

DIVISORS ON GENERIC COMPLETE INTERSECTIONS IN PROJECTIVE SPACE

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ABSTRACT. Let V be a generic complete intersection of hypersurfaces of degree d_1, d_2, \dots, d_m in n -dimensional projective space. We study the question when a divisor on V is nonrational or of general type, and give an alternative proof of a result of Ein. We also give some improvement of Ein's result in the case $d_1 + d_2 + \dots + d_m = n + 2$.

0. INTRODUCTION

Let V be a generic complete intersection of hypersurfaces of degree d_1, d_2, \dots, d_m in \mathbf{P}^n . A conjecture of Kobayashi (cf. [L]) states that V is hyperbolic if $d = d_1 + d_2 + \dots + d_m \geq n + 2$. In general, S. Lang [L] has conjectured that a variety X is hyperbolic if and only if every subvariety of X is of general type. In this paper, we will prove the following

Theorem 1. *Let V be a complete intersection of m generic hypersurfaces of degree d_1, d_2, \dots, d_m in \mathbf{P}^n , $M \subset V$ a reduced and irreducible divisor, $p_g(M)$ the geometric genus of the desingularization of M . Assume that $1 \leq m \leq n - 3$ and $d_i \geq 2$ for all i . Then*

- (1) $p_g(M) \geq n - 1$ if $d = d_1 + d_2 + \dots + d_m \geq n + 2$,
- (2) M is of general type if $d = d_1 + d_2 + \dots + d_m > n + 2$.

In [E1,E2], Ein has shown that M is nonrational if $d \geq n + 2$, and is of general type if $d > n + 2$. Here we are going to give an alternative proof of it. Ein also proved that every subvariety of V of dimension l is nonrational if $d \geq 2n - m - l + 1$, and is of general type if $d > 2n - m - l + 1$. Therefore the improvement we made here is in the case $d = n + 2$ and $l = n - m - 1$. In particular, we conclude that the divisor M can not be an abelian variety. If a variety X is hyperbolic, then every rational map of an abelian variety or \mathbf{P}^1 into X is constant. On the other hand, Lang [L] conjectured that this condition is also sufficient for X to be hyperbolic.

If V is a generic hypersurface in \mathbf{P}^n , it was first shown by Clemens [CKM] that V contains no rational curves, if $\deg V \geq n - 1$. In [X1], we study generic surfaces in \mathbf{P}^3 , obtain that every curve C on S has geometric genus $g(C) \geq \frac{1}{2}d(d - 3) - 2$ ($d = \deg S$), and the bound is sharp. We also obtain results about divisors on a generic hypersurface in \mathbf{P}^n . In [X2], we generalize these results to some nongeneric cases.

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When V is a generic quintic 3-fold in \mathbf{P}^4 , a conjecture of Clemens says that V should contain only finitely many rational curves of given degree, which is equivalent to the statement that every divisor on V must have a nonnegative Kodaira dimension. Chang and Ran [CR] has proved that V does not contain a reduced and irreducible divisor which admits a desingularization having a numerically effective anticanonical bundle.

To establish Theorem 1, we need to get control over the singularities of the divisor M on V . The method we use here is deformation of singularity as we did in [X1].

Throughout this paper we work over the complex number field \mathbb{C} .

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1. DEFORMATION OF SINGULARITIES

For simplicity of notations, we will give a proof of Theorem 1 in the case $m = 2$.

First of all, we recall some definitions from [X1].

Let V be an n -dimensional smooth variety, and $M \subset V$ be a reduced and irreducible divisor. According to Hironaka [H], there is a desingularization of M :

$$V_{m+1} \xrightarrow{\pi_{m+1}} V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = V,$$

so that the proper transform \tilde{M} of M in V_{m+1} is smooth. Here $V_j \xrightarrow{\pi_j} V_{j-1}$ is the blow-up of V_{j-1} along a ν_{j-1} -dimensional submanifold X_{j-1} with $E_{j-1} \subset V_j$ the exceptional divisor. If X_{j-1} is a μ_{j-1} -fold singular submanifold of the proper transform of M in V_{j-1} , we say that M has a *type* $\mu = (\mu_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ *singularity*.

