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CESÀRO SUMMABILITY OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. We introduce p-quasi-local operators and the two-dimensional dyadic Hardy spaces H_p defined by the dyadic squares. It is proved that, if a sublinear operator T is p-quasi-local and bounded from L_∞ to L_∞, then it is also bounded from H_p to L_p $(0 < p \le 1)$. As an application it is shown that the maximal operator of the Cesàro means of a martingale is bounded from H_p to L_p $(1/2 < p \le \infty)$ and is of weak type (1,1) provided that the supremum in the maximal operator is taken over a positive cone. So we obtain the dyadic analogue of a summability result with respect to two-dimensional trigonometric Fourier series due to Marcinkievicz and Zygmund; more exactly, the Cesàro means of a function $f \in L_1$ converge a.e. to the function in question, provided again that the limit is taken over a positive cone. Finally, it is verified that if we take the supremum in a cone, but for two-powers, only, then the maximal operator of the Cesàro means is bounded from H_p to L_p for every $0 < p \le \infty$.

1. Introduction

For double trigonometric Fourier series Marcinkievicz and Zygmund [8] proved that the Cesàro means $\sigma_{n,m} f$ of a function $f \in L_1$ converge a.e. to f as $n, m \to \infty$ provided that the pairs (n, m) are in a positive cone, i.e., provided that $m/n \le \alpha$ and $n/m \le \alpha$.

The corresponding result for one-parameter Walsh-Fourier series has been shown by Fine [4].

The following results are known for double Walsh-Fourier series. For $f \in L_p$ $(1 \le p < \infty)$ one has that $\sigma_{n,m} f \to f$ in L_p norm as $\min(n, m) \to \infty$. Móricz, Schipp and Wade [9] have proved that if $f \in L \log^+ L$ then the Cesàro summability holds with the only restriction that $\min(n, m) \to \infty$.

In this paper the result of Marcinkievicz and Zygmund is proved for double Walsh-Fourier series. The trigonometric techniques are not powerful enough for the Walsh system because the classical Fejér kernels are dominated by decreasing functions whose integrals are bounded but this property does not hold for the Walsh-Fejér kernels. We obtain the result with considering the so-called p-quasi-local operators. An operator T is p-quasi-local $(0 < p \le 1)$ if for all p-atoms a

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the integral of $|Ta|^p$ over $[0,1)^2 \setminus I$ is less than an absolute constant where I is the support of the atom a.

For a martingale we define a maximal function f^* such that the supremum is taken over the diagonal of $\mathbf{N} \times \mathbf{N}$. f^* can also be defined by the help of dyadic squares. The Hardy-Lorentz spaces $H_{p,q}$ are introduced with the $L_{p,q}$ Lorentz norms of f^* $(0 < p, q \leq \infty)$. Of course, $H_p = H_{p,p}$ is the usual Hardy space $(0 < p \leq \infty)$.

We shall verify that a sublinear, p-quasi-local operator T which is bounded from L_∞ to L_∞ is also bounded from H_p to L_p $(0 < p \leq 1)$. With interpolation we get that T is bounded from $H_{p,q}$ to $L_{p,q}$ as well $(0 < p < \infty, 0 < q \leq \infty)$ and is of weak type (1,1), i.e.

$$\sup_{\alpha>0} \alpha\lambda(|Tf| > \alpha) \leq C\|f\|_1$$

whenever $f \in L_1$.

Two maximal operators of the Cesàro means are investigated: $\sigma^* f$ resp. σf is defined by the supremum over a positive cone of $|\sigma_{n,m} f|$ resp. $|\sigma_{2^n,2^m} f|$. In the one-dimensional case it is known that σ^* is bounded from H_1 to L_1 and is of weak type (1,1) (see Fujii [6] and Schipp [11]). It will be shown, in the two-parameter case, that σ^* is p-quasi-local for each $1/2 < p \leq 1$. Consequently, σ^* is bounded from $H_{p,q}$ to $L_{p,q}$ for $1/2 < p < \infty$ and $0 < q \leq \infty$ and is of weak type (1,1). The analogous result for two-dimensional trigonometric-Fourier series is verified by the author [16]. A usual density argument implies then the Walsh analogue of the result of Marcinkievicz and Zygmund. Finally, it is proved that the operator σ is p-quasi-local for each $0 < p \leq 1$ and so, by interpolation, it is bounded from $H_{p,q}$ to $L_{p,q}$ for every $0 < p < \infty$ and $0 < q \leq \infty$.

