''.$$

Lemma 5. *Let $\mathcal{S}$ and $\mathcal{T}$ be equicontinuous semigroups of operators on the lctvs X and Y, respectively, and let $\mathcal{S} \otimes \mathcal{T} := \{u \otimes v : u \in \mathcal{S}, v \in \mathcal{T}\}$. For each $u \in \overline{\mathcal{S}}$ and $v \in \overline{\mathcal{T}}$ there exists a unique member $u \otimes v$ of $\overline{\mathcal{S} \otimes \mathcal{T}}$ (weak* operator closures) such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$ (= the right continuous tensor product of $u(x)$ and $v(y)$). Moreover,*

$$(11) \qquad (u \otimes v)(u_1 \otimes v_1) = uu_1 \otimes vv_1 \qquad (u \in \overline{\mathcal{S}} \cap \mathcal{L}(X),\ u_1 \in \overline{\mathcal{S}},\ v, v_1 \in \overline{\mathcal{T}}).$$

Proof. Let $\{u_\alpha\} \subset \mathcal{S}$ and $\{v_\beta\} \subset \mathcal{T}$ be nets such that $u_\alpha \overset{w^*op}{\to} u$ and $v_\beta \overset{w^*op}{\to} v$. Since the mapping $(x, y) \mapsto u(x) \otimes v(y) : X \times Y \mapsto (X \otimes Y)''$ is bilinear it factors through a linear mapping $u \otimes v : X \odot Y \mapsto (X \otimes Y)''$. If $z = \sum_i x_i \otimes y_i \in X \odot Y$ and $z' \in (X \otimes Y)'$ then from (a) and (b) of Lemma 4

$$\langle z', (u \otimes v)(z) \rangle = \Big\langle z', \sum_i u(x_i) \otimes v(y_i) \Big\rangle$$

$$(12) \qquad\qquad = \lim_\alpha \lim_\beta \Big\langle z', \sum_i u_\alpha(x_i) \otimes v_\beta(y_i) \Big\rangle$$

$$= \lim_\alpha \lim_\beta \langle (u_\alpha \otimes v_\beta)(z), z' \rangle.$$

Let C' be an equicontinuous subset of $(X \otimes Y)'$ and choose equicontinuous subsets $A' \subset X'$ and $B' \subset Y'$ such that $p_{C'} \leq p_{A' \otimes B'}$. Then $A_1' := \mathcal{S}'(A')$ and $B_1' := \mathcal{T}'(B')$ are equicontinuous and $|\langle (u_\alpha \otimes v_\beta)(z), z' \rangle| \leq p_{A_1' \otimes B_1'}(z)$ for all α, β and all $z' \in C'$. From (12), $q_{C'}((u \otimes v)(z)) \leq p_{A_1' \otimes B_1'}(z)$ for all $z \in X \odot Y$ and all $u \in \overline{\mathcal{S}}, v \in \overline{\mathcal{T}}$. Thus $u \otimes v$ is continuous on $X \odot Y$ and hence may be extended continously to $X \otimes Y$. By (12) and the equicontinuity of $\mathcal{S} \otimes \mathcal{T}$, $(u \otimes v)(z) = w^*\text{-}\lim_\alpha \lim_\beta (u_\alpha \otimes v_\beta)(z)$ for $z \in X \otimes Y$ so $u \otimes v = \lim_\alpha \lim_\beta u_\alpha \otimes v_\beta \in \overline{\mathcal{S} \otimes \mathcal{T}}$.

To verify (11), let $U_{s_\alpha} \overset{w^*op}{\to} u_1$ and $V_{t_\beta} \overset{w^*op}{\to} v_1$. Then in the weak* operator topology

$$(u \otimes v)(u_1 \otimes v_1) = \lim_\alpha \lim_\beta (u \otimes v)(U_{s_\alpha} \otimes V_{t_\beta})$$

$$= \lim_\alpha \lim_\beta (uU_{s_\alpha}) \otimes (vV_{t_\beta})$$

$$= uu_1 \otimes vv_1,$$

the last equality because $uU_{s_\alpha} \in \mathcal{L}(X)$.

Lemma 6. *Let A and B be relatively weakly compact subsets of X and Y, respectively. Then $A \otimes B := \{x \otimes y : x \in A, y \in B\}$ is relatively weakly compact.*

