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PRESENTATION AND CENTRAL EXTENSIONS OF MAPPING CLASS GROUPS

SYLVAIN GERVAIS

Abstract. We give a presentation of the mapping class group $\mathcal{M}$ of a (possibly bounded) surface, considering either all twists or just non-separating twists as generators. We also study certain central extensions of $\mathcal{M}$. One of them plays a key role in studying TQFT functors, namely the mapping class group of a p_1-structure surface. We give a presentation of this extension.

Introduction

All surfaces that we shall consider are connected and oriented. We shall denote by $\Sigma_{g,r}$ a surface with genus g and r boundary components, or simply Σ when there is no ambiguity. We shall denote its mapping class group by $\mathcal{M}_{g,r}$ (or $\mathcal{M}_\Sigma$). Recall that $\mathcal{M}_\Sigma$ is the group of diffeomorphisms of Σ which leave fixed its boundary, modulo those which are isotopic to the identity. The aim of this article is to find a presentation of $\mathcal{M}_{g,r}$ with very simple geometrical relations and to describe more precisely this presentation as central extensions.

These groups are generated by Dehn twists ([D]). Lickorish ([L1, L2]) proved that $3g-1$ of these twists generate $\mathcal{M}_{g,0}$. Humphries ([Hu]) reduced this number to $2g+1$ and showed that it was minimal.

Using the presentation of $\mathcal{M}_{g,0}$ found by Hatcher and Thurston in [H-T], Wajnryb gave in [W] an explicit presentation of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,0}$ with Humphries generators. We shall also use their result to obtain a presentation of $\mathcal{M}_{g,r}$ with the set of all Dehn twists for generators.

Four types of relations will appear in this presentation: "braids", "lanterns", "stars" and "chains".

The braids. It is well known that if α is a simple closed curve (s.c.c.) in Σ and h is a diffeomorphism of Σ, then $\tau_{h(\alpha)} = h\tau_\alpha h^{-1}$ (where τ_α represents the twist along α). More specifically, if h is a twist τ_β, one has the relation (called a braid relation)

$$(T) \qquad \tau_\gamma = \tau_\beta \tau_\alpha \tau_\beta^{-1},$$

where $\gamma = \tau_\beta(\alpha)$. We will denote this relation by T_n when $|\alpha \cap \beta| = n$ and by T_{2_0} when $|\alpha \cap \beta| = 2$ and the algebraic intersection is zero (denoted $|\alpha \cap \beta| = 2_0$).

Remark. If G is a multiplicative group, $\overline{x}$ will denote x^{-1} and $y\,(x)$ will denote the conjugate of x by y (i.e. $y\,x\,\overline{y}$).

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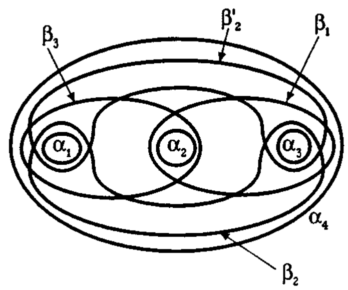

FIGURE 1

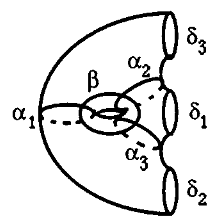

FIGURE 2

The lanterns. These relations were discovered by Johnson ([J]). Let us consider a subsurface of $\Sigma_{g,r}$ which is homeomorphic to a sphere with four holes. Let α_1, α_2, α_3 and α_4 be the boundary components and β_1, β_2 and β_3 curves as shown in Figure 1. The relation (called a lantern relation) is:

$$(L) \qquad \tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4} = \tau_{\beta_3}\tau_{\beta_2}\tau_{\beta_1}.$$

Remark. By a homology calculation, one can check that if two curves α and β in a surface Σ are such that $|\alpha \cap \beta| = 2_0$, then a neighborhood of $\alpha \cup \beta$ in Σ is homeomorphic to a sphere with four holes. So, a lantern (L) is completely determined by the curves β_1 and β_3. The curves α_i are the four boundary components of a neighborhood of $\beta_1 \cup \beta_3$. There are two choices for β_2, but only one gives the equality (L) (the other choice, the curve β_2' shown in Figure 1, gives the equality $\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4} = \tau_{\beta_1}\tau_{\beta_2'}\tau_{\beta_3}$).

The stars. We consider a subsurface of $\Sigma_{g,r}$ which is homeomorphic to a surface of genus one with three boundary components. The relation is

$$(E) \qquad (\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\beta})^3 = \tau_{\delta_1}\tau_{\delta_2}\tau_{\delta_3},$$

where the curves are described in Figure 2.

Remark. The star (E) is uniquely determined by four curves α_1, α_2, α_3 and β such that $|\alpha_i \cap \alpha_j| = 0$ if $i \neq j$ and $|\beta \cap \alpha_i| = 1$ for all $i \in \{1,2,3\}$ (a neighborhood of these curves in Σ is homeomorphic to $\Sigma_{1,3}$).

The chains. These are just a particular case of the stars: we suppose that the curve δ_1 bounds a disc in Σ. Then, we have $\tau_{\delta_1} = 1$, $\tau_{\alpha_2} = \tau_{\alpha_3}$, and the relation (E) becomes:

$$(C) \qquad (\tau_{\alpha_1}\tau_{\beta}\tau_{\alpha_2})^4 = \tau_{\delta_2}\tau_{\delta_3}.$$

Remark. Using relations T_0 and T_1, it is easy to show that $(\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_2}\tau_\beta)^3 = (\tau_{\alpha_1}\tau_\beta\tau_{\alpha_2})^4$. This kind of calculation will be often done in this article, and the reader can refer to them. Note that a chain is determined by the curves α_1, β and α_2, with $|\alpha_i \cap \beta| = 1$ and $\alpha_1 \cap \alpha_2 = \emptyset$.

Theorem A. *For any oriented surface Σ of genus $g \geq 0$, with r boundary components $(r \geq 0)$, $\mathcal{M}_\Sigma$ has the following presentation:*
generators: $\{\tau_\alpha \,/\, \alpha$ *s.c.c. in* $\Sigma\}$ (*if α bounds a disc in Σ, then $\tau_\alpha = 1$*).
relations: (I) *All the braids T_0, T_1 and T_{2_0}.*
(II) *All the chains.*
(III) *All the lanterns.*
If $g \geq 1$, one can replace (II) *and* (III) *by* (IV)*:*
(IV) *All the stars.*

Remark. If $g = 0$, only the relations T_0, T_{2_0} and (III) remain.

Harer proved in [H] that when $g \geq 3$, $\mathcal{M}_{g,r}$ is generated by the twists along the non-separating curves for any r. We shall see that this result is still true when $g = 2$, but not when $g = 1$. We shall also give a presentation of $\mathcal{M}_{g,r}$ $(g \geq 2)$ with these generators.