2. Hardy spaces and atomic decomposition

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^2$ be its Cartesian product $\mathbf{X} \times \mathbf{X}$ taken with itself. An element from $\mathbf{N}^2$ will be denoted by (n,m). In this paper the unit square $[0,1)^2$ and the two-dimensional Lebesgue measure λ are to be considered.

By a dyadic interval we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbf{N}$, $0 \leq k < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0,1)$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. If I_1 and I_2 are dyadic intervals and $\lambda(I_1) = \lambda(I_2)$ then the set

$$I := I_1 \times I_2$$

is a dyadic square. Clearly, the dyadic square of area 2^{-2n} containing $(x,y) \in [0,1)^2$ is given by

$$I_n(x) \times I_n(y).$$

We write for this set also $I_{n,n}(x,y)$.

The σ-algebra generated by the dyadic squares $\{I_{n,n}(x) : x \in [0,1)^2\}$ will be denoted by $\mathcal{F}_{n,n}$ $(n \in \mathbf{N})$, more precisely,

$$\mathcal{F}_{n,n} = \sigma\{[k2^{-n}, (k+1)2^{-n}) \times [l2^{-n}, (l+1)2^{-n}) : 0 \leq k < 2^n, 0 \leq l < 2^n\}.$$

The expectation operator and the conditional expectation operator relative to $\mathcal{F}_{n,n}$ $(n \in \mathbf{N})$ are denoted by E and $E_{n,n}$, respectively. We briefly write L_p instead of the real $L_p([0,1)^2, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \le \infty)$. For simplicity, we assume that for a function $f \in L_1$ we have $E_{0,0}f = 0$.

We investigate one-parameter martingales of the form $f = (f_{n,n}, n \in \mathbf{N})$ with respect to $(\mathcal{F}_{n,n}, n \in \mathbf{N})$ and suppose that $f_{0,0} = 0$. The martingale $f = (f_{n,n}, n \in \mathbf{N})$ is said to be *L_p-bounded* $(0 < p \le \infty)$ if $f_{n,n} \in L_p$ $(n \in \mathbf{N})$ and

$$\|f\|_p := \sup_{n\in\mathbf{N}} \|f_{n,n}\|_p < \infty.$$

If $f \in L_1$ then it is easy to show that the sequence $\tilde{f} = (E_{n,n}f, n \in \mathbf{N})$ is a martingale. Moreover, if $1 \le p < \infty$ and $f \in L_p$ then $\tilde{f}$ is L_p-bounded and

$$\lim_{n\to\infty} \|E_{n,n}f - f\|_p = 0,$$

consequently, $\|\tilde{f}\|_p = \|f\|_p$ (see Neveu [10]). The converse of this result holds also if $1 < p < \infty$ (see Neveu [10]): for an arbitrary martingale $f = (f_{n,n}, n \in \mathbf{N})$ there exists a function $g \in L_p$ for which $f_{n,n} = E_{n,n}g$ if and only if f is L_p-bounded. If $p = 1$ then there exists a function $g \in L_1$ of the preceding type if and only if f is *uniformly integrable* (Neveu [10]), namely, if

$$\lim_{\alpha\to\infty} \sup_{n\in\mathbf{N}} \int_{\{|f_{n,n}|>\alpha\}} |f_{n,n}|\, d\lambda = 0.$$

Thus the map $f \mapsto \tilde{f} := (E_{n,n}f, n \in \mathbf{N})$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p < \infty$. Consequently, these two spaces can be identified with each other. Similarly, the L_1 space can be identified with the space of uniformly integrable martingales. For this reason a function $f \in L_1$ and the corresponding martingale $(E_{n,n}f, n \in \mathbf{N})$ will be denoted by the same symbol f.