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j \mid j \in \Gamma)$ singularity, $\Omega \subset V$ is an open set, we localize our definition by saying that M has a type $\mu_\Omega = (\mu_j, X_j, E_j \mid j \in \Gamma_\Omega = \{j \mid \exists q \in E_j, q \text{ is an infinitely near point of some } p \in \Omega\})$ singularity on Ω .

Given any resolution of the singularity of $M \subset V$ as above, if $D \subset V$ is a divisor, such that

$$\pi_j^*(\cdots (\pi_2^*(\pi_1^*(D) - \delta_0 E_0) - \delta_1 E_1) - \cdots) - \delta_{j-1} E_{j-1}$$

is an effective divisor for all $j = 1, 2, \dots, m+1$, then we say that D has a *weak type* $\delta = (\delta_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ *singularity*. It is easy to see that a type μ singularity implies a weak type μ singularity.

Assume that $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ singularity. The following lemma describes the connection between the singularities of M and the canonical bundle of the desingularization \tilde{M} of M .

Lemma 2. *A section of $K_V \otimes M$ with a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ singularity induces a section of $K_{\tilde{M}}$.*

Proof. Proposition 1.1 in [X1]. q.e.d.

Definition. Let $T \subset \mathbb{C}^N$ be an open neighborhood of the origin $0 \in T$. Assuming that $\sigma: M \rightarrow T$ is a family of reduced equidimensional algebraic varieties, $M_t = \sigma^{-1}(t)$, then we say that the family M_t is μ -equisingular at $t = 0$ in the sense

that we can resolve the singularity of M_t simultaneously, that is, there is a proper morphism $\pi: \tilde{M} \longrightarrow M$, so that $\sigma \circ \pi: \tilde{M} \longrightarrow T$ is a flat map and

$$\sigma \circ \pi: \tilde{M}_t = (\sigma \circ \pi)^{-1}(t) \longrightarrow M_t$$

is a resolution of the singularities of M_t . Moreover, if M_t has a type $\mu(t) = (\mu_j(t), X_j(t), E_j(t) \mid j \in \Gamma(t))$ singularity with the above resolution, then $\mu_j(t) = \mu_j$ and $\Gamma(t) = \Gamma$ are independent of t , and the exceptional divisors and the singular loci of the desingularization $\tilde{M}_t \longrightarrow M_t$ have the same configuration for all t .

Now we state a lemma concerning the local deformation theory of singular divisors.

Lemma 3. *If $M_t = \{g_t(z_1, \dots, z_n) = 0\}$ is a μ -equisingular family of varieties defined in an open set $\Omega \subset \mathbb{C}^n$, and M_t has a type $\mu(t)_\Omega = (\mu_j, X_j(t), E_j(t) \mid j \in \{0, \dots, m\})$ singularity on Ω , then the variety $\{\frac{dg_t}{dt} \mid_{t=0} = 0\}$ has a weak type $\mu(0)_\Omega - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \{0, \dots, m\})$ singularity on Ω .*

Proof. Lemma 4.4 in [X1]. q.e.d.

Let $\{Z_i\}$ be some homogeneous coordinates of \mathbf{P}^n , $F \in H^0(\mathbf{P}^n, \mathcal{O}(r))$ and $G \in H^0(\mathbf{P}^n, \mathcal{O}(l))$ be homogeneous polynomials. We define

$$\frac{\partial(F, G)}{\partial(Z_i, Z_j)} = \det \begin{vmatrix} \frac{\partial F}{\partial Z_i} & \frac{\partial F}{\partial Z_j} \\ \frac{\partial G}{\partial Z_i} & \frac{\partial G}{\partial Z_j} \end{vmatrix}.$$

The next lemma tells us how to use deformation of singularities to produce special homogeneous polynomials.