Notation. If the curves involved in a lantern L (resp. a star E or a chain C) are all non-separating curves, we will denote L by L_0 (resp. E by E_0 and C by C_0).

Theorem B. *For any orientable surface Σ of genus $g \geq 2$, with r boundary components $(r \geq 0)$, $\mathcal{M}_\Sigma$ has the following presentation:*
generators: $\{\tau_\alpha \,/\, \alpha$ *non-separating s.c.c. in* $\Sigma\}$.
relations: (I)' *All the braids T_0 and T_1 (with non-separating curves).*
(II)' *One chain C_0.*
(III)' *One lantern L_0.*
When $g \geq 3$, one can replace (II)' *or* (III)' *by a star E_0.*

Remark. In the case of genus two, there are only the relations (I)' and (II)' (the lanterns L_0 and the stars E_0 exist only if $g \geq 3$).

Let $S_{g,r}$ be the group generated by $\{a_\alpha \,/\, \alpha$ non-separating s.c.c. in $\Sigma\}$ and defined by the relations T_0 and T_1 between these curves. Theorem B shows that $\mathcal{M}_{g,r}$ is a quotient of $S_{g,r}$ when $g \geq 2$. We shall prove the following more precise result.

Theorem C. *For any orientable surface Σ, there exists a central extension*

$$(*) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_\Sigma \longrightarrow \mathcal{M}_\Sigma \longrightarrow 1$$

satisfying the following properties:

1) *$\Sigma \mapsto \mathcal{M}_\Sigma$, $\Sigma \mapsto S_\Sigma$ and $\Sigma \mapsto \mathcal{T}_\Sigma$ are three functors from the category of surfaces (where morphisms are embeddings) to the category of groups, and the extension $(*)$ is natural. When $g \geq 3$ (g denotes the genus of Σ), this extension is the universal central extension of $\mathcal{M}_\Sigma$.*
Furthermore, there exist two natural morphisms $\varphi : S_\Sigma \to \mathcal{T}_\Sigma$ and $\psi : S_\Sigma \to \mathbb{Z}$ such that:
2) *if $g \geq 3$, φ and ψ induce an isomorphism from S_Σ to $\mathcal{T}_\Sigma \times \mathbb{Z}$.*
3) *if $g = 2$, φ is an isomorphism from S_Σ to $\mathcal{T}_\Sigma$.*

1. Preliminary results

In this section, Σ is a surface of genus $g \geq 0$ with r boundary components ($r \geq 0$).

Let R_Σ (resp. S_Σ) be the group generated by $\{a_\alpha \,/\, \alpha$ a s.c.c. in $\Sigma\}$ (resp. $\{a_\alpha \,/\, \alpha$ a non-separating s.c.c. in $\Sigma\}$) and defined by the relations T_0 , T_1 and T_{2_0} (resp. the relations T_0 and T_1 between non-separating curves). The goal of this section will be to prove that all the braid relations are satisfied in R and S and to find an equality between the lanterns, the stars and the chains.

We start by proving two lemmas that will be used often.

Lemma 1.1. *(i) Let α and β be two disjoint s.c.c. (resp. non-separating s.c.c.) in Σ. Then, a_α commutes with a_β in R (resp. S).*

(ii) If α and β are two s.c.c. in Σ such that $|\alpha \cap \beta| = 1$, then $a_\alpha \, a_\beta \, a_\alpha = a_\beta \, a_\alpha \, a_\beta$ in R and S.

Notation. In the following, we will refer to these relations respectively by (0) and (1).

Proof. (i) is exactly the relation T_0 because $\tau_\beta(\alpha) = \alpha$. (ii) is a consequence of T_1 and the fact that $\tau_\beta(\alpha) = \tau_\alpha^{-1}(\beta)$. □

Lemma 1.2. *Let $(\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star in Σ and set*

$$X_1 = a_{\alpha_1} a_{\alpha_2}, \qquad X_2 = a_\beta \, X_1 \, a_\beta, \qquad X_3 = a_{\alpha_3} X_2 \, a_{\alpha_3}.$$

Then, one has in R and S:

(i) $X_i \, X_j = X_j \, X_i$,
(ii) $(a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta)^3 = X_1 \, X_2 \, X_3$,
(iii) $(a_{\alpha_1} a_\beta \, a_{\alpha_2})^4 = X_1^2 \, X_2^2$.

Proof. (i) Using (1), we obtain

$$\text{(2)} \quad a_{\alpha_1} X_2 = a_{\alpha_1} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta = a_\beta \, a_{\alpha_1} a_\beta \, a_{\alpha_2} a_\beta = a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_2} = X_2 \, a_{\alpha_2}$$

and in the same way, $a_{\alpha_2} X_2 = X_2 \, a_{\alpha_1}$. From this, we conclude that $X_1 \, X_2 = X_2 \, X_1$ and, since a_{α_3} commutes with a_{α_1} and a_{α_2} (by (0)), $X_1 \, X_3 = X_3 \, X_1$.

One has by (0) and (1)

$$\begin{aligned} a_\beta \, X_3 &= a_\beta \, a_{\alpha_3} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} = a_{\alpha_3} a_\beta \, a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} \\ &= a_{\alpha_3} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_\beta = X_3 \, a_\beta \end{aligned}$$

so X_2 commutes with X_3.

(ii)

$$\begin{aligned} X_1 \, X_2 \, X_3 &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta \, a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_{\alpha_3} a_\beta \, a_{\alpha_3} \quad \text{by (0)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_{\alpha_1} a_\beta \, a_{\alpha_2} a_\beta \, a_{\alpha_1} a_\beta \, a_{\alpha_3} a_\beta \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_{\alpha_1} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_\beta \, a_{\alpha_1} a_{\alpha_3} a_\beta \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_3} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_{\alpha_3} a_\beta \quad \text{by (2)} \\ &= (a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta)^3 \quad \text{by (0) and (1)}. \end{aligned}$$

(iii)

$$\begin{aligned} X_1^2 \, X_2^2 &= a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \quad \text{by } (i) \\ &= a_{\alpha_1} a_\beta \, a_{\alpha_2} a_\beta \, a_{\alpha_1} a_\beta \, a_{\alpha_2} a_\beta \, a_{\alpha_1} a_\beta \, a_{\alpha_2} a_\beta \quad \text{by (0) and (1)} \\ &= a_{\alpha_1} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_\beta \, a_{\alpha_2} a_{\alpha_1} a_\beta \, a_{\alpha_1} a_{\alpha_2} a_\beta \, a_{\alpha_2} \quad \text{by (0) and (1)} \\ &= (a_{\alpha_1} a_\beta \, a_{\alpha_2})^4 \quad \text{by (0)}. \end{aligned}$$

□

1.1. **The braids.**

Theorem 1.3. *For any simple closed curves α and β in Σ, one has in R_Σ $a_\gamma = a_\beta\, a_\alpha\, \overline{a_\beta}$ where $\gamma=\tau_\beta(\alpha)$.*

If α and β do not separate, this relation is also true in S_Σ.

