TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 7, July 1996

ON JACOBIAN IDEALS INVARIANT BY A REDUCIBLE $s\ell(2,\mathbf{C})$ ACTION

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Abstract. This paper deals with a reducible $s\ell(2,\mathbf{C})$ action on the formal power series ring. The purpose of this paper is to confirm a special case of the Yau Conjecture: suppose that $s\ell(2,\mathbf{C})$ acts on the formal power series ring via (0.1). Then $I(f)=(\ell_{i_1})\oplus(\ell_{i_2})\oplus\cdots\oplus(\ell_{i_s})$ modulo some one dimensional $s\ell(2,\mathbf{C})$ representations where (ℓ_i) is an irreducible $s\ell(2,\mathbf{C})$ representation of dimension ℓ_i or empty set and $\{\ell_{i_1},\ell_{i_2},\dots,\ell_{i_s}\}\subseteq\{\ell_1,\ell_2,\dots,\ell_r\}$. Unlike classical invariant theory which deals only with irreducible action and 1–dimensional representations, we treat the reducible action and higher dimensional representations succesively.

0. Introduction

In 1983, [Ya1] had a spectacular discovery which relates arbitrary isolated hypersurface singularities (the same principle applies to arbitrary isolated singularities) to finite dimensional Lie algebras for the first time. These Yau (Lie) algebras are very useful in studying isolated hypersurface singularities. For example, Seeley and Yau showed in [Se–Ya] that one can construct a continuous numerical invariant from Yau algebras. Recently Xu and Yau [Xu–Ya] showed that Yau algebras can also be used to detect the quasi–homogeneity of the original singularities. Yau algebras are not arbitrary finite dimensional Lie algebras. It was shown in [Ya2] that these algebras are solvable Lie algebras. Since every Lie algebra is a semidirect product of semi–simple Lie algebra and a solvable Lie algebra, in proving his Lie algebras are solvable, Yau only needs to show that his Lie algebras do not contain $s\ell(2,\mathbf{C})$. This leads him to study the $s\ell(2,\mathbf{C})$ action via derivations preserving m–adic filtration on the formal power series ring. In [Ya3], Yau classifies all these actions.

Theorem (Yau). *Let $L=s\ell(2,\mathbf{C})$ act on the formal power series ring via derivations preserving m–adic filtration where m is the maximal ideal (i.e., $L(m^k)\subseteq m^k$). Then there exists a coordinate $x_1,x_2,\dots,x_{\ell_1},x_{\ell_1+1},\dots,x_{\ell_1+\ell_2},x_{\ell_1+\cdots+\ell_{r-1}+1},\dots,$ $x_{\ell_1+\cdots+\ell_r},x_{\ell_1+\cdots+\ell_{r+1}},\dots,x_n$ such that the action of L is given by*

$$\begin{aligned}\tau&=D_{\tau,1}+\cdots+D_{\tau,r},\\X_+&=D_{X_+,1}+\cdots+D_{X_+,r},\\X_-&=D_{X_-,1}+\cdots+D_{X_-,r},\end{aligned}\tag{0.1}$$

Received by the editors April 28, 1995.
1991 *Mathematics Subject Classification.* Primary 17B10, 17B20; Secondary 14B05.
Key words and phrases. Invariant polynomial, weight, irreducible submodule, representation, completely reducible.
Research partially supported by N.S.C.

where

$$\begin{aligned}
D_{\tau,i} &= (\ell_i - 1)x_{\ell_1+\cdots+\ell_{i-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+1}} \\
&\quad + (\ell_i - 3)x_{\ell_1+\cdots+\ell_{i-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+2}} \\
&\quad + \cdots + (-(\ell_i - 3))x_{\ell_1+\cdots+\ell_i-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i-1}} \\
&\quad + (-(\ell_i - 1))x_{\ell_1+\cdots+\ell_i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i}}, \\
D_{X_+,i} &= (\ell_i - 1)x_{\ell_1+\cdots+\ell_{i-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+2}} \\
&\quad + \cdots + j(\ell_i - j)x_{\ell_1+\cdots+\ell_{i-1}+j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+j+1}} \\
&\quad + \cdots + (\ell_i - 1)x_{\ell_1+\cdots+\ell_i-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i}}, \\
D_{X_-,i} &= x_{\ell_1+\cdots+\ell_{i-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+1}} \\
&\quad + \cdots + x_{\ell_1+\cdots+\ell_{i-1}+j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+j-1}} \\
&\quad + \cdots + x_{\ell_1+\cdots+\ell_i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i-1}}.
\end{aligned}$$

Let f be a homogeneous polynomial of degree $k+1 \geq 3$ in n variables. Let $I(f)$ be the vector space spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$. In 1985, Yau gave the following conjecture about the structure of $I(f)$ when it is a $s\ell(2, \mathbf{C})$ module.

Yau's conjecture. *Suppose that $s\ell(2, \mathbf{C})$ acts on the formal power series ring via* (0.1)*. Then $I(f) = (\ell_{i_1}) \oplus (\ell_{i_2}) \oplus \cdots \oplus (\ell_{i_s})$ modulo some one dimensional $s\ell(2, \mathbf{C})$ representations where (ℓ_i) is an irreducible $s\ell(2, \mathbf{C})$ representation of dimension ℓ_i, or empty set and $\{\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_s}\} \subseteq \{\ell_1, \ell_2, \ldots, \ell_r\}$.*

If the $s\ell(2, \mathbf{C})$ action is irreducible, i.e., $\ell_1 = n$ in the above theorem of Yau, then Yau's conjecture was confirmed by Sampson–Yau–Yu [Sa–Ya–Yu]. In fact, they proved that f must be an invariant polynomial if $I(f)$ is an $s\ell(2, \mathbf{C})$ module. In [Ya4], Yau's conjecture was proved for any $s\ell(2, \mathbf{C})$–action for $n \leq 5$. The purpose of this paper is to confirm this conjecture for a special case of $n = 6$.

Theorem. *Let $s\ell(2, \mathbf{C})$ act on the formal power series ring in 6 variables via*

$$\begin{aligned}
\tau &= (3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4}) + (x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6}), \\
X_+ &= (3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4}) + (x_5\frac{\partial}{\partial x_6}), \\
X_- &= (x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}) + (x_6\frac{\partial}{\partial x_5}).
\end{aligned}$$

Let f be a homogeneous polynomial of degree $k+1$ in 6 variables where $k \geq 2$. If $I = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_6} \rangle$ is an $s\ell(2, \mathbf{C})$ submodule then either (i) *f is an $s\ell(2, \mathbf{C})$ invariant polynomial in $x_1, x_2, \ldots, x_6$ variables and $I = (4) \oplus (2)$, or* (ii) $f = g + c_1x_5x_6^3 +$

$c_2x_5^3x_6$ *where* $g = d(2x_1x_6^3 - \frac{1}{3}x_4x_5^3 - 2x_2x_5x_6^2 + x_3x_5^2x_6)$ *is an* $s\ell(2,\mathbf{C})$ *invariant polynomial with* $(c_1,c_2) \neq (0,0)$ *and* $d \neq 0$, *and* $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}, \frac{\partial g}{\partial x_6}\rangle = (4)\oplus(2) = \langle x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\rangle \oplus \langle 6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2, 2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5^2\rangle$, *or* (iii) f *is an* $s\ell(2,\mathbf{C})$ *invariant polynomial in* x_1, x_2, x_3, x_4 *variables and* $I = (4)$, *where* (ℓ) *denotes* ℓ*–dimensional irreducible representation of* $s\ell(2,\mathbf{C})$.

We would like to thank Professor Stephen Yau for many fruitful discussions and suggestions.

1. The proof of the theorem

Lemma 1.1. *Suppose* $s\ell(2,\mathbf{C})$ *acts on the space of homogeneous polynomials of degree* $k \geq 2$ *in* $x_1, x_2, x_3, x_4, x_5, x_6$ *via*

$$\begin{aligned}
\tau &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6},\\
X_+ &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_6},\\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_5}.
\end{aligned}$$

Suppose the weight of x_i *is given by the corresponding coefficient in the expression of* τ *above, i.e.,*

$$wt(x_1) = 3,\ wt(x_2) = 1,\ wt(x_3) = -1,\ wt(x_4) = -3,\ wt(x_5) = 1,\ wt(x_6) = -1.$$

Let I *be the complex vector subspace spanned by* $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}$, *and* $\frac{\partial f}{\partial x_6}$ *where* f *is a homogeneous polynomial of degree* $k+1$. *If* I *is an* $s\ell(2,\mathbf{C})$*–submodule and* $\dim I = 6$, *then either* (i) f *is an* $s\ell(2,\mathbf{C})$ *invariant polynomial in* $x_1, x_2, x_3, x_4, x_5, x_6$ *variables and* $I = (4)\oplus(2)$, *or* (ii) $f = g + c_1x_5x_6^3 + c_2x_5^3x_6$ *where* $g = d(2x_1x_6^3 - \frac{1}{3}x_4x_5^3 - 2x_2x_5x_6^2 + x_3x_5^2x_6)$ *is an* $s\ell(2,\mathbf{C})$ *invariant polynomial with* $(c_1,c_2) \neq (0,0)$ *and* $d \neq 0$. $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}, \frac{\partial g}{\partial x_6}\rangle = (4)\oplus(2) = \langle x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\rangle \oplus \langle 6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2, 2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5^2\rangle$.

Proof. Case 1. $I = (6)$.

By the classification theorem of $s\ell(2,\mathbf{C})$ representations, every element in I is a linear combination of homogeneous polynomials of degree k and weights $5, 3, 1, -1, -3, -5$. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f_{k+1}^i is a homogeneous ploynomial of degree $k+1$ and weight i.

For $|i| \geq 9$

$$\begin{aligned}
&\left|wt\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geq 6, \qquad 1 \leq j \leq 6,\\
&\Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0, \qquad 1 \leq j \leq 6,\\
&\Rightarrow f_{k+1}^i = 0.
\end{aligned}$$

For $i = \pm 1, \pm 3, \pm 5, \pm 7$

$$\begin{aligned} &wt\frac{\partial f_{k+1}^i}{\partial x_j} \text{ are even integers for } \ 1 \le j \le 6 \\ &\Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0, \qquad 1 \le j \le 6, \\ &\Rightarrow f_{k+1}^i = 0. \end{aligned}$$

For $i = 8$

$$\begin{aligned} &wt\frac{\partial f_{k+1}^8}{\partial x_1} = 5, \quad wt\frac{\partial f_{k+1}^8}{\partial x_j} \ge 7 \text{ for } 2 \le j \le 6 \\ &\Rightarrow f_{k+1}^8 \text{ depends only on the } x_1 \text{ variable} \\ &\Rightarrow f_{k+1}^8 = 0 \text{ because } wt(x_1) = 3. \end{aligned}$$

Similar arguments show that $f_{k+1}^{-8} = 0$.

For $i = 6$

$$\begin{aligned} &wt\frac{\partial f_{k+1}^6}{\partial x_1} = 3, \quad wt\frac{\partial f_{k+1}^6}{\partial x_2} = 5, \quad wt\frac{\partial f_{k+1}^6}{\partial x_3} = 7, \\ &wt\frac{\partial f_{k+1}^6}{\partial x_4} = 9, \quad wt\frac{\partial f_{k+1}^6}{\partial x_5} = 5, \quad wt\frac{\partial f_{k+1}^6}{\partial x_6} = 7. \\ &\Rightarrow f_{k+1}^6 \text{ depends only on the } x_1, x_2, x_5 \text{ variables.} \end{aligned}$$

If f_{k+1}^6 were not zero, then either $\frac{\partial f_{k+1}^6}{\partial x_1}$ or $\frac{\partial f_{k+1}^6}{\partial x_2}$ or $\frac{\partial f_{k+1}^6}{\partial x_5}$ would generate I because I is an irreducible $s\ell(2,\mathbf{C})$ module. Hence I would involve only the x_1, x_2, x_5 variables. It follows that $\frac{\partial f}{\partial x_j}$, $1 \le j \le 6$, involves only the x_1, x_2, x_5 variables and hence so does f. This implies that $\frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_6} = 0$, which contradicts the fact that dim $I = 6$. Thus we have $f_{k+1}^6 = 0$. Similar argument shows that $f_{k+1}^{-6} = 0$ and that $f_{k+1}^4 = f_{k+1}^{-4} = 0$. Hence $f = f_{k+1}^{-2} + f_{k+1}^0 + f_{k+1}^2$.

For $i = 2$

$$\begin{aligned} &wt\frac{\partial f_{k+1}^2}{\partial x_1} = -1, \quad wt\frac{\partial f_{k+1}^2}{\partial x_2} = 1, \quad wt\frac{\partial f_{k+1}^2}{\partial x_3} = 3, \\ &wt\frac{\partial f_{k+1}^2}{\partial x_4} = 5, \quad wt\frac{\partial f_{k+1}^2}{\partial x_5} = 1, \quad wt\frac{\partial f_{k+1}^2}{\partial x_6} = 3. \end{aligned}$$

Since $wt\frac{\partial f_{k+1}^2}{\partial x_2} = wt\frac{\partial f_{k+1}^2}{\partial x_5} = 1$ and $wt\frac{\partial f_{k+1}^2}{\partial x_3} = wt\frac{\partial f_{k+1}^2}{\partial x_6} = 3$, in view of Lemma 5.1 of [Ya4], there exist constants r_1, r_2, r_3, r_4 such that

$$f_{k+1}^2 = \sum_{a,b} c_{a,b} x_1^a x_4^b (r_1 x_2 + r_2 x_5)^{\frac{k+3-4a+2b}{2}} (r_3 x_3 + r_4 x_6)^{\frac{k-1+2a-4b}{2}}.$$

Similarly, we can write

$$f_{k+1}^0 = \sum_{a,b} d_{a,b} x_1^a x_4^b (r_5 x_2 + r_6 x_5)^{\frac{k+1-4a+2b}{2}} (r_7 x_3 + r_8 x_6)^{\frac{k+1+2a-4b}{2}}$$

and

$$f_{k+1}^{-2}=\sum_{a,b}e_{a,b}x_1^ax_4^b(r_9x_2+r_{10}x_5)^{\frac{k-1-4a+2b}{2}}(r_{11}x_3+r_{12}x_6)^{\frac{k+3+2a-4b}{2}}.$$

Assuming $\frac{\partial f_{k+1}^2}{\partial x_2}\neq 0$ or $\frac{\partial f_{k+1}^2}{\partial x_5}\neq 0$, then

$$\frac{\partial f_{k+1}^2}{\partial x_2}=\sum_{a,b}\frac{k+3-4a+2b}{2}r_1c_{a,b}x_1^ax_4^b(r_1x_2+r_2x_5)^{\frac{k+1-4a+2b}{2}}(r_3x_3+r_4x_6)^{\frac{k-1+2a-4b}{2}}$$

or

$$\frac{\partial f_{k+1}^2}{\partial x_5}=\sum_{a,b}\frac{k+3-4a+2b}{2}r_2c_{a,b}x_1^ax_4^b(r_1x_2+r_2x_5)^{\frac{k+1-4a+2b}{2}}(r_3x_3+r_4x_6)^{\frac{k-1+2a-4b}{2}}$$

is a nonzero element of weight 1 in I. This implies that

$$\begin{aligned}\frac{\partial f_{k+1}^0}{\partial x_3}&=\sum_{a,b}\frac{k+1+2a-4b}{2}r_7d_{a,b}x_1^ax_4^b(r_5x_2+r_6x_5)^{\frac{k+1-4a+2b}{2}}\\&\qquad\times(r_7x_3+r_8x_6)^{\frac{k-1+2a-4b}{2}},\\\frac{\partial f_{k+1}^0}{\partial x_6}&=\sum_{a,b}\frac{k+1+2a-4b}{2}r_8d_{a,b}x_1^ax_4^b(r_5x_2+r_6x_5)^{\frac{k+1-4a+2b}{2}}\\&\qquad\times(r_7x_3+r_8x_6)^{\frac{k-1+2a-4b}{2}},\\\frac{\partial f_{k+1}^{-2}}{\partial x_4}&=\sum_{a,b}be_{a,b}x_1^ax_4^{b-1}(r_9x_2+r_{10}x_5)^{\frac{k-1-4a+2b}{2}}\\&\qquad\times(r_{11}x_3+r_{10}x_6)^{\frac{k+3+2a-4b}{2}},\end{aligned}$$

are constant multiples of $\frac{\partial f_{k+1}^2}{\partial x_2}$ or $\frac{\partial f_{k+1}^2}{\partial x_5}$. It follows that $(r_5x_2+r_6x_5)$ and $(r_9x_2+r_{10}x_5)$ are constant multiples of $(r_1x_2+r_2x_5)$ and $(r_7x_3+r_8x_6)$ and $(r_{11}x_3+r_{12}x_6)$ are constant multiples of $(r_3x_3+r_4x_6)$. Thus

$$\begin{aligned}f=&f_{k+1}^2+f_{k+1}^0+f_{k+1}^{-2}\\=&\sum_{a,b}c_{a,b}x_1^ax_4^b(r_1x_2+r_2x_5)^{\frac{k+3-4a+2b}{2}}(r_3x_3+r_4x_6)^{\frac{k-1+2a-4b}{2}}\\&+\sum_{a,b}\widetilde{d_{a,b}}\,x_1^ax_4^b(r_1x_2+r_2x_5)^{\frac{k+1-4a+2b}{2}}(r_3x_3+r_4x_6)^{\frac{k+1+2a-4b}{2}}\\&+\sum_{a,b}\widetilde{e_{a,b}}\,x_1^ax_4^b(r_1x_2+r_2x_5)^{\frac{k-1-4a+2b}{2}}(r_3x_3+r_4x_6)^{\frac{k+3+2a-4b}{2}}.\end{aligned}$$

This implies that $\dim I\le 4$, which contradicts our hypothesis that $\dim I=6$. Hence $\frac{\partial f_{k+1}^2}{\partial x_2}=\frac{\partial f_{k+1}^2}{\partial x_5}=0$.