Proof. We use the double limit criterion of Grothendieck [11, p. 187]. Let $\{x_n \otimes y_n\}$ be a sequence in $A \otimes B$ and let $\{z'_n\}$ be an equicontinuous sequence in $(X \otimes Y)'$ such that the limits $a := \lim_m \lim_n \langle x_n \otimes y_n, z'_m \rangle$ and $b := \lim_n \lim_m \langle x_n \otimes y_n, z'_m \rangle$ exist. Let x_0 and y_0 be weak cluster points of $\{x_n\}$ and $\{y_n\}$, respectively, and let z'_0 be a weak* cluster point of $\{z'_n\}$ (Alaoglu's Theorem). There exist weak* closed, equicontinuous subsets A' of X' and B' of Y' and a bounded sequence of complex Borel measures μ_m on $A' \times B'$ (where the latter has the product of the relative weak* topologies) such that

$$\langle x \otimes y, z'_m \rangle = \int_{A' \times B'} \langle x, x' \rangle \langle y, y' \rangle \, d\mu_m(x', y') \qquad (x \in X, y \in Y, m \geq 0)$$

[11, 9.2]. Choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $\langle x_{n_k}, x' \rangle \to \langle x_0, x' \rangle$ for all $x' \in A'$ and $\langle y_{n_k}, y' \rangle \to \langle y_0, y' \rangle$ for all $y' \in B'$. This is possible by the weak* compactness of A' and B' (see [11, p. 186]). By Lebesgue's dominated convergence theorem,

$$\begin{aligned}
a &= \lim_m \lim_n \int_{A' \times B'} \langle x_n, x' \rangle \langle y_n, y' \rangle \, d\mu_m(x', y') \\
&= \lim_m \int_{A' \times B'} \langle x_0, x' \rangle \langle y_0, y' \rangle \, d\mu_m(x', y') \\
&= \lim_m \langle x_0 \otimes y_0, z'_m \rangle = \langle x_0 \otimes y_0, z'_0 \rangle \\
&= \lim_n \int_{A' \times B'} \langle x_n, x' \rangle \langle y_n, y' \rangle \, d\mu_0(x', y') \\
&= \lim_n \langle x_n \otimes y_n, z'_0 \rangle \\
&= b,
\end{aligned}$$

as required.

The following theorem generalizes Theorem 2.2 of [7].

Theorem 6. *Let U and V be weakly continuous, equicontinuous representations of stsgs S and T on lctvs X and Y, respectively. Suppose that one of the following holds:*
  (a) *U and V are weakly almost periodic;*
  (b) *Y'_β is separable;*
  (c) *V is almost periodic.*
Then
  (i) *$W_{(s,t)} := U_s \otimes V_t$ defines a weakly continuous, equicontinuous representation $W = U \otimes V$ of $S \times T$ on $X \otimes Y$ such that $\mathcal{W} = \mathcal{U} \otimes \mathcal{V}$;*
  (ii) *$K(\mathcal{W}) = K(\mathcal{U}) \otimes K(\mathcal{V})$;*
  (iii) *if U and V are (weakly) almost periodic then W is (weakly) almost periodic;*
  (iv) *if $\mathcal{P}$ and $\mathcal{Q}$ are regular properties of compactifications of S and T, respectively, and if U is a $\mathcal{P}$-representation and V is a $\mathcal{Q}$-representation then the compactification $\mathcal{U} \otimes \mathcal{V}$ of $S \times T$ is a factor of the direct product compactification $S^{\mathcal{P}} \times T^{\mathcal{Q}}$.*

Proof. We claim that under condition (a), (b), or (c) the mapping $(u, v) \mapsto u \otimes v : \mathcal{U} \times \mathcal{V} \mapsto \mathcal{U} \otimes \mathcal{V}$ is jointly continuous in the weak* operator topologies. Assuming this for the moment, we see immediately that (i) and (iii) hold, the latter by Lemma 6. Moreover, if $\theta_S : S^{\mathcal{P}} \mapsto \mathcal{U}$ and $\theta_T : T^{\mathcal{Q}} \mapsto \mathcal{V}$ are the compactification homomorphisms in (iv) then $(s', t') \mapsto \theta_S(s') \otimes \theta_T(t') : S^{\mathcal{P}} \times T^{\mathcal{Q}} \mapsto \mathcal{U} \otimes \mathcal{V}$ is continuous in the weak* operator topology and hence is a compactification homomorphism.

To prove the claim, let $u_\alpha \to u$ in $\mathcal{U}$ and $v_\alpha \to v$ in $\mathcal{V}$. Suppose that (a) holds. Then $\mathcal{W} = \mathcal{U} \otimes \mathcal{V} \subset \mathcal{L}(X \otimes Y)$. We show that $u \otimes v$ is the unique limit point of the net $\{u_\alpha \otimes v_\alpha\}$. Let a subnet $\{u_\beta \otimes v_\beta\}$ converge in the weak operator topology to some $w \in \mathcal{W}$. For $x \in X$, $y \in Y$, $x' \in X'$ and $y' \in Y'$ we have

$$\langle w(x \otimes y), x' \otimes y' \rangle = \lim_\beta \langle u_\beta(x) \otimes v_\beta(y), x' \otimes y' \rangle$$
$$= \lim_\beta \langle u_\beta(x), x' \rangle \langle v_\beta(y), y' \rangle$$
$$= \langle u(x), x' \rangle \langle v(y), y' \rangle$$
$$= \langle u(x) \otimes v(y), x' \otimes y' \rangle.$$

Therefore $w = u \otimes v$, which proves the claim in case (a).