Lemma 4. *Let $F_{1,t} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1))$, $F_{2,t} \in H^0(\mathbf{P}^n, \mathcal{O}(d_2))$, $G_t \in H^0(\mathbf{P}^n, \mathcal{O}(k))$, and $M_t = \{F_{1,t} = 0\} \cap \{F_{2,t} = 0\} \cap \{G_t = 0\}$ be a μ -equisingular family of varieties with a type $\mu(t) = (\mu_j, X_j(t), E_j(t) \mid j \in \Gamma)$ singularity. Setting*

$$\frac{dF_{1,t}}{dt} \mid_{t=0} = F'_1, \quad \frac{dF_{2,t}}{dt} \mid_{t=0} = F'_2, \quad \frac{dG_t}{dt} \mid_{t=0} = G',$$

and assuming that both the varieties $\{F_{i,t} = 0\}$ ($i = 1, 2$) and $\{F_{1,t} = 0\} \cap \{F_{2,t} = 0\}$ are smooth for t in a neighborhood of 0. Then the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 = 0 \right\}$$

($i, j = 0, 1, \dots, n$) on $V = \{F_{1,0} = 0\} \cap \{F_{2,0} = 0\}$ has a weak type $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma)$ singularity, where $\{Z_0, Z_1, \dots, Z_n\}$ are homogeneous coordinates of \mathbf{P}^n .

Proof. For any point $P \in M_0$, we can find an open set $\Omega \ni P$ of V , and generic homogeneous coordinates $\{Z'_i\}$ with

$$Z'_i = \sum_{j=0}^n l_{ij} Z_j \quad (i = 0, 1, \dots, n),$$

so that

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_i, Z'_j)} \neq 0$$

on Ω for all $i \neq j$ ($i, j = 0, 1, \dots, n$). Assuming M_0 has a type $\mu_\Omega(0) = (\mu_j, X_j(0), E_j(0) \mid j \in \Gamma_\Omega)$ singularity on Ω . Denoting

$$\{z_1, z_2, \dots, z_n\} = \left\{ \frac{Z'_1}{Z'_0}, \frac{Z'_2}{Z'_0}, \dots, \frac{Z'_n}{Z'_0} \right\},$$

if we solve the equation

$$F_{1,t}(1, z_1, z_2, \dots, z_n) = 0, \quad F_{2,t}(1, z_1, z_2, \dots, z_n) = 0$$

near the point $P(t)$, where $P(0) = P$, and get

$$z_1 = \varphi_{1,t}(z_3, \dots, z_n), \quad z_2 = \varphi_{2,t}(z_3, \dots, z_n),$$

then on some open set of \mathbb{C}^{n-2} , M_t is a μ -equisingular family of divisors locally defined by the equation

$$G_t(1, \varphi_{1,t}, \varphi_{2,t}, z_3, \dots, z_n) = 0.$$

By Lemma 3, the divisor locally defined by the equation

$$\frac{dG_t}{dt}(1, \varphi_{1,t}(z_3, \dots, z_n), \varphi_{2,t}(z_3, \dots, z_n), z_3, \dots, z_n) \big|_{t=0} = 0$$

on Ω has a weak type $\mu_\Omega(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma_\Omega)$ singularity.

Now a detailed computation shows that

$$\begin{aligned} \frac{dG_t}{dt}(1, \varphi_{1,t}, \varphi_{2,t}, z_3, \dots, z_n) \big|_{t=0} &= G' + \frac{\partial G_0}{\partial Z'_1} \frac{d\varphi_{1,t}}{dt} \big|_{t=0} + \frac{\partial G_0}{\partial Z'_2} \frac{d\varphi_{2,t}}{dt} \big|_{t=0} \\ &= \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_1, Z'_2)} \right\}^{-1} \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_1, Z'_2)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z'_1, Z'_2)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z'_1, Z'_2)} F'_2 \right\}. \end{aligned}$$

Then the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_1, Z'_2)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z'_1, Z'_2)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z'_1, Z'_2)} F'_2 = 0 \right\}$$

has a weak type $\mu_\Omega(0) - 1$ singularity on Ω . Similarly, the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_i, Z'_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z'_i, Z'_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z'_i, Z'_j)} F'_2 = 0 \right\} \quad (i, j = 0, 1, \dots, n)$$

has a weak type $\mu_\Omega(0) - 1$ singularity on Ω . Finally, since the expression

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2$$

is a linear combination of expressions

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z'_s, Z'_l)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z'_s, Z'_l)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z'_s, Z'_l)} F'_2, \quad (s, l = 0, 1, \dots, n)$$

and weak type $\mu_\Omega(0) - 1$ singularity is additive (cf. section 1 in [X1]), we conclude that the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 = 0 \right\}$$

has a weak type $\mu_\Omega(0) - 1$ singularity on Ω , hence it has a weak type $\mu(0) - 1$ singularity on V . q.e.d.

