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ON VANISHING OF CHARACTERISTIC NUMBERS IN POINCARÉ COMPLEXES

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ABSTRACT. Let $G_r(X) \subset \pi_r(X)$ be the evaluation subgroup as defined by Gottlieb. Assume the Hurewicz map $G_r(X) \to H_r(X;R)$ is non-trivial and R is a field. We will prove: if X is a Poincaré complex oriented in R-coefficient, all the characteristic numbers of X in R-coefficient vanish. Similarly, if $R = Z_p$ and X is a Z_p-Poincaré complex, then all the mod p Wu numbers vanish. We will also show that the existence of a non-trivial derivation on $H^*(X;Z_p)$ with some suitable conditions implies vanishing of mod p Wu numbers.

1. INTRODUCTION

D. H. Gottlieb proved the following interesting fact ([3]): *If X is a connected finite aspherical complex and $Z(\pi_1(X)) \neq 1$, then $\chi(X) = 0$.* Here $Z(G)$ means the center for any group G and $\chi(X)$ is the Euler-Poincaré number of X. This paper is largely motivated by the question of whether the condition $Z(\pi_1(X)) \neq 1$ also implies the vanishing of the Stiefel-Whitney numbers if X is a compact closed aspherical manifold. The question is only partially answered, but in a more general setting.

The paper is mainly about the relation between the characteristic numbers of a Poincaré complex X and *its evaluation subgroup*, which is the subgroup $G_r(X, x_0)$ of $\pi_r(X, x_0)$, x_0 being a base point of X, defined as follows by Gottlieb ([4]):

Definition. Let S^r denote the r-sphere. Consider a continuous function

$$F : S^r \times X \to X$$

such that $F(z_0, x) = x$, where $x \in X$ and z_0 is a basepoint of S^r. Then $G_r(X, x_0)$ consists of all the elements in $\pi_r(X, x_0)$ represented by the map $f : (S^r, z_0) \to (X, x_0)$ defined by $f(z) = F(z, x_0)$ for some F such as the above.

Let R be a commutative ring with a unit. Then, by an n-dimensional R-Poincaré complex, we will mean a CW complex X for which there is a homology class $[X] \in H_n(X;R)$ such that the map $\bigcap[X] : H^*(X;R) \to H_{n-*}(X;R)$ is an isomorphism.

In sections 2 and 3, we will deal with Z_p-Poincaré complexes X for any prime integer p. Then some mod p characteristic numbers of X can be defined using the mod p Steenrod powers as in section 2 without referring to any vector bundle or spherical fibration. We call them the *mod p Wu numbers* of X. In particular, when

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$p = 2$ and X is a manifold, these characteristic numbers are none other than the usual Stiefel-Whitney numbers of X.

The following answers the question asked above:

Theorem 1. *Let p be a prime and X a connected aspherical Z_p-Poincaré complex. If there is an element in $Z(\pi_1(X))$ whose image in $H_1(X; Z_p)$ under the Hurewicz map is not zero, then all the mod p Wu numbers of X vanish.*

Note that more than just $Z(\pi_1(X)) \neq 1$ is required in the above for the mod p Wu numbers to vanish. Therefore a question can be raised, for which the author does not know the answer at the moment: *Is there an example of a connected aspherical Z_p-Poincaré complex X such that $Z(\pi_1(X)) \neq 1$ but some of the mod p Wu numbers are nonzero?*

Definition. Let X be a space and R a commutative ring with identity. A *derivation* of degree $-r$ on $H^*(X; R)$ is a group homomorphism $D : H^*(X; R) \to H^{*-r}(X; R)$ such that $D(u \cup v) = Du \cup v + (-1)^{r\,|u|} u \cup Dv$ for any $u, v \in H^*(X)$. Here $|u|$ means $dim\, u$.

On the way to proving Theorem 1, we will establish the following lemma which relates the derivation to the vanishing of the mod p Wu numbers:

Lemma 2. *Let X be a Z_p-Poincaré space for a prime p. Assume there is a derivation D of deg $-r$, $r > 0$, on $H^*(X)$ which commutes with the mod p Steenrod powers. Also assume there is a class $\varphi \in H^r(X)$ such that $D\varphi = 1$, $\wp^i \varphi = 0$ for any i such that $0 < i < r$ if $p = 2$ and $0 < i < [r/2]$ if $p > 2$, where the $\wp^i$'s are the mod p i-th Steenrod powers. Then all the mod p Wu numbers of X vanish.*

In particular, the existence of a nontrivial derivation of deg -1 commuting with the mod p Steenrod powers on a Z_p-Poincaré complex implies vanishing of all of its mod p Wu numbers.

This lemma will be directly applied to projective planes to give a succinct argument to decide which of them are boundaries (see the Remark that follows the proof in section 3).