For the remaining cases note that by virtue of (a) of Lemma 4 and the identity

$$u_\alpha(x) \otimes v_\alpha(y) - u(x) \otimes v(y) = (u_\alpha(x) - u(x)) \otimes v(y) + u_\alpha(x) \otimes (v_\alpha(y) - v(y))$$

it suffices to show that for $z' \in (X \otimes Y)$, $c_\alpha = c_\alpha(x, y, z') := \langle z', u_\alpha(x) \otimes (v_\alpha(y) - v(y)) \rangle \to 0$. This is obvious if (c) holds. Suppose that (b) holds. Let $A' \subset X'$ and $B' \subset Y'$ be weak* closed and equicontinuous and let μ be a complex Borel measure on $A' \times B'$ such that

$$z'(x \otimes y) = \int_{A' \times B'} \langle x, x' \rangle \langle y, y' \rangle \, d\mu(x', y') \qquad (x \in X, \ y \in Y).$$

Let A_1' denote the equicontinuous set $U_{S'}(A')$ and let $p_{A_1'}$ be the corresponding seminorm. If $w \in \mathcal{V}$ then, since $\mathcal{V}(y)$ is weak* metrizable, $w(y)$ is the weak* limit of a sequence $\{V_{t_n} y\}$. The function $y' \mapsto \langle y', w(y) \rangle$ is therefore Borel measurable on B', and by (b) of Lemma 4 and the dominated convergence theorem,

$$\langle z', U_s x \otimes w(y) \rangle = \int_{A' \times B'} \langle U_s x, x' \rangle \langle y', w(y) \rangle \, d\mu(x', y') \qquad (s \in S).$$

Thus, for all α and all $s \in S$,

$$|\langle z', U_s x \otimes (v_\alpha(y) - v(y)) \rangle| \leq p_{A_1'}(x) \int_{A' \times B'} |\langle y', v_\alpha(y) - v(y) \rangle| \, d|\mu|(x', y'),$$

so by (a) of Lemma 4

$$|c_\alpha| \leq p_{A_1'}(x) \int_{A' \times B'} |\langle y', v_\alpha(y) - v(y) \rangle| \, d|\mu|(x', y').$$

Since $\mathcal{V}(y)$ is weak* metrizable, we may assume that $\{v_\alpha(y)\}$ is a sequence. The dominated convergence theorem then implies that $c_\alpha \to 0$. This completes the proof of the claim.

It remains to prove (ii). Since $K(\mathcal{U}) \otimes K(\mathcal{V})$ is obviously an ideal of $\mathcal{W}$, $K(\mathcal{W}) \subset K(\mathcal{U}) \otimes K(\mathcal{V})$. To show the reverse inclusion it suffices to show that if $d \in K(\mathcal{U})$ and $e \in K(\mathcal{V})$ are idempotents then $d \otimes e \in K(\mathcal{W})$. This follows easily from the fact that the idempotents in the minimal ideal are precisely those which are minimal with respect to the natural partial order on the set of all idempotents.

A similar theorem holds if X is separable or if U is almost periodic, provided one uses the left continuous tensor product.

By induction we have

Corollary 1. *For $i = 1, 2, \ldots, n$ let U^i be a weakly continuous, equicontinuous representation of a stsg S_i on a lctvs X_i such that one of the following holds:*
  (a) *U^i is weakly almost periodic for every i;*
  (b) *for each $i \geq 2$, either $X'_{i\beta}$ is separable or U^i is almost periodic.*
Then
  (i) *the tensor product representation $U_{(s_1, s_2, \ldots, s_n)} := U^1_{s_1} \otimes U^2_{s_2} \otimes \cdots \otimes U^n_{s_n}$ is a weakly continuous, equicontinuous representation of $S_1 \times S_2 \times \cdots \times S_n$ on $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ such that $\mathcal{U} = \mathcal{U}^1 \otimes \mathcal{U}^2 \otimes \cdots \otimes \mathcal{U}^n$ and $K(\mathcal{U}) = K(\mathcal{U}^1) \otimes K(\mathcal{U}^2) \otimes \cdots \otimes K(\mathcal{U}^n)$;*
  (ii) *if each U^i is (weakly) almost periodic then U is (weakly) almost periodic;*
  (iii) *if $\mathcal{P}_i$ is a regular property of compactifications of S_i and if U_i is a $\mathcal{P}_i$-representation then the compactification $\mathcal{U}$ of $S_1 \times S_2 \times \cdots \times S_n$ is a factor of the direct product compactification $S_1^{\mathcal{P}_1} \times S_2^{\mathcal{P}_2} \times \cdots \times S_n^{\mathcal{P}_n}$; in particular,*
  (iv) *if $\mathcal{P}$ is a regular property of compactifications which is inherited by finite direct products then $\bigotimes_{i=1}^n X_{i\mathcal{P}} \subset X_{\mathcal{P}}$.*