Since $wt(X_+ f_{k+1}^2) = 4$, so $X_+ f_{k+1}^2 = 0$ by previous argument. Now

$$0 = \frac{\partial}{\partial x_2} X_+ f_{k+1}^2 = X_+ \frac{\partial f_{k+1}^2}{\partial x_2} + 4\frac{\partial f_{k+1}^2}{\partial x_3} \Rightarrow \frac{\partial f_{k+1}^2}{\partial x_3} = 0.$$

Similarly,

$$0 = \frac{\partial}{\partial x_3} X_+ f_{k+1}^2 = X_+ \frac{\partial f_{k+1}^2}{\partial x_3} + 3\frac{\partial f_{k+1}^2}{\partial x_4} \Rightarrow \frac{\partial f_{k+1}^2}{\partial x_4} = 0,$$
$$0 = \frac{\partial}{\partial x_5} X_+ f_{k+1}^2 = X_+ \frac{\partial f_{k+1}^2}{\partial x_5} + \frac{\partial f_{k+1}^2}{\partial x_6} \Rightarrow \frac{\partial f_{k+1}^2}{\partial x_6} = 0.$$

Thus f_{k+1}^2 depends only on the x_1 variable $\Rightarrow f_{k+1}^2 = 0$. Similar arguments show that $f_{k+1}^{-2} = 0$. So $f = f_{k+1}^0$.

$$wt\frac{\partial f}{\partial x_1} = -3, \quad wt\frac{\partial f}{\partial x_2} = -1, \quad wt\frac{\partial f}{\partial x_3} = 1,$$
$$wt\frac{\partial f}{\partial x_4} = 3, \quad wt\frac{\partial f}{\partial x_5} = -1, \quad wt\frac{\partial f}{\partial x_6} = 1.$$

This implies that dim $I \leq 4$, which contradicts our hypothesis that dim $I = 6$. We conclude that Case 1 cannot occur.

Case 2. $I = (5) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $4, 2, 0, -2$, and -4.

By the same argument as in the beginning of Case 1 we have $f_{k+1}^i = 0$ for $i = 0, \pm 2, \pm 4$ and $|i| \geq 6$.

For $i = 5$

$$wt\frac{\partial f_{k+1}^5}{\partial x_1} = 2, \quad wt\frac{\partial f_{k+1}^5}{\partial x_2} = 4, \quad wt\frac{\partial f_{k+1}^5}{\partial x_3} = 6,$$
$$wt\frac{\partial f_{k+1}^5}{\partial x_4} = 8, \quad wt\frac{\partial f_{k+1}^5}{\partial x_5} = 4, \quad wt\frac{\partial f_{k+1}^5}{\partial x_6} = 6.$$
$$\Rightarrow f_{k+1}^5 \text{ depends only on the } x_1, x_2, x_5 \text{ variables.}$$

Since $wt\frac{\partial f_{k+1}^5}{\partial x_2} = wt\frac{\partial f_{k+1}^5}{\partial x_5} = 4$, in view of Lemma 5.1 of [Ya4], there exist constants r_1, r_2 such that

$$f_{k+1}^5 = c x_1^{\frac{4-k}{2}} (r_1 x_2 + r_2 x_5)^{\frac{3k-2}{2}}.$$

If $f_{k+1}^5 \neq 0$, then $\frac{\partial f_{k+1}^5}{\partial x_2} \neq 0$ or $\frac{\partial f_{k+1}^5}{\partial x_5} \neq 0$. Without loss generality, we may assume that $\frac{\partial f_{k+1}^5}{\partial x_2} \neq 0$. Then

$$\frac{\partial f_{k+1}^5}{\partial x_2} = r_1 c \frac{3k-2}{2} x_1^{\frac{4-k}{2}} (r_1 x_2 + r_2 x_5)^{\frac{3k-4}{2}}$$

is a nonzero element of weight 4 in I.

$$X_+ \frac{\partial f_{k+1}^5}{\partial x_2} = 3 r_1^2 c \frac{3k-2}{2} \frac{3k-4}{2} x_1^{\frac{6-k}{2}} (r_1 x_2 + r_2 x_5)^{\frac{3k-6}{2}}.$$

Since $wt\big(X_+\frac{\partial f^5_{k+1}}{\partial x_2}\big)=6$, so $X_+\frac{\partial f^5_{k+1}}{\partial x_2}=0\Rightarrow r_1^2c=0\Rightarrow \frac{\partial f^5_{k+1}}{\partial x_2}=0$. Thus $f^5_{k+1}=0$. Similar arguments shows that $f^{-5}_{k+1}=0$.

For $i=3$

$$\begin{aligned}
&wt\frac{\partial f^3_{k+1}}{\partial x_1}=0,\quad wt\frac{\partial f^3_{k+1}}{\partial x_2}=2,\quad wt\frac{\partial f^3_{k+1}}{\partial x_3}=4,\\
&wt\frac{\partial f^3_{k+1}}{\partial x_4}=6,\quad wt\frac{\partial f^3_{k+1}}{\partial x_5}=2,\quad wt\frac{\partial f^3_{k+1}}{\partial x_6}=4.\\
&\Rightarrow f^3_{k+1}\quad\text{is independent of the } x_4 \text{ variable.}
\end{aligned}$$

Since $wt\frac{\partial f^3_{k+1}}{\partial x_2}=wt\frac{\partial f^3_{k+1}}{\partial x_5}=2$ and $wt\frac{\partial f^3_{k+1}}{\partial x_3}=wt\frac{\partial f^3_{k+1}}{\partial x_6}=4$, in view of Lemma 5.1 of [Ya4], there exist constants r_1,r_2,r_3,r_4 such that

$$f^3_{k+1}=\sum_{a\geq 0}c_ax_1^b(r_1x_2+r_2x_5)^c(r_3x_3+r_4x_6)^a$$

where $b=\frac{2a-k+2}{2}$, $c=\frac{-4a+3k}{2}$.

Assuming $\frac{\partial f^3_{k+1}}{\partial x_3}\neq 0$. Since $wt(X_+f^3_{k+1})=5$, so $X_+f^3_{k+1}=0$ by previous argument. Now $0=\frac{\partial}{\partial x_2}X_+f^3_{k+1}=X_+\frac{\partial f^3_{k+1}}{\partial x_2}+4\frac{\partial f^3_{k+1}}{\partial x_3}\Rightarrow\frac{\partial f^3_{k+1}}{\partial x_2}\neq 0$. Since $wt(X_-\frac{\partial f^3_{k+1}}{\partial x_3})=wt(\frac{\partial f^3_{k+1}}{\partial x_2})=2$, there exists a constant d such that $X_-\frac{\partial f^3_{k+1}}{\partial x_3}=d\frac{\partial f^3_{k+1}}{\partial x_2}$. Differentiating this equation with respect to the x_4 variable, we get

$$\frac{\partial^2 f^3_{k+1}}{\partial x_3^2}+X_-\frac{\partial^2 f^3_{k+1}}{\partial x_4\partial x_3}=d\frac{\partial^2 f^3_{k+1}}{\partial x_4\partial x_2}\Rightarrow\frac{\partial^2 f^3_{k+1}}{\partial x_3^2}=0.$$

Hence $\frac{\partial f^3_{k+1}}{\partial x_3}$ is independent of the x_3 variable. Thus

$$\frac{\partial f^3_{k+1}}{\partial x_3}=\sum_{a\geq 1}ar_3c_ax_1^b(r_1x_2+r_2x_5)^c(r_3x_3+r_4x_6)^{a-1}\Rightarrow a=1.$$

So

$$\frac{\partial f^3_{k+1}}{\partial x_3}=r_3c_1x_1^{\frac{4-k}{2}}(r_1x_2+r_2x_5)^{\frac{-4+3k}{2}}\Rightarrow r_3c_1\neq 0.$$

Since $wt(X_+\frac{\partial f^3_{k+1}}{\partial x_3})=6$, so $0=X_+\frac{\partial f^3_{k+1}}{\partial x_3}=3(\frac{-4+3k}{2})c_1r_1r_3x_1^{\frac{6-k}{2}}(r_1x_2+r_2x_5)^{\frac{-6+3k}{2}}$ $\Rightarrow r_1=0\Rightarrow\frac{\partial f^3_{k+1}}{\partial x_2}=0$.

Thus $\frac{\partial f^3_{k+1}}{\partial x_3}=0$. Therefore,

$$f^3_{k+1}=\sum_{a=\ell_1}^{\ell_2}c_ax_1^bx_6^a(r_1x_2+r_2x_5)^c$$

where $b=\frac{2a-k+2}{2}$, $c=\frac{-4a+3k}{2}$, $\ell_1=\lceil\frac{k-2}{2}\rceil$, $\ell_2=\lfloor\frac{3k}{4}\rfloor$.

Assuming $\frac{\partial f^3_{k+1}}{\partial x_6}\neq 0$. Then $a\geq 1$, $c_a\neq 0$ for some $a\geq 1$ and $0=\frac{\partial}{\partial x_5}X_+f^3_{k+1}=X_+\frac{\partial f^3_{k+1}}{\partial x_5}+\frac{\partial f^3_{k+1}}{\partial x_6}\Rightarrow\frac{\partial f^3_{k+1}}{\partial x_5}\neq 0\Rightarrow r_2\neq 0$, $c\geq 1$, $c_a\neq 0$ for some $a\geq 0$. Since

$wt(X_+\frac{\partial f^3_{k+1}}{\partial x_5}) = wt(\frac{\partial f^3_{k+1}}{\partial x_6}) = 4$, there exists a constant d such that $X_+\frac{\partial f^3_{k+1}}{\partial x_5} = d\frac{\partial f^3_{k+1}}{\partial x_6}$. If $d = 0$, then $X_+\frac{\partial f^3_{k+1}}{\partial x_5} = 0$. Since $wt(X_-^4\frac{\partial f^3_{k+1}}{\partial x_5}) = -6$, so either $(4) = \langle\frac{\partial f^3_{k+1}}{\partial x_5}, X_-\frac{\partial f^3_{k+1}}{\partial x_5}, X_-^2\frac{\partial f^3_{k+1}}{\partial x_5}, X_-^3\frac{\partial f^3_{k+1}}{\partial x_5}\rangle$ or $(3) = \langle\frac{\partial f^3_{k+1}}{\partial x_5}, X_-\frac{\partial f^3_{k+1}}{\partial x_5}, X_-^2\frac{\partial f^3_{k+1}}{\partial x_5}\rangle$ or $(2) = \langle\frac{\partial f^3_{k+1}}{\partial x_5}, X_-\frac{\partial f^3_{k+1}}{\partial x_5}\rangle$ or $(1) = \langle\frac{\partial f^3_{k+1}}{\partial x_5}\rangle$ in I. This contradicts $I = (5) \oplus (1)$ because $wt\frac{\partial f^3_{k+1}}{\partial x_5} = 2$. Thus $d \neq 0$.

Now

$$\begin{aligned}
0 &= X_+\frac{\partial f^3_{k+1}}{\partial x_5} - d\frac{\partial f^3_{k+1}}{\partial x_6}\\
&= \sum_{a=\ell_1}^{\ell_2}[3r_1r_2c(c-1)c_ax_1^{b+1}x_6^a(r_1x_2+r_2x_5)^{c-2}\\
&\qquad + cr_2c_ax_1^bx_5x_6^{a-1}(r_1x_2+r_2x_5)^{c-1} - d_ac_ax_1^bx_6^{a-1}(r_1x_2+r_2x_5)^c].
\end{aligned}$$

Suppose $\ell_1 \geq 1$. Then $k \geq 3 \Rightarrow \ell_2 \geq 2$ and

$$\begin{aligned}
0 = \sum_{a=\ell_1}^{\ell_2-1} & x_1^{b+1}x_6^a\{(r_1x_2+r_2x_5)^{c-2}[3r_1r_2c(c-1)c_a - d(a+1)c_{a+1}]\\
&\qquad\qquad + x_5(r_1x_2+r_2x_5)^{c-3}cr_2(a+1)c_{a+1}\}\\
&+ (-d)\ell_1c_{\ell_1}x_1^{\frac{2\ell_1-k+2}{2}}x_6^{\ell_1-1}(r_1x_2+r_2x_5)^{\frac{-4\ell_1+3k}{2}}\\
&+ \frac{-4\ell_1+3k}{2}r_2\ell_1c_{\ell_1}x_1^{\frac{2\ell_1-k+2}{2}}x_5x_6^{\ell_1-1}(r_1x_2+r_2x_5)^{\frac{-4\ell_1+3k-2}{2}}.
\end{aligned}$$

(Note that $a = \ell_2 - 1 \Rightarrow c = 2$ whereas $a = \ell_2 \Rightarrow c = 0$). Therefore,

(1) $r_1^{c-2}[3r_1r_2c(c-1)c_a - d(a+1)c_{a+1}] = 0$ for $\ell_1 \leq a \leq \ell_2 - 1$.

(2) $-d\ell_1c_{\ell_1}r_1^{\ell_3} = 0$ where $\ell_3 = \frac{-4\ell_1+3k}{2} \Rightarrow c_{\ell_1}r_1^{\ell_3} = 0$.

If $\ell_3 = 0$, then $\ell_1 = \frac{3k}{4} \Rightarrow k = -4$. So $\ell_3 \geq 1 \Rightarrow c_{\ell_1}r_1 = 0$,

$$\begin{aligned}
&a = \ell_1 \overset{(1)}{\Rightarrow} -d(\ell_1+1)r_1^{\frac{-4\ell_1+3k-4}{2}}c_{\ell_1+1} = 0 \Rightarrow c_{\ell_1+1}r_1 = 0.\\
&a = \ell_1+1 \overset{(1)}{\Rightarrow} -d(\ell_1+2)r_1^{\frac{-4\ell_1+3k-8}{2}}c_{\ell_1+2} = 0 \Rightarrow c_{\ell_1+2}r_1 = 0.\\
&\quad\vdots\\
&a = \ell_2-1 \overset{(1)}{\Rightarrow} -d\ell_2c_{\ell_2} = 0 \Rightarrow c_{\ell_2} = 0.
\end{aligned}$$

Thus $r_1 = 0$. Suppose $k = 2$. Then $f^3_3 = c_0(r_1x_2+r_2x_5)^3 + c_1x_1x_6(r_1x_2+r_2x_5)$,

$$\begin{aligned}
0 &= X_+\frac{\partial f^3_3}{\partial x_5} - d\frac{\partial f^3_3}{\partial x_6}\\
&= (-dr_1c_1 + 18c_0r_1^2r_2)x_1x_2 + (-dr_2c_1 + 18c_0r_1r_2^2 + r_2c_1)x_1x_5.
\end{aligned}$$

So

(3) $0 = -dr_1c_1 + 18c_0r_1^2r_2 = r_1(-dc_1 + 18c_0r_1r_2)$.