Corollary 1.4. *Let h be a diffeomorphism of Σ and suppose that it is decomposed into a product of Dehn twists; denote by m the corresponding word in R_Σ. Let α be a s.c.c. in Σ and $\gamma=h(\alpha)$. Then, in R_Σ one has the relation $a_\gamma = m\, a_\alpha\, \overline{m}$.*

Furthermore, if the twists involved in the decomposition of h are all twists along non-separating curves and if α does not separate, then the relation is also true in S_Σ.

Proof. Write $h=\tau_{\beta_n}^{\varepsilon_n}\dots\tau_{\beta_1}^{\varepsilon_1}$ ($\varepsilon_i=\pm1$); then, $m=a_{\beta_n}^{\varepsilon_n}\dots a_{\beta_1}^{\varepsilon_1}$ in R (or S).

Denote, for $k\in\{0,\dots,n\}$, $\gamma_k=\tau_{\beta_k}^{\varepsilon_k}\dots\tau_{\beta_1}^{\varepsilon_1}(\alpha)$ ($\gamma_0=\alpha$ and $\gamma_n=\gamma$). Then, Theorem 1.3 implies

$$a_{\gamma_k} = a_{\beta_k}^{\varepsilon_k}\, a_{\gamma_{k-1}}\, a_{\beta_k}^{-\varepsilon_k}\,,$$

thus

$$a_\gamma = a_{\gamma_n} = a_{\beta_n}^{\varepsilon_n}\, a_{\gamma_{n-1}}\, a_{\beta_n}^{-\varepsilon_n} = \dots = m\, a_\alpha\, \overline{m}\,.$$

□

Before we prove Theorem 1.3, let us show that the relations T_{2_0} are true in S:

Lemma 1.5. *Let α and β be two non-separating curves in Σ such that $|\alpha\cap\beta|=2_0$. Then, if $\gamma=\tau_\beta(\alpha)$, one has in S*

$$a_\gamma = a_\beta\, a_\alpha\, \overline{a_\beta}\,.$$

Proof. A neighborhood of $\alpha\cup\beta$ in Σ is a lantern where α and β are respectively identified with the curves β_1 and β_2 of the Figure 1. Since these curves do not separate, there exists a curve δ such that

$$|\delta\cap\alpha|=|\delta\cap\beta|=|\delta\cap\alpha_1|=|\delta\cap\alpha_2|=1\,.$$

(We might have to replace $\{\alpha_1,\alpha_2\}$ by $\{\alpha_3,\alpha_4\}$.)

It can be seen that

$$\tau_\delta\tau_{\alpha_1}\tau_{\alpha_2}\tau_\delta(\beta)=\alpha \quad\text{and}\quad \tau_\delta^{-1}\tau_{\alpha_1}^{-1}\tau_{\alpha_2}^{-1}\tau_\delta^{-1}(\beta)=\gamma.$$

And since $|\delta\cap\beta|=|\alpha_2\cap\tau_\delta(\beta)|=\dots=|\delta\cap\tau_{\alpha_1}^{-1}\tau_{\alpha_2}^{-1}\tau_\delta^{-1}(\beta)|=1$, the relation T_1 implies

$$X\, a_\beta\, \overline{X} = a_\alpha \quad\text{and}\quad \overline{X}\, a_\beta\, X = a_\gamma, \quad\text{where}\quad X = a_\delta\, a_{\alpha_1}\, a_{\alpha_2}\, a_\delta$$

and so $a_\beta\, a_\alpha\, \overline{a_\beta} = a_\beta\, X\, a_\beta\, \overline{X}\, \overline{a_\beta}$.

Since $(\alpha_1,\alpha_2,\beta,\delta)$ defines a star in Σ, Lemma 1.2 shows that X commutes with $a_\beta\, X\, a_\beta$. Thus, we obtain:

$$a_\beta\, a_\alpha\, \overline{a_\beta} = \overline{X}\, a_\beta\, X\, a_\beta\, \overline{a_\beta} = a_\gamma\,. \quad \square$$

Proof of Theorem 1.3. Suppose inductively that the braid relations T_k are satisfied in R and S for $k\le n-1$, let α and β be two s.c.c. in Σ such that $|\alpha\cap\beta|=n$, and set $\gamma=\tau_\beta(\alpha)$ (for S, we suppose also that α and β do not separate). Let us fix an orientation on α and β.

First case. There exist two points A and B in $\alpha\cap\beta$ which are consecutive on α and such that the signs of the intersection of α and β at these points are the same.

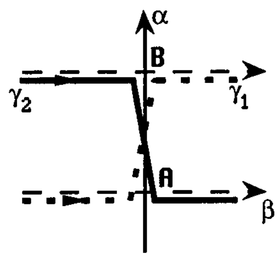

Figure 3

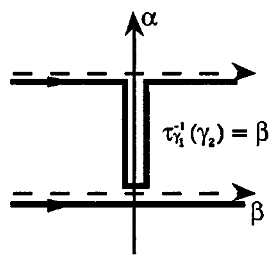

Figure 4

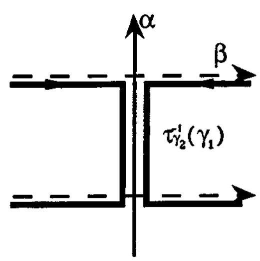

Figure 5

Let us consider the curves γ_1 and γ_2 shown on Figure 3. As shown on Figure 4, one has $\tau_{\gamma_1}^{-1}(\gamma_2)=\beta$ and so, using T_1 and (1),

$$a_\beta=\overline{a_{\gamma_1}}\,a_{\gamma_2}\,a_{\gamma_1}=\overline{a_{\gamma_1}}\,\overline{a_{\gamma_1}}\,a_{\gamma_1}\,a_{\gamma_2}\,a_{\gamma_1}=\overline{a_{\gamma_1}}\,\overline{a_{\gamma_1}}\,a_{\gamma_2}\,a_{\gamma_1}\,a_{\gamma_2}. \tag{3}$$

Let us denote by $\delta_1,\ \delta_2,\ \delta_3$ and δ_4 the curves

$$\delta_1=\tau_{\gamma_2}(\alpha)\ ,\quad \delta_2=\tau_{\gamma_1}(\delta_1)\ ,\quad \delta_3=\tau_{\gamma_2}(\delta_2)\ ,\quad \delta_4=\tau_{\gamma_1}^{-1}(\delta_3)\,.$$

Then, we have $\gamma=\tau_{\gamma_1}^{-1}(\delta_4)$. We will successively show that $|\gamma_2\cap\alpha|$, $|\gamma_1\cap\delta_1|$, $|\gamma_2\cap\delta_2|$, $|\gamma_1\cap\delta_3|$ and $|\gamma_1\cap\delta_4|$ are smaller than $n-1$. Then, by the induction hypothesis we obtain:

$$a_\gamma=\overline{a_{\gamma_1}}(a_{\delta_4})=\overline{a_{\gamma_1}}\,\overline{a_{\gamma_1}}(a_{\delta_3})=\ldots=\overline{a_{\gamma_1}}\,\overline{a_{\gamma_1}}\,a_{\gamma_2}\,a_{\gamma_1}\,a_{\gamma_2}(a_\alpha),$$

that is to say, $a_\gamma=a_\beta\,a_\alpha\,\overline{a_\beta}$ using (3).