In section 4, we will deal the same issue in a more geometric way, confining ourselves to the Poincaré complexes in the sense of Wall([15]), in which case the characteristic numbers of X come from its Spivak normal fibration (see the definition and the comments following it, before the proof of Theorem 3). We will prove:

Theorem 3. *Let M be a connected Poincaré complex of formal dimension n, oriented with respect to a field coefficient R. Assume there is an $\alpha \in G_k(M, x_0)$ such that $h_R(\alpha) \neq 0$, where $h_R : \pi_k(M, x_0) \to H_k(M; R)$ is the Hurewicz homomorphism and x_0 is a base point. Then all the characteristic numbers in R-coefficients coming from the Spivak fibration vanish.*

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2. The mod p Wu numbers of Z_p-Poincaré spaces

From now on, to simplify some statements, we will denote the k-th Steenrod square by $\wp^k$, specifying Z_2 coefficients if necessary, even though it is customary to denote it by Sq^k.

Let p be a prime and assume ξ is a sphere fibration over X orientable in Z_p-coefficients. Then the mod p k-th Wu class $q_k(\xi)$ is defined as follows ([9]):

$$q_k(\xi) = \Phi^{-1}\wp^k\Phi(1) \in H^{k\{p\}}(X; Z_p)$$

Here Φ is the Thom isomorphism and $\wp^k$ is the k-th mod p Steenrod power, and $\{p\} = 1$ if $p = 2$ and $\{p\} = 2(p-1)$ if $p > 2$. Note that, if $p = 2$, $q_k(\xi)$ is the Stiefel-Whitney class of ξ.

If $p > 2$ and ξ is a vector bundle, we have the following characterization of the mod p Wu classes ([8], pp. 227-230):

Theorem [Wu]. *Let ξ be a vector bundle and $u_i(\xi)$ its Pontrjagin classes reduced at an odd prime $p = 2s+1$. Then*

$$q_k(\xi) = K_{sk}(u_1(\xi), \cdots, u_{sk}(\xi))$$

where $\{K_i\}$ is the multiplicative sequence belonging to the power series $f(t) = 1 + t^s \in Z_p[[t]]$

In particular, if $p = 3$, then $s = 1$ and it follows that $q_k(\xi)$ is equal to the Pontrjagin class $u_k(\xi)$ reduced mod 3.

Now let us consider an n-dimensional Z_p-Poincaré complex X with a prime integer p. Then, exploiting Poincaré duality, the class $v_k(X) = v_k \in H^{\{p\}k}(X; Z_p)$ can be defined by the condition

$$x \cup v_k = \wp^k x$$

for any $x \in H^{n-k\{p\}}(X; Z_p)$. We set $v(X) = v = 1 + v_1 + v_2 + \cdots$.

Let $\wp = 1 + \wp_1 + \wp_2 + \cdots$ be the mod p total Steenrod power. Then we define the mod p total Wu class $q(X) = q$ and the k-th Wu class q_k by the equation

$$\wp v = q(X) = 1 + q_1 + \cdots, \qquad q_k \in H^{\{p\}k}(X; Z_p)$$

Then the mod p Wu numbers are defined as $q_I(X) = \langle q_{\iota_1} q_{\iota_2} \cdots q_{\iota_k}, [X]\rangle$ for any sequence of natural numbers $I = (\iota_1, \iota_2, \cdots, \iota_k)$ such that $\{p\}(\iota_1 + \iota_2 + \cdots + \iota_k) = n$. (When $p = 2$, the classical terminology ([8]) calls q_k the Stiefel-Whitney class of X and $v_k(X)$ the Wu class of X. Our terminology is motivated by [9].)

Note that the classes $v_k(X)$ are defined in the above without referring to any specific sphere fibration. However, if X is a Poincaré complex, a theorem of W. Browder ([2], p. 64) shows that $v(X) \cup U = \wp^{-1}U$, when $p = 2$, where U is the Thom class of the Spivak normal fibration ξ of X and $\wp^{-1}$ is the multiplicative inverse of $\wp$. The proof can easily be modified for any prime p. In particular, this theorem implies that $q_k(X) = q_k(\eta)$, where η is the fibration stably inverse to ξ.

3. Vanishing of the mod p Wu numbers

Proof of Lemma 2. We may assume that $n \equiv 0 \bmod 2(p-1)$ if p is an odd prime; otherwise the lemma is trivially true.

Claim. $\varphi \cup Dq = 0$.

Let us assume the claim, which will be proved in a moment.

First of all, we observe that, for any even dimensional cohomology class u such that $D(\varphi \cup u) = 0$, we have: $u = (-1)^{r+1}\varphi \cup Du$. In particular, $\varphi \cup u = 0$ implies $u = (-1)^{r+1}\varphi \cup Du$.

Let $I = (\iota_1, \iota_2, \cdots, \iota_l)$ be a sequence of natural numbers such that $\{p\}(\iota_1 + \iota_2 + \cdots + \iota_l) = n$. Set $q_I = q_{\iota_1} q_{\iota_2} \cdots q_{\iota_l}$. Then, $q_I = (-1)^{r+1}\varphi \cup Dq_I = (-1)^{r+1}\varphi \cup \sum_j Dq_{\iota_j} q_{\iota_1} \cdots \widehat{q_{\iota_j}} \cdots q_{\iota_l} = (-1)^{r+1}\sum_j (\varphi \cup Dq_{\iota_j}) q_{\iota_1} \cdots \widehat{q_{\iota_j}} \cdots q_{\iota_l} = 0$ (the last equality follows from the claim).