(4) $0=-dr_2c_1+18c_0r_1r_2^2+r_2c_1=r_2(-dc_1+18c_0r_1r_2+c_1)$.

$\overset{(4)}{\Rightarrow}-dc_1+18c_0r_1r_2=-c_1\overset{(3)}{\Rightarrow}0=-c_1r_1\Rightarrow c_1r_1=0\overset{(3)}{\Rightarrow}18c_0r_1^2r_2=0\Rightarrow c_0r_1=0$.

Thus $r_1=0$.

Thus $f_{k+1}^3=\sum_{a\geq 0}c_ax_1^bx_5^cx_6^a$ where $b=\frac{2a-k+2}{2}$, $c=\frac{-4a+3k}{2}$. Since $wt(X_+\frac{\partial f_{k+1}^3}{\partial x_6})=6$, so $0=X_+\frac{\partial f_{k+1}^3}{\partial x_6}=\sum_{a\geq 2}c_aa(a-1)x_1^bx_5^{c+1}x_6^{a-2}\Rightarrow c_a=0$ for all $a\geq 2$. So, $f_{k+1}^3=c_0x_1^{\frac{2-k}{2}}x_5^{\frac{3k}{2}}+c_1x_1^{\frac{4-k}{2}}x_5^{\frac{-4+3k}{2}}x_6$.

$k=2,\ f_3^3=c_0x_5^3+c_1x_1x_5x_6\quad(\Rightarrow c_1\neq 0)$.

$k=4,\ f_5^3=c_1x_5^4x_6\quad(\Rightarrow c_1\neq 0)$.

Suppose $f_3^3=c_0x_5^3+c_1x_1x_5x_6$. Since $X_+^2\frac{\partial f_3^3}{\partial x_1}=0$ and $X_-^2\frac{\partial f_3^3}{\partial x_1}=0$, so

$$(3)=\langle\frac{\partial f_3^3}{\partial x_1},X_+\frac{\partial f_3^3}{\partial x_1},X_-\frac{\partial f_3^3}{\partial x_1}\rangle=\langle x_5x_6,x_5^2,x_6^2\rangle\subseteq I.$$

This contradicts $I=(5)\oplus(1)$.

Suppose $f_5^3=c_1x_5^4x_6$. Since $X_+\frac{\partial f_5^3}{\partial x_6}=0$ and $X_-^5\frac{\partial f_5^3}{\partial x_6}=0$, so

$$\begin{aligned}(5)&=\langle\frac{\partial f_5^3}{\partial x_6},X_-\frac{\partial f_5^3}{\partial x_6},X_-^2\frac{\partial f_5^3}{\partial x_6},X_-^3\frac{\partial f_5^3}{\partial x_6},X_-^4\frac{\partial f_5^3}{\partial x_6}\rangle\\&=\langle x_5^4,x_5^3x_6,x_5^2x_6^2,x_5x_6^3,x_6^4\rangle.\end{aligned}$$

Now

$$\begin{aligned}&wt\frac{\partial f_5^1}{\partial x_1}=-2,\quad wt\frac{\partial f_5^1}{\partial x_2}=0,\quad wt\frac{\partial f_5^1}{\partial x_3}=2,\\&wt\frac{\partial f_5^1}{\partial x_4}=4,\quad wt\frac{\partial f_5^1}{\partial x_5}=0,\quad wt\frac{\partial f_5^1}{\partial x_6}=2.\\&\Rightarrow\frac{\partial f_5^1}{\partial x_1}=d_1x_5x_6^3\\&\Rightarrow f_5^1=d_1x_1x_5x_6^3+g_1(x_2,x_3,x_4,x_5,x_6)\\&\Rightarrow\frac{\partial f_5^1}{\partial x_3}=\frac{\partial g_1}{\partial x_3}.\end{aligned}$$

On the other hand, $\frac{\partial f_5^1}{\partial x_3}=d_3x_5^3x_6$. Thus

$$\begin{aligned}&\frac{\partial g_1}{\partial x_3}=d_3x_5^3x_6\\&\Rightarrow g_1=d_3x_3x_5^3x_6+g_3(x_2,x_4,x_5,x_6)\\&\Rightarrow f_5^1=d_1x_1x_5x_6^3+d_3x_3x_5^3x_6+g_3(x_2,x_4,x_5,x_6)\\&\Rightarrow\frac{\partial f_5^1}{\partial x_6}=3d_1x_1x_5x_6^2+d_3x_3x_5^3+\frac{\partial g_3}{\partial x_6}.\end{aligned}$$

Furthermore, $\frac{\partial f_5^1}{\partial x_6} = d_6 x_5^3 x_6$. Thus

$$\begin{aligned}
&d_1 = d_3 = 0 \text{ and } \frac{\partial g_3}{\partial x_6} = d_6 x_5^3 x_6 \\
&\Rightarrow g_3 = \frac{1}{2} d_6 x_5^3 x_6^2 + g_6(x_2, x_4, x_5) \\
&\Rightarrow f_5^1 = \frac{1}{2} d_6 x_5^3 x_6^2 + g_6(x_2, x_4, x_5) \\
&\Rightarrow \frac{\partial f_5^1}{\partial x_4} = \frac{\partial g_6}{\partial x_4}.
\end{aligned}$$

Moreover, $\frac{\partial f_5^1}{\partial x_4} = d_4 x_5^4$. Thus

$$\begin{aligned}
&\frac{\partial g_6}{\partial x_4} = d_4 x_5^4 \\
&\Rightarrow g_6 = d_4 x_4 x_5^4 + g_4(x_2, x_5) \\
&\Rightarrow f_5^1 = \frac{1}{2} d_6 x_5^3 x_6^2 + d_4 x_4 x_5^4 + g_4(x_2, x_5).
\end{aligned}$$

Since $g_4(x_2, x_5)$ is a homogeneous polynomial in x_2, x_5 of degree 5 and weight 1, so $g_4(x_2, x_5) = 0$. Thus

$$f_5^1 = \frac{1}{2} d_6 x_5^3 x_6^2 + d_4 x_4 x_5^4.$$

Since $wt\big(X_+^3 \frac{\partial f_5^1}{\partial x_5}\big) = 6$, so

$$\begin{aligned}
0 = X_+^3 \frac{\partial f_5^1}{\partial x_5} &= 144 d_4 x_1 x_5^3. \\
\Rightarrow d_4 = 0 \Rightarrow f_5^1 &= \frac{1}{2} d_6 x_5^3 x_6^2.
\end{aligned}$$

Similarly, we can show that $f_5^{-3} = c x_5 x_6^4$ and $f_5^{-1} = d x_5^2 x_6^3$. So

$$\begin{aligned}
f &= f_5^{-3} + f_5^{-1} + f_5^1 + f_5^3 \\
&= c x_5 x_6^4 + d x_5^2 x_6^3 + \frac{1}{2} d_6 x_5^3 x_6^2 + c_1 x_5^4 x_6 \\
&\Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \\
&\Rightarrow \dim I \leq 2.
\end{aligned}$$

This contradicts $\dim I = 6$. Thus $\frac{\partial f_{k+1}^3}{\partial x_6} = 0$. Therefore,

$$\begin{aligned}
&f_{k+1}^3 \text{ depends only on the } x_1, x_2, x_5 \text{ variables} \\
&\Rightarrow f_{k+1}^3 = c x_1^{\frac{2-k}{2}} (r_1 x_2 + r_2 x_5)^{\frac{3k}{2}} \\
&\Rightarrow f_{k+1}^3 = (r_1 x_2 + r_2 x_5)^3.
\end{aligned}$$

Assuming $\frac{\partial f_{k+1}^3}{\partial x_2} \neq 0$, then $\frac{\partial f_{k+1}^3}{\partial x_2} = 3r_1(r_1x_2+r_2x_5)^2$ is a nonzero element of weight 2 in I. Since $wt\big(X_+^2\frac{\partial f_{k+1}^3}{\partial x_2}\big) = 6$, so

$$0 = X_+^2\frac{\partial f_{k+1}^3}{\partial x_2} = 54r_1^3x_1^2$$
$$\Rightarrow r_1 = 0 \Rightarrow \frac{\partial f_{k+1}^3}{\partial x_2} = 0.$$

Thus f_{k+1}^3 depends only on the x_5 variable.

$$\Rightarrow f_{k+1}^3 = cx_5^3.$$

If $c \neq 0$, then since $X_+\frac{\partial f_{k+1}^3}{\partial x_5} = 0$ and $X_-^3\frac{\partial f_{k+1}^3}{\partial x_5} = 0$, so

$$(3) = \langle\frac{\partial f_{k+1}^3}{\partial x_5}, X_-\frac{\partial f_{k+1}^3}{\partial x_5}, X_-^2\frac{\partial f_{k+1}^3}{\partial x_5}\rangle = \langle x_5^2, x_5x_6, x_6^2\rangle \subseteq I.$$

This contradicts $I = (5)\oplus(1)$. Thus $f_{k+1}^3 = 0$. Similarly, we can show that $f_{k+1}^{-3} = 0$.

For $i = 1$

$$wt\frac{\partial f_{k+1}^1}{\partial x_1} = -2, \quad wt\frac{\partial f_{k+1}^1}{\partial x_2} = 0, \quad wt\frac{\partial f_{k+1}^1}{\partial x_3} = 2,$$
$$wt\frac{\partial f_{k+1}^1}{\partial x_4} = 4, \quad wt\frac{\partial f_{k+1}^1}{\partial x_5} = 0, \quad wt\frac{\partial f_{k+1}^1}{\partial x_6} = 2.$$

Since $wt\frac{\partial f_{k+1}^1}{\partial x_3} = wt\frac{\partial f_{k+1}^1}{\partial x_6} = 2$, in view of Lemma 5.1 [Ya4], there exist constants r_1, r_2 such that

$$f_{k+1}^1 = \sum_{b,c,e} w_{b,c,e}x_1^ax_2^bx_4^cx_5^d(r_1x_3+r_2x_6)^e$$

where $a = \frac{4c-k+2e}{2}$, $d = -\frac{2b+6c-3k+4e-2}{2}$, assuming $\frac{\partial f_{k+1}^1}{\partial x_4} \neq 0$.

Now $0 = \frac{\partial}{\partial x_3}X_+f_{k+1}^1 = X_+\frac{\partial f_{k+1}^1}{\partial x_3} + 3\frac{\partial f_{k+1}^1}{\partial x_4} \Rightarrow \frac{\partial f_{k+1}^1}{\partial x_3} \neq 0$. Then $c \geq 1, e \geq 1, r_1 \neq 0$, $w_{b,c,e} \neq 0$ for some $c \geq 1, e \geq 1$. Since $wt(X_+\frac{\partial f_{k+1}^1}{\partial x_3}) = wt\frac{\partial f_{k+1}^1}{\partial x_4} = 4$, there exists a constant ℓ such that $X_+ \frac{\partial f_{k+1}^1}{\partial x_3} = \ell \frac{\partial f_{k+1}^1}{\partial x_4}$. As before, we must make $\ell \neq 0$. Thus

$$\begin{aligned}
0 &= X_+ \frac{\partial f_{k+1}^1}{\partial x_3} - \ell \frac{\partial f_{k+1}^1}{\partial x_4} \\
&= \sum_{b,c,e} \{ w_{b,c,e} e r_1 [3b x_1^{a+1} x_2^{b-1} x_4^c x_5^d (r_1 x_3 + r_2 x_6)^{e-1} \\
&\qquad + 4 r_1 (e-1) x_1^a x_2^{b+1} x_4^c x_5^d (r_1 x_3 + r_2 x_6)^{e-2} \\
&\qquad + 3c x_1^a x_2^b x_4^{c-1} x_5^d x_3 (r_1 x_3 + r_2 x_6)^{e-1} \\
&\qquad + r_2 (e-1) x_1^a x_2^b x_4^c x_5^{d+1} (r_1 x_3 + r_2 x_6)^{e-2}] \\
&\qquad\qquad - \ell w_{b,c,e} c x_1^a x_2^b x_4^{c-1} x_5^d (r_1 x_3 + r_2 x_6)^e \} \\
&= \sum_{\substack{b\geq 0\\ c\geq 0\\ e\geq 1}} x_1^{a+2} x_2^{b+1} x_4^c x_5^{d-1} \{ (r_1 x_3 + r_2 x_6)^e [w_{b+2,c,e+1} 3(b+2)(e+1) r_1 \\
&\qquad + w_{b,c,e+2} 4(e+1)(e+2) r_1^2 \\
&\qquad + w_{b+1,c,e+2} (e+1)(e+2) r_1 r_2 - \ell w_{b+1,c+1,e}(c+1)] \\
&\qquad + x_3 (r_1 x_3 + r_2 x_6)^{e-1} w_{b+1,c+1,e} 3(c+1) e r_1 \} \\
&\quad + \sum_{c\geq 0} x_1^{\frac{4c-k+4}{2}} x_4^c x_5^{\frac{-6c+3k-4}{2}} [w_{1,c,1} 3 r_1 + w_{0,c,2} 2 r_1 r_2 - \ell w_{0,c+1,0}(c+1)] \\
&\quad + \sum_{\substack{c\geq 0\\ e\geq 2}} x_1^{a+1} x_4^c x_5^{\frac{-6c-4e+3k}{2}} \{ (r_1 x_3 + r_2 x_6)^{e-1} [w_{1,c,e} 3 e r_1 + w_{0,c,e+1} e(e+1) r_1 r_2 \\
&\qquad - \ell w_{0,c+1,e-1}(c+1)] \\
&\qquad + x_3 (r_1 x_3 + r_2 x_6)^{e-2} w_{0,c+1,e-1} 3(c+1)(e-1) r_1 \} \\
&\quad + \sum_{\substack{b\geq 0\\ c\geq 0}} x_1^{\frac{4c-k+4}{2}} x_2^{b+1} x_4^c x_5^{\frac{-2b-6c+3k-6}{2}} [w_{b+2,c,1} 3(b+2) r_1 + w_{b,c,2} 8 r_1^2 \\
&\qquad + w_{b+1,c,2} 2 r_1 r_2 - \ell w_{b+1,c+1,0}(c+1)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& r_1^e [w_{b+2,c,e+1} 3(b+2)(e+1) r_1 \\
&\quad + w_{b,c,e+2} 4(e+1)(e+2) r_1^2 \\
&\quad + w_{b+1,c,e+2} (e+1)(e+2) r_1 r_2 \\
&\quad - \ell w_{b+1,c+1,e}(c+1) + w_{b+1,c+1,e} 3(c+1) e] = 0, \qquad b \geq 0,\ c \geq 0,\ e \geq 1.
\end{aligned} \tag{5}$$

$$\begin{aligned}
& r_2^e [w_{b+2,c,e+1} 3(b+2)(e+1) r_1 \\
&\quad + w_{b,c,e+2} 4(e+1)(e+2) r_1^2 \\
&\quad + w_{b+1,c,e+2} (e+1)(e+2) r_1 r_2 \\
&\quad - \ell w_{b+1,c+1,e}(c+1)] = 0, \qquad b \geq 0,\ c \geq 0,\ e \geq 1.
\end{aligned} \tag{6}$$

$$w_{1,c,1} 3 r_1 + w_{0,c,2} 2 r_1 r_2 - \ell w_{0,c+1,0}(c+1) = 0,\ c \geq 0. \tag{7}$$

(8)
$$r_1^{e-1}[w_{1,c,e}3er_1 + w_{0,c,e+1}e(e+1)r_1r_2 - \ell w_{0,c+1,e-1}(c+1) + w_{0,c+1,e-1}3(c+1)(e-1)] = 0, c \geq 0,\ e \geq 2.$$