The distribution function of a Borel-measurable function f is defined by

$$\lambda(\{|f| > \alpha\}) := \lambda(\{x : |f(x)| > \alpha\}) \qquad (\alpha \ge 0).$$

The *weak* L_p spaces L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\alpha>0} \alpha\lambda(\{|f| > \alpha\})^{1/p} < \infty$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\tilde{f}(t) := \inf\{\alpha : \lambda(\{|f| > \alpha\}) \le t\}.$$

Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := (\int_0^\infty \tilde{f}(t)^q t^{q/p} \frac{dt}{t})^{1/q}$$

while for $0 < p \le \infty$

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \tilde{f}(t).$$

Let

$$L_{p,q} := L_{p,q}([0,1)^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities

$$L_{p,p} = L_p, \quad L_{p,\infty} = L_p^* \qquad (0 < p \le \infty)$$

(see e.g. Bennett, Sharpley [1] or Bergh, Löfström [2]).

The *maximal function* of a martingale $f = (f_{n,n}, n \in \mathbf{N})$ is defined by

$$f^* := \sup_{n\in\mathbf{N}} |f_{n,n}|.$$

It is easy to see that, in case $f \in L_1$, the maximal function can also be given by

$$f^*(x,y) = \sup_{n\in\mathbf{N}} \frac{1}{\lambda(I_{n,n}(x,y))} \left|\int_{I_{n,n}(x,y)} f \, d\lambda\right|.$$

For $0 < p, q \le \infty$ the *martingale Hardy-Lorentz space* $H_{p,q}$ consists of all martingales for which

$$\|f\|_{H_{p,q}} := \|f^*\|_{p,q} < \infty.$$

Note that in case $p = q$ the usual definition of Hardy space $H_{p,p} = H_p$ is obtained.

It is well-known that for a martingale $f = (f_{n,n}, n \in \mathbf{N})$,

$$\sup_{\alpha>0} \alpha\lambda(f^* > \alpha) \le \sup_{n\in\mathbf{N}} \|f_{n,n}\|_1 \tag{1}$$

and

$$\|f^*\|_p \le \frac{p}{p-1}\|f\|_p \qquad (1 < p \le \infty), \tag{2}$$

hence $H_p \sim L_p$ whenever $1 < p \le \infty$ (see Neveu [10]), where $\sim$ denotes the equivalence of the norms and spaces. Moreover, it is proved in Weisz [18] that $H_{p,q} \sim L_{p,q}$ if $1 < p \le \infty$. Note that the $H_{p,q}$ spaces can also be defined by the norms

$$\|f\| = \|S(f)\|_{p,q}$$

where

$$S(f) := (\sum_{n=1}^{\infty} |f_{n,n} - f_{n-1,n-1}|^2)^{1/2}.$$

For this and for other equivalent definitions see [18].

The atomic decomposition is a useful characterization of Hardy spaces. To demonstrate this let us introduce first the concept of an atom. A bounded measurable function a is a *p-atom* if there exists a dyadic square I such that

$$\begin{array}{ll} \text{(i)} & \displaystyle\int_I a \, d\lambda = 0 \\ \text{(ii)} & \|a\|_\infty \le \lambda(I)^{-1/p} \\ \text{(iii)} & \{a \ne 0\} \subset I. \end{array}$$

The basic result of the atomic decomposition is stated as follows (see Coifman, Weiss [3] and Weisz [19]).

Theorem A. *A martingale $f = (f_{n,n}, n \in \mathbf{N})$ is in H_p $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k E_{n,n} a^k = f_{n,n} \quad \text{for all} \quad n \in \mathbf{N}, \tag{3}$$
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, the following equivalence of norms holds:

$$\|f\|_{H_p} \sim \inf(\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p} \tag{4}$$

where the infimum is taken over all decompositions of f of the form (3)*.*

We shall also use the next interpolation result (Weisz [18]).