We now prove an infinite tensor product version of Corollary 1 for commutative C^*-algebras. Let $\{X_i : i \in I\}$ be an arbitrary family of unital abelian C^*-algebras and let I_f denote the collection of all nonempty finite subsets of I. The C^*-algebra tensor product $X := \bigotimes_{i \in I} X_i$ is the C^*-algebra inductive limit of the family $\{Y_a := \bigotimes_{i \in a} X_i : a \in I_f\}$ relative to the maps $\Phi_{a,b} : Y_b \mapsto Y_a$, $b \subset a$, defined by $\Phi_{a,b}(y) = y \otimes \bigotimes_{i \in a \setminus b} e_i$, where e_i is the unit of X_i [10, Section 1.23]. X may be viewed as the closure of $Y := \bigcup_{a \in I_f} Y_a$. For each $i \in I$ let U^i be a representation of a stsg S_i by unital*-homomorphisms on X_i. For $s = (s_i)_i \in S := \prod_{i \in I} S_i$ define an operator U_s on Y by $U_s|_{Y_a} = \bigotimes_{i \in a} U^i_{s_i}$. U_s is well-defined and may be extended to an operator on $X = \overline{Y}$. It is easy to see that U_s is a *-homomorphism and that $s \mapsto U_s$ is a unital representation of S on X. U is called the *tensor product* of the representations U^i and is denoted by $\bigotimes_{i \in I} U^i$.

Corollary 2. *For each $i \in I$ let U^i be a weakly continuous representation of a stsg S_i by unital *-homomorphisms on a unital abelian C^*-algebra X_i. Suppose that either*
  (a) *U^i is weakly almost periodic for every i, or*
  (b) *for each i, either $X'_{i\beta}$ is separable or U^i is almost periodic.*
Then
  (i) *$U := \bigotimes_{i \in I} U^i$ is a weakly continuous representation of $S = \prod_{i \in I} S_i$ by unital * - homomorphisms on the C^*-algebra $X = \bigotimes_{i \in I} X_i$ such that $\mathcal{U} = \varprojlim_{a \in I_f} \bigotimes_{i \in a} \mathcal{U}^i$ and $K(\mathcal{U}) = \varprojlim_{a \in I_f} \bigotimes_{i \in a} K(\mathcal{U}^i)$;*
  (ii) *if each U^i is (weakly) almost periodic then U is (weakly) almost periodic;*

(iii) *if, for each i, $\mathcal{P}_i$ is a regular property of compactifications of S_i and U^i is a $\mathcal{P}_i$-representation, then $\mathcal{U}$ is a factor of the direct product compactification $\prod_{i \in I} S_i^{\mathcal{P}_i}$;*

(iv) *if $\mathcal{P}$ is a regular property of compactifications which is inherited by finite direct products then $\bigotimes_{i \in I} X_{i\mathcal{P}} \subset X_{\mathcal{P}}$.*

Proof. Since X is the closure of $\bigcup_{a \in I_f} Y_a$, it follows from Corollary 1 that U is weakly continuous and from Corollary 2 of Lemma 2 that $\mathcal{U} \cong \varprojlim_{a \in I_f} \mathcal{U}_{Y_a}$. To complete the proof of part (i), use Corollary 1 and Lemma 2.

To prove (iii) note that by Corollary 1, $\prod_{i \in I} S_i^{\mathcal{P}_i} \geq \prod_{i \in a} S_i^{\mathcal{P}_i} \geq \bigotimes_{i \in a} \mathcal{U}^i$ for all $a \in I_f$ hence, by Lemma 2 and part (i), $\prod_{i \in I} S_i^{\mathcal{P}_i} \geq \mathcal{U}$. Part (iv) follows from part (i) applied to the C^*-algebras $X_{i\mathcal{P}}$ and from Corollary 1 of Lemma 2.

Now assume that each U^i is weakly almost periodic. Since X is commutative, X_{wap} is a C^*-subalgebra of X. (For example, to see that X_{wap} is closed under multiplication, express X as $C(\Omega)$ for some compact Hausdorff topological space Ω and use the fact that a norm bounded subset of $C(\Omega)$ is relatively compact in the weak topology if and only if it is relatively compact in the topology of pointwise convergence of $C(\Omega)$.) Since X is generated by the X_i and since $X_i \subset X_{wap}$, it follows that $X = X_{wap}$.

In the next section we show that the inclusion in part (iv) of Corollaries 1 and 2 may be proper (Example 8) but that equality holds for the right translation representation on certain subalgebras of $C(S_i)$ (Corollary to Theorem 8). The following result provides another instance of equality.