(9) $$r_2^{e-1}[w_{1,c,e}3er_1 + w_{0,c,e+1}e(e+1)r_1r_2 - \ell w_{0,c+1,e-1}(c+1)] = 0,\ c \geq 0,\ e \geq 2.$$

(10)
$$w_{b+2,c,1}3(b+2)r_1 + w_{b,c,2}8r_1^2 + w_{b+1,c,2}2r_1r_2 - \ell w_{b+1,c+1,0}(c+1) = 0,\ b \geq 0,\ c \geq 0.$$

Since $wt(X_+\frac{\partial f_{k+1}^1}{\partial x_4}) = 6$, so

$$\begin{aligned}
0 &= X_+\frac{\partial f_{k+1}^1}{\partial x_4} \\
&= \sum_{b,c,e} w_{b,c,e}[3cbx_1^{a+1}x_2^{b-1}x_4^{c-1}x_5^d(r_1x_3+r_2x_6)^e \\
&\qquad + 4cer_1x_1^ax_2^{b+1}x_4^{c-1}x_5^d(r_1x_3+r_2x_6)^{e-1} \\
&\qquad + 3c(c-1)x_1^ax_2^bx_3x_4^{c-2}x_5^d(r_1x_3+r_2x_6)^e \\
&\qquad + cer_2x_1^ax_2^bx_4^{c-1}x_5^{d+1}(r_1x_3+r_2x_6)^{e-1}] \\
&\quad + \sum_{\substack{b\geq 1\\ c\geq 1\\ e\geq 1}} x_1^{a+1}x_2^bx_4^{c-1}x_5^{d-1}\{(r_1x_3+r_2x_6)^e[w_{b+1,c,e}3c(b+1) \\
&\qquad + w_{b-1,c,e+1}4c(e+1)r_1 + w_{b,c,e+1}c(e+1)r_2] \\
&\qquad + x_3(r_1x_3+r_2x_6)^{e-1}w_{b,c+1,e-1}3c(c+1)\} \\
&\quad + \sum_{c\geq 1} x_1^{\frac{4c-k+2}{2}}x_4^{c-1}x_5^{\frac{-6c+3k}{2}}(w_{1,c,0}3c + w_{0,c,1}cr_2) \\
&\quad + \sum_{\substack{c\geq 1\\ e\geq 1}} x_1^{a+1}x_4^{c-1}x_5^{\frac{-6c-4e+3k}{2}}\{(r_1x_3+r_2x_6)^e[w_{1,c,e}3c + w_{0,c,e+1}c(e+1)r_2] \\
&\qquad + x_3(r_1x_3+r_2x_6)^{e-1}w_{0,c+1,e-1}3c(c+1)\} \\
&\quad + \sum_{\substack{b\geq 0\\ c\geq 1}} x_1^{\frac{4c-k+2}{2}}x_2^{b+1}x_4^{c-1}x_5^{\frac{-2b-6c+3k-2}{2}}[w_{b+2,c,0}3(b+2)c + w_{b,c,1}4cr_1 + w_{b+1,c,1}cr_2].
\end{aligned}$$

Thus

(11)
$$\begin{aligned}
&r_1^e[w_{b+1,c,e}3c(b+1) + w_{b-1,c,e+1}4c(e+1)r_1 \\
&\qquad + w_{b,c,e+1}c(e+1)r_2] + r_1^{e-1}w_{b,c+1,e-1}3c(c+1) = 0, \\
&\qquad b \geq 1,\ c \geq 1,\ e \geq 1.
\end{aligned}$$

(12)
$$\begin{aligned}
&r_2^e[w_{b+1,c,e}3c(b+1) + w_{b-1,c,e+1}4c(e+1)r_1 \\
&\qquad + w_{b,c,e+1}c(e+1)r_2] = 0,\ b \geq 1,\ c \geq 1,\ e \geq 1.
\end{aligned}$$

$$w_{1,c,0}3c + w_{0,c,1}cr_2 = 0,\ c \geq 1. \tag{13}$$

$$r_1^e[w_{1,c,e}3c + w_{0,c,e+1}c(e+1)r_2] + r_1^{e-1}w_{0,c+1,e-1}3c(c+1) = 0,\ c \geq 1,\ e \geq 1. \tag{14}$$

$$r_2^e[w_{1,c,e}3c + w_{0,c,e+1}c(e+1)r_2] = 0,\ c \geq 1,\ e \geq 1. \tag{15}$$

$$w_{b+2,c,0}3(b+2)c + w_{b,c,1}4cr_1 + w_{b+1,c,1}cr_2 = 0,\ b \geq 0,\ c \geq 1. \tag{16}$$

Suppose $r_2 \neq 0$. Then from equations (5), (6), we get, $w_{b+1,c+1,e} = 0$, $b \geq 0, c \geq 0, e \geq 1$. That is,

$$w_{b,c,e} = 0,\ b \geq 1,\ c \geq 1,\ e \geq 1. \tag{17}$$

From equations (8), (9), $w_{0,c+1,e-1} = 0$, $c \geq 0, e \geq 2$. That is,

$$w_{0,c,e} = 0,\ c \geq 1,\ e \geq 1. \tag{18}$$

Equations (17), (18) imply

$$w_{b,c,e} = 0,\ b \geq 0,\ c \geq 1,\ e \geq 1. \tag{19}$$

From equations (7), (19), we obtain $w_{0,c+1,0} = 0$, $c \geq 1$. That is,

$$w_{0,c,0} = 0,\ c \geq 2. \tag{20}$$

Equations (11), (12) imply $w_{b,c+1,e-1} = 0$, $b \geq 1$, $c \geq 1$, $e \geq 1$. That is,

$$w_{b,c,e} = 0,\ b \geq 1,\ c \geq 2,\ e \geq 0. \tag{21}$$

Equations (20), (21) imply

$$w_{b,c,e} = 0,\ b \geq 0,\ c \geq 2,\ e \geq 0. \tag{22}$$

Equations (13), (19) imply

$$w_{1,c,0} = 0,\ c \geq 1. \tag{23}$$

Equations (16), (19) imply $w_{b+2,c,0} = 0$, $b \geq 0$, $c \geq 1$. That is,

$$w_{b,c,0} = 0,\ b \geq 2,\ c \geq 1. \tag{24}$$

Equations (23), (24) imply

$$w_{b,c,0} = 0,\ b \geq 1,\ c \geq 1. \tag{25}$$

From equations (19), (22), (25), we have

$$f_{k+1}^1 = \sum_{\substack{b \geq 0 \\ e \geq 0}} w_{b,0,e} x_1^{\frac{2e-k}{2}} x_2^b x_5^{\frac{-2b-4e+3k+2}{2}} (r_1x_3 + r_2x_6)^e + w_{0,1,0}x_1^{\frac{4-k}{2}} x_4 x_5^{\frac{3k-4}{2}}.$$

So $\frac{\partial f^1_{k+1}}{\partial x_4} = w_{0,1,0}x_1^{\frac{4-k}{2}}x_5^{\frac{3k-4}{2}} \quad (\Rightarrow w_{0,1,0} \neq 0)$.

Case a: $k = 2 \Rightarrow \frac{\partial f^1_3}{\partial x_4} = w_{0,1,0}x_1x_5$.

Case b: $k = 4 \Rightarrow \frac{\partial f^1_5}{\partial x_4} = w_{0,1,0}x_5^4$.

If $\frac{\partial f^1_3}{\partial x_4} = w_{0,1,0}x_1x_5$, then

$$
\begin{aligned}
(5) &= \langle \frac{\partial f^1_3}{\partial x_4}, X_- \frac{\partial f^1_3}{\partial x_4}, X_-^2 \frac{\partial f^1_3}{\partial x_4}, X_-^3 \frac{\partial f^1_3}{\partial x_4}, X_-^4 \frac{\partial f^1_3}{\partial x_4} \rangle \\
&= \langle x_1x_5, x_1x_6 + x_2x_5, 2x_2x_6 + x_3x_5, 3x_3x_6 + x_4x_5, x_4x_6 \rangle
\end{aligned}
$$

because $X_+ \frac{\partial f^1_3}{\partial x_4} = 0$, and $X_-^5 \frac{\partial f^1_3}{\partial x_4} = 0$. Now

$$
\begin{aligned}
&f^1_3 = w_{0,1,0}x_1x_4x_5 + g_4(x_1, x_2, x_3, x_5, x_6) \\
\Rightarrow &\frac{\partial f^1_3}{\partial x_1} = w_{0,1,0}x_4x_5 + \frac{\partial g_4}{\partial x_1}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\frac{\partial f^1_3}{\partial x_1} = d_1(3x_3x_6 + x_4x_5) \\
\Rightarrow &\frac{\partial g_4}{\partial x_1} = 3d_1x_3x_6 \text{ and } d_1 = w_{0,1,0} \\
\Rightarrow &g_4 = 3d_1x_1x_3x_6 + g_1(x_2, x_3, x_5, x_6) \\
\Rightarrow &f^1_3 = w_{0,1,0}x_1x_4x_5 + 3w_{0,1,0}x_1x_3x_6 + g_1(x_2, x_3, x_5, x_6) \\
\Rightarrow &\frac{\partial f^1_3}{\partial x_3} = 3w_{0,1,0}x_1x_6 + \frac{\partial g_1}{\partial x_3}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\frac{\partial f^1_3}{\partial x_3} = d_3(x_1x_6 + x_2x_5) \\
\Rightarrow &\frac{\partial g_1}{\partial x_3} = d_3x_2x_5 \text{ and } d_3 = 3w_{0,1,0} \\
\Rightarrow &g_1 = d_3x_2x_3x_5 + g_3(x_2, x_5, x_6) \\
\Rightarrow &f^1_3 = w_{0,1,0}x_1x_4x_5 + 3w_{0,1,0}x_1x_3x_6 + 3w_{0,1,0}x_2x_3x_5 + g_3(x_2, x_5, x_6) \\
\Rightarrow &\frac{\partial f^1_3}{\partial x_6} = 3w_{0,1,0}x_1x_3 + \frac{\partial g_3}{\partial x_6}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{\partial f^1_3}{\partial x_6} = d_6(x_1x_6 + x_2x_5) \\
\Rightarrow &w_{0,1,0} = 0 \Rightarrow \frac{\partial f^1_3}{\partial x_4} = 0.
\end{aligned}
$$

If $\frac{\partial f^1_3}{\partial x_4} = w_{0,1,0}x_5^4$, then

$$(5) = \langle x_5^4, x_5^3x_6, x_5^2x_6^2, x_5x_6^3, x_6^4 \rangle.$$

Now

$$\begin{aligned}&f_5^1 = w_{0,1,0}x_4x_5^4 + g_4(x_1,x_2,x_3,x_5,x_6)\\ \Rightarrow&\frac{\partial f_5^1}{\partial x_3} = \frac{\partial g_4}{\partial x_3}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{\partial f_5^1}{\partial x_3} = d_3x_5^3x_6\\ \Rightarrow&\frac{\partial g_4}{\partial x_3} = d_3x_5^3x_6\\ \Rightarrow&g_4 = d_3x_3x_5^3x_6 + g_3(x_1,x_2,x_5,x_6)\\ \Rightarrow&f_5^1 = w_{0,1,0}x_4x_5^4 + d_3x_3x_5^3x_6 + g_3(x_1,x_2,x_5,x_6)\\ \Rightarrow&\frac{\partial f_5^1}{\partial x_6} = d_3x_3x_5^3 + \frac{\partial g_3}{\partial x_6}.\end{aligned}$$

Furthermore,

$$\frac{\partial f_5^1}{\partial x_6} = d_6x_5^3x_6 \Rightarrow d_3 = 0 \Rightarrow \frac{\partial f_5^1}{\partial x_3} = 0.$$

Thus

$$\frac{\partial f_5^1}{\partial x_4} = 0.$$

Suppose $r_2 = 0$. So f_{k+1}^1 is independent of the x_6 variable. We may write $f_{k+1}^1 = \sum\limits_{c,d,e} w_{c,d,e}x_1^ax_2^bx_3^cx_4^dx_5^e$ where $a = \frac{2c+4d-k}{2}$, $b = \frac{-4c-6d-2e+3k+2}{2}$.

Since $wt(X_-\frac{\partial f_{k+1}^1}{\partial x_4}) = wt\frac{\partial f_{k+1}^1}{\partial x_3} = 2$, there exists a constant ℓ such that $X_-\frac{\partial f_{k+1}^1}{\partial x_4} = \ell\frac{\partial f_{k+1}^1}{\partial x_3}$. If $\ell = 0$, then $X_-\frac{\partial f_{k+1}^1}{\partial x_4} = 0$. Since $wt(X_+\frac{\partial f_{k+1}^1}{\partial x_4}) = 6$, so $(1) = \langle\frac{\partial f_{k+1}^1}{\partial x_4}\rangle \subseteq I$. Since $wt\frac{\partial f_{k+1}^1}{\partial x_4} = 4$, this contradicts $I = (5)\oplus(1)$. Thus $\ell \neq 0$. So

$$\begin{aligned}0 &= X_-\frac{\partial f_{k+1}^1}{\partial x_4} - \ell\frac{\partial f_{k+1}^1}{\partial x_3}\\ &= \sum_{c,d,e}\{w_{c,d,e}d[ax_1^{a-1}x_2^{b+1}x_3^cx_4^{d-1}x_5^e + bx_1^ax_2^{b-1}x_3^{c+1}x_4^{d-1}x_5^e\\ &\qquad + cx_1^ax_2^bx_3^{c-1}x_4^dx_5^e + ex_1^ax_2^bx_3^cx_4^{d-1}x_5^{e-1}x_6]\\ &\qquad - \ell w_{c,d,e}cx_1^ax_2^bx_3^{c-1}x_4^dx_5^e\}\\ &= \sum_{\substack{c\geq 1\\ d\geq 1\\ e\geq 1}} x_1^{a+1}x_2^{b-2}x_3^cx_4^dx_5^e[w_{c,d+1,e}(a+2)(d+1) + w_{c-1,d+1,e}(b-1)(d+1)\\ &\qquad + w_{c+1,d,e}(c+1)d - \ell w_{c+1,d,e}(c+1)]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{c\geq 0\\ d\geq 1\\ e\geq 1}} w_{c,d,e}de x_1^a x_2^b x_3^c x_4^{d-1} x_5^{e-1} x_6\\
&+\sum_{e\geq 0} x_1^{\frac{2-k}{2}} x_2^{\frac{-2e+3k-2}{2}} x_5^e\left(w_{0,1,e}\frac{4-k}{2}-w_{1,0,e}\ell\right)\\
&+\sum_{\substack{d\geq 1\\ e\geq 0}} x_1^{\frac{4d-k+2}{2}} x_2^{\frac{-6d-2e+3k-2}{2}} x_4^d x_5^e\left[w_{0,d+1,e}\frac{4d-k+4}{2}(d+1)+w_{1,d,e}(d-\ell)\right]\\
&+\sum_{\substack{c\geq 0\\ e\geq 0}} x_1^{\frac{2c-k+4}{2}} x_2^{\frac{-4c-2e+3k-6}{2}} x_3^{c+1} x_5^e\Big[w_{c,1,e}\frac{-4c-2e+3k-4}{2}\\
&\qquad\qquad + w_{c+1,1,e}\frac{2c-k+6}{2}-w_{c+2,0,e}\ell(c+2)\Big].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
w_{c,d+1,e}&(a+2)(d+1)+w_{c-1,d+1,e}(b-1)(d+1)\\
&+w_{c+1,d,e}(c+1)d-\ell w_{c+1,d,e}(c+1)=0,\ c\geq 1, d\geq 1, e\geq 0.
\end{aligned}
\tag{26}
$$

$$
w_{c,d,e}de=0,\ c\geq 0, d\geq 1, e\geq 1. \tag{27}
$$

$$
w_{0,1,e}\frac{4-k}{2}-w_{1,0,e}\ell=0,\ e\geq 0. \tag{28}
$$

$$
w_{0,d+1,e}\frac{4d-k+4}{2}(d+1)+w_{1,d,e}(d-\ell)=0,\ d\geq 1, e\geq 0. \tag{29}
$$

$$
w_{c,1,e}\frac{-4c-2e+3k-4}{2}+w_{c+1,1,e}\frac{2c-k+6}{2}w_{c+2,0,e}\ell(c+2)=0,\ c\geq 0, e\geq 0. \tag{30}
$$