Theorem B. *If a sublinear operator T is bounded from H_{p_0} to L_{p_0} and from L_∞ to L_∞ then it is bounded from $H_{p,q}$ to $L_{p,q}$ as well, if $p_0 < p < \infty$ and $0 < q \leq \infty$.*

3. Quasi-local operators

Motivated by the definition in Móricz, Schipp, Wade [9], the quasi-local operators are introduced. Their definition is weakened and extended here. For each dyadic interval I let I^r be the dyadic interval for which $I \subset I^r$ and

$$\lambda(I^r) = 2^r \lambda(I).$$

If $I := I_1 \times I_2$ is a dyadic square then set

$$I^r := I_1^r \times I_2^r.$$

An operator T which maps the set of martingales into the collection of measurable functions, will be called *p-quasi-local* if there exist $r \in \mathbf{N}$ and a constant $C_p > 0$ such that

$$\int_{[0,1)^2 \setminus I^r} |Ta|^p \, d\lambda \leq C_p$$

for every p-atom a where I is the support of the atom.

The quasi-local operators were defined in [9] only for $p = 1$ and for L_1 functions instead of atoms.

The following result gives sufficient conditions for T to be bounded from H_p to L_p. For the sake of the completeness it is verified here.

Theorem 1. *Suppose that the operator T is sublinear and p-quasi-local for any $0 < p \leq 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Proof. Suppose that a is a p-atom with support I. By the p-quasi-locality and L_∞ boundedness of T we obtain

$$\begin{aligned}\int_{[0,1)^2} |Ta|^p \, d\lambda &= \int_{I^r} |Ta|^p \, d\lambda + \int_{[0,1)^2 \setminus I^r} |Ta|^p \, d\lambda \\ &\le \|T\|_\infty^p \|a\|_\infty^p \lambda(I^r) + C_p = C_p\end{aligned}$$

where the symbol C_p may denote different constants in different contexts. Applying Theorem A,

$$\|Tf\|_p^p \le \sum_{k=0}^{\infty} |\mu_k|^p \|Ta^k\|_p^p \le C_p \|f\|_{H_p}^p$$

which proves the theorem. □

Taking into account Theorem B and (1) we have

Corollary 1. *Suppose that the operator T is sublinear and p-quasi-local for each $0 < p \le 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $0 < p < \infty$ and $0 < q \le \infty$. Specially, T is of weak type $(1,1)$, i.e., if $f \in L_1$ then

$$\|Tf\|_{1,\infty} = \sup_{\alpha > 0} \alpha \lambda(|Tf| > \alpha) \le C_1 \|f\|_{H_{1,\infty}} = C_1 \sup_{\alpha > 0} \alpha \lambda(f^* > \alpha) \le C_1 \|f\|_1.$$

4. Cesàro summability of double Walsh-Fourier series

It is proved by Marcinkievicz and Zygmund [8] that the Cesàro means of the double trigonometric series of $f \in L_1$ converge in a cone to f; more exactly,

$$\sigma_{n,m} f \to f \qquad \text{a.e.} \tag{5}$$

as $\min(n,m) \to \infty$ and $2^{-\alpha} \le n/m \le 2^\alpha$ $(\alpha \ge 0)$.

Móricz, Schipp and Wade claimed in [9] that, for the double Walsh-Fourier series,

$$\sigma_{2^n, 2^m} f \to f \qquad \text{a.e.}$$

as $\min(n,m) \to \infty$ and $|n-m| \le \alpha$ $(\alpha \ge 0)$. The authors raised the question in [9] whether the original convergence (5) holds for the double Walsh-Fourier series. In this section we solve this problem.

First the Walsh system is to be introduced. Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \qquad 0 \le x_k < 2, \ x_k \in \mathbf{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \qquad (n \in \mathbf{N})$$

are called *Rademacher functions*.

The product system generated by these functions is the *one-dimensional Walsh system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $0 \le n_k < 2$ and $n_k \in \mathbf{N}$.