Proof of the claim. Let v_k be as defined in the previous section. Then

$$\wp^k x = x \cup v_k$$

for any x such that $|x| = n - \{p\}k$. Note that $|x|$ is even if p is odd, because $n \equiv 0 \mod \{p\}$. Apply D to the both sides of the equation and then multiply them by φ to get

$$\varphi \cup \wp^k Dx = x \cup (\varphi \cup Dv_k) + (\varphi \cup Dx) \cup v_k. \tag{1}$$ □

Case 1. Assume $p = 2$ or r is even. Then, noting that $\wp\varphi = \varphi + \varphi^p$ from the hypothesis, we obtain, by Cartan's formula,

$$\wp^k(\varphi \cup Dx) = \sum_i \wp^{k-i}\varphi \cup \wp^i Dx = \varphi \cup \wp^k Dx + \varphi^p \cup \wp^{k-\kappa} Dx, \tag{2}$$

where $\kappa = r$ if $p = 2$ and $\kappa = r/2$ if $p > 2$, and $\wp^j \equiv 0$ is understood if $j < 0$. By the equations (1) and (2), we have

$$\wp^k(\varphi \cup Dx) - \varphi^p \cup \wp^{k-\kappa} Dx = x \cup (\varphi \cup Dv_k) + (\varphi \cup Dx) \cup v_k \ . \tag{3}$$

However, $\wp^k(\varphi \cup Dx) = (\varphi \cup Dx) \cup v_k$, by the defining property of v_k. Together with equation (3), we have

$$\varphi^p \cup \wp^{k-\kappa} Dx = -x \cup (\varphi \cup Dv_k) \ . \tag{4}$$

Next note that $\varphi^p \cup \wp^{k-\kappa} x = 0$ for dimensional reasons and that $D\varphi^p = p\varphi^{p-1} = 0$. We therefore have

$$0 = D(\varphi^p \cup \wp^{k-\kappa} x) = D\varphi^p \cup \wp^{k-\kappa} x + \varphi^p \cup D\wp^{k-r} x = \varphi^p \cup \wp^{k-\kappa} Dx \ . \tag{5}$$

Plugging the extreme equation of (5) into equation (4), we get $(\varphi \cup Dv_k) \cup x = 0$ for any x such that $|x| = n - \{p\}k$. By Poincaré duality, we conclude that $\varphi \cup Dv_k = 0$ and, therefore, that $\varphi \cup Dv = 0$.

Now we apply the mod p total Steenrod power to each side of the equation $\varphi \cup Dv = 0$, to get $0 = \wp(\varphi \cup Dv) = (\varphi + \varphi^p) \cup \wp Dv$

It follows that $\varphi \cup \wp Dv = -\varphi^p \cup \wp Dv = \cdots = (-1)^N \varphi^{pN} \cup \wp Dv = 0$, where N is any integer large enough.

Note that $\wp Dv = D\wp v = Dq$. So we conclude $\varphi \cup Dq = 0$, as required.

Case 2. Assume both p and r are odd.

Then $\wp\varphi = \varphi$ from the hypothesis and, therefore, $\wp^k(\varphi \cup Dx) = \varphi \cup \wp^k Dx$. Together with equation (1), we obtain

$$\wp^k(\varphi \cup Dx) = x \cup (\varphi \cup Dv_k) + (\varphi \cup Dx) \cup v_k \ .$$

Since $\wp^k(\varphi \cup Dx) = (\varphi \cup Dx) \cup v_k$ by the defining property of v_k, we conclude that

$x \cup (\varphi \cup Dv_k) = 0$ for any x such that $|x| = n - 2(p-1)k$. Therefore, $\varphi \cup Dv_k = 0$ and $\varphi \cup Dv = 0$. So $0 = \wp(\varphi \cup Dv) = \varphi \cup D\wp v = \varphi \cup Dq$. This proves Lemma 2.

□

Remark. It is a well known fact that *the projective planes FP^n are boundaries if and only if n is odd*, where F is the real, the complex, or the quaternion field. In fact, if n is even, FP^n is not a boundary since $\chi(FP^n) \not\equiv 0 \bmod 2$. For the same reason the Cayley projective plane does not bound.

If n is odd, Lemma 2 provides a unified argument to prove FP^n bounds: Note that $H^*(FP^n; Z_2) = Z_2[x]/ < x^{n+1} >$ with $|x| = 1, 2$, or 4. The formal derivation $D(x^k) = \overline{k}x^{k-1}$ on $Z_2[x]$ gives rise to a well defined derivation on $Z_2[x]/ < x^{n+1} >$ because n is odd. By inspection, $Sq^k D = DSq^k$ for any integer k. Furthermore, $Dx = 1$ and $Sq\, x = x + x^2$. By Lemma 2, all the Stiefel-Whitney numbers of FP^n vanish. Therefore FP^n bounds.