By (27), we have

$$
w_{c,d,e}=0,\ c\geq 0, d\geq 1, e\geq 1. \tag{31}
$$

From equations (28), (31), we get

$$
w_{1,0,e}=0,\ e\geq 1. \tag{32}
$$

Equations (30), (31) imply $w_{c+2,0,e}=0,\ c\geq 0, e\geq 1$. That is,

$$
w_{c,0,e}=0,\ c\geq 2, e\geq 1. \tag{33}
$$

From equations (31), (32), (33), we have

$$f_{k+1}^1=\sum_{\substack{c\geq 0\\ d\geq 0}} w_{c,d,0}x_1^a x_2^{\frac{-4c-6d+3k+2}{2}}x_3^c x_4^d+\sum_{e\geq 1} w_{0,0,1}x_1^{\frac{-k}{2}}x_2^{\frac{-2e+3k+2}{2}}x_5^e.$$

So f_{k+1}^1 is independent of the x_5 variable. Thus we get either $\frac{\partial f_{k+1}^1}{\partial x_4}=0$ or f_{k+1}^1 is independent of the x_5 and x_6 variables. Similar arguments show that either $\frac{\partial f_{k+1}^{-1}}{\partial x_1}=0$ or f_{k+1}^{-1} is independent of the x_5 and x_6 variables.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_4}=0$.

$$wt\frac{\partial f_{k+1}^{-1}}{\partial x_1}=-4,\quad wt\frac{\partial f_{k+1}^{-1}}{\partial x_2}=-2,\quad wt\frac{\partial f_{k+1}^{-1}}{\partial x_3}=0,$$
$$wt\frac{\partial f_{k+1}^{-1}}{\partial x_4}=2,\quad wt\frac{\partial f_{k+1}^{-1}}{\partial x_5}=-2,\quad wt\frac{\partial f_{k+1}^{-1}}{\partial x_6}=0.$$

$$\begin{aligned}
f&=f_{k+1}^{-1}+f_{k+1}^1,\\
I&=\langle\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},\frac{\partial f}{\partial x_5},\frac{\partial f}{\partial x_6}\rangle\\
&=\langle\frac{\partial f_{k+1}^{-1}}{\partial x_1}+\frac{\partial f_{k+1}^1}{\partial x_1},\frac{\partial f_{k+1}^{-1}}{\partial x_2}+\frac{\partial f_{k+1}^1}{\partial x_2},\frac{\partial f_{k+1}^{-1}}{\partial x_3}\\
&\qquad+\frac{\partial f_{k+1}^1}{\partial x_3},\frac{\partial f_{k+1}^{-1}}{\partial x_4},\frac{\partial f_{k+1}^{-1}}{\partial x_5}+\frac{\partial f_{k+1}^1}{\partial x_5},\frac{\partial f_{k+1}^{-1}}{\partial x_6}+\frac{\partial f_{k+1}^1}{\partial x_6}\rangle\\
&\subseteq\langle\frac{\partial f_{k+1}^{-1}}{\partial x_1},\frac{\partial f_{k+1}^1}{\partial x_1},\frac{\partial f_{k+1}^{-1}}{\partial x_2},\frac{\partial f_{k+1}^1}{\partial x_2},\frac{\partial f_{k+1}^{-1}}{\partial x_3},\\
&\qquad\frac{\partial f_{k+1}^1}{\partial x_3},\frac{\partial f_{k+1}^{-1}}{\partial x_4},\frac{\partial f_{k+1}^{-1}}{\partial x_5},\frac{\partial f_{k+1}^1}{\partial x_5},\frac{\partial f_{k+1}^{-1}}{\partial x_6},\frac{\partial f_{k+1}^1}{\partial x_6}\rangle.
\end{aligned}$$

$\dim\langle\frac{\partial f_{k+1}^1}{\partial x_2},\frac{\partial f_{k+1}^1}{\partial x_5},\frac{\partial f_{k+1}^{-1}}{\partial x_3},\frac{\partial f_{k+1}^{-1}}{\partial x_6}\rangle\leq 2$ because they have same weight 0.
$\dim\langle\frac{\partial f_{k+1}^1}{\partial x_1},\frac{\partial f_{k+1}^{-1}}{\partial x_2},\frac{\partial f_{k+1}^{-1}}{\partial x_5}\rangle\leq 1$ because they have same weight -2.
$\dim\langle\frac{\partial f_{k+1}^1}{\partial x_3},\frac{\partial f_{k+1}^1}{\partial x_6},\frac{\partial f_{k+1}^{-1}}{\partial x_4},\rangle\leq 1$ because they have same weight 2.
$\Rightarrow\ \dim I\leq 5$. This contradicts $\dim I=6$.

Similar arguments show that $\frac{\partial f_{k+1}^{-1}}{\partial x_1}=0$ is impossible. Finally, suppose f_{k+1}^1 and f_{k+1}^{-1} are independent of the x_5,x_6 variables. Then $\frac{\partial f}{\partial x_5}=\frac{\partial f}{\partial x_6}=0$. So dim $I\leq 4$. This contradicts dim $I=6$. We conclude that Case 2 cannot occur.

Case 3. $I=(4)\oplus(2)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $3,1,-1,-3$.

By the same argument as in the begining of Case 1 we have $f_{k+1}^i=0$ for $i=\pm 1,\pm 3$ and $|i|\geq 5$.

For $i=4$

$$wt\frac{\partial f_{k+1}^4}{\partial x_1}=1,\quad wt\frac{\partial f_{k+1}^4}{\partial x_2}=3,\quad wt\frac{\partial f_{k+1}^4}{\partial x_3}=5,$$
$$wt\frac{\partial f_{k+1}^4}{\partial x_4}=7,\quad wt\frac{\partial f_{k+1}^4}{\partial x_5}=3,\quad wt\frac{\partial f_{k+1}^4}{\partial x_6}=5.$$

$\Rightarrow f_{k+1}^4$ depends only on the x_1,x_2,x_5 variables.

Similar arguments as in Case 2 in the proof of "For $i=5$" show that $f^4_{k+1}=0=f^{-4}_{k+1}$.

For $i=2$

$$
\begin{aligned}
&wt\frac{\partial f^2_{k+1}}{\partial x_1}=-1,\quad wt\frac{\partial f^2_{k+1}}{\partial x_2}=1,\quad wt\frac{\partial f^2_{k+1}}{\partial x_3}=3,\\
&wt\frac{\partial f^2_{k+1}}{\partial x_4}=5,\quad wt\frac{\partial f^2_{k+1}}{\partial x_5}=1,\quad wt\frac{\partial f^2_{k+1}}{\partial x_6}=3.\\
&\Rightarrow f^2_{k+1}\quad \text{is independent of the } x_4 \text{ variable.}
\end{aligned}
$$

Since $wt\frac{\partial f^2_{k+1}}{\partial x_3}=wt\frac{\partial f^2_{k+1}}{\partial x_6}=3$, in view of Lemma 5.1 of [Ya4], there exist constants r_1,r_2 such that

$$f^2_{k+1}=\sum_{b,d}w_{b,d}x_1^ax_2^bx_5^c(r_1x_3+r_2x_6)^d$$

where $a=\frac{2d-k+1}{2}$, $c=\frac{-2b-4d+3k+1}{2}$.

Assuming $\frac{\partial f^2_{k+1}}{\partial x_3}\neq 0$. Then as before, $\frac{\partial f^2_{k+1}}{\partial x_2}\neq 0$. Thus $r_1\neq 0$, $b\geq 1, d\geq 1$, $w_{b,d}\neq 0$ for some $b\geq 1, d\geq 1$. Since $wt(X_-^3\frac{\partial f^2_{k+1}}{\partial x_2})=-5$, so $X_-^3\frac{\partial f^2_{k+1}}{\partial x_2}=0$. The coefficient of $x_1^ax_2^{b-1}x_4^3x_5^c(r_1x_3+r_2x_6)^{d-3}$ in $X_-^3\frac{\partial f^2_{k+1}}{\partial x_2}=0$ is $\sum_{\substack{b\geq 1\\ d\geq 3}}w_{b,d}br_1^3d(d-1)(d-2)$.

Thus $w_{b,d}=0$ for $b\geq 1, d\geq 3$. Therefore,

$$
\begin{aligned}
\frac{\partial f^2_{k+1}}{\partial x_2}&=\sum_{\substack{b\geq 1\\ d\leq 2}}w_{b,d}bx_1^ax_2^{b-1}x_5^c(r_1x_3+r_2x_6)^d\\
&=\sum_{b\geq 1}\Big[w_{b,0}bx_1^{\frac{1-k}{2}}x_2^{b-1}x_5^{\frac{-2b+3k+1}{2}}+w_{b,1}bx_1^{\frac{3-k}{2}}x_2^{b-1}x_5^{\frac{-2b+3k-3}{2}}(r_1x_3+r_2x_6)\\
&\qquad +w_{b,2}bx_1^{\frac{5-k}{2}}x_2^{b-1}x_5^{\frac{-2b+3k-7}{2}}(r_1x_3+r_2x_6)^2\Big].
\end{aligned}
$$

Case a: $k=3\Rightarrow f^2_4=(w_{0,1}x_5^3+w_{1,1}x_2x_5^2+w_{2,1}x_2^2x_5+w_{3,1}x_2^3)(r_1x_3+r_2x_6)$
$+(w_{0,2}x_1x_5+w_{1,2}x_1x_2)(r_1x_3+r_2x_6)^2$.

Case b: $k=5\Rightarrow f^2_6=(w_{0,2}x_5^4+w_{1,2}x_2x_5^3+w_{2,2}x_2^2x_5^2+w_{3,2}x_2^3x_5+w_{4,2}x_2^4)$
$\times(r_1x_3+r_2x_6)^2+w_{0,3}x_1x_5^2(r_1x_3+r_2x_6)^3+w_{0,4}x_1^2(r_1x_3+r_2x_6)^4$.

Suppose $k=3$. Since $wt(X_+\frac{\partial f^2_4}{\partial x_3})=5$, so $0=X_+\frac{\partial f^2_4}{\partial x_3}=6r_1^2w_{1,2}x_1^2x_3+6r_1r_2w_{1,2}x_1^2x_6+(8r_1w_{1,2}+9w_{3,1})r_1x_1x_2^2+(2r_2w_{1,2}+6w_{2,1}+8r_1w_{0,2})r_1x_1x_2x_5+(3w_{1,1}+2r_2w_{0,2})r_1x_1x_5^2$. Thus $w_{1,2}=w_{3,1}=0$, $w_{2,1}=-\frac{4}{3}r_1w_{0,2}$, $w_{1,1}=-\frac{2}{3}r_2w_{0,2}$. Hence

$$
\begin{aligned}
f^2_4=&(w_{01,}x_5^3-\frac{2}{3}r_2w_{0,2}x_2x_5^2-\frac{4}{3}r_1w_{0,2}x_2^2x_5)(r_1x_3+r_2x_6)\\
&+w_{0,2}x_1x_5(r_1x_3+r_2x_6)^2.
\end{aligned}
$$

Since $wt(X_+^2\frac{\partial f^2_4}{\partial x_2})=5$, so $0=X_+^2\frac{\partial f^2_4}{\partial x_2}=-96r_1^2w_{0,2}x_1x_2x_5-24r_1r_2w_{0,2}x_1x_5^2\Rightarrow w_{0,2}=0\Rightarrow f^2_4=w_{0,1}x_5^3(r_1x_3+r_2x_6)$. So $\frac{\partial f^2_4}{\partial x_2}=0\Rightarrow\frac{\partial f^2_4}{\partial x_3}=0$.

Suppose $k=5$. Since $wt(X_+\frac{\partial f^2_6}{\partial x_3})=5$, so $0=X_+\frac{\partial f^2_6}{\partial x_3}=0\Rightarrow f^2_6=0\Rightarrow\frac{\partial f^2_6}{\partial x_3}=0$. Hence $f^2_{k+1}=\sum_{b,d}w_{b,d}x_1^ax_2^bx_5^cx_6^d$ where $a=\frac{2d-k+1}{2}$, $c=\frac{-2b-4d+3k+1}{2}$.

Since $wt(X_-^4\frac{\partial f_{k+1}^2}{\partial x_6})=-5$, so $X_-^4\frac{\partial f_{k+1}^2}{\partial x_6}=0$. The coefficient of $x_1^a x_2^{b-4}x_3^4x_5^cx_6^{d-1}$ in $X_-^4\frac{\partial f_{k+1}^2}{\partial x_6}$ is $\sum_{b,d} w_{b,d}db(b-1)(b-2)(b-3)$. Thus $w_{b,d}=0$ for $b\geq 4, d\geq 1$. So

$$\frac{\partial f_{k+1}^2}{\partial x_6}=\sum_{d\geq 1}\sum_{b=0}^{3} w_{b,d}dx_1^ax_2^bx_5^cx_6^{d-1}.$$

Since $wt(X_+\frac{\partial f_{k+1}^2}{\partial x_6})=5$, so $X_+\frac{\partial f_{k+1}^2}{\partial x_6}=0$. Now

$$\begin{aligned}X_+\frac{\partial f_{k+1}^2}{\partial x_6} &=\sum_{d\geq 1}\sum_{b=0}^{3} w_{b,d}d[3bx_1^{a+1}x_2^{b-1}x_5^cx_6^{d-1}+(d-1)x_1^ax_2^bx_5^{c+1}x_6^{d-1}]\\ &=\sum_{d\geq 1}\sum_{b=0}^{3} w_{b,d}x_1^{a+1}x_2^bx_5^{\frac{-2b-4d+3k-1}{2}}x_6^{d-1}[w_{b+1,d}3(b+1)d+w_{b,d+1}d(d-1)].\end{aligned}$$

Therefore, $w_{b+1,d}3(b+1)d+w_{b,d+1}d(d+1)=0$ for $b=0,1,2,3,\ d\geq 1$.

$$\Rightarrow w_{3,d}=0,\ d\geq 2;\ w_{2,d}=0,\ d\geq 3;\ w_{1,d}=0,\ d\geq 4;\ w_{0,d}=0,\ d\geq 5.$$

Case a: $k=3\Rightarrow f_4^2=w_{0,1}x_5^3x_6+w_{0,2}x_1x_5x_6^2+w_{1,1}x_2x_5^2x_6+w_{1,2}x_1x_2x_6^2+w_{2,1}x_2^2x_5x_6+w_{3,1}x_2^3x_6$.

Case b: $k=5\Rightarrow f_6^2=w_{0,2}x_5^4x_6^2+w_{0,3}x_1x_5^2x_6^3+w_{0,4}x_1^2x_6^4+w_{1,2}x_2x_5^3x_6^2+w_{1,3}x_1x_2x_5x_6^3+w_{2,2}x_2^2x_5^2x_6^2$.

Case c: $k=7\Rightarrow f_8^2=w_{0,3}x_5^5x_6^3+w_{0,4}x_1x_5^3x_6^4+w_{1,3}x_2x_5^4x_6^3$.

Case d: $k=9\Rightarrow f_{10}^2=w_{0,4}x_5^6x_6^4$.

Suppose $k=3$. Since $wt(X_+\frac{\partial f_4^2}{\partial x_6})=5$, so $0=X_+\frac{\partial f_4^2}{\partial x_6}=6w_{1,2}x_1^2x_6+9w_{3,1}x_1x_2^2+(2w_{1,2}+6w_{2,1})x_1x_2x_5+(2w_{0,2}+3w_{1,1})x_1x_5^2\Rightarrow w_{1,2}=w_{3,1}=w_{2,1}=0,\ w_{1,1}=-\frac{2}{3}w_{0,2}\Rightarrow f_4^2=w_{0,1}x_5^3x_6+w_{0,2}x_1x_5x_6^2-\frac{2}{3}w_{0,2}x_2x_5^2x_6$. Since $wt(X_+^2\frac{\partial f_4^2}{\partial x_5})=5$, so $0=X_+^2\frac{\partial f_4^2}{\partial x_5}=-6w_{0,2}x_1x_5^2\Rightarrow w_{0,2}=0\Rightarrow f_4^2=w_{0,1}x_5^3x_6$.