The Kronecker product $(w_{n,m}; n, m \in \mathbf{N})$ of two Walsh systems is said to be the *two-dimensional Walsh system*. Thus

$$w_{n,m}(x, y) := w_n(x) w_m(y).$$

The *Walsh-Dirichlet kernels* are defined by

$$D_n := \sum_{k=0}^{n-1} w_k, \qquad D_{n,m} := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} w_{k,l} \qquad (n, m \in \mathbf{N}).$$

Obviously, $D_{n,m}(x, y) = D_n(x) D_m(y)$. Recall that

$$D_{2^n}(x) := \begin{cases} 2^n & \text{if } x \in I_n(0), \\ 0 & \text{if } x \in [0,1) \setminus I_n(0) \end{cases} \tag{6}$$

for $n \in \mathbf{N}$ (see Fine [5]).

If $f \in L_1$ then the number

$$\hat{f}(n, m) := E(f w_{n,m})$$

is said to be the (n, m)-th *Walsh-Fourier coefficients* of f $(n, m \in \mathbf{N})$. Let us extend this definition to martingales as well. If $f = (f_{k,k}, k \in \mathbf{N})$ is a martingale then let

$$\hat{f}(n, m) := \lim_{k \to \infty} E(f_{k,k} w_{n,m}) \qquad (n, m \in \mathbf{N}).$$

Since $w_{n,m}$ is $\mathcal{F}_{k,k}$ measurable for $n, m < 2^k$, it can immediately be seen that this limit does exist. Note that if $f \in L_1$ then $E_{k,k} f \to f$ in L_1 norm as $k \to \infty$, hence

$$\hat{f}(n, m) = \lim_{k \to \infty} E((E_{k,k} f) w_{n,m}) \qquad (n, m, k \in \mathbf{N}).$$

Thus the Walsh-Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_{k,k} f, k \in \mathbf{N})$ obtained from f.

Denote by $R_{n,m} f$ the (n, m)-th partial sum of the Walsh-Fourier series of a martingale f, namely,

$$R_{n,m} f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \hat{f}(k, l) w_{k,l}.$$

If $f \in L_1$ and $f_{n,n} := R_{2^n,2^n} f$ then it is easy to prove that

$$f_{n,n}(x,y) = 2^{2n} \int_{I_{n,n}(x,y)} f \, d\lambda = E_{n,n} f(x,y), \tag{7}$$

that is to say, $(f_{n,n}; n \in \mathbf{N})$ is the martingale relative to $(\mathcal{F}_{n,n})$ obtained from f. Moreover, a sequence $f = (f_{n,n}, n \in \mathbf{N})$ is a martingale if and only if there exist complex numbers $c_{k,l}$ such that

$$f_{n,n} = \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} c_{k,l} w_{k,l}.$$

Of course, $c_{k,l} = \hat{f}(k,l)$.

Recall that the *Walsh-Fejér kernels*

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_n \qquad (n \in \mathbf{N})$$

satisfy

$$|K_n(x)| \le \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} \Big(D_{2^i}(x) + D_{2^i}(x \dot{+} 2^{-j-1}) \Big) \tag{8}$$

for $x \in [0,1)$, $n, N \in \mathbf{N}$ and $2^{N-1} \le n < 2^N$ and

$$K_{2^n}(x) = \frac{1}{2} \Big(2^{-n} D_{2^n}(x) + \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x \dot{+} 2^{-j-1}) \Big) \tag{9}$$

for $x \in [0,1)$ and $n \in \mathbf{N}$, where $\dot{+}$ denotes the dyadic addition (see e.g. Schipp, Wade, Simon, Pál [13]).

For $n, m \in \mathbf{N}$ and a martingale f the *Cesàro mean* of order (n,m) of the double Walsh-Fourier series of f is given by

$$\sigma_{n,m} f := \frac{1}{nm} \sum_{k=1}^{n} \sum_{l=1}^{m} R_{k,l} f.$$

It is simple to show that

$$\sigma_{n,m} f(x,y) = \int_0^1 \int_0^1 f(t,u) K_n(x \dot{+} t) K_m(y \dot{+} u) \, dt \, du$$

if $f \in L_1$.