Let R be a commutative ring with a unit and $F : S^r \times X \to X$ be as in section 1. Then it can be shown that for any $u \in H^*(X; R)$, $F^*(u) = 1 \times u + \overline{[S^r]} \times u' \in H^*(S^r \times X; R)$ with a unique $u' \in H^{*-r}(X; R)$, where $\overline{[S^r]} \in H^r(S^r; R)$ is the class such that $\langle\overline{[S^r]}, [S^r]\rangle = 1$. Then the *derivation D_F associated to F* is the group homomorphism $D_F : H^*(X; R) \to H^{*-r}(X; R)$, defined by $D_F u = u'$.

Lemma 4 proves, in particular, that D_F is a derivation of degree $-r$.

For simplicity of notation, we set $D = D_F$.

Lemma 4. (i) *$D(u \cup v) = Du \cup v + (-1)^{r\,|u|} u \cup Dv$ for any $u, v \in H^*(X; R)$.*

(ii) *$Du = \langle u, f_*[S^r]\rangle 1$ for any $u \in H^r(X; R)$, where f is the map defined by $f(z) = F(z, x_0)$ and 1 the identity of the cohomology ring $H^*(X; R)$.*

(iii) *If $R = Z_p$, we have $D\wp^k = \wp^k D$ for any integer k.*

Proof. (i) We have $F^*(u \cup v) = 1 \times (u \cup v) + \overline{[S^r]} \times D(u \cup v)$. On the other hand, $F^*(u \cup v) = F^*(u) \cup F^*(v) = (1 \times u + \overline{[S^r]} \times Du) \cup (1 \times v + \overline{[S^r]} \times Dv) = 1 \times (u \cup v) + \overline{[S^r]} \times (Du \cup v + (-1)^{r\,|u|} u \cup Dv)$.

Thus, $D(u \cup v) = D(u \cup v) + (-1)^{r\,|u|} u \cup Dv$.

(ii) Let $[x_0]$ be the generator of $H_0(X)$ and $u \in H^r(X)$. Then, $\langle F^*(u), [S^r] \times [x_0]\rangle = \langle 1 \times u + \overline{[S^r]} \times Du, [S^r] \times [x_0]\rangle = \langle\overline{[S^r]}, [S^r]\rangle\langle Du, [x_0]\rangle = \langle Du, [x_0]\rangle$.

On the other hand,

$$\langle F^*(u), [S^r] \times [x_0]\rangle = \langle u, F_*([S^r] \times [x_0])\rangle = \langle u, f_*[S^r]\rangle.$$

Therefore, we conclude that $Du = \langle Du, [x_0]\rangle 1 = \langle u, f_*[S^r]\rangle 1$.

(iii) Note that $\wp^k F^*(u) = F^* \wp^k u$. We have $\wp^k F^*(u) = \wp^k(1 \times u + \overline{[S^r]} \times Du) = 1 \times (\wp^k u) + \overline{[S^r]} \times (\wp^k Du)$ and $F^*(\wp^k u) = 1 \times (\wp^k u) + \overline{[S^r]} \times (D\wp^k u)$.

Hence, $\wp^k Du = D\wp^k u$. This completes the proof of Lemma 5. □

Lemmas 2 and 4 together prove

Theorem 5. *Let p be a prime and X a path connected Z_p-Poincaré space with a base point x_0. Assume there is $f : S^r \to X$ which represents an element of $G_r(X, x_0)$ such that there is a class $\varphi \in H^r(X; Z_p)$ satisfying $\langle \varphi, f_*[S^r]\rangle = 1$ and $\wp^i \varphi = 0$ for any i such that $0 < i < [r/2]$ if $p > 2$ and $0 < i < r$ if $p = 2$. Then all the mod p Wu numbers of X vanish.* □

Note that, if we concentrate on the first evaluation subgroup $G_1(X, x_0)$, Theorem 5 takes a much simpler form:

Corollary 6. *Let X be a path connected Z_p-Poincaré space with a base point x_0. If there is an element $\alpha \in G_1(X, x_0)$ whose image under the Hurewicz map $h : \pi_1(X, x_0) \to H_1(X; Z_p)$ is not zero, then all the mod p Wu numbers vanish.* □

Proof. In view of Theorem 5, it is enough to show the existence of $\varphi \in H^1(X; Z_p)$ such that $\langle \varphi, h(\alpha) \rangle = 1$. But the map from $H^1(X; Z_p)$ into $Hom(H_1(X), Z_p)$ defined by the Kronecker index is an isomorphism, which completes the proof. □

The following lemma is due to D. H. Gottlieb ([3]).