If $\frac{\partial f_4^2}{\partial x_5}\neq 0$, then $w_{0,1}\neq 0$.

$$\begin{aligned}&wt\frac{\partial f_4^{-2}}{\partial x_1}=-5,\quad wt\frac{\partial f_4^{-2}}{\partial x_2}=-3,\quad wt\frac{\partial f_4^{-2}}{\partial x_3}=1,\\ &wt\frac{\partial f_4^{-2}}{\partial x_4}=1,\quad wt\frac{\partial f_4^{-2}}{\partial x_5}=-3,\quad wt\frac{\partial f_4^{-2}}{\partial x_6}=1.\\ &\Rightarrow f_4^{-2}\text{ is independent of the } x_1\text{variable.}\end{aligned}$$

Now

$$\begin{aligned}(4)&=\langle X_+\frac{\partial f_4^2}{\partial x_5},\frac{\partial f_4^2}{\partial x_5},X_-\frac{\partial f_4^2}{\partial x_5},X_-^2\frac{\partial f_4^2}{\partial x_5}\rangle\\ &=\langle x_5^3,x_5^2x_6,x_5x_6^2,x_6^3\rangle\end{aligned}$$

because $X_+^2\frac{\partial f_4^2}{\partial x_5}=0$ and $X_-^3\frac{\partial f_4^2}{\partial x_5}=0$.

$$\frac{\partial f_4^{-2}}{\partial x_2}=c_2x_6^3\Rightarrow f_4^{-2}=c_2x_2x_6^3+g_2(x_3,x_4,x_5,x_6)\Rightarrow\frac{\partial f_4^{-2}}{\partial x_5}=\frac{\partial g_2}{\partial x_5}.$$

On the other hand,

$$\begin{aligned}&\frac{\partial f_4^{-2}}{\partial x_5}=c_5x_6^3\Rightarrow\frac{\partial g_2}{\partial x_5}=c_5x_6^3\Rightarrow g_2=c_5x_5x_6^3+g_5(x_3,x_4,x_6)\\&\Rightarrow f_4^{-2}=c_2x_2x_6^3+c_5x_5x_6^3+g_5(x_3,x_4,x_6).\end{aligned}$$

Since $wt(g_5)=-2$ and deg $g_5=4$, so $g_5=0$, hence

$$f_4^{-2}=c_2x_2x_6^3+c_5x_5x_6^3.$$

Since $wt(X_-^2\frac{\partial f_4^{-2}}{\partial x_6})=-5$, so $0=X_-^2\frac{\partial f_4^{-2}}{\partial x_6}=3c_2x_4x_6^2\Rightarrow c_2=0$. Hence $f_4^{-2}=c_5x_5x_6^3$.

$$\begin{aligned}&wt\frac{\partial f_4^0}{\partial x_1}=-3,\quad wt\frac{\partial f_4^0}{\partial x_2}=-1,\quad wt\frac{\partial f_4^0}{\partial x_3}=1,\\&wt\frac{\partial f_4^0}{\partial x_4}=3,\quad wt\frac{\partial f_4^0}{\partial x_5}=-1,\quad wt\frac{\partial f_4^0}{\partial x_6}=1.\\&\frac{\partial f_4^0}{\partial x_1}=e_1x_6^3\Rightarrow f_4^0=e_1x_1x_6^3+h_1(x_2,x_3,x_4,x_5,x_6)\Rightarrow\frac{\partial f_4^0}{\partial x_4}=\frac{\partial h_1}{\partial x_4}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\frac{\partial f_4^0}{\partial x_4}=e_4x_5^3\Rightarrow\frac{\partial h_1}{\partial x_4}=e_4x_5^3\Rightarrow h_1=e_4x_4x_5^3+h_4(x_2,x_3,x_5,x_6)\\&\Rightarrow f_4^0=e_1x_1x_6^3+e_4x_4x_5^3+h_4(x_2,x_3,x_5,x_6).\end{aligned}$$

$$\begin{aligned}h_4(x_2,x_3,x_5,x_6)&=\sum_{\substack{0\le a\le 2\\0\le b\le 2}}u_{a,b}x_2^ax_3^bx_5^{2-a}x_6^{2-b}\\&=u_{0,0}x_5^2x_6^2+u_{0,1}x_3x_5^2x_6+u_{0,2}x_3^2x_5^2\\&\quad+u_{1,0}x_2x_5x_6^2+u_{1,1}x_2x_3x_5x_6+u_{1,2}x_2x_3^2x_5\\&\quad+u_{2,0}x_2^2x_6^2+u_{2,1}x_2^2x_3x_6+u_{2,2}x_2^2x_3^2.\end{aligned}$$

$$\begin{aligned}X_+\frac{\partial f_4^0}{\partial x_3}&=X_+\frac{\partial h_4}{\partial x_3}\\&=u_{0,1}x_5^2+(8u_{0,2}+u_{1,1})x_2x_5^2+(8u_{1,2}+u_{2,1})x_2^2x_5\\&\quad+3u_{1,1}x_1x_5x_6+6u_{1,2}x_1x_3x_5\\&\quad+6u_{2,1}x_1x_2x_6+12u_{2,2}x_1x_2x_3+8u_{2,2}x_2^3.\end{aligned}$$

Moreover, since $wt(X_+\frac{\partial f_4^0}{\partial x_3})=3$, so $X_+\frac{\partial f_4^0}{\partial x_3}=ex_5^3$. Thus $e=u_{0,1}$, $u_{0,2}=u_{1,1}=u_{1,2}=u_{2,1}=u_{2,2}=0$.

$$h_4(x_2,x_3,x_5,x_6)=u_{0,0}x_5^2x_6^2+u_{0,1}x_3x_5^2x_6+u_{1,0}x_2x_5x_6^2+u_{2,0}x_2^2x_6^2.$$

Hence

$$f_4^0 = e_1x_1x_6^3 + e_4x_4x_5^3 + u_{0,0}x_5^2x_6^2 + u_{0,1}x_3x_5^2x_6 + u_{1,0}x_2x_5x_6^2 + u_{2,0}x_2^2x_6^2.$$

Since $wt(X_+^2\frac{\partial f_4^0}{\partial x_6}) = 5 = wt(X_+^3\frac{\partial f_4^0}{\partial x_5})$, so

$$\begin{aligned}
&0 = X_+^2\frac{\partial f_4^0}{\partial x_6} = 36u_{2,0}x_1^2x_6 + 24u_{2,0}x_1x_2x_5 + (6e_1 + 12u_{1,0} + 12u_{0,1})x_1x_5^2.\\
&0 = X_+^3\frac{\partial f_4^0}{\partial x_5} = 18(6e_4 + u_{0,1} + 4u_{0,1})x_1x_5^2.\\
&\Rightarrow u_{2,0} = 0, e_1 = -2u_{1,0} - 2u_{0,1}, e_4 = \frac{-u_{1,0} - 4u_{0,1}}{6}.\\
&\Rightarrow f_4^0 = (-2u_{1,0} - 2u_{0,1})x_1x_6^3 + \frac{-u_{1,0} - 4u_{0,1}}{6}x_4x_5^3 + u_{0,0}x_5^2x_6^2\\
&\qquad\quad + u_{1,0}x_2x_5x_6^2 + u_{0,1}x_3x_5^2x_6.
\end{aligned}$$

Since $wt(X_+\frac{\partial f_4^0}{\partial x_6}) = 3$, so

$$\begin{aligned}
&X_+\frac{\partial f_4^0}{\partial x_6} = d_1x_5^3\\
&\Rightarrow (-6u_{1,0} - 12u_{0,1})x_1x_5x_6 + (2u_{1,0} + 4u_{0,1})x_2x_5^2 + 2u_{0,0}x_5^3 = d_1x_5^3\\
&\Rightarrow u_{1,0} = -2u_{0,1}.
\end{aligned}$$

Hence

$$f_4^0 = 2u_{0,1}x_1x_6^3 - \frac{1}{3}u_{0,1}x_4x_5^3 + u_{0,0}x_5^2x_6^2 - 2u_{0,1}x_2x_5x_6^2 + u_{0,1}x_3x_5^2x_6.$$

Assuming $\frac{\partial f_4^0}{\partial x_3} = \frac{\partial h_4}{\partial x_3} \neq 0$. Then $u_{0,1} \neq 0$. Now

$$\begin{aligned}
&\frac{\partial f_4^0}{\partial x_6} = 6u_{0,1}x_1x_6^2 - 4u_{0,1}x_2x_5x_6 + u_{0,1}x_3x_5^2 + 2u_{0,0}x_5^2x_6.\\
&X_+\frac{\partial f_4^0}{\partial x_6} = 2u_{0,0}x_5^3.\\
&X_+^2\frac{\partial f_4^0}{\partial x_6} = 0.\\
&X_-\frac{\partial f_4^0}{\partial x_6} = 2u_{0,1}x_2x_6^2 - 2u_{0,1}x_3x_5x_6 + u_{0,1}x_4x_5^2 + 4u_{0,0}x_5x_6.\\
&X_-^2\frac{\partial f_4^0}{\partial x_6} = 4u_{0,0}x_6^3.\\
&X_-^3\frac{\partial f_4^0}{\partial x_6} = 0.
\end{aligned}$$

If $u_{0,0} \neq 0$, then

$$\begin{aligned}(4) =&\langle X_+\frac{\partial f_4^0}{\partial x_6}, \frac{\partial f_4^0}{\partial x_6}, X_-\frac{\partial f_4^0}{\partial x_6}, X_-^2\frac{\partial f_4^0}{\partial x_6}\rangle\\
=&\langle x_5^3, 6u_{0,1}x_1x_6^2 - 4u_{0,1}x_2x_5x_6 + u_{0,1}x_3x_5^2 + 2u_{0,0}x_6^2x_6,\\
&2u_{0,1}x_2x_6^2 - 2u_{0,1}x_3x_5x_6 + u_{0,1}x_4x_5^2 + 4u_{0,0}x_5x_6^2, x_6^3\rangle\\
\Rightarrow&6u_{0,1}x_1x_6^2 - 4u_{0,1}x_2x_5x_6 + u_{0,1}x_3x_5^2 + 2u_{0,0}x_5^2x_6 = d_2x_5^2x_6\\
&2u_{0,1}x_2x_6^2 - 2u_{0,1}x_3x_5x_6 + u_{0,1}x_4x_5^2 + 4u_{0,0}x_5x_6^2 = d_3x_5x_6\\
\Rightarrow&u_{0,1} = 0.\end{aligned}$$

Thus $u_{0,0} = 0$. Then

$$\begin{aligned}(2) =&\langle\frac{\partial f_4^0}{\partial x_6}, X_-\frac{\partial f_4^0}{\partial x_6}\rangle\\
=&\langle 6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2, 2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5^2\rangle.\end{aligned}$$

So $f_4^0 = 2u_{0,1}x_1x_6^3 - \frac{1}{3}u_{0,1}x_4x_5^3 - 2u_{0,1}x_2x_5x_6^2 + u_{0,1}x_3x_5^2x_6$. Now

$$\begin{aligned}f =&f_4^{-2} + f_4^0 + f_4^2\\
=&c_5x_5x_6^3 + 2u_{0,1}x_1x_6^3 - \frac{1}{3}u_{0,1}x_4x_5^3 - 2u_{0,1}x_2x_5x_6^2 + u_{0,1}x_3x_5^2x_6 + w_{0,1}x_5^3x_6.\\
\frac{\partial f}{\partial x_1} =&2u_{0,1}x_6^3.\\
\frac{\partial f}{\partial x_2} =&-2u_{0,1}x_5x_6^2.\\
\frac{\partial f}{\partial x_3} =&u_{0,1}x_5^2x_6.\\
\frac{\partial f}{\partial x_4} =&-\frac{1}{3}u_{0,1}x_5^3.\\
\frac{\partial f}{\partial x_5} =&-u_{0,1}(2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5) + 3w_{0,1}x_5^2x_6 + c_5x_6^3.\\
\frac{\partial f}{\partial x_6} =&u_{0,1}(6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2) + w_{0,1}x_5^3 + 3c_5x_5x_6^2.\end{aligned}$$

Consider the coefficient matrix Φ of $\frac{\partial f}{\partial x_j}$ with respect to the ordered basis $\{x_5^3, x_5^2x_6, x_5x_6^2, x_6^3,\ 6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2, 2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5^2\}$:

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 2u_{0,1} & 0 & 0\\ 0 & 0 & -2u_{0,1} & 0 & 0 & 0\\ 0 & u_{0,1} & 0 & 0 & 0 & 0\\ -\frac{1}{3}u_{0,1} & 0 & 0 & 0 & 0 & 0\\ 0 & 3w_{0,1} & 0 & c_5 & 0 & -u_{0,1}\\ w_{0,1} & 0 & 3c_5 & 0 & u_{0,1} & 0\end{bmatrix}.$$

Since $\det(\Phi) = \frac{3}{4}u_{0,1}^6 \neq 0$, so $\dim\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_6}\rangle = 6$. Let $g = f_4^0$. Since $X_+g = 0 = X_-g$, so g is an $s\ell(2,\mathbf{C})$ invariant polynomial. Note that $f = g + c_5x_5x_6^3 + w_{0,1}r_2x_5^3x_6$ and $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_6}\rangle$ since $\dim\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_6}\rangle = 6$ and $I \subseteq \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_6}\rangle$.

If $\frac{\partial f_4^0}{\partial x_3} = \frac{\partial h_4}{\partial x_3} = 0$, then $u_{0,1} = 0 \Rightarrow f_4^0 = u_{0,0}x_5^2x_6^2$. Now $f = f_4^{-2} + f_4^0 + f_4^2 = c_5x_5x_6^3 + u_{0,0}x_5^2x_6^2 + w_{0,1}x_5^3x_6 \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leq 2$. This contradicts $\dim I = 6$.

If $\frac{\partial f_4^2}{\partial x_5} = 0$, then $w_{0,1} = 0 \Rightarrow f_4^2 = 0$.

Suppose $k = 5$. Since $wt(X_+\frac{\partial f_6^2}{\partial x_6}) = 5$, so $0 = X_+\frac{\partial f_6^2}{\partial x_6} = (9w_{1,3} + 12w_{0,4})x_5x_6^2x_1^2 + (6w_{1,3} + 12w_{2,2})x_6x_5^2x_1x_2 + (6w_{1,2} + 6w_{0,3})x_5^3x_6x_1 + 2w_{2,2}x_5^2x_2^2 + 2w_{1,2}x_5^4x_2 + 2w_{0,2}x_5^5 \Rightarrow f_6^2 = 0$.

Suppose $k = 7$. Since $wt(X_+\frac{\partial f_8^2}{\partial x_6}) = 5$, so $0 = X_+\frac{\partial f_8^2}{\partial x_6} = (12w_{0,4}+9w_{1,3})x_1x_5^4x_6^2 + 6w_{1,3}x_2x_5^5x_6 + 6w_{0,3}x_5^6x_6 \Rightarrow f_8^2 = 0$.

Suppose $k = 9$. Since $wt(X_+\frac{\partial f_{10}^2}{\partial x_6}) = 5$, so $0 = X_+\frac{\partial f_{10}^2}{\partial x_6} = 12w_{0,4}x_4x_5^7x_6^2 \Rightarrow f_{10}^2 = 0$.