- By definition, $|\gamma_2\cap\alpha|<n$.
- $|\gamma_1\cap\delta_1|=|\gamma_1\cap\tau_{\gamma_2}(\alpha)|=|\tau_{\gamma_2}^{-1}(\gamma_1)\cap\alpha|=n-2$ as shown on Figure 5.
- $|\gamma_2\cap\delta_2|=|\gamma_2\cap\tau_{\gamma_1}\tau_{\gamma_2}(\alpha)|=|\tau_{\gamma_2}^{-1}\tau_{\gamma_1}^{-1}(\gamma_2)\cap\alpha|=|\tau_{\gamma_2}^{-1}\tau_{\gamma_2}(\gamma_1)\cap\alpha|=|\gamma_1\cap\alpha|<n$.

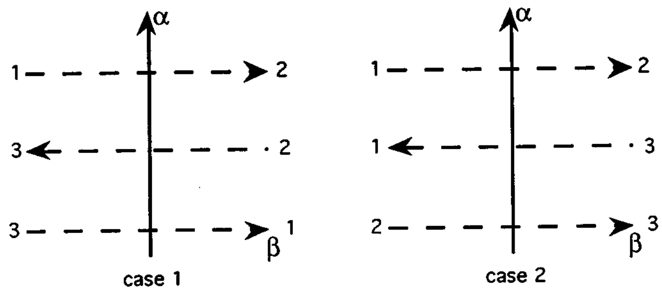

FIGURE 6

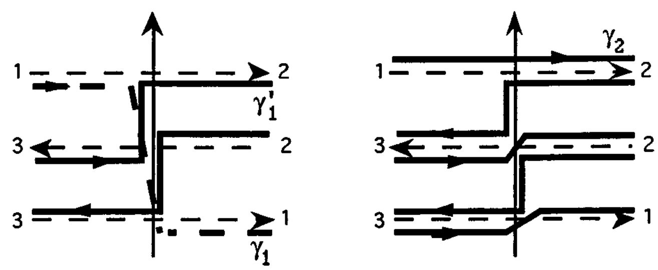

FIGURE 7

- $|\gamma_1 \cap \delta_3| = |\gamma_1 \cap \tau_{\gamma_2}\tau_{\gamma_1}\tau_{\gamma_2}(\alpha)| = |\tau_{\gamma_2}^{-1}\tau_{\gamma_1}^{-1}\tau_{\gamma_2}^{-1}(\gamma_1) \cap \alpha| = |\tau_{\gamma_2}^{-1}\tau_{\gamma_1}^{-1}\tau_{\gamma_1}(\gamma_2) \cap \alpha| = |\gamma_2 \cap \alpha| < n$.
- $|\gamma_1 \cap \delta_4| = |\tau_{\gamma_1}(\gamma_1) \cap \tau_{\gamma_1}(\delta_4)| = |\gamma_1 \cap \delta_3| < n$.

Remark. Note that the curves γ_1 and γ_2 do not separate. So, the proof above remains valid for S.

Second case. There are not two points in $\alpha \cap \beta$ consecutive on α and with the same intersection sign.

If n is equal to two, then T_n is the relation T_{2_0} (wich is true in S by Lemma 1.5); so, suppose that $n \geq 3$. As we can see in Figure 6 there are only two different ways to join the strands in order to obtain the simple closed curve β. Denote by n_k the number of points in $\alpha \cap \beta$ which are on the arc $k-k$ of β (k=1,2 or 3).

Let us consider the first case. If γ_1 and γ_2 are the curves shown in Figure 7, one has $|\gamma_1 \cap \beta| = 2_0$ and $\tau_\beta(\gamma_1) = \gamma_2$. So, the relation T_{2_0} yields

$$a_{\gamma_2} = a_\beta \, a_{\gamma_1} \, \overline{a_\beta}, \qquad \text{i.e.,} \qquad a_\beta = \overline{a_{\gamma_2}} \, a_\beta \, a_{\gamma_1} \, .$$

Now, if $\delta_1 = \tau_{\gamma_1}(\alpha)$ and $\delta_2 = \tau_\beta(\delta_1)$, then $\gamma = \tau_{\gamma_2}^{-1}(\delta_2)$.

Since $|\gamma_1 \cap \alpha| = n_1 + 1 \leq n - 2 < n$, one has by the induction hypothesis that $a_{\delta_1} = a_{\gamma_1}(a_\alpha)$. We have (Figure 8 shows $\tau_{\gamma_1}^{-1}(\beta)$)

$$|\beta \cap \delta_1| = |\beta \cap \tau_{\gamma_1}(\alpha)| = |\tau_{\gamma_1}^{-1}(\beta) \cap \alpha| = n - 2 < n$$

so $a_{\delta_2} = a_\beta(a_{\delta_1}) = a_\beta \, a_{\gamma_1}(a_\alpha)$. Finally,

$$|\gamma_2 \cap \delta_2| = |\gamma_2 \cap \tau_\beta\tau_{\gamma_1}(\alpha)| = |\tau_\beta^{-1}(\gamma_2) \cap \tau_{\gamma_1}(\alpha)| = |\gamma_1 \cap \tau_{\gamma_1}(\alpha)| = |\gamma_1 \cap \alpha| = n_1 + 1 < n$$