Lemma 7. *Let X be an aspherical complex. If $f : S^1 \to X$ represents a central element in $\pi_1(X, x_0)$, then f represents an element of the first evaluation subgroup.*

Proof. Consider f as a map from $(I, \{0, 1\})$ into (X, x_0). We may also assume $x_0 \in X$ is a nondegenerate base point. Then $1_X \cup f : \{0\} \times X \cup I \times \{x_0\} \to X$ extends to a map $F : I \times X \to X$.

Set $g = F |_{\{1\} \times X}$. Then g preserves the base point and $g_\sharp(\alpha) = [f]^{-1}\alpha[f]$ for any $\alpha \in \pi_1(X, x_0)$. However, $[f]$ is central. So $g_\sharp$ is the identity homomorphism of $\pi_1(X, x_0)$. Since X is an aspherical complex, g is homotopic to the identity by a base point preserving homotopy ([12], Theorem 9, pp. 424-432). Hence there is no loss of generality in assuming $g = F |_{\{1\} \times X}$ is the identity map. Thus F can be considered as a map from $S^1 \times X$ into X which satisfies $F(z, x_0) = f(z)$ and $F(z_0, x) = x$ for all $x \in X$ for some base point $z_0 \in S^1$. □

Finally, Corollary 6 and Lemma 7 together prove Theorem 1.

4. The Poincaré complexes

Let (Y, X) be a Poincaré pair in the sense of C.T.C. Wall ([15]). Note, in particular, that both Y and X are finitely dominated complexes, and are allowed to be infinite. Of course, we do not exclude the case $X = \phi$ and will identify (Y, ϕ) with Y for any topological space Y. We shall say Y is a Poincaré complex if and only if (Y, ϕ) is a Poincaré pair.

For any Poincaré pair (Y, X) of formal dimension n, if k is a sufficiently large integer, there exists a fibration ν^k over Y, called the Spivak fibration of (Y, X), such that its fiber is S^{k-1} and there is a degree one map $\psi : S^{n+k} \to T(\nu)/T(\nu|_X)$. Here $T(\xi)$ means the Thom space of ξ for any spherical fibration ξ, and ψ is a degree one map in the sense it induces an isomorphism between the top dimensional homology groups with integer coefficients. The Spivak fibration of a Poincaré pair is well-defined up to stable fiber homotopy equivalence: any two spherical fibrations with degree one map over a Poincaré pair are stably fiber homotopy equivalent to each other by a stable fiber homotopy equivalence preserving the degree one maps ([13] and also [15]).

Now we shall introduce the notion of Poincaré embedding, essentially following Wall's definition in [14], p. 113.

Let M^m, V^{m+q} be Poincaré complexes and $\iota : M \to V$ be a continuous map. Then the realization of ι as a Poincaré embedding shall consist of:

a $(q-1)$-spherical fibration ν_ι with projection $p : E \to M$,

a Poincaré pair (C, E), and

a homotopy equivalence $h : C \cup M(p) \to V$, where $M(p)$ is the mapping cylinder of p and $C \cap M(p) = E$, so that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & V \\ s\downarrow & & h\uparrow \\ M(p) & \hookrightarrow & C \cup M(p) \end{array}$$

where s is the natural inclusion.

We shall call ν_ι in the above a normal fibration of ι.

From now on, for any Poincaré pair (Y, X), ν_Y will denote the stable Spivak fibration of (Y, X). The following statement is due to C.T.C. Wall ([14], p. 115).

Lemma 8. *Let M and V be Poincaré complexes. Assume $\iota : M \to V$ is a continuous map which admits a realization as a Poincaré embedding with normal fibration ν_ι. Then,*

$$\nu_M = \nu_\iota + \iota^* \nu_V \ .$$

Remark. Lemma 8 implies that the normal fibration of a continuous map is unique up to stable fiber homotopy equivalence.

Consider a fibration $F^m \xrightarrow{i} Y^{m+n} \xrightarrow{f} X^n$, in which F, Y, X are connected Poincaré complexes of formal dimensions $m, m+n, n$, respectively. Let $C \cup M(p) = X$ be a realization of the inclusion $\iota : M^{n-k} \hookrightarrow X^n$ as a Poincaré embedding with a normal fibration ν_ι whose projection is $p : E \to M$. Here we identify M with a subspace of $M(p)$ by the natural inclusion $s : M \hookrightarrow M(p)$ as in the above. Note that $f^{-1}M$ is a Poincaré complex by a theorem of Gottlieb ([6]) (even if Gottlieb proves the theorem for the fibration sequence of spaces which are homotopic to finite complexes, his proof can be extended to general Poincaré complexes, exploiting a result of Mather ([7]) that X *is a finitely dominated complex if and only if* $X \times S^1$ *is homotopic to a finite complex* together with a theorem of Wall ([15]) that $A \times B$ *is a Poincaré complex if and only if A and B are*).