Corollary 3. *Suppose that, in addition to the hypotheses of Corollary 2, $WAP(S_i)$ is left amenable for each i. Then $X_{sap} = \bigotimes_i X_{i\,sap}$ and X has the splitting property (see Example 6).*

Proof. Let V and W be weakly continuous, weakly almost periodic, equicontinuous representations of stsgs S and T on lctvs Y and Z, respectively, such that $Y = Y_{umri}$ and $Z = Z_{umri}$. By Corollary 1, $Y \otimes Z = (Y \otimes Z)_{umri}$ and $Y_{sap} \otimes Z_{sap} \subset (Y \otimes Z)_{sap}$. Let $x = \lim_n \sum_i y_{in} \otimes z_{in} \in (Y \otimes Z)_{sap}$, where $y_{in} \in Y$ and $z_{in} \in Z$, and let $d \in \mathcal{V}$ and $e \in \mathcal{W}$ be minimal idempotents. Then $d \otimes e$ is a minimal idempotent in $\mathcal{V} \otimes \mathcal{W}$, and by Example 6 $x = (d \otimes e)x = \lim_n \sum_i d(y_{in}) \otimes e(z_{in}) \in Y_{md} \otimes Z_{md} = Y_{sap} \otimes Z_{sap}$. Thus, $(Y \otimes Z)_{sap} = Y_{sap} \otimes Z_{sap}$.

Now, since $WAP(S_i)$ is left amenable, $\mathcal{U}^i$ has unique minimal right ideal [2, 6.2.14], [3]. Therefore, by Corollary 2, $X = X_{umri}$ hence X has the splitting property. By the same corollary, $\bigotimes_i X_{i\,sap} \subset X_{sap}$. For the reverse inclusion, let $x = \lim_n y_n \in X_{sap}$, where $y_n \in Y_n := \bigotimes_{i \in a_n} X_i$, $a_n \in I_f$. By the first part of the proof, $Y_{n\,sap} = \bigotimes_{i \in a_n} X_{i\,sap}$. Let e be any minimal idempotent in $\mathcal{U}$. Then $x = e(x) = \lim_n e(y_n)$ and $e(y_n) \in Y_{n\,sap}$ so $x \in \bigotimes_i X_{i\,sap}$.

6. Compactifications of Infinite Direct Products of Semigroups

Let $\{S_i \ : \ i \in I\}$ be an infinite family of stsgs with identity. In this section we consider tensor products of right m-introverted C^*-subalgebras of $C(S_i)$ by examining compactifications of the direct product semigroup $S := \prod_{i \in I} S_i$. The following notion is fundamental in the construction of such compactifications.

Definition 2. A left topological compactification (φ, T') of a stsg T with identity is said to have the *local joint continuity property* if the mapping

$$(t', t) \mapsto t'\varphi(t) : T' \times T \mapsto T'$$

is continuous at each point of $T' \times \varphi(1)$.

For example, if T' is semitopological or if T is locally compact or complete metrizable, then, by a theorem of Lawson [9], (φ, T') has the local joint continuity property. The local joint continuity property is clearly inherited by factors, direct products, and subcompactifications.

We shall use the following notation: For each $a \subset I$ let $S_a := \prod_{i \in a} S_i$ and let $p_a : S \mapsto S_a$ and $q_a : S_a \mapsto S$ denote the projection and canonical injection mappings. We set $r_a := q_a \circ p_a$. The directed set of all nonempty finite subsets of I is denoted by I_f.

Lemma 7. *Let (φ, S') be a left topological compactification of S. If S has the local joint continuity property then $\varphi(s) = \lim_{a \in I_f} \varphi(r_a(s))$ uniformly in $s \in S$ (relative to the unique uniformity on S' compatible with the topology).*

Proof. A standard compactness argument shows that if O is a neighborhood of the diagonal in $S' \times S'$ then there exists a neighborhood N of the identity in S such that

$$(s'\varphi(s), s') \in O \qquad (s \in N, s' \in S').$$

Now choose $a \in I_f$ and a neighborhood N_a of the identity in S_a such that $p_a^{-1}(N_a) \subset N$. For any $s \in S$ and $b \in I_f$ with $b \supset a$ we have $r_{I \setminus b}(s) \in p_a^{-1}(N_a)$ and $r_b(s) \cdot r_{I \setminus b}(s) = s$ hence

$$(\varphi(s), \varphi(r_b(s))) = (\varphi(r_b(s)) \cdot \varphi(r_{I \setminus b}(s)), \varphi(r_b(s))) \in O.$$

Definition 3. Let $\mathcal{P}_i$ be a regular property of semigroup compactifications of S_i, $i \in I$. A compactification (φ, S') of S will be called a $\mathcal{P}_I$-*compactification* if it satisfies the following conditions:
  (a) For each $i \in I$, the subcompactification $(\varphi_i, S_i') := (\varphi \circ q_i, \overline{\varphi \circ q_i(S_i)})$ of S' is a $\mathcal{P}_i$-compactification of S_i.
  (b) For each $a \in I_f$ and each choice of $s_i' \in S_i'$, the set $\{s_i' : i \in a\}$ is commutative and the multiplication map $\mu_a' : S_a' := \prod_{i \in a} S_i' \mapsto S', \mu_a'((s_i')_{i \in a}) = \prod_{i \in a} s_i'$, is continuous.
  (c) $\varphi(s) = \lim_{a \in I_f} \varphi(r_a(s))$ uniformly in $s \in S$.