For a martingale f we consider the maximal operators

$$\sigma^* f := \sup_{2^{-\alpha} \le n/m \le 2^{\alpha}} |\sigma_{n,m} f|, \qquad \sigma f := \sup_{|n-m| \le \alpha} |\sigma_{2^n,2^m} f| \qquad (\alpha \ge 0)$$

and prove our following main result.

Theorem 2. *There are absolute constants C and $C_{p,q}$ such that*

$$\|\sigma^* f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q}) \tag{10}$$

for every $1/2 < p < \infty$ and $0 < q \le \infty$. Especially, if $f \in L_1$ then

$$\lambda(\sigma^* f > \alpha) \le \frac{C}{\alpha} \|f\|_1 \qquad (\alpha > 0). \tag{11}$$

Note that, in the one-parameter case, (10) was proved by Fujii [6] for $p = q = 1$ (see also Schipp, Simon [12]) and (11) was shown by Schipp [11]. For double trigonometric-Fourier series the analogous theorem is verified by the author [16].

By (7) it is easy to show that the two-dimensional Walsh polynomials are dense in L_1. Hence (11) and the usual density argument (see Marcinkievicz, Zygmund [8]) imply

Corollary 2. *If $f \in L_1$ then*

$$\sigma_{n,m} f \to f \qquad a.e.$$

as $\min(n, m) \to \infty$ and $2^{-\alpha} \le n/m \le 2^{\alpha}$ $(\alpha \ge 0)$.

The analogous one-dimensional result can be found in Fine [4]. While this paper was undergoing publication we found that this corollary was also proved by Gát [21] in another way.

Proof of Theorem 2. By Corollary 1 the proof of Theorem 2 will be complete if we show that the operator σ^* is p-quasi-local $(1/2 < p \le 1)$ and bounded from L_∞ to L_∞.

The boundedness follows from

$$\sup_{n \in \mathbf{N}} \int_0^1 |K_n(x)|\, dx < \infty \tag{12}$$

(see (6) and (8)).

We are going to verify the p-quasi-locality for $1/2 < p \le 1$. Let a be an arbitrary p-atom with support $I \times J$ and $\lambda(I) = \lambda(J) = 2^{-K}$ $(K \in \mathbf{N})$. It is easy to see that $\hat{a}(n, m) = 0$ if $n < 2^K$ and $m < 2^K$, so, in this case, $\sigma_{n,m} a = 0$. Therefore we can suppose that $n \ge 2^K$ or $m \ge 2^K$. Choose $r \in \mathbf{N}$ such that $r - 1 < \alpha \le r$. If $n \ge 2^K$ then, by the hypothesis,

$$m \ge 2^{-\alpha} n \ge 2^{K-r}.$$

Let $n \ge 2^{K-r}$ and $m \ge 2^{K-r}$.

To prove the quasi-locality of σ^* we have to integrate $|\sigma^* a|^p$ over $[0,1)^2 \setminus (I^r \times J^r)$. We do this in three steps.

Step 1. Integrating over $([0,1) \setminus I^r) \times J^r$**.**

Using (12) and the fact that $|a| \le 2^{2K/p}$ we have

$$\begin{aligned}
|\sigma_{n,m} a(x,y)| &= \left| \int_I \int_J a(t,u) K_n(x \dot{+} t) K_m(y \dot{+} u)\, dt\, du \right| \\
&\le \int_I \int_J |a(t,u)| |K_m(y \dot{+} u)|\, du |K_n(x \dot{+} t)|\, dt \\
&\le C 2^{2K/p} \int_I |K_n(x \dot{+} t)|\, dt.
\end{aligned}$$

By (8) we conclude that

$$|\sigma_{n,m}a(x,y)| \le C2^{2K/p}2^{-N}\sum_{j=0}^{N-1}2^j\sum_{i=j}^{N-1}\int_I D_{2^i}(x\dot{+}t)+D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt.$$

If $j \ge K-r$ and $x \notin I^r$ then $x\dot{+}2^{-j-1} \notin I^r$. Consequently, for $x \notin I^r$ and $i \ge j \ge K-r$ we have

$$\int_I D_{2^i}(x\dot{+}t)\,dt = \int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt = 0. \tag{13}$$