Let $\hat{p}$ be the composite $f^{-1}E \hookrightarrow f^{-1}M(p) \simeq f^{-1}M$ and consider the following diagram which commutes up to homotopy:

$$\begin{array}{ccc} f^{-1}E & \xrightarrow{f''} & E \\ \hat{p}\downarrow & & \downarrow p \\ f^{-1}M & \xrightarrow{f'} & M \end{array}$$

where f', f'' are f restricted at the corresponding subspaces. Since the homotopy fibers of f' and f'' are both F, the above diagram is a homotopy pull-back square. Therefore, $\hat{p}$ has the same homotopy fiber as p, that is, S^{k-1}. Thus we conclude that $\hat{p}$ is the projection of the spherical fibration $f'^* \nu_\iota$ up to homotopy. We also assume $(f^{-1}C, f^{-1}E)$ is a Poincaré pair (which is redundant if $k \geq 3$ by Wall [15], Lemma 11.1, but, even when $k \leq 2$, we still conjecture the assumption is not necessary, presuming a relative version of the theorem by Gottlieb mentioned above holds). These data give a realization of the inclusion $\bar{\iota} : f^{-1}M \hookrightarrow Y$ as a Poincaré embedding with normal fibration $f'^* \nu_\iota$. And it follows that the Spivak fibration of $f^{-1}M$ is $f'^* \nu_\iota + \bar{\iota}^* \nu_Y$ by Lemma 8. In particular, we have:

Theorem 9. *Let $F^m \xrightarrow{\iota} Y^{n+m} \xrightarrow{f} X^n$ be a fibration sequence in which F, Y, X are connected Poincaré complexes of formal dimension m, $n+m$, n, respectively. Furthermore, assume $X = D^n \cup K$, $D^n \cap K = S^{n-1}$, and (K, S^{n-1}), $(f^{-1}K, f^{-1}S^{n-1})$*

are Poincaré pairs. Then

$$\nu_F = \iota^* \nu_Y \ .$$

Remark. If $n \geq 3$, the conditions of Theorem 9 to ensure that the inclusions of a point and the fiber into X and Y respectively can be realized as Poincaré embeddings in a consistent way are redundant by results of Wall ([14], Theorem 2.4. and [15], Lemma 11.1).

Theorem 10 is a consequence of a theorem by Gottlieb ([5]) if the projection is a smooth map. However, his technique can be applied directly to a corresponding homotopy theoretic situation. In the rest of this paper, the homology groups are the usual ones with a commutative ring with identity as their coefficients. Furthermore, $[A]$ ($\overline{[A]}$) will denote a generator of the top dimensional homology (cohomology) for any space A, and we define $[A, B]$ ($\overline{[A, B]}$) similarly for any pair (A, B).

Theorem 10 (D. Gottlieb)**.** *Let $F^m \xrightarrow{\iota} Y^{n+m} \xrightarrow{f} X^n$ be a fibration sequence in which F, Y, X are connected Poincaré complexes of formal dimension m, $n+m$, n, respectively. Assume $X = D^n \cup K$, where $D^n \cap K = S^{n-1}$ and (K, S^{n-1}), $(f^{-1}K, f^{-1}S^{n-1})$ are Poincaré pairs. Then*

$$\iota_*[F] = f^*\overline{[X]} \cap [Y] \ .$$

Proof. We introduce the following two commutative diagrams:

$$\begin{array}{ccc} & & Y \\ & \iota \nearrow & \\ F & & \varphi \uparrow \\ & \iota_1 \searrow & \\ & & D^n \times F \end{array} \quad , \qquad \begin{array}{ccc} Y & \xrightarrow{f} & X \\ \bar{j} \downarrow & & j \downarrow \\ (Y, f^{-1}(K)) & \xrightarrow{f'} & (X, K) \\ \bar{k} \uparrow & & k \uparrow \\ (D^n, S^{n-1}) \times F & \xrightarrow{p_1} & (D^n, S^{n-1}) \end{array} \ .$$

In the first diagram, φ is a fiber preserving map which restricts to a homotopy equivalence at each fiber, and ι, ι_1 are the inclusions defined by a base point x_0. In the second diagram, f' is the map defined by f, p_1 is the projection, and all the vertical maps are inclusions except for $\bar{k}$ which is defined by φ. Note that the vertical maps in the second diagram induce isomorphisms between the top dimensional homology groups. In particular, to prove j and $\bar{j}$ are degree one maps, we exploit the assumption that (K, S^{n-1}) and $(f^{-1}K, f^{-1}S^{n-1})$ are Poincaré pairs, respectively.