Remark. In general, if

$$V_t = \{F_{1,t} = 0\} \cap \{F_{2,t} = 0\} \cap \dots \cap \{F_{m,t} = 0\}$$

is a complete intersection of m hypersurfaces, and $M_t^* = V_t \cap \{G_t = 0\}$ is a μ -equisingular family of divisors. Then one can state and prove an analogy of Lemma 4 with the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 = 0 \right\}$$

replaced by a divisor of the form

$$\begin{aligned} & \left\{ \frac{\partial(F_{1,0}, F_{2,0}, F_{3,0}, \dots, F_{m,0})}{\partial(Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_m})} G' - \frac{\partial(G_0, F_{2,0}, F_{3,0}, \dots, F_{m,0})}{\partial(Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_m})} F'_1 \right. \\ & - \frac{\partial(F_{1,0}, G_0, F_{3,0}, \dots, F_{m,0})}{\partial(Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_m})} F'_2 - \frac{\partial(F_{1,0}, F_{2,0}, G_0, \dots, F_{m,0})}{\partial(Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_m})} F'_3 \\ & \left. - \dots - \frac{\partial(F_{1,0}, F_{2,0}, F_{3,0}, \dots, G_0)}{\partial(Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_m})} F'_m = 0 \right\}, \end{aligned}$$

here $i_1, \dots, i_m = 0, 1, \dots, n$.

2. PROOF OF THEOREM 1

Let $V = \{F_1 = 0\} \cap \{F_2 = 0\} \subset \mathbf{P}^n$ be a complete intersection of generic hypersurfaces $\{F_1 = 0\}$ and $\{F_2 = 0\}$ of degree d_1 and d_2 . By our assumption $m \leq n - 3$, that is $\dim V \geq 3$, we know that $\text{Pic } V = \mathbb{Z}$ and it is generated by $\mathcal{O}_V(1)$, thanks to the Lefschetz theorem. Now if $M \subset V$ is a reduced and irreducible divisor, then it is a complete intersection of V with another hypersurface $\{G = 0\}$ of degree k . Here F_1, F_2 and G are homogeneous polynomials.

Proposition 5. *Let V be a complete intersection of m generic hypersurfaces of degree d_1, d_2, \dots, d_m in \mathbf{P}^n , and $M \subset V$ a reduced and irreducible divisor. Assume that $d = d_1 + d_2 + \dots + d_m \geq n + 2$, $1 \leq m \leq n - 3$ and $d_i \geq 2$ for all i . Then there is a desingularization $\sigma : \tilde{M} \rightarrow M$ of M , and we have*

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) \geq n - 1.$$

Remark. This is an improvement of an early result of L. Ein [E2] which states that

$$H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) \neq 0.$$

Assuming Proposition 5, now we can give the

Proof of Theorem 1. (1) If $d \geq n+2$, then $H^0(M, \mathcal{O}(d-n-2)) \neq 0$, by Proposition 5,

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-d+n+2)) \geq n-1.$$

Hence we have

$$\begin{aligned} p_g(M) &= \dim H^0(\tilde{M}, K_{\tilde{M}}) \\ &\geq \dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-d+n+2)) + \dim H^0(\tilde{M}, \sigma^* \mathcal{O}(d-n-2)) - 1 \\ &\geq n-1, \end{aligned}$$

thanks to Hopf's theorem.

(2) If $d > n+2$, then $d-n-2 \geq 1$. From

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) \geq n-1 > 0,$$

we conclude that M is of general type. q.e.d.

We now begin the proof of Proposition 5. For simplicity of notation, we will assume that $m = 2$.