Thus we get either $f_{k+1}^2 = w_{0,1}x_5^3x_6$, $w_{0,1} \neq 0$ or $f_{k+1}^2 = 0$. Similar arguments show that either $f_{k+1}^{-2} = cx_5x_6^3$, $c \neq 0$ or $f_{k+1}^{-2} = 0$. If $f_{k+1}^{-2} = cx_5x_6^3$, then we have the same result as one of $f_{k+1}^2 = w_{0,1}x_5^3x_6$. If $f_{k+1}^2 = 0$ and $f_{k+1}^{-2} = 0$, then $f = f_{k+1}^0$ is of weight 0. Since $wt(X_+f) = 2$ and $wt(X_-f) = -2$, so $X_+f = X_-f = 0$ by previous arguments. Thus f is an invariant polynomial in the $x_1, x_2, x_3, x_4, x_5, x_6$ variables. Moreover, $0 = \frac{\partial}{\partial x_1}X_+f = X_+\frac{\partial f}{\partial x_1} + 3\frac{\partial f}{\partial x_2} \Rightarrow X_+\frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}$. Similarly, we have $X_+\frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}$, $X_+\frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4}$, $X_+\frac{\partial f}{\partial x_4} = 0$, $X_+\frac{\partial f}{\partial x_5} = -\frac{\partial f}{\partial x_6}$, $X_+\frac{\partial f}{\partial x_6} = 0$, $X_-\frac{\partial f}{\partial x_1} = 0$, $X_-\frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_-\frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$, $X_-\frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}$, $X_-\frac{\partial f}{\partial x_5} = 0$, $X_-\frac{\partial f}{\partial x_6} = -\frac{\partial f}{\partial x_5}$.

Case 4. $I = (4) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $3, 1, -1, -3, 0$.

By the same argument as in the beginning of Case 1 we have $f_{k+1}^i = 0$ for $i = \pm 1, \pm 3$ and $|i| \geq 5$.

For $i = \pm 4$.

Similar arguments as in Case 2 in the proof of "For $i = 5$" show that $f_{k+1}^i = 0$.

Thus $f = f_{k+1}^{-2} + f_{k+1}^0 + f_{k+1}^2$.

Similar arguments as in Case 3 show that $f_{k+1}^2 = cx_5^3x_6$. If $c \neq 0$, then again similar arguments as in Case 3 and $(2) \not\subseteq I$ show that $f_{k+1}^{-2} = c_1x_5x_6^3$ and $f_{k+1}^0 = c_2x_5^2x_6^2$. So $f = c_1x_5x_6^3 + c_2x_5^2x_6^2 + cx_5^3x_6 \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leq 2$. This contradicts $\dim I = 6$. Hence

$$c = 0 \Rightarrow f_{k+1}^2 = 0.$$

Similar arguments show that $f_{k+1}^{-2} = 0$. Thus $f = f_{k+1}^0$.

$$\begin{aligned}
&wt\frac{\partial f^0_{k+1}}{\partial x_1}=-3,\ \ wt\frac{\partial f^0_{k+1}}{\partial x_2}=-1,\ \ wt\frac{\partial f^0_{k+1}}{\partial x_3}=1,\\
&wt\frac{\partial f^0_{k+1}}{\partial x_4}=3,\ \ wt\frac{\partial f^0_{k+1}}{\partial x_5}=-1,\ \ wt\frac{\partial f^0_{k+1}}{\partial x_6}=1.\\
&\Rightarrow \dim I\leq 4,
\end{aligned}$$

which contradicts dim $I=6$. We conclude that Case 4 cannot occur.

Case 5. $I=(3)\oplus(3)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $2,0,-2$.

By the same argument as in the beginning of Case 1 we have $f^i_{k+1}=0$ for $i=0,\pm2$ and $|i|\geq 4$.
Thus $f=f^{-3}_{k+1}+f^{-1}_{k+1}+f^1_{k+1}+f^3_{k+1}$.

$$\begin{aligned}
&wt\frac{\partial f^3_{k+1}}{\partial x_1}=0,\quad wt\frac{\partial f^3_{k+1}}{\partial x_2}=2,\quad wt\frac{\partial f^3_{k+1}}{\partial x_3}=4,\\
&wt\frac{\partial f^3_{k+1}}{\partial x_4}=6,\quad wt\frac{\partial f^3_{k+1}}{\partial x_5}=2,\quad wt\frac{\partial f^3_{k+1}}{\partial x_6}=4.\\
&\Rightarrow f^3_{k+1}\text{ depends only on the }x_1,x_2,x_5\text{ variables}\\
&\Rightarrow f^3_{k+1}=c_1x_2^3+c_2x_2^2x_5+c_3x_2x_5^2+c_4x_5^3.\\
&\frac{\partial f^3_{k+1}}{\partial x_2}=3c_1x_2^2+2c_2x_2x_5+c_3x_5^2.\\
&X_+\frac{\partial f^3_{k+1}}{\partial x_2}=18c_1x_1x_2+6c_2x_1x_5.
\end{aligned}$$

Since $wt(X_+\frac{\partial f^3_{k+1}}{\partial x_2})=4$, so

$$X_+\frac{\partial f^3_{k+1}}{\partial x_2}=0\Rightarrow c_1=c_2=0\Rightarrow f^3_{k+1}=c_3x_2x_5^2+c_4x_5^3.$$

Since $wt(X_+\frac{\partial f^3_{k+1}}{\partial x_5})=4$, so

$$0=X_+\frac{\partial f^3_{k+1}}{\partial x_5}=6c_3x_1x_5\Rightarrow c_3=0\Rightarrow f^3_{k+1}=c_4x_5^3.$$

$$\begin{aligned}
&wt\frac{\partial f^1_{k+1}}{\partial x_1}=-2,\quad wt\frac{\partial f^1_{k+1}}{\partial x_2}=0,\quad wt\frac{\partial f^1_{k+1}}{\partial x_3}=2,\\
&wt\frac{\partial f^1_{k+1}}{\partial x_4}=4,\quad wt\frac{\partial f^1_{k+1}}{\partial x_5}=0,\quad wt\frac{\partial f^1_{k+1}}{\partial x_6}=2.\\
&\Rightarrow f^1_{k+1}\text{ is independent of the }x_4\text{ variable}\\
&\Rightarrow f^1_{k+1}=\sum_{b,c,e}w_{b,c,e}x_1^ax_2^bx_3^cx_5^dx_6^e
\end{aligned}$$

where $a=c+e-1$, $d=-b-2c-2e+4$.

Thus $f^1_{k+1} = w_{0,0,1}x_5^2x_6 + w_{0,0,2}x_1x_6^2 + w_{0,1,0}x_3x_5^2 + w_{0,1,1}x_1x_3x_6 + w_{0,2,0}x_1x_3^2 + w_{1,0,1}x_2x_5x_6 + w_{1,1,0}x_2x_3x_5 + w_{2,0,1}x_2^2x_6 + w_{2,1,0}x_2^2x_3$.

Since $wt(X_+\frac{\partial f^1_{k+1}}{\partial x_3}) = wt(X_+\frac{\partial f^1_{k+1}}{\partial x_6}) = 4$, so

$$
\begin{aligned}
&0 = X_+\frac{\partial f^1_{k+1}}{\partial x_3} = (8w_{0,2,0} + 6w_{2,1,0})x_1x_2 + (w_{0,1,1} + 3w_{1,1,0})x_1x_5.\\
&0 = X_+\frac{\partial f^1_{k+1}}{\partial x_6} = (4w_{0,1,1} + 6w_{2,0,1})x_1x_2 + (2w_{0,0,2} + 3w_{1,0,1})x_1x_5.\\
&\Rightarrow w_{2,1,0} = -\frac{4}{3}w_{0,2,0}, w_{1,1,0} = -\frac{1}{3}w_{0,1,1}, w_{2,0,1} = -\frac{2}{3}w_{0,1,1}, w_{1,0,1} = -\frac{2}{3}w_{0,0,2}.
\end{aligned}
$$

Since $wt(X_+^2\frac{\partial f^1_{k+1}}{\partial x_2}) = wt(X_+^2\frac{\partial f^1_{k+1}}{\partial x_5}) = 4$, so

$$
\begin{aligned}
&0 = X_+^2\frac{\partial f^1_{k+1}}{\partial x_2} = -96w_{0,2,0}x_1x_2 - 12w_{0,1,1}x_1x_5.\\
&0 = X_+^2\frac{\partial f^1_{k+1}}{\partial x_5} = -12w_{0,1,1}x_1x_2 + (-4w_{0,0,2} + 24w_{0,1,0})x_1x_5.\\
&\Rightarrow w_{0,2,0} = w_{0,1,1} = 0, w_{0,1,0} = \frac{1}{6}w_{0,0,2}.
\end{aligned}
$$

Thus $f^1_{k+1} = w_{0,0,1}x_5^2x_6 + w_{0,0,2}x_1x_6^2 + \frac{1}{6}w_{0,0,2}x_3x_5^2 - \frac{2}{3}w_{0,0,2}x_2x_5x_6$.

Similarly, we can show that $f^{-3}_{k+1} = d_4x_6^3$ and $f^{-1}_{k+1} = u_{0,0,2}x_5x_6^2 + u_{0,0,0}x_4x_5^2 + 2u_{0,0,0}x_2x_6^2 - 2u_{0,0,0}x_3x_5x_6$.

If $w_{0,0,2} \neq 0$, then

$$
\begin{aligned}
(3) \oplus (3) &= \langle X_+\frac{\partial f^1_{k+1}}{\partial x_2}, \frac{\partial f^1_{k+1}}{\partial x_2}, X_-\frac{\partial f^1_{k+1}}{\partial x_2}\rangle \oplus \langle X_+\frac{\partial f^1_{k+1}}{\partial x_5}, \frac{\partial f^1_{k+1}}{\partial x_5}, X_-\frac{\partial f^1_{k+1}}{\partial x_5}\rangle\\
&= \langle x_5^2, x_5x_6, x_6^2\rangle \oplus \langle -3x_1x_6 + x_2x_5, -2x_2x_6 + x_3x_5, -x_3x_6 + x_4x_5\rangle.
\end{aligned}
$$

Now

$$
\begin{aligned}
&f = f^3_{k+1} + f^1_{k+1} + f^{-1}_{k+1} + f^{-3}_{k+1}.\\
&\frac{\partial f}{\partial x_1} = w_{0,0,2}x_6^2.\\
&\frac{\partial f}{\partial x_2} = -\frac{2}{3}w_{0,0,2}x_5x_6 + 2u_{0,0,0}x_6^2.\\
&\frac{\partial f}{\partial x_3} = \frac{1}{6}w_{0,0,2}x_5^2 - 2u_{0,0,0}x_5x_6.\\
&\frac{\partial f}{\partial x_4} = u_{0,0,0}x_5^2.\\
&\frac{\partial f}{\partial x_5} = 3c_4x_5^2 + 2w_{0,0,1}x_5x_6 + \frac{1}{3}w_{0,0,2}x_3x_5 - \frac{2}{3}w_{0,0,2}x_2x_6\\
&\qquad + u_{0,0,2}x_6^2 + 2u_{0,0,0}x_4x_5 - 2u_{0,0,0}x_3x_6.\\
&\frac{\partial f}{\partial x_6} = w_{0,0,1}x_5^2 + 2w_{0,0,2}x_1x_6 - \frac{2}{3}w_{0,0,2}x_2x_5 + 3d_4x_6^2\\
&\qquad + 2u_{0,0,2}x_5x_6 + 4u_{0,0,0}x_2x_6 - 2u_{0,0,0}x_3x_5.
\end{aligned}
$$

Consider the coefficient matrix Φ of $\frac{\partial f}{\partial x_j}$ with respect to the ordered basis $\{x_5^2, x_5x_6, x_6^2, -3x_1x_6+x_2x_5, -2x_2x_6+x_3x_5, -x_3x_6+x_4x_5\}$:

$$\Phi = \begin{bmatrix} 0 & 0 & w_{0,0,2} & 0 & 0 & 0 \\ 0 & -\dfrac{2}{3}w_{0,0,2} & 2u_{0,0,0} & 0 & 0 & 0 \\ \dfrac{1}{6}w_{0,0,2} & -2u_{0,0,0} & 0 & 0 & 0 & 0 \\ u_{0,0,0} & 0 & 0 & 0 & 0 & 0 \\ 3c_4 & 2w_{0,0,1} & u_{0,0,2} & 0 & \dfrac{1}{3}w_{0,0,2} & 2u_{0,0,0} \\ w_{0,0,1} & 2u_{0,0,2} & 3d_4 & -\dfrac{2}{3}w_{0,0,2} & -2u_{0,0,0} & 0 \end{bmatrix}.$$

Since $\det(\Phi)=0$, so $\dim I \le 5$. This contradicts $\dim I = 6$. So $w_{0,0,2}=0$. $f_{k+1}^1 = w_{0,0,1}x_5^2x_6$. Similarly, $u_{0,0,0}$ must be zero. So $f_{k+1}^{-1} = u_{0,0,2}x_5x_6^2$. Therefore, $f = c_4x_5^3 + w_{0,0,1}x_5^2x_6 + u_{0,0,2}x_5x_6^2 + d_4x_6^3 \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \Rightarrow$ $\dim I \le 2$. This contradicts $\dim I = 6$. We conclude that Case 5 connot occur.

Case 6. $I = (3)\oplus(2)\oplus(1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $2, 0, -2, 1, -1$.

By the same argument as in the begining of Case 1 we have $f_{k+1}^i = 0$ for $i=\pm 2$ and $|i|\ge 4$. Thus $f = f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^0 + f_{k+1}^1 + f_{k+1}^3$. Similar arguments as in Case 5 and $(3)\oplus(3)\not\subseteq I$ show that

$$f_{k+1}^3 = c_1x_5^3,\ f_{k+1}^1 = c_2x_6x_5^2,\ f_{k+1}^{-1} = c_3x_6^2x_5,\ f_{k+1}^{-3} = c_4x_6^3.$$

$$\begin{aligned} & wt\frac{\partial f_{k+1}^0}{\partial x_1} = -3, \quad wt\frac{\partial f_{k+1}^0}{\partial x_2} = -1, \quad wt\frac{\partial f_{k+1}^0}{\partial x_3} = 1, \\ & wt\frac{\partial f_{k+1}^0}{\partial x_4} = 3, \quad wt\frac{\partial f_{k+1}^0}{\partial x_5} = -1, \quad wt\frac{\partial f_{k+1}^0}{\partial x_6} = 1. \\ & \Rightarrow f_{k+1}^0 \text{ is independent of the } x_1, x_4 \text{ variables} \\ & \Rightarrow f_{k+1}^0 = \sum_{a,b} w_{a,b}x_2^a x_3^b x_5^{\frac{3}{2}-a} x_6^{\frac{3}{2}-b} \\ & \Rightarrow f_{k+1}^0 = 0. \end{aligned}$$

Thus

$$\begin{aligned} & f = f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^1 + f_{k+1}^3 = c_1x_5^3 + c_2x_5^2x_6 + c_3x_5x_6^2 + c_4x_6^3 \\ & \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \\ & \Rightarrow \dim I \le 2. \end{aligned}$$

This contradicts $\dim I = 6$. We conclude that Case 6 connot occur.

Case 7. $I = (3)\oplus(1)\oplus(1)\oplus(1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $2, 0, -2$.

Similar arguments as in Case 5 and $(3) \oplus (3) \not\subseteq I$ show that this case cannot occur.

Case 8. $I = (2) \oplus (2) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $1, -1$.

By the same argument as in the beginning of Case 1 we have $f_{k+1}^i = 0$ for $|i| \geq 1$. Thus $f = f_{k+1}^0$

$$\begin{aligned}
& wt\frac{\partial f_{k+1}^0}{\partial x_1} = -3, \quad wt\frac{\partial f_{k+1}^0}{\partial x_2} = -1, \quad wt\frac{\partial f_{k+1}^0}{\partial x_3} = 1, \\
& wt\frac{\partial f_{k+1}^0}{\partial x_4} = 3, \quad wt\frac{\partial f_{k+1}^0}{\partial x_5} = -1, \quad wt\frac{\partial f_{k+1}^0}{\partial x_6} = 1. \\
& \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial f_{k+1}^0}{\partial x_4} = 0 \\
& \Rightarrow \dim I \leq 4.
\end{aligned}$$

This contradicts dim $I = 6$. We conclude that Case 8 cannot occur.

Case 9. $I = (2) \oplus (2) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $1, -1, 0$.

Similar arguments as in Case 8 show that this case cannot occur.

Case 10. $I = (2) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $1, -1, 0$.

Similar arguments as in Case 9 show that this case cannot occur.

Case 11. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights 0.

The same argument as in the beginning of Case 1 shows that this case cannot occur.

Q.E.D.