In the calculations to follow, we shall repeatedly use the following, which we call *rule** ([12], p. 254): *Let $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$, and let $g : X \to Y$ map A_i to B_i, $i = 1, 2$. Let $g_i : (X, A_i) \to (Y, B_i)$, $i = 1, 2$, and $\bar{g} : (X, A_1 \cup A_2) \to (Y, B_1 \cup B_2)$ be the maps defined by g. Then, for any $u \in H^q(Y, B_1)$ and $z \in H_n(X, A_1 \cup A_2)$, we have*

$$g_{2*}(g_1^* u \cap z) = u \cap \bar{g}_* z \in H_{n-q}(Y, B_2).$$

First of all, we observe that

$$f^*\overline{[X]} = \bar{j}^* f'^* \overline{[X, K]} \in H^{m+n}(Y).$$

Therefore, we have

$$\begin{aligned} f^*\overline{[X]} \cap [Y] &= \bar{j}^* f'^* \overline{[X, K]} \cap [Y] \\ &= f'^* \overline{[X, K]} \cap \bar{j}_*[Y] \\ &= f'^* \overline{[X, K]} \cap [Y, f^{-1}K] \in H_m(Y) \end{aligned}$$

in which we apply rule* above with $g_1 = \bar{g} = \bar{j}$ and $g_2 = 1_Y$.

Furthermore, noting $\bar{k}$ is defined by φ, we have

$$\begin{aligned}
&\varphi_*(p_1^*\overline{[D^n, S^{n-1}]} \cap ([D^n, S^{n-1}] \times [F])) \\
&\quad = \varphi_*(\bar{k}^* f'^* \overline{[X, K]} \cap ([D^n, S^{n-1}] \times [F])) \\
&\quad = f'^* \overline{[X, K]} \cap \bar{k}_*(([D^n, S^{n-1}] \times [F]) \\
&\quad = f'^* \overline{[X, K]} \cap [Y, f^{-1}K] \in H_m(Y),
\end{aligned}$$

in which, for the second equality, again we use rule*.

On the other hand,

$$\varphi_*((p_1^*\overline{[D^n, S^{n-1}]}) \cap ([D^n, S^{n-1}] \times [F])) = \varphi_*(\iota_{1*}[F]) = \iota_*[F]$$

To summarize,

$$\begin{aligned}
i_*[F] &= \varphi_*(p_1^*\overline{[D^n, S^{n-1}]} \cap ([D^n, S^{n-1}] \times [F])) \\
&= f'^* \overline{[X, K]} \cap [Y, f^{-1}K] \\
&= f^* \overline{[X]} \cap [Y] \in H_m(Y),
\end{aligned}$$

which proves the theorem. □

We will need the following ([10], Lemma 2).

Lemma 11 (V. Puppe)**.** *Consider the following diagram which commutes up to homotopy:*

$$\begin{array}{ccccc}
Y_1 & \longleftarrow & Y_0 & \longrightarrow & Y_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \longleftarrow & X_0 & \longrightarrow & X_2.
\end{array}$$

Assume that the maps at the top row of the above diagram induce homotopy equivalences between the homotopy fibers. Then, the homotopy fiber of the map from hocolim$(Y_1 \leftarrow Y_0 \rightarrow Y_2)$ *into hocolim*$(X_1 \leftarrow X_0 \rightarrow X_2)$ *defined by the above diagram is the same homotopy type as those of* $Y_i \rightarrow X_i, i = 0, 1, 2$*, and the natural maps from* Y_i*'s to hocolim*$(Y_1 \leftarrow Y_0 \rightarrow Y_2)$ *induce the homotopy equivalences between the homotopy fibers.*

Definition. Given a spherical fibration ξ over a complex X, let $\bar{\xi} : X \rightarrow BG$ be the continuous map classifying ξ up to stable fiber homotopy equivalence. If R is a commutative ring with identity, for any $\alpha \in H^i(BG; R)$, we shall call $\bar{\xi}^*\alpha \in H^i(X; R)$ the (α-)*characteristic class* coming from ξ.

In particular, if $i = n$ and X is a Poincaré complex of formal dimension n, oriented with respect to the coefficient R, that is, $H_n(X; R)$ is isomorphic to R with a preferred generator $[X] \in H_n(X; R)$, $\bar{\xi}^*\alpha$ will be called the (α-)*characteristic number*, identifying it with $\langle \bar{\xi}^*\alpha, [X] \rangle \in R$.

If X is a Poincaré complex of formal dimension n, oriented with respect to the coefficient R and ν, the Spivak normal fibration over X, then ν gives rise to the characteristic numbers as in the above. Note that, by the naturality of the cup product and the Steenrod squares, the equality $v(X) \cup U = \wp^{-1}U$ in the last paragraph of section 2 implies that the classes $v_k(X) \in H^k(X; Z_p)$ and, therefore, the $q_k(X)$'s are the characteristic classes coming from the Spivak normal fibration. Therefore, Theorem 3 implies vanishing of all the mod p Wu numbers.

Proof of Theorem 3. Let z_0, x_0 be the base points of S^k, M, respectively. We choose a map $F : S^k \times M \to M$ such that $F(z_0, \cdot) : M \to M$ is the identity and $F(\cdot, x_0) : S^k \to M$ represents α. We shall denote the $(k+1)$-dimensional upper and lower hemispheres by D_+^{k+1}, D_-^{k+1}, respectively. Then we define $F' : S^k \times M \to S^k \times M$ by the rule $F'(z, x) = (z, F(z, x))$ and, subsequently, $\overline{F}$ as the composite $S^k \times M \xrightarrow{F'} S^k \times M \hookrightarrow D_+^{k+1} \times M$.