Recall that $I^r \supset I$ and $\lambda(I^r) = 2^{-K+r}$. Since $n \ge 2^{K-r}$ and $2^N > n \ge 2^{N-1}$, one has $N-1 \ge K-r$. Henceforth, for $x \notin I^r$,

$$\begin{aligned}|\sigma_{n,m}a(x,y)| &\le C2^{2K/p}2^{-K}\sum_{j=0}^{K-r-1}2^j\sum_{i=j}^{K-1}\int_I D_{2^i}(x\dot{+}t)+D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\\ &\quad + C2^{2K/p}\sum_{j=0}^{K-r-1}2^j\sum_{i=K}^{N-1}2^{-i}\int_I D_{2^i}(x\dot{+}t)+D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\\ &\le C2^{2K/p-K}\sum_{j=0}^{K-r-1}2^j\sum_{i=j}^{K-1}\int_I D_{2^i}(x\dot{+}t)+D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\\ &\quad + C2^{2K/p}\sum_{j=0}^{K-r-1}2^j\sum_{i=K}^{\infty}2^{-i}\int_I D_{2^i}(x\dot{+}t)+D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt\end{aligned}$$

where C is depending on r. Observe that the right hand side is independent of n and m. Since the dyadic addition is a group operation and measure preserving we may assume that $I = [0, 2^{-K})$. Using (6) we can verify that, for $x \notin I^r$,

$$\int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt = 2^{i-K}1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}(x) \tag{14}$$

if $j \le i \le K-1$,

$$\int_I D_{2^i}(x\dot{+}t)\,dt = 2^{i-K}1_{[2^{-K+r},2^{-i})}(x) \tag{15}$$

if $i \in \mathbf{N}$ and

$$\int_I D_{2^i}(x\dot{+}t\dot{+}2^{-j-1})\,dt = 1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-K})}(x) \tag{16}$$

if $i \ge K$. Therefore

$$\begin{aligned}\sigma^* a(x,y) &\le C2^{2K/p-K}\sum_{j=0}^{K-r-1}2^j\sum_{i=j}^{K-1}\Big(2^{i-K}(1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}(x)+1_{[2^{-K+r},2^{-i})}(x))\Big)\\ &\quad + C2^{2K/p}\sum_{j=0}^{K-r-1}2^j\sum_{i=K}^{\infty}2^{-i}1_{[2^{-j-1},2^{-j-1}\dot{+}2^{-K})}(x).\end{aligned}$$

Applying the inequality

$$(\sum_{k=0}^{\infty} a_k)^p \le \sum_{k=0}^{\infty} a_k^p \qquad (a_k \ge 0, k \in \mathbf{N}; 0 < p \le 1)$$

we obtain

$$\begin{aligned}\int_{[0,1)\setminus I^r}\int_{J^r} |\sigma^* a(x,y)|^p \, dx\, dy &\le C_p 2^{-K+r} 2^{2K-2Kp} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} 2^{i(p-1)} \\ &\quad + C_p 2^{-K+r} 2^{2K} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} 2^{-K} \\ &=: (A) + (B).\end{aligned}$$

For $1/2 < p < 1$ we have

$$(A) = C_p 2^{K-2Kp} \sum_{j=0}^{K-r-1} 2^{j(2p-1)} = C_p 2^{K-2Kp} 2^{(K-r)(2p-1)} = C_p,$$

while for $p = 1$,

$$(A) = C_p \sum_{j=0}^{K-r-1} 2^{j-K}(K-j) \le C_p \sum_{k=0}^{\infty} k 2^{-k} = C_p.$$

On the other hand,

$$(B) = C_p 2^{(K-r)p} 2^{-Kp} = C_p.$$

Hence we can establish that

$$\int_{[0,1)\setminus I^r}\int_{J^r} |\sigma^* a(x,y)|^p \, dx\, dy \le C_p. \tag{17}$$

Note that C_p depends only on p and on r.