Lemma 1.2. *With the same hypothesis as Lemma* 1.1*, if* $\dim I = 5$*, then* I *cannot be an* $s\ell(2, \mathbf{C})$ *submodule.*

Proof. We assume on the contrary that I is an $s\ell(2, \mathbf{C})$ submolule.

Case 1. $I = (5)$.

By the same argument as in Case 2 of Lemma 1.1 we can write

$$f = f_{k+1}^{-1} + f_{k+1}^1.$$

Again, the same argument as in Case 1 in the proof "For $i = 2$" of Lemma 1.1 will prove that $f = 0$. Thus Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

This case cannot happen by the same argument as Case 4 of Lemma 1.1.

Case 3. $I = (3) \oplus (2)$.

This case cannot happen by the same argument as Case 6 of Lemma 1.1.

Case 4. $I = (3) \oplus (1) \oplus (1)$.
This case cannot happen by the same argument as Case 7 of Lemma 1.1.
Case 5. $I = (2) \oplus (2) \oplus (1)$.
This case cannot happen by the same argument as Case 9 of Lemma 1.1.
Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.
This case cannot happen by the same argument as Case 9 of Lemma 1.1.
Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.
This case cannot happen by the same argument as Case 11 of Lemma 1.1.

Q.E.D.

Lemma 1.3. *With the same hypothesis as Lemma* 1.1*, if* $\dim I = 4$*, then* f *is an* $s\ell(2,\mathbf{C})$ *invariant polynomial in the* x_1, x_2, x_3, x_4 *variables and* $I = (4)$.

Proof. Case 1. $I = (4)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $3, 1, -1, -3$.

By the same argument as in the beginning of Case 1 in Lemma 1.1 we have $f^i_{k+1} = 0$ for $i = \pm 1, \pm 3$ and $|i| \geq 5$.

For $i = 4$

$$\begin{aligned} &wt\frac{\partial f^4_{k+1}}{\partial x_1} = 1, \quad wt\frac{\partial f^4_{k+1}}{\partial x_2} = 3, \quad wt\frac{\partial f^4_{k+1}}{\partial x_3} = 5, \\ &wt\frac{\partial f^4_{k+1}}{\partial x_4} = 7, \quad wt\frac{\partial f^4_{k+1}}{\partial x_5} = 3, \quad wt\frac{\partial f^4_{k+1}}{\partial x_6} = 5. \\ &\Rightarrow f^4_{k+1} \text{ depends only on the } x_1, x_2, x_5 \text{ variables.} \end{aligned}$$

If f^4_{k+1} were not zero, then either $\frac{\partial f^4_{k+1}}{\partial x_1}$ or $\frac{\partial f^4_{k+1}}{\partial x_2}$ or $\frac{\partial f^4_{k+1}}{\partial x_5}$ would generate I because I is an irreducible $s\ell(2,\mathbf{C})$ module. Hence I would involve only the x_1, x_2, x_5 variables. It follows that $\frac{\partial f}{\partial x_j}$, $1 \leq j \leq 6$, involves only the x_1, x_2, x_5 variables and hence so does f. This implies that $\frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_6} = 0$. Thus dim $I \leq 3$, which contradicts the fact that dim $I = 4$. Similar arguments show that $f^{-4}_{k+1} = 0$.

For $i = 2$

$$\begin{aligned} &wt\frac{\partial f^2_{k+1}}{\partial x_1} = -1, \quad wt\frac{\partial f^2_{k+1}}{\partial x_2} = 1, \quad wt\frac{\partial f^2_{k+1}}{\partial x_3} = 3, \\ &wt\frac{\partial f^2_{k+1}}{\partial x_4} = 5, \quad wt\frac{\partial f^2_{k+1}}{\partial x_5} = 1, \quad wt\frac{\partial f^2_{k+1}}{\partial x_6} = 3. \\ &\Rightarrow f^2_{k+1} \quad \text{is independent of the } x_4 \text{ variable.} \end{aligned}$$

Since $wt\frac{\partial f^2_{k+1}}{\partial x_2} = wt\frac{\partial f^2_{k+1}}{\partial x_5} = 1$ and $wt\frac{\partial f^2_{k+1}}{\partial x_3} = wt\frac{\partial f^2_{k+1}}{\partial x_6} = 3$, in view of Lemma 5.1 of [Ya4], there exist constants r_1, r_2, r_3, r_4 such that

$$f^2_{k+1} = \sum_{a \geq 0} c_a x_1^b (r_1 x_2 + r_2 x_5)^c (r_3 x_3 + r_4 x_6)^a$$

where $b = \frac{2a-k+1}{2}$, $c = \frac{-4a+3k+1}{2}$.

Assuming $\frac{\partial f^2_{k+1}}{\partial x_3} \neq 0$. Since $wt(X_+ f^2_{k+1}) = 4$, so $X_+ f^2_{k+1} = 0$ by previous arguments. Now $0 = \frac{\partial}{\partial x_2} X_+ f^2_{k+1} = X_+ \frac{\partial f^2_{k+1}}{\partial x_2} + 4\frac{\partial f^2_{k+1}}{\partial x_3} \Rightarrow \frac{\partial f^2_{k+1}}{\partial x_3} \neq 0$. Since

$wt(X_-\frac{\partial f_{k+1}^2}{\partial x_1}) = 1 = wt(\frac{\partial f_{k+1}^2}{\partial x_2})$, there exists a constant d such that $X_-\frac{\partial f_{k+1}^2}{\partial x_3} = d\frac{\partial f_{k+1}^2}{\partial x_2}$. Differentiating this equation with respect to the x_4 variable, we get

$$\frac{\partial^2 f_{k+1}^2}{\partial x_3^2} + X_-\frac{\partial^2 f_{k+1}^2}{\partial x_4 \partial x_3} = d\frac{\partial f_{k+1}^2}{\partial x_4 \partial x_2} \Rightarrow \frac{\partial^2 f_{k+1}^2}{\partial x_3^2} = 0.$$

Hence $\frac{\partial f_{k+1}^2}{\partial x_3}$ is independent of the x_3 variable. Thus

$$\frac{\partial f_{k+1}^2}{\partial x_3} = \sum_{a\geq 1} ar_3c_a x_1^b(r_1x_2 + r_2x_5)^c(r_3x_3 + r_4x_6)^{a-1} \Rightarrow a = 1.$$

So $\frac{\partial f_{k+1}^2}{\partial x_3} = r_3c_1x_1^{\frac{3-k}{2}}(r_1x_2 + r_2x_5)^{\frac{-3+3k}{2}} \Rightarrow r_3c_1 \neq 1$. Since $wt(X_+\frac{\partial f_{k+1}^2}{\partial x_3}) = 5$, so

$$0 = X_+\frac{\partial f_{k+1}^2}{\partial x_3} = 3(\frac{-3+3k}{2})c_1r_1r_3x^{\frac{5-k}{2}}(r_1x_2+r_2x_5)^{\frac{-5+3k}{2}} \Rightarrow r_1 = 0 \Rightarrow \frac{\partial f_{k+1}^2}{\partial x_2} = 0.$$

Thus $\frac{\partial f_{k+1}^2}{\partial x_3} = 0$. Similar arguments show that $\frac{\partial f_{k+1}^2}{\partial x_2} = 0$. Therefore, f_{k+1}^2 depends only on the x_1, x_5, x_6 variables. Similar arguments as in the proof of "For $i = 4$" shows that $f_{k+1}^2 = 0$. Similar arguments show that $f_{k+1}^{-2} = 0$. Thus $f = f_{k+1}^0$ is of weight 0. Since $wt(X_+f) = 2$ and $wt(X_-f) = -2$, so $X_+f = X_-f = 0$ by previous arguments. Now

$$wt\frac{\partial f}{\partial x_1} = -3, \quad wt\frac{\partial f}{\partial x_2} = -1, \quad wt\frac{\partial f}{\partial x_3} = 1,$$
$$wt\frac{\partial f}{\partial x_4} = 3, \quad wt\frac{\partial f}{\partial x_5} = -1, \quad wt\frac{\partial f}{\partial x_6} = 1.$$

Now that $\frac{\partial f}{\partial x_1} \neq 0$ and $\frac{\partial f}{\partial x_4} \neq 0$, otherwise $\dim I \leq 2 < 4$. $\frac{\partial f}{\partial x_2} \neq 0$, otherwise $0 = \frac{\partial}{\partial x_2}(X_-f) = \frac{\partial f}{\partial x_1} + X_-\frac{\partial f}{\partial x_2} \Rightarrow \frac{\partial f}{\partial x_1} = 0$. $\frac{\partial f}{\partial x_3} \neq 0$, otherwise $0 = \frac{\partial}{\partial x_3}(X_+f) = 3\frac{\partial f}{\partial x_4} + X_+\frac{\partial f}{\partial x_3} \Rightarrow \frac{\partial f}{\partial x_4} = 0$. If $\frac{\partial f}{\partial x_5} \neq 0$, then there exists a constant $d \neq 0$ such that $\frac{\partial f}{\partial x_2} = d\frac{\partial f}{\partial x_5}$. Since $0 = \frac{\partial}{\partial x_2}(X_-f) = \frac{\partial f}{\partial x_1} + X_-\frac{\partial f}{\partial x_2}$ and $0 = \frac{\partial}{\partial x_5}(X_-f) = X_-\frac{\partial f}{\partial x_5} \Rightarrow -\frac{\partial f}{\partial x_1} = X_-\frac{\partial f}{\partial x_2} = d(X_-\frac{\partial f}{\partial x_5}) = 0$. Thus $\frac{\partial f}{\partial x_5} = 0$. Similar arguments show that $\frac{\partial f}{\partial x_6} = 0$. Therefore f is an invariant polynomial in the x_1, x_2, x_3, x_4 variables. Moreover, $X_+\frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}$, $X_+\frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}$, $X_+\frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4}$, $X_+\frac{\partial f}{\partial x_4} = 0$, $X_-\frac{\partial f}{\partial x_1} = 0$, $X_-\frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_-\frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$, $X_-\frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}$.

Case 2. $I = (3) \oplus (1)$.

This case cannot happen by the same argument as Case 7 of Lemma 1.1.

Case 3. $I = (2) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $1, -1$. By the same argument as in the beginning of Case 1 in Lemma 1.1 we have

$f = f^0_{k+1}$. Now

$$
\begin{aligned}
& wt\frac{\partial f}{\partial x_1} = wt\frac{\partial f^0_{k+1}}{\partial x_1} = -3.\\
& wt\frac{\partial f}{\partial x_4} = wt\frac{\partial f^0_{k+1}}{\partial x_4} = 3\\
& \Rightarrow \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_4} = 0.\\
& \frac{\partial}{\partial x_j}X_+ f = X_+\frac{\partial}{\partial x_j}f + a_j\frac{\partial f}{\partial x_{j+1}} \in I.\\
& wt(\frac{\partial}{\partial x_j}X_+ f) = -1,1,3,5,1,3 \text{ for } j = 1,2,3,4,4,5,6 \text{ respectively}\\
& \Rightarrow X_+ f \text{ depends only on the } x_1, x_2, x_5 \text{ variables}\\
& \Rightarrow X_+ f = c_1x_2^2 + c_2x_2x_5 + c_3x_5^2 \text{ for } wt(X_+ f) = 2\\
& \Rightarrow X_+ f = 0 \text{ because } k \geq 2.
\end{aligned}
$$

Now $0 = \frac{\partial}{\partial x_1}X_+ f = X_+\frac{\partial f}{\partial x_1} + 3\frac{\partial f}{\partial x_2} \Rightarrow \frac{\partial f}{\partial x_2} = 0$. Similarly, $0 = \frac{\partial}{\partial x_2}X_+ f = X_+\frac{\partial f}{\partial x_2} + 4\frac{\partial f}{\partial x_3} \Rightarrow \frac{\partial f}{\partial x_3} = 0$. Thus dim $I = \langle \frac{\partial f}{\partial x_5}, \frac{\partial f}{\partial x_6}\rangle \leq 2$. This contradicts dim $I = 4$. So Case 3 cannot occur.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

By the same argument as in the beginning of Case 1 in Lemma 1.1 we have $f^i_{k+1} = 0$ for $|i| \geq 1$. Thus $f = f^0_{k+1}$ is of weight 0 and $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_4} = 0$. Similar arguments as in Case 3 show that Case 4 cannot occur.

Case 5. $I = (1) \oplus (1) \oplus (1)$.

This case cannot happen by the same argument as Case 11 of Lemma 1.1.

Q.E.D.

Lemma 1.4. *With the same hypothesis as Lemma* 1.1, *if* $\dim I \leq 3$, *then* I *cannot be an* $s\ell(2,\mathbf{C})$ *submodule.*

Proof. We assume on the contrary that I is an $s\ell(2,\mathbf{C})$ submodule.

Case 1. $I = (3)$.

This case cannot happen by the same argument as Case 7 of Lemma 1.1.

Case 2. $I = (2) \oplus (1)$.

This case cannot happen by the same argument as Case 4 of Lemma 1.3.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

This case cannot happen by the same argument as Case 11 of Lemma 1.1.

Case 4. $I = (2)$.

By the same argument as in Case 3 of Lemma 1.3 we have $f = f^0_{k+1}$ is of weight 0 and $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0$. Thus f depends only on the x_5, x_6 variables $\Rightarrow f = cx_5^{\frac{k+1}{2}}x_6^{\frac{k+1}{2}}$. If $f \neq 0$, then $c \neq 0$, $\frac{\partial f}{\partial x_6} = \frac{k+1}{2}cx_5^{\frac{k+1}{2}}x_6^{\frac{k+1}{2}}$, $X_+\frac{\partial f}{\partial x_6} = \frac{k+1}{2}\frac{k-1}{2}cx_5^{\frac{k+3}{2}}x_6^{\frac{k+3}{2}}$. Since $wt(X_+\frac{\partial f}{\partial x_6}) = 3$, so $X_+\frac{\partial f}{\partial x_6} = 0 \Rightarrow c = 0$. We conclude that Case 4 cannot occur.

Case 5. $I = (1) \oplus (1)$.

This case cannot happen by the same argument as Case 11 of Lemma 1.1.

Case 6. $I = (1)$.

This case cannot happen by the same argument as Case 11 of Lemma 1.1.

Q.E.D.

Theorem. *Suppose $s\ell(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in $x_1, x_2, x_3, x_4, x_5, x_6$ via*

$$\begin{aligned}
\tau &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6},\\
X_+ &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_6},\\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_5}.
\end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.,

$$wt(x_1) = 3, wt(x_2) = 1, wt(x_3) = -1, wt(x_4) = -3, wt(x_5) = 1, wt(x_6) = -1.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}$, and $\frac{\partial f}{\partial x_6}$ where f is a homogeneous polynomial of degree $k+1$. If I is an $s\ell(2,\mathbf{C})$–submodule then either

(i) *f is an $s\ell(2,\mathbf{C})$ invariant polynomial in the $x_1, x_2, x_3, x_4, x_5, x_6$ variables and $I = (4) \oplus (2)$, or*

(ii) *$f = g + c_1x_5x_6^3 + c_2x_5^3x_6$ where $g = d(2x_1x_6^3 - \frac{1}{3}x_4x_5^3 - 2x_2x_5x_6^2 + 2x_3x_5^2x_6)$ is an $s\ell(2,\mathbf{C})$ invariant polynomial with $(c_1, c_2) \neq (0,0)$ and $d \neq 0$. $I =$*

$$\begin{aligned}
&\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}, \frac{\partial g}{\partial x_6}\rangle = (4) \oplus (2)\\
&\quad = \langle x_5^3, x_5^2x_6, x_5x_6^2, x_6^3\rangle \oplus \langle 6x_1x_6^2 - 4x_2x_5x_6 + x_3x_5^2, 2x_2x_6^2 - 2x_3x_5x_6 + x_4x_5^2\rangle,
\end{aligned}$$

or

(iii) *f is an $s\ell(2,\mathbf{C})$ invariant polynomial in the x_1, x_2, x_3, x_4 variables and $I = (4)$.*

Proof. This is an immediate consequence of Lemma 1.1 through Lemma 1.4.

Q.E.D.

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