The following diagram commutes:

$$\begin{array}{ccccc} D_+^{k+1} \times M & \xleftarrow{\bar{F}} & S^k \times M & \xhookrightarrow{i} & D_-^{r+1} \times M \\ p_1 \downarrow & & p_0 \downarrow & & p_2 \downarrow \\ D_+^{k+1} & \hookleftarrow & S^k & \hookrightarrow & D_-^{k+1} \end{array}$$

in which $p_i, i = 0, 1, 2$, are the projections and the hooked arrows are inclusions. Let E be the push-out of the top row and $p : E \to S^{k+1}$ be the push-out of $p_1 \leftarrow p_0 \hookrightarrow p_2$. Note that E is in fact the homotopy push-out, for the inclusions on the right hand side are cofibrations.

The vertical arrows are bundle projections with fiber M and the horizontal arrows in the top row preserve the fibers. In particular, $\bar{F}$ restricts to a homotopy equivalence at each fiber. Therefore, by Lemma 11, we obtain a fibration sequence

$$M \xrightarrow{\iota} E \xrightarrow{p} S^{k+1}$$

up to homotopy. Since the fiber and the base space are Poincaré complexes, E is also a Poincaré complex by the same argument we used in the discussion to derive Theorem 9.

Clearly, the assumptions of Theorem 9 are satisfied for the fibration $M \xrightarrow{\iota} E \xrightarrow{p} S^{k+1}$. Thus the Spivak fibration of M is the pull-back of the Spivak fibration of E by ι, and therefore

$$\bar{\nu}_M = \bar{\nu}_E\ \iota,$$

where $\bar{\nu}_M, \bar{\nu}_E$ are the classifying maps of the Spivak fibrations of M, E, respectively, into BG.

Therefore, to complete the proof of Theorem 3, it is enough to show:

Claim. With a field of coefficients R, $\iota^* : H^n(E) \to H^n(M)$ is the null homomorphism.

Proof. We may assume $M \xrightarrow{\iota} E \xrightarrow{p} S^{k+1}$ is already a fibration.

Consider the following portion of the exact sequence (see [12], sec. 8.5):

$$H_{k+1}(M) \xrightarrow{\iota_*} H_{k+1}(E) \xrightarrow{j} H_{k+1}((D_-^{k+1}, S^k) \times M) \xrightarrow{F_*\partial} H_k(M)\ ,$$

which we shall call *exact sequence**. Note that $\partial : H_{k+1}((D_-^{k+1}, S^k) \times M) \to H_k(S^k \times M)$ is an isomorphism. Furthermore, consider the commutative diagram

$$\begin{array}{ccc} S^k \times M & \xrightarrow{F} & M \\ \uparrow & \nearrow \overline{\alpha} & \\ S^k \times \{x_0\} & & \end{array} ,$$

where $\overline{\alpha} : S^k = S^k \times \{x_0\} \to M$ is the map defined by $\overline{\alpha}(z) = F(z, x_0)$, which represents α. Thus, the image by $F_*\partial : H_{k+1}((D_+^{k+1}, S^k) \times M) \to H_k(M)$ cannot be trivial since it must contain $h(\alpha) \neq 0$.

However, $H_{k+1}((D_-^{k+1}, S^k) \times M)$ is a vector space with one generator over R. Therefore, $F_*\partial$ is a monomorphism and j in exact sequence* is the zero homomorphism.

Now we introduce the following commutative diagram:

$$\begin{array}{ccccc} H_{k+1}(E) & \stackrel{\bar{q}}{\to} & H_{k+1}(E, D_+^{k+1} \times M) & \stackrel{\overline{exc}}{\leftarrow} & H_{k+1}((D_-^{k+1}, S^k) \times M) \\ p_* \downarrow & & p'_* \downarrow & & p_{1*} \downarrow \\ H_{k+1}(S^{k+1}) & \stackrel{q}{\to} & H_{k+1}(S^{k+1}, D_+^{k+1}) & \stackrel{exc}{\leftarrow} & H_{k+1}(D_-^{k+1}, S^k), \end{array}$$

where all the horizontal arrows are induced by inclusions, and the vertical arrows by restrictions of the projection $p : E \to S^{k+1}$. By inspection, we know that all the arrows are isomorphisms except for, possibly, $\bar{q}$ and p_*. Note that the homomorphism j in exact sequence* is none other than $\overline{exc}^{-1}\bar{q}$. Therefore, $p_* = q^{-1} exc\, p_{1*} j = 0$. By a universal coefficient theorem, $p^* : H^{k+1}(S^{k+1}) \to H^{k+1}(E)$ is also the the null homomorphism. However, $\iota_*[M] = p^*[S^{k+1}] \cap [E]$ by Theorem 10. Therefore, $\iota_* = 0$ on $H_n(M)$ and it follows that $\iota^* = 0$ on $H^n(E)$, which completes the proof of the claim. □

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