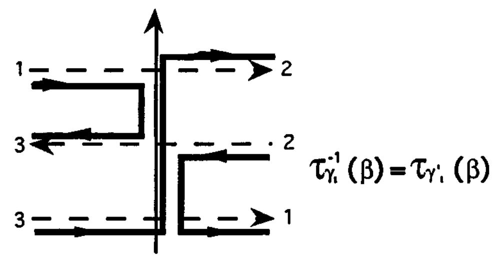

FIGURE 8

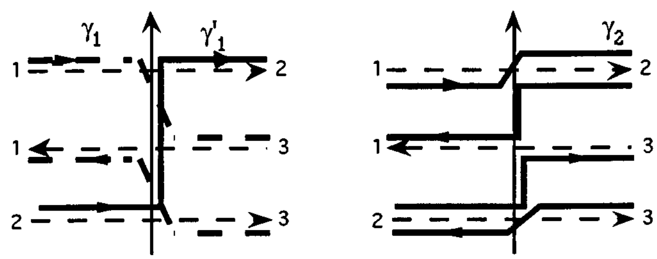

FIGURE 9

and we obtain

$$a_\gamma = \overline{a_{\gamma_2}}(a_{\delta_2}) = \overline{a_{\gamma_2}}\, a_\beta\, a_{\gamma_1}(a_\alpha) = a_\beta(a_\alpha).$$

Note that the curve γ_1 (and also γ_2) may separate Σ, and in this case the proof above is incorrect for S. But, if γ_1' is the curve shown in Figure 7, then $[\gamma_1] + [\gamma_1'] = [\beta]$ (where $[\lambda]$ is the homology class modulo two of the curve λ) and so γ_1' is a non-separating curve if β does not separate and γ_1 does.

Let us set $\gamma_2' = \tau_\beta(\gamma_1')$; then, since $|\beta \cap \gamma_1'| = 2_0$, one has by Lemma 1.5

$$a_{\gamma_2'} = a_\beta\, a_{\gamma_1'}\, \overline{a_\beta}, \qquad \text{and thus} \qquad a_\beta = a_{\gamma_2'}\, a_\beta\, \overline{a_{\gamma_1'}}\,.$$

If δ_1' and δ_2' are respectively the curves $\tau_{\gamma_1'}^{-1}(\alpha)$ and $\tau_\beta(\delta_1')$, then $\gamma = \tau_{\gamma_2'}(\delta_2')$.

- $|\gamma_1' \cap \alpha| = 2 + n_2 + n_3 < n$.
- $|\beta \cap \delta_1'| = |\beta \cap \tau_{\gamma_1'}^{-1}(\alpha)| = |\tau_{\gamma_1'}(\beta) \cap \alpha|$.

Since $\tau_{\gamma_1'}(\beta) = \tau_{\gamma_1}^{-1}(\beta)$ (see Figure 8), one has $|\beta \cap \delta_1'| = n - 2$.

- $|\gamma_2' \cap \delta_2'| = |\gamma_2' \cap \tau_\beta \tau_{\gamma_1'}^{-1}(\alpha)| = |\tau_{\gamma_1'} \tau_\beta^{-1}(\gamma_2') \cap \alpha| = |\gamma_1' \cap \alpha| < n$.

So, by the induction hypothesis,

$$a_\gamma = a_{\gamma_2'}(a_{\delta_2'}) = a_{\gamma_2'}\, a_\beta(a_{\delta_1'}) = a_{\gamma_2'}\, a_\beta\, \overline{a_{\gamma_1'}}(a_\alpha) = a_\beta(a_\alpha)\,.$$

For the second case of the Figure 6, the proof is the same using the curves γ_1, γ_1', and γ_2 of Figure 9 □

1.2. **A relation between the lanterns, the stars and the chains.** For any lantern determined by the curves β_1 and β_3, we denote by $L(\beta_1, \beta_3)$ the element $a_{\alpha_1}\, a_{\alpha_2}\, a_{\alpha_3}\, a_{\alpha_4} \overline{a_{\beta_1}}\, \overline{a_{\beta_2}}\, \overline{a_{\beta_3}}$ of R, and by $\mathcal{L}$ the subgroup of R generated by all these elements. Similarly, we define elements $E(\alpha_1, \alpha_2, \alpha_3, \beta)$ and $C(\alpha_1, \beta, \alpha_2)$ associated to the stars and the chains and the subgroups $\mathcal{E}$ and $\mathcal{C}$. When the curves involved do not separate, we will also consider these elements in S, but we will denote them with a subscript "0".

Theorem 1.6. a) *(i) The subgroups $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{C}$ are in the center of R.*

(ii) If $g \geq 1$, then $\mathcal{E} = \mathcal{C}\mathcal{L}$, where $\mathcal{C}\mathcal{L}$ is the subgroup generated by $\mathcal{C}$ and $\mathcal{L}$.

b) *(i) The elements $L_0(.,.)$, $E_0(.,.,.,.)$ and $C_0(.,.,.)$ are central in S.*

(ii) For any lanterns (β_1, β_3) and (β'_1, β'_3) with non-separating curves, one has in S

$$L_0(\beta_1, \beta_3) = L_0(\beta'_1, \beta'_3).$$

The same result holds for stars and chains. Let us denote by L_0, E_0 and C_0 the three elements of S obtained this way.

(iii) $C_0 = L_0\, E_0$ in S.

Remark. Part b) makes sense only if the elements L_0, E_0 and C_0 exists, that is to say, when $g \geq 2$ for C_0 and when $g \geq 3$ for L_0 and E_0.

Proof. The assertion (i) of a) and b) is an easy consequence of Corollary 1.4, and (ii) of b) is a consequence of (i) and the classification of surfaces. Thus, let us look at the relation $\mathcal{E} = \mathcal{C}\,\mathcal{L}$. To do this, we first consider a lantern $L = L(\beta_1, \beta_3)$ where the curves are denoted as shown in Figure 1.

First case: At least one of the curves α_i does not separate. Then, there is a second curve α_j which does not separate. Without loss of generality, one can suppose that these two curves are α_1 and α_2. Then, if β and δ are the curves shown in Figure 10, one has by Lemma 1.2

$$C = C(\alpha_1, \beta, \alpha_2) = X_1^2\, X_2^2\, \overline{a_\delta}\, \overline{a_{\beta_3}}$$

and

$$E = E(\alpha_1, \beta_1, \alpha_2, \beta) = X_1\, X_2\, X_3\, \overline{a_\delta}\, \overline{a_{\alpha_4}}\, \overline{a_{\alpha_3}}$$

where $X_1 = a_{\alpha_1}\, a_{\alpha_2}$, $X_2 = a_\beta\, X_1\, a_\beta$ and $X_3 = a_{\beta_1}\, X_2\, a_{\beta_1}$. Thus, we obtain by (0), Lemma 1.2 and (i)

$$C\,\overline{E} = a_{\alpha_1}\, a_{\alpha_2}\, a_{\alpha_3}\, a_{\alpha_4}\, \overline{a_{\beta_3}}\, \overline{a_{\beta_1}}\, \overline{X_2}\, \overline{a_{\beta_1}}\, X_2\,.$$

But $(\tau_\beta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\beta)^{-1}(\beta_1) = \beta_2$, so, by Corollary 1.4, $a_{\beta_2} = \overline{X_2}(a_{\beta_1})$ and we obtain $C\,\overline{E} = L$. Thus, since $\mathcal{C} \subset \mathcal{E}$, we have $L \in \mathcal{E}$ (by (i), $L = a_{\alpha_1}\, a_{\alpha_2}\, a_{\alpha_3}\, a_{\alpha_4} \overline{a_{\beta_1}}\, \overline{a_{\beta_2}}\, \overline{a_{\beta_3}}$ $= a_{\alpha_1}\, a_{\alpha_2}\, a_{\alpha_3}\, a_{\alpha_4} \overline{a_{\beta_2}}\, \overline{a_{\beta_3}}\, \overline{a_{\beta_1}} = a_{\alpha_1}\, a_{\alpha_2}\, a_{\alpha_3}\, a_{\alpha_4} \overline{a_{\beta_3}}\, \overline{a_{\beta_1}}\, \overline{a_{\beta_2}}$).