Assume the contrary; namely, for any generic complete intersection of 2 hyper-surfaces of degree d_1, d_2 , there is a reduced and irreducible divisor on it with

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) < n-1.$$

Set

$$\begin{aligned} B &= \{ \{F_1, F_2\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \mid \text{both varieties} \\ &\quad \{F_i = 0\} (i = 1, 2) \text{ and } \{F_1 = 0\} \cap \{F_2 = 0\} \text{ are smooth} \}, \\ A_k &= \{ \{F_1, F_2, G\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \times H^0(\mathbf{P}^n, \mathcal{O}(k)) \mid \\ &\quad \{F_1, F_2\} \in B, M = \{G = 0\} \cap V \text{ is a reduced and irreducible divisor on} \\ &\quad V = \{F_1 = 0\} \cap \{F_2 = 0\}, \dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) < n-1 \}. \end{aligned}$$

Then the map

$$\bigcup_{k=1}^{\infty} A_k \rightarrow B$$

is dominant by assumption. Hence the map $A_k \rightarrow B$ is dominant for some k . Therefore at some regular point $\{F_1, F_2\}$ of B , we can find a smooth section $B \rightarrow A_k$, that is, there is a triple

$$\{F_1, F_2, G\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \times H^0(\mathbf{P}^n, \mathcal{O}(k)),$$

which has the following property: both varieties $\{F_i = 0\}$ ($i = 1, 2$) and $V = \{F_1 = 0\} \cap \{F_2 = 0\}$ are smooth, the divisor $M = V \cap \{G = 0\}$ is reduced and irreducible,

and for any deformation $F_{1,t}$ of $F_1 = F_{1,0}$ and $F_{2,t}$ of $F_2 = F_{2,0}$, there is a unique deformation G_t of $G = G_0$, so that the divisor

$$M_t = \{F_{1,t} = 0\} \cap \{F_{2,t} = 0\} \cap \{G_t = 0\}$$

on $\{F_{1,t} = 0\} \cap \{F_{2,t} = 0\}$ has

$$\dim H^0(\tilde{M}_t, K_{\tilde{M}_t} \otimes \sigma_t^* \mathcal{O}(-(d-n-2))) < n-1.$$

Here $\sigma_t^* : \tilde{M}_t \rightarrow M_t$ is a desingularization of M_t . Moreover, we can assume that the family M_t is μ -equisingular, and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) \mid j \in \Gamma)$ singularity.

Let $\{Z_i\}$ be fixed homogeneous coordinates of \mathbf{P}^n . By Lemma 4, for any deformation $F'_1 \in H^0(\mathbf{P}^n, \mathcal{O}(d_1))$ of F_1 and $F'_2 \in H^0(\mathbf{P}^n, \mathcal{O}(d_2))$ of F_2 , there is a unique deformation $G' \in H^0(\mathbf{P}^n, \mathcal{O}(k))$ of G , so that the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 = 0 \right\} \quad (i, j = 0, 1, \dots, n)$$

on $V = \{F_{1,0} = 0\} \cap \{F_{2,0} = 0\} = \{F_1 = 0\} \cap \{F_2 = 0\}$ has a weak type $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma)$ singularity. Denote $G' = \Phi(F'_1, F'_2)$. Then we have a map

$$\Phi : H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \longrightarrow H^0(\mathbf{P}^n, \mathcal{O}(k)) / (F_1, F_2, G),$$

here (F_1, F_2, G) is the ideal generated by F_1, F_2, G .

Lemma 6. Φ is linear in $F_1, F_2 \bmod (F_1, F_2, G)$.

Proof. Otherwise, since

$$\begin{aligned} & \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} (\Phi(aF'_1 + b\tilde{F}'_1, F'_2) - a\Phi(F'_1, 0) - b\Phi(\tilde{F}'_1, 0) - \Phi(0, F'_2)) \\ &= \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(aF'_1 + b\tilde{F}'_1, F'_2) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} (aF'_1 + b\tilde{F}'_1) - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 \right\} \\ & \quad - a \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(F'_1, 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F'_1 \right\} \\ & \quad - b \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(\tilde{F}'_1, 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} \tilde{F}'_1 \right\} \\ & \quad - \left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(0, F'_2) - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F'_2 \right\}, \end{aligned}$$