Step 2. Integrating over $([0,1)\setminus I^r) \times ([0,1)\setminus J^r)$**.**

Similarly to Step 1 we get for $x \notin I^r$ and $y \notin J^r$ that

$$\begin{aligned}|\sigma_{n,m} a(x,y)| &\le 2^{2K/p} \int_I |K_n(x \dot{+} t)| \, dt \int_J |K_m(y \dot{+} u)| \, du \\ &\le C 2^{2K/p} \Big[2^{-K} \sum_{j=0}^{K-r-1} 2^j \sum_{i=j}^{K-1} \int_I D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1}) \, dt \\ &\qquad + \sum_{j=0}^{K-r-1} 2^j \sum_{i=K}^{\infty} 2^{-i} \int_I D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1}) \, dt \Big] \\ &\qquad \Big[2^{-K} \sum_{k=0}^{K-r-1} 2^k \sum_{l=k}^{K-1} \int_J D_{2^l}(y \dot{+} u) + D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1}) \, du \\ &\qquad + \sum_{k=0}^{K-r-1} 2^k \sum_{l=K}^{\infty} 2^{-l} \int_J D_{2^l}(y \dot{+} u) + D_{2^l}(y \dot{+} u \dot{+} 2^{-k-1}) \, du \Big].\end{aligned}$$

Using the results of Step 1 we conclude that

$$\int_{[0,1)\setminus I^r}\int_{[0,1)\setminus J^r} |\sigma^* a(x,y)|^p \, dx\, dy \le C_p 2^{2K} 2^{-K} 2^{-K} = C_p. \tag{18}$$

Step 3. Integrating over $I^r \times ([0,1) \setminus J^r)$.

This case is analogous to Step 1.

Combining (17) and (18) we can see that

$$\int_{[0,1)^2 \setminus (I^r \times J^r)} |\sigma^* a(x,y)|^p \, dx \, dy \le C_p \tag{19}$$

which completes the proof. □

Considering the operator σ one can extend Theorem 2 to every $0 < p < \infty$.

Theorem 3. *There are absolute constants $C_{p,q}$ such that*

$$\|\sigma f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $0 < p < \infty$ and $0 < q \le \infty$.

Proof. We only have to prove that the operator σ is p-quasi-local for every $0 < p \le 1$. Let a be an arbitrary p-atom with support $I \times J := [0, 2^{-K}) \times [0, 2^{-K})$ ($K \in \mathbf{N}$). Again, we can suppose that $2^n \ge 2^{K-r}$ and $2^m \ge 2^{K-r}$ where $r - 1 < \alpha \le r$. We are going to prove Step 1, only. By (9), (12) and (13),

$$\begin{aligned}
|\sigma_{2^n,2^m} a(x,y)| &\le C 2^{2K/p} \int_I |K_{2^n}(x \dot{+} t)| \, dt \\
&\le C 2^{2K/p} \sum_{j=0}^{n} 2^{j-n} \int_I D_{2^n}(x \dot{+} t \dot{+} 2^{-j-1}) \, dt \\
&\le C 2^{2K/p} \sum_{j=0}^{K-r-1} 2^{j-n} \int_I D_{2^n}(x \dot{+} t \dot{+} 2^{-j-1}) \, dt
\end{aligned}$$

whenever $x \notin I^r$. Since $2^n \ge 2^{K-r}$,

$$|\sigma_{2^n,2^m} a(x,y)| \le C 2^{2K/p} \sum_{j=0}^{K-r-1} 2^{j-K+r} \int_I D_{2^{K-r}}(x \dot{+} t \dot{+} 2^{-j-1}) \, dt.$$

Taking into account (14) and integrating, we get that

$$\int_{[0,1) \setminus I^r} \int_{J^r} |\sigma a(x,y)|^p \, dx \, dy \le C_p 2^{-K} 2^{2K} \sum_{j=0}^{K-r-1} 2^{jp-Kp} 2^{-(K-r)} = C_p$$

for every $0 < p \le 1$.

The other two steps can be proved as in Theorem 2. □

Observe that the results of this paper can be proved in the same way for each dimension.

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