Note that this shows the relation (iii) of b) because if β_3 does not separate, then δ does not separate either.

Second case: All the curves α_i separate in Σ. Since g is greater than or equal to one, there exists i such that the component of $\Sigma \backslash \alpha_i$ which does not contain α_j (for all $j \neq i$) is of genus greater than or equal to one. We can suppose, without loss of generality, that $i = 3$ and consider the curves $\gamma_1, \ldots, \gamma_6$ shown in Figure 11.

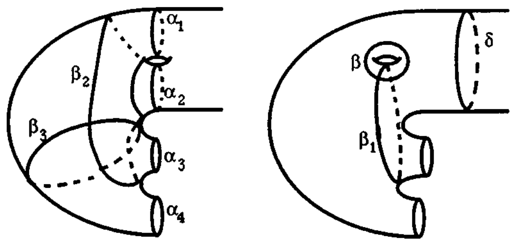

FIGURE 10

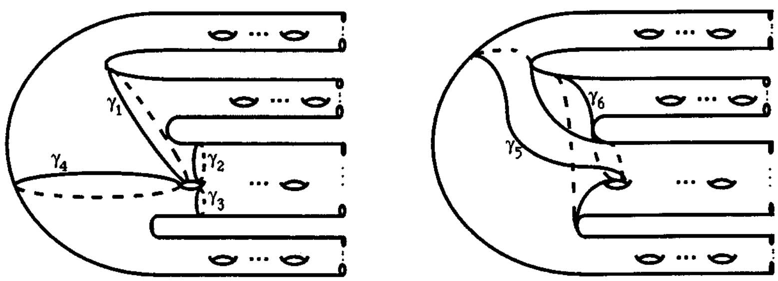

FIGURE 11

Denote respectively by L_1, L_2 and L_3 the lanterns $L(\gamma_6,\gamma_1)$, $L(\gamma_6,\gamma_4)$ and $L(\gamma_5,\gamma_1)$. Then, using (0) and (i), one has

$$\begin{aligned}
\overline{L_1}\,L_2\,L_3 &= (a_{\alpha_2}\,a_{\beta_1}\,a_{\gamma_2}\,a_{\gamma_3}\,\overline{a_{\gamma_6}}\,\overline{a_{\alpha_3}}\,\overline{a_{\gamma_1}})^{-1}(a_{\alpha_2}\,a_{\alpha_4}\,a_{\gamma_3}\,a_{\gamma_5}\,\overline{a_{\gamma_6}}\,\overline{a_{\beta_2}}\,\overline{a_{\gamma_4}}) \\
&\qquad\qquad\qquad\qquad \times (a_{\alpha_1}\,a_{\alpha_2}\,a_{\gamma_2}\,a_{\gamma_4}\,\overline{a_{\gamma_5}}\,\overline{a_{\beta_3}}\,\overline{a_{\gamma_1}}) \\
&= a_{\alpha_3}\,\overline{a_{\gamma_3}}\,\overline{a_{\gamma_2}}\,\overline{a_{\beta_1}}\,a_{\alpha_4}\,a_{\gamma_3}\,\overline{a_{\beta_2}}\,a_{\alpha_1}\,a_{\alpha_2}\,a_{\gamma_2}\,\overline{a_{\beta_3}} \\
&= a_{\alpha_3}\,\overline{a_{\beta_1}}\,a_{\alpha_4}\,\overline{a_{\beta_2}}\,a_{\alpha_1}\,a_{\alpha_2}\,\overline{a_{\beta_3}} \\
&= L.
\end{aligned}$$

The first case proves that $L_k\in\mathcal{E}$, so we have $L\in\mathcal{E}$.

Thus, we have proved that $\mathcal{L}\subset\mathcal{E}$ and so $\mathcal{LC}\subset\mathcal{E}$. Now, let $(\alpha_1,\alpha_2,\alpha_3,\beta)$ be a star E. We shall find a lantern L and a chain C such that $E=C\overline{L}$ in R. Considering the curves shown in Figure 12, one has

$$E = X_1\,X_2\,X_3\,\overline{a_{\delta_1}}\,\overline{a_{\delta_2}}\,\overline{a_{\delta_3}},$$
$$C(\alpha_1,\beta,\alpha_2) = X_1^2\,X_2^2\,\overline{a_{\delta_3}}\,\overline{a_{\gamma_1}} = C,$$
$$L(\gamma_2,\gamma_1) = a_{\alpha_1}\,a_{\alpha_2}\,a_{\delta_1}\,a_{\delta_2}\,\overline{a_{\alpha_3}}\,\overline{a_{\gamma_2}}\,\overline{a_{\gamma_1}} = L,$$

where $X_1=a_{\alpha_1}\,a_{\alpha_2}$, $X_2=a_\beta\,X_1\,a_\beta$ and $X_3=a_{\alpha_3}\,X_2\,a_{\alpha_3}$.

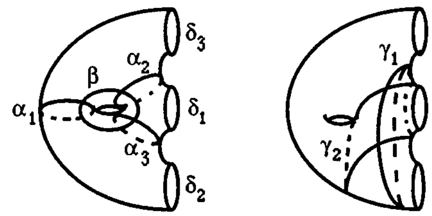

FIGURE 12

Then, using (0) and (i), we obtain

$$\begin{aligned} C\overline{L} &= X_1\, X_2\, \overline{a_{\delta_3}}\, X_1\, X_2\, \overline{a_{\gamma_1}}\, a_{\gamma_1}\, a_{\gamma_2}\, a_{\alpha_3}\, \overline{a_{\alpha_1}}\, \overline{a_{\alpha_2}}\, \overline{a_{\delta_1}}\, \overline{a_{\delta_2}} \\ &= X_1\, X_2\, \overline{a_{\delta_3}}\, X_2\, a_{\gamma_2}\, a_{\alpha_3}\, \overline{a_{\delta_1}}\, \overline{a_{\delta_2}} \\ &= X_1\, X_2\, X_2\, a_{\gamma_2}\, a_{\alpha_3}\, \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}}. \end{aligned}$$

But one can see that $\tau_\beta \tau_{\alpha_2} \tau_{\alpha_1} \tau_\beta(\gamma_2) = \alpha_3$, so $a_{\alpha_3} = X_2\, a_{\gamma_2}\, \overline{X_2}$ by Corollary 1.4. Thus,

$$\begin{aligned} C\overline{L} &= X_1\, X_2\, a_{\alpha_3}\, X_2\, a_{\alpha_3}\, \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}} \\ &= X_1\, X_2\, X_3\, \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}} \\ &= E\,. \end{aligned}$$

This shows that $\mathcal{E} \subset \mathcal{CL}$ and concludes the proof of Theorem 1.6. □

We conclude this section by a lemma which gives us a new relation in $\mathcal{M}$.