and for any point $P \in V$,

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)}(P) \neq 0$$

for some i, j . By Lemma 4 and the additivity of weak type $\mu(0) - 1$ singularity, the divisor

$$\{\Phi(aF'_1 + b\tilde{F}'_1, F'_2) - a\Phi(F'_1, 0) - b\Phi(\tilde{F}'_1, 0) - \Phi(0, F'_2) = 0\}$$

will have a weak type $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) | j \in \Gamma)$ singularity on V . On the other hand, by the adjunction formula, we have

$$K_V \otimes M = \mathcal{O}(d + k - n - 1).$$

If Φ is not linear mod (F_1, F_2, G) , then

$$\Phi(aF'_1 + b\tilde{F}'_1, F'_2) - a\Phi(F'_1, 0) - b\Phi(\tilde{F}'_1, 0) - \Phi(0, F'_2)$$

will generate a section of $K_{\tilde{M}} \otimes \mathcal{O}(-(d - n - 2) - 1)$ by Lemma 2, which will imply that

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) \geq n - 1$$

because $\dim H^0(M, \mathcal{O}(1)) \geq n - 1$. Here we use the fact that $\deg F_i = d_i \geq 2$. q.e.d.

Let $\{Y_i\}$ be another homogeneous coordinate of \mathbf{P}^n . Now we take a special deformation $F'_1 = Y_p U$ ($p = 0, 1, \dots, n$) of F_1 with $U \in H^0(\mathbf{P}^n, \mathcal{O}(d_1 - 1))$. Since

$$\begin{aligned} & \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} (Y_p \Phi(Y_q U, 0) - Y_q \Phi(Y_p U, 0)) \\ &= Y_p \left(\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(Y_q U, 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} Y_q U \right) \\ & - Y_q \left(\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(Y_p U, 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} Y_p U \right), \end{aligned}$$

by Lemma 4 we conclude that the divisor $\{Y_p \Phi(Y_q U, 0) - Y_q \Phi(Y_p U, 0) = 0\}$ on V has a weak type $\mu(0) - 1$ singularity.

Lemma 7. *If $\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) < n - 1$, then there is a linear map*

$$\Phi_1 : H^0(\mathbf{P}^n, \mathcal{O}(d_1 - 1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2 - 1)) \rightarrow H^0(\mathbf{P}^n, \mathcal{O}(k - 1)) / (F_1, F_2, G),$$

so that for any $U \in H^0(\mathbf{P}^n, \mathcal{O}(d_1 - 1))$, and $W \in H^0(\mathbf{P}^n, \mathcal{O}(d_2 - 1))$, the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi_1(U, W) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} U - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} W = 0 \right\}$$

($i, j = 0, 1, \dots, n$) on V has a weak type $\mu(0) - 1$ singularity.

Proof. Let $Y, H \in H^0(\mathbf{P}^n, \mathcal{O}(1))$ be 2 hyperplanes, and $U \in H^0(\mathbf{P}^n, \mathcal{O}(d_1 - 1))$ be a fixed polynomial. By the argument before Lemma 7 (choose $Y_p = Y, Y_q = H$), we know that the divisor $\{Y \Phi(HU, 0) - H \Phi(YU, 0) = 0\}$ on V has a weak type $\mu(0) - 1$ singularity. Since we have

$$K_V \otimes M = \mathcal{O}(d + k - n - 1),$$

and $Y \Phi(HU, 0) - H \Phi(YU, 0) \in H^0(\mathbf{P}^n, \mathcal{O}(k + 1))$, if

$$Y \Phi(HU, 0) - H \Phi(YU, 0) \neq 0$$

on M , that is $Y\Phi(HU, 0) - H\Phi(YU, 0) \notin (F_1, F_2, G)$, then it will induce a section of $K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))$ by Lemma 2. Denote

$$\Lambda_H = \{Y|Y\Phi(HU, 0) - H\Phi(YU, 0) \in (F_1, F_2, G)\} \subset H^0(\mathbf{P}^n, \mathcal{O}(1)).$$

The linearity of Φ implies that Λ_H is a linear subspace of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. We conclude that $\dim \Lambda_H \geq 2$ by our assumption that

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) \leq n-2.$$

Hence there is a nontrivial hyperplane $Y_H \in \Lambda_H$ such that

$$Y_H \notin (H, F_1, F_2, G),$$

thanks to the fact that $\deg F_i \geq 2$.