Note that the commutativity condition in (b) always holds for elements $s_i' = \varphi(q_i(s_i))$. Hence, if (φ, S') is a semitopological compactification of S then the commutativity condition holds for arbitrary $s_i' \in S_i'$.

Lemma 8. *Let $\mathcal{P}_i$ be a regular property of semigroup compactifications of S_i and let $(\psi_i, S_i^{\mathcal{P}_i})$ be the universal $\mathcal{P}_i$-compactification of S_i, $i \in I$. If each $(\psi_i, S_i^{\mathcal{P}_i})$ has the local joint continuity property then $\mathcal{P}_I$ is regular and the direct product compactification*

$$(\psi, \tilde{S}) := \left(\prod_{i \in I} \psi_i, \prod_{i \in I} S_i^{\mathcal{P}_i} \right)$$

is the universal $\mathcal{P}_I$-compactification of S.

Proof. Denote the projection and canonical injection mappings by $\tilde{p}_a : \tilde{S} \mapsto \tilde{S}_a$ and $\tilde{q}_a : \tilde{S}_a \mapsto \tilde{S}$, respectively. Let (φ, S') be a factor of $(\psi, \tilde{S})$ with compactification homomorphism $\theta : (\psi, \tilde{S}) \mapsto (\varphi, S')$. We show that (φ, S') is a $\mathcal{P}_I$-compactification of S. Let $\theta_i := \theta \circ \tilde{q}_i : S_i^{\mathcal{P}_i} \mapsto S'$. Then $\theta_i \circ \psi_i = \theta \circ \tilde{q}_i \circ \psi_i = \theta \circ \psi \circ q_i = \varphi \circ q_i$, so $S_i' = \theta_i(S_i^{\mathcal{P}_i})$. Since $\mathcal{P}_i$ is regular, S_i' is a $\mathcal{P}_i$-compactification of S_i. Therefore, (φ, S') satisfies (a) of Definition 3. To verify (b), let $a \in I_f$ and note that the multiplication map

$$\tilde{\mu}_a : \tilde{S}_a \mapsto \tilde{S}, \ (\tilde{s}_i)_{i \in a} \mapsto \prod_{i \in a} \tilde{q}_i(\tilde{s}_i)$$

is well defined and continuous. Since $\{\theta_i(\tilde{s}_i) = \theta \circ \tilde{q}_i(\tilde{s}_i) : i \in a\}$ is commutative and $\theta \circ \tilde{\mu}_a((\tilde{s}_i)_{i \in a}) = \mu_a'((\theta_i(\tilde{s}_i))_{i \in a})$, it follows that μ_a' is well defined and continuous. To verify (c) of Definition 3 use the fact that S' has the local joint continuity property (because $\tilde{S}$ does) and apply Lemma 7.

We have shown that any factor of $(\psi, \tilde{S})$ is a $\mathcal{P}_I$-compactification of S. To complete the proof of the lemma we need to verify that the converse holds. So let (φ, S') be a $\mathcal{P}_I$-compactifcation of S. By (a) of Definition 3, there exists for each $i \in I$ a continuous homomorphism $\theta_i : S_i^{\mathcal{P}_i} \mapsto S_i'$ such that $\theta_i \circ \psi_i = \varphi \circ q_i$. For $a \in I_f$ let $(\psi_a, \tilde{S}_a) := \left(\prod_{i \in a} \psi_i, \prod_{i \in a} S_i^{\mathcal{P}_i} \right)$ and define $\theta_a : \tilde{S}_a \mapsto S'$ by

$$\theta_a((u_i)_{i \in a}) = \mu_a'((\theta_i(u_i))_{i \in a}).$$

By (b) of Definition 3, θ_a is well defined and continuous. Also, from $\theta_i \circ \psi_i = \varphi \circ q_i$ we have $\theta_a \circ \psi_a = \varphi \circ q_a$, so θ_a is a homomorphism on a dense subsemigroup of $\tilde{S}_a$ and hence on $\tilde{S}_a$. By (c) of Definition 3, if W is a closed neighborhood of the diagonal in $S' \times S'$, then for some $a \in I_f$ and any $b, c \in I_f$ containing a,

$$(\theta_b \circ \tilde{p}_b \circ \psi(s), \theta_c \circ \tilde{p}_c \circ \psi(s)) = (\varphi \circ r_b(s), \varphi \circ r_c(s)) \in W \qquad (s \in S).$$