Lemma 1.7. *Let Σ' be a subsurface of Σ which is diffeomorphic to a surface of genus one with four boundary components. Consider the curves shown on Figure* 13 *and set (in R and S)*

$$X_1 = a_{\alpha_1}\, a_{\alpha_3}\,, \quad X_2 = a_\beta\, X_1\, a_\beta\,, \quad X_3 = a_{\alpha_2}\, X_2\, a_{\alpha_2}, \quad X_3' = a_{\alpha_4}\, X_2\, a_{\alpha_4}\,.$$

Then, one can find two stars E_1, E_2, and a chain C such that

$$X_3\, X_3' = a_{\delta_1}\, a_{\delta_2}\, a_{\delta_3}\, a_{\delta_4}\, E_1\, E_2\, C^{-1} \quad \text{in } R.$$

The same result holds in S if $\delta_1, \ldots, \delta_4$ do not separate in Σ.

Proof. By Lemma 1.2, one has

$$\begin{aligned} E_1 = E(\alpha_1, \alpha_3, \alpha_2, \beta) &= X_1\, X_2\, X_3\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}}\, \overline{a_\delta}, \\ E_2 = E(\alpha_1, \alpha_3, \alpha_4, \beta) &= X_1\, X_2\, X_3'\, \overline{a_{\delta_1}}\, \overline{a_{\delta_4}}\, \overline{a_{\delta'}}, \\ C = C(\alpha_1, \beta, \alpha_3) &= X_1^2\, X_2^2\, \overline{a_\delta}\, \overline{a_{\delta'}}, \end{aligned}$$

where δ and δ' are the curves shown on Figure 14. Thus, we obtain

$$\begin{aligned} E_1\, E_2\, C^{-1} &= \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}}\, \overline{a_{\delta_4}}\, X_1\, X_2\, X_3\, \overline{a_\delta}\, X_1\, X_2\, X_3'\, \overline{a_{\delta'}}\, a_\delta\, a_{\delta'}\, \overline{X_1^2}\, \overline{X_2^2} \quad \text{by } (0) \\ &= \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}}\, \overline{a_{\delta_4}}\, X_3\, \overline{a_\delta}\, X_3'\, a_\delta \quad \text{by Lemma 1.2 and } (0) \\ &= \overline{a_{\delta_1}}\, \overline{a_{\delta_2}}\, \overline{a_{\delta_3}}\, \overline{a_{\delta_4}}\, X_3\, X_3' \quad \text{by } (0) \text{ and } (i) \text{ of Theorem 1.6.} \end{aligned}$$

Suppose now that the four curves $\delta_1, \ldots, \delta_4$ do not separate. The proof above remains valid for S only if both δ and δ' do not separate Σ. But if they separate, the curves γ and γ' shown on Figure 14 do not separate. Then, one can conclude in the

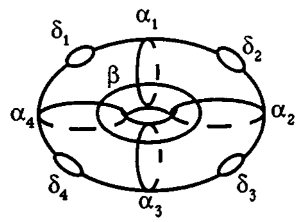

FIGURE 13

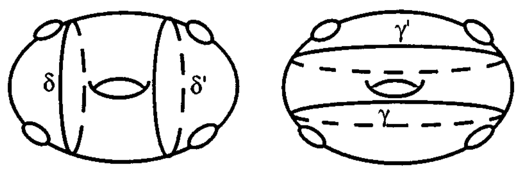

FIGURE 14

same way using the stars $(\alpha_2, \alpha_4, \alpha_1, \beta)$, $(\alpha_2, \alpha_4, \alpha_3, \beta)$ and the chain $(\alpha_2, \beta, \alpha_4)$. □

2. PROOF OF THEOREMS A AND B

Let Σ be a surface of genus $g \geq 0$ with r boundary components $(r \geq 1)$, $\mathcal{M} = \mathcal{M}_{g,r}$ its mapping class group (we consider Σ to be a disc with g handles attached and $r-1$ holes).

In order to simplify notation, we will denote by β the twist along a curve β. First, we shall recall the results obtained by Hatcher-Thurston ([H-T]) and Harer ([H]) about presentations of $\mathcal{M}$. To do this, we will consider the curves shown on Figures 15, 16 and 17. If $D_1, D_2, \ldots, D_{2g+r-1}$ are the holes obtained when we cut Σ along the curves α_i or corresponding to $\partial\Sigma$ (exept δ, the boundary of the disc on which the handles were attached), then $\delta_{ij,\, 1 \leq i < j \leq 2g+r-1}$ is a curve in $\Sigma \backslash (\bigcup \alpha_i)$ which encircles the holes D_i, and D_j and $\varphi_{k,\, 3 \leq k \leq 2g+r-1}$ a curve which encircles D_1, D_2 and D_k (see Figure 17).