Let $\sigma : \tilde{M} \rightarrow M$ be a desingularization of M . Then the linear system $|\sigma^*\mathcal{O}(1)|$ on \tilde{M} is base point free. Since $\dim M = \dim V - 1 \geq 2$, and M is reduced and irreducible, Bertini's theorem implies that the generic hyperplane section of \tilde{M} is irreducible. Therefore we can choose a generic hyperplane H , so that $H \cap M$ is irreducible and reduced. By our construction of Y_H , we have

$$Y_H\Phi(HU, 0) - H\Phi(Y_HU, 0) \in (F_1, F_2, G),$$

that is $Y_H\Phi(HU, 0) \in (H, F_1, F_2, G)$. The fact that $Y_H \notin (H, F_1, F_2, G)$ and that $H \cap M$ is irreducible now gives us $\Phi(HU, 0) \in (H, F_1, F_2, G)$. Therefore,

$$\Phi(HU, 0) = HU^* \mod (F_1, F_2, G)$$

for some $U^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))$, and U^* is unique mod (F_1, F_2, G) because M is reduced and irreducible. Similarly, for any $W \in H^0(\mathbf{P}^n, \mathcal{O}(d_2-1))$, there is a $W^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))$, such that

$$\Phi(0, HW) = HW^* \mod (F_1, F_2, G).$$

Now we define

$$\Phi_1(U, W) = U^* + W^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))/(F_1, F_2, G),$$

then Φ_1 is independent of the choice of the generic hyperplane H .

From Lemma 4, we know that the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(HU, HW) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} HU - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} HW = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity. Using the fact that

$$\Phi(HU, HW) = \Phi(HU, 0) + \Phi(0, HW) = H\Phi_1(U, W) \mod (F_1, F_2, G),$$

we find that the divisor

$$\left\{ H \left(\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi_1(U, W) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} U - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} W \right) = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity. Therefore we know that the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi_1(U, W) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} U - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} W = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity if we choose the generic hyperplane H such that it is in general position with respect to the singular locus of M . Again, we may assume that Φ_1 to be linear mod (F_1, F_2, G) as we did for Φ . q.e.d.

We continue the proof of Theorem 1. If

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) < n - 1,$$

we can repeat the argument in the proof of Lemma 7 again on the triple

$$(U, W, \Phi_1(U, W)) \in H^0(\mathbf{P}^n, \mathcal{O}(d_1 - 1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2 - 1)) \times H^0(\mathbf{P}^n, \mathcal{O}(k - 1))$$

instead of the triple

$$(F'_1, F'_2, \Phi(F'_1, F'_2)) \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \times H^0(\mathbf{P}^n, \mathcal{O}(k)),$$

and using Lemma 7 instead of Lemma 4. After repeating this process for several times, eventually we arrive at the following situation.

Case (1). $d_1 \leq k$ and $d_2 \leq k$. There are

$$R_{ij} \in H^0(\mathbf{P}^n, \mathcal{O}(k - d_1)) \text{ and } S_{ij} \in H^0(\mathbf{P}^n, \mathcal{O}(k - d_2)),$$

so that both the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} R_{ij} - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} \cdot 1 = 0 \right\}$$

and the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} S_{ij} - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} \cdot 1 = 0 \right\}$$

on V have weak type $\mu(0) - 1$ singularities. Moreover,

$$R_{ij} \equiv R, S_{ij} \equiv S \text{ mod } (F_1, F_2, G)$$

are independent of i, j , because we assume that the deformation $G' = \Phi(F'_1, F'_2)$ is unique for given F'_1, F'_2 (the reason is the same as we assume that Φ is linear).