(Here we have used the identities $\tilde{p}_b \circ \psi = \psi_b \circ p_b$ and $\theta_b \circ \psi_b = \varphi \circ q_b$, $b \in I_f$.) It follows that $(\theta_b \circ \tilde{p}_b(u), \theta_c \circ \tilde{p}_c(u)) \in W$ for all $u \in \tilde{S}$. Therefore, $\{\theta_b \circ \tilde{p}_b : b \in I_f\}$ is a uniform Cauchy net, so the limit

$$(13) \qquad \theta(u) := \lim_b \theta_b \circ \tilde{p}_b(u) = \lim_b \mu_b'((\theta_i(u_i))_{i \in b}) \qquad (u \in \tilde{S})$$

exists and defines a continuous mapping $\theta : \tilde{S} \mapsto S'$ such that $\theta \circ \psi = \varphi$. Therefore S' is a factor of $\tilde{S}$.

The next theorem gives sufficient conditions for a factor of the direct product compactification $(\psi, \tilde{S})$ to be a direct product of compactifications.

Theorem 7. *For each $i \in I$ let $\mathcal{P}_i$ be a regular property of semigroup compactifications of S_i and let $S_i^{\mathcal{P}_i}$ have the local joint continuity property. If a factor (φ, S') of $(\psi, \tilde{S})$ is universal with respect to a regular property $\mathcal{P}$ which is inherited by subcompactifications, then S' is isomorphic to a direct product of $\mathcal{P}$-compactifications. If $\mathcal{P}$ is also inherited by direct products then $S' \cong \prod_{i \in I} S_i^{\mathcal{P}}$.*

Proof. We show that $(\varphi, S') \cong \left(\prod_{i \in I} \varphi_i, \prod_{i \in I} S_i' \right)$ (notation as in (a) of Definition 3). Let $\theta : (\psi, \tilde{S}) \mapsto (\varphi, S')$ be thecompactification homomorphism. Then $\theta_i := \theta \circ$

$\tilde{q}_i : (\psi_i, S_i^{\mathcal{P}_i}) \mapsto (\varphi_i, S_i')$ is a compactification homomorphism, where $\tilde{q}_i : S_i^{\mathcal{P}_i} \mapsto \tilde{S}$ is the canonical injection. Define a mapping $\Psi : S' \mapsto \prod_{i \in I} S_i'$ by $\Psi(\theta(u)) = (\theta_i(u_i))_i$, where $u = (u_i)_i \in \tilde{S}$. To see that Ψ is well defined, consider the compactification $(\varphi \circ r_i, \overline{\varphi \circ r_i(S)})$ of S, which is a subcompactification of S' and hence, by hypothesis, a $\mathcal{P}$-compactification. Thus there exists a continuous mapping $\phi_i : S' \mapsto S'$ such that $\phi_i \circ \varphi = \varphi \circ r_i$. If $\tilde{p}_i : \tilde{S} \mapsto S_i^{\mathcal{P}_i}$ denotes the projection mapping then $\phi_i \circ \theta(\psi(s)) = \phi_i \circ \varphi(s) = \varphi \circ r_i(s) = \varphi \circ q_i(s_i) = \theta_i \circ \psi_i(s_i) = \theta_i \circ \tilde{p}_i(\psi(s))$, hence $\phi_i \circ \theta(u) = \theta_i(u_i)$. This shows that Ψ is well defined and continuous. By (13), Ψ is also 1-1 and hence is a compactification isomorphism. Since $S' \geq S_i'$, S_i' is a $\mathcal{P}$-compactification.

Finally, if $\mathcal{P}$ is inherited by direct products then $\prod_{i \in I} S_i^{\mathcal{P}}$ is a $\mathcal{P}$-compactification of S so $\prod_{i \in I} S_i' \geq \prod_{i \in I} S_i^{\mathcal{P}}$ and hence $S_i' \cong S_{i\mathcal{P}}$.

The next theorem, which generalizes results of [4] and [8], gives sufficient conditions for a compactification of S to be a direct product of compactifications.

Theorem 8. *Let $\mathcal{P}_i = \mathcal{P}$ for all i, where $\mathcal{P}$ is a regular property of compactifications which is inherited by subcompactifications and direct products. If $\mathcal{P}$-compactifications are topological then $S^{\mathcal{P}_I} \cong S^{\mathcal{P}}$.*

Proof. Since $\mathcal{P}$ is inherited by direct products, $S^{\mathcal{P}_I}$ $(= \tilde{S})$ is a $\mathcal{P}$-compactification of S hence $S^{\mathcal{P}} \geq S^{\mathcal{P}_I}$. Since $S^{\mathcal{P}}$ is obviously a $\mathcal{P}_I$-compactification of S, $S^{\mathcal{P}_I} \geq S^{\mathcal{P}}$.