Consider the following linear equation

$$\begin{aligned} \alpha \frac{\partial F_{1,0}}{\partial Z_i} + \beta \frac{\partial F_{2,0}}{\partial Z_i} &= \frac{\partial G_0}{\partial Z_i} - \frac{\partial F_{1,0}}{\partial Z_i} R - \frac{\partial F_{2,0}}{\partial Z_i} S, \\ \alpha \frac{\partial F_{1,0}}{\partial Z_j} + \beta \frac{\partial F_{2,0}}{\partial Z_j} &= \frac{\partial G_0}{\partial Z_j} - \frac{\partial F_{1,0}}{\partial Z_j} R - \frac{\partial F_{2,0}}{\partial Z_j} S. \end{aligned}$$

When we solve this equation, we get

$$\begin{aligned} \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \alpha &= \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} \cdot 1 - \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} R, \\ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \beta &= \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} \cdot 1 - \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} S. \end{aligned}$$

Hence the divisor

$$\left\{ \frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \left(\frac{\partial G_0}{\partial Z_i} - \frac{\partial F_{1,0}}{\partial Z_i} R - \frac{\partial F_{2,0}}{\partial Z_i} S \right) = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity. For any point $P \in V$, we can choose generic homogeneous coordinates so that

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \neq 0$$

near P for all $i \neq j$. Then the divisor

$$\left\{ \frac{\partial G}{\partial Z_i} - \frac{\partial F_1}{\partial Z_i} R - \frac{\partial F_2}{\partial Z_i} S = 0 \right\} = \left\{ \frac{\partial G_0}{\partial Z_i} - \frac{\partial F_{1,0}}{\partial Z_i} R - \frac{\partial F_{2,0}}{\partial Z_i} S = 0 \right\}$$

has a weak type $\mu(0) - 1$ singularity in a neighborhood of P . Now let $\{Y_i\}$ be another homogeneous coordinate of \mathbf{P}^n . Since

$$\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S$$

is a linear combination of expressions

$$\frac{\partial G}{\partial Z_i} - \frac{\partial F_1}{\partial Z_i} R - \frac{\partial F_2}{\partial Z_i} S \quad (i = 0, 1, \dots, n)$$

and weak type $\mu(0) - 1$ singularity is additive, we conclude that the divisor

$$\left\{ \frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity. If

$$\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S = 0 \quad \text{mod } (F_1, F_2, G)$$

for all j , then

$$\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S = 0 \quad \text{mod } (F_1, F_2)$$

because its degree $k - 1 < k$, and the Euler equation will imply that $G \in (F_1, F_2)$, which is impossible.

Therefore

$$\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S \notin (F_1, F_2, G)$$

for some j , that is,

$$\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j} R - \frac{\partial F_2}{\partial Y_j} S \neq 0$$

on M . Now we can choose

$$H_1, \dots, H_{n-1} \in H^0(\mathbf{P}^n, \mathcal{O}(1)),$$

so that H_i generates a linear subspace of dimension $n - 1$ and G is not there (in case $\deg G = 1$). Then

$$\left(\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j}R - \frac{\partial F_2}{\partial Y_j}S\right)H_1H_i \quad (i = 1, 2, \dots, n-1)$$

will induce $n - 1$ linear independent sections of $K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))$ by Lemma 2. A contradiction.

Case (2). $d_1 \leq k < d_2$. Then Lemma 7 implies that the divisor

$$\left\{ \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity. The argument in case (1) (take $S = 0$) shows that for some j , the divisor

$$\left\{ \frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j}R = 0 \right\}$$

on V has a weak type $\mu(0) - 1$ singularity, and it is nontrivial on M . Again we get

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) \geq n - 1.$$

Case (3). $d_1 > k$ and $d_2 > k$. This time, we conclude that both divisors

$$\left\{ \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} = 0 \right\} \quad \text{and} \quad \left\{ \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} = 0 \right\}$$

on V have weak type $\mu(0) - 1$ singularities. The argument in case (1) (take $R = S = 0$) shows that for some j , the divisor $\left\{ \frac{\partial G}{\partial Y_j} = 0 \right\}$ on V has a weak type $\mu(0) - 1$ singularity, and it is nontrivial on M . We conclude again that

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d - n - 2))) \geq n - 1.$$

This completes the proof of Proposition 7.

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