Corollary. *Let X_i be a C^*-subalgebra of $WAP(S_i)$ containing $AP(S_i)$ and let X be the C^*-algebra tensor product $\bigotimes_{i \in I} X_i$. If $\mathcal{P}$ is a regular property inherited by subcompacifications and direct products and if $\mathcal{P}$ compactifications are topological then, with respect to the right translation representation, $X_{\mathcal{P}} = \bigotimes_{i \in I} X_{i\mathcal{P}}$.*

Proof. By Theorem 2, $X_{\mathcal{P}}$ is a unital C^*-subalgebra of $X \subset WAP(S)$. Also, since $L_s \circ R_t = R_t \circ L_s$ it follows from the corollary to Lemma 3 that $L_S X_{\mathcal{P}} \subset X_{\mathcal{P}}$. Therefore $X_{\mathcal{P}}$ is left and right introverted [2, 4.2.7] so we may form the canonical $X_{\mathcal{P}}$-compactification $S^{X_{\mathcal{P}}}$. Since S has an identity, $S^{X_{\mathcal{P}}} \cong \mathcal{R}_{X_{\mathcal{P}}}$ (Example 1). We claim that $X_{\mathcal{P}} = \varphi^*(C(S^{\mathcal{P}}))$. Indeed, since $\mathcal{P}$-compactifications are topological, $Y := \varphi^*(C(S^{\mathcal{P}})) \subset AP(S) = \bigotimes_i AP(S_i) \subset X$, and since $R_Y \cong S^{\mathcal{P}}$, $Y \subset X_{\mathcal{P}}$. On the other hand, since $S^{X_{\mathcal{P}}} \cong \mathcal{R}_{X_{\mathcal{P}}}$, $S^{X_{\mathcal{P}}}$ has property $\mathcal{P}$ hence $X_{\mathcal{P}} \subset Y$. Therefore, $X_{\mathcal{P}} = \varphi^*(C(S^{\mathcal{P}}))$, as claimed. Similarly, $X_{i\mathcal{P}} = \varphi_i^*(C(S_i^{\mathcal{P}}))$. Since $S^{\mathcal{P}} = \prod_i S_i^{\mathcal{P}}$ (Theorem 8), $X_{\mathcal{P}} = \varphi^*(C(S^{\mathcal{P}})) = \bigotimes_i \varphi_i^*(C(S_i^{\mathcal{P}})) = \bigotimes_i X_{i\mathcal{P}}$.

The conclusions of Theorem 7, Theorem 8 and the above corollary hold if $\mathcal{P}$ is any of the following properties of compactifications (φ, T'): T' is a topological group; T' is a topological semilattice; T' is a left simple, right simple, or simple topological semigroup [2, p. 32]; T' is a left zero or right zero semigroup; more generally, T' is a topological semigroup which satisfies a family of identities [2, p. 119].

The conclusions of Theorems 7 and 8 fail in general if $\mathcal{P}$ is not inherited by subcompactifications, as the following example demonstrates.

Example 8. Let $\mathcal{P}$ be the property of compactifications (φ, S') of locally compact stsgs S that multiplication in S' is jointly continuous and $\lim_{s \to \infty} \varphi(s)$ exists. If T is the interval $[0, \infty)$ under addition, then $(\varphi, (T \times T)^{\mathcal{P}}) \cong (\epsilon, (T \times T)^F)$, where F is the C^*-algebra of almost periodic functions f on $T \times T$ with the property that

$\lim\limits_{(s,t)\to(\infty,\infty)} f(s,t)$ exists. Similarly, $T^{\mathcal{P}} \times T^{\mathcal{P}} = T^{\infty} \times T^{\infty} \cong (T \times T)^{G}$, where G is contained in the C*-algebra of almost periodic functions f on $T \times T$ with the property that $\lim\limits_{s\to\infty} \lim\limits_{t\to\infty} f(s,t)$ and $\lim\limits_{t\to\infty} \lim\limits_{s\to\infty} f(s,t)$ exist. Since there are almost periodic functions f on $T \times T$ for which the joint limit exists but the iterated limits do not (e.g., $f(s,t) = e^{i(s+t)}[(s+1)^{-1} + (t+1)^{-1}]$), $(T \times T)^{\mathcal{P}} > T^{\mathcal{P}} \times T^{\mathcal{P}}$. Therefore $(T \times T)^{P}$ is not a direct product of compactifications, hence the conclusion of Theorem 8 fails. Note that $(T \times T)^{AP} \cong T^{AP} \times T^{AP} > (T \times T)^{\mathcal{P}}$, hence the conclusion of Theorem 7 also fails, as does that of the above corollary.

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