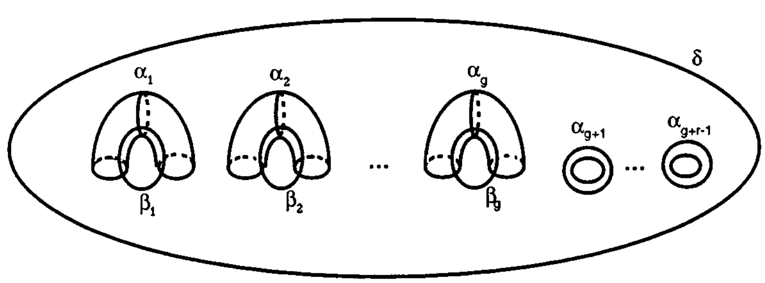

FIGURE 15

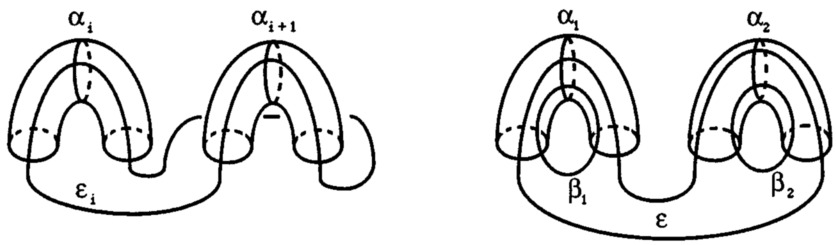

FIGURE 16

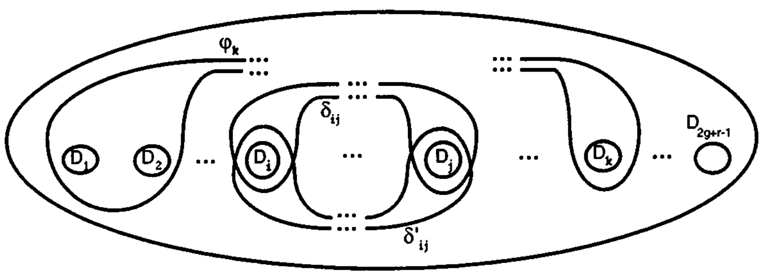

FIGURE 17

Denote by H_0 the subgroup of elements of $\mathcal{M}$ which leave the curves α_i fixed. Hatcher and Thurston have proved in [H-T] that H_0 is isomorphic to $P_{2g+r-1} \times \mathbb{Z}^{g+r-1}$, where P_n is the pure braid group on n strands. It is generated by α_i's and $\delta_{i,j}$'s, and admits a presentation with relations (see [B] for a presentation of P_n)

$(i)_a$ $\alpha_i\,\alpha_j = \alpha_j\,\alpha_i \quad \forall\, i, j \in \langle 1, g+r-1\rangle$,

$(i)_b$ $\alpha_k\,\delta_{ij} = \delta_{ij}\,\alpha_k \quad \forall\, i < j \in \langle 1, 2g+r-1\rangle$ and $k \in \langle 1, g+r-1\rangle$,

$(i)_c$ $\delta_{ij}\,\lambda_{ijkl} = \lambda_{ijkl}\,\delta_{ij}$, with

$$\lambda_{ijkl} = \begin{cases} \delta_{kl} & \text{if } 1 \le k < l < i < j \le 2g+r-1 \\ & \text{or } 1 \le i < k < l < j \le 2g+r-1, \\ \delta_{kj}\delta_{kl} & \text{if } 1 \le k < l = i < j \le 2g+r-1, \\ \delta_{lj}\delta_{ij}\delta_{kl} & \text{if } 1 \le i = k < l < j \le 2g+r-1, \\ [\delta_{lj}^{-1}, \delta_{kj}^{-1}]\delta_{kl} & \text{if } 1 \le k < i < l < j \le 2g+r-1. \end{cases}$$

H is the subgroup of elements of $\mathcal{M}$ which leave the set $\{\alpha_1, \dots, \alpha_g\}$ fixed (an element of H can permute the α_i's and reverse their orientations). So H is defined by the following exact sequences (see [W]):

$$1 \to H_0 \to H \xrightarrow{\theta} \pm\mathcal{S}_n \to 1 \;\;, \qquad 1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \pm\mathcal{S}_n \to \mathcal{S}_n \to 1,$$

where $\mathcal{S}_n$ is the symmetric group of rank n and θ the morphism which assigns to each element of H the induced signed permutation on the α_i's. Therefore, H is generated by H_0, $\xi = \beta_1\,\alpha_1\,\alpha_1\,\beta_1$ and $(\theta_i = \varepsilon_i\,\alpha_i\,\alpha_{i+1}\,\varepsilon_i)_{1 \le i \le g-1}$, and is defined by the relations (i) and

$(ii)_a$ $\theta_{i+1}\,\theta_i\,\theta_{i+1}=\theta_i\,\theta_{i+1}\,\theta_i$ for $1\le i\le g-2$, and $\theta_i\,\theta_j=\theta_j\,\theta_i$ for $|i-j|\ge 2$,
$(ii)_b$ $\theta_i^2\in H_0$ for $1\le i\le g-1$,
$(ii)_c$ $\xi^2\in H_0$,
$(ii)_d$ $[\xi_i,\xi_j]\in H_0$ for $1\le i<j\le g$ where $\xi_i=\theta_{i-1}\,\theta_{i-2}\dots\theta_1\,\xi\,\overline{\theta_1}\dots\overline{\theta_{i-2}}\,\overline{\theta_{i-1}}$,
$(ii)_e$ $[\theta_i,\xi_j]\in H_0$ for $i\ne j,\ j-1$,

$(iii)_a$ $\xi(\alpha_i)\in H_0$ for $1\le i\le g+r-1$,
$(iii)_b$ $\xi(\delta_{ij})\in H_0$ for $1\le i<j\le 2g+r-1$,
$(iii)_c$ $\theta_k(\alpha_i)\in H_0$ for $1\le i\le g+r-1$ and $1\le k\le g-1$,
$(iii)_d$ $\theta_k(\delta_{ij})\in H_0$ for $1\le i<j\le 2g+r-1$ and $1\le k\le g-1$.

Remark. θ_i permutes α_i and α_{i+1}, and ξ_i reverses the orientation of α_i ($\xi_1=\xi$).

Theorem 2.1 ([H-T][H])**.** *$\mathcal{M}$ is generated by H and $\sigma=\alpha_1\,\beta_1\,\alpha_1$. All relations involving σ come from* (A)–(E) *below and, together with (i)–(iii), they suffice for the presentation of $\mathcal{M}$.*

(A) *σ commutes with $H(\alpha_1,\beta_1)$, where $H(\alpha_1,\beta_1)$ is the subgroup of elements of H which leave α_1 and β_1 fixed.*
(B) *$\sigma^2\in H$.*
(C) *Consider the curves $\gamma_1,\dots,\gamma_N$ shown on Figure* 18*. For any γ among these curves and any $h\in H$ such that $h\sigma(\alpha_1)=\gamma$, $h\sigma(\gamma)=\beta_1$ and $h\sigma(\beta_1)=\alpha_1$, one has $\sigma h\sigma h\sigma\in H$. In the presence of* (A)*, the choice of one h suffices.*
(D) *For any $h\in H$ such that $h(\beta_2)=\beta_1$, σ commutes with $\overline{h}(\sigma)$. In the presence of* (A)*, the choice of one h suffices.*
(E) *If ε is the curve shown on Figure* 16*, then $\sigma\,h_5\,\sigma\,h_4\,\sigma\,h_3\,\sigma\,h_2\,\sigma\,h_1\,\sigma\in H$ whenever there are $h_1,h_2,h_3,h_4,h_5\in H$ such that $h_1\,\sigma(\delta_{2,3})=\beta_1$, $h_2\,\sigma\,h_1\,\sigma(\varepsilon)=\beta_1$, $h_3\,\sigma\,h_2\,\sigma\,h_1\,\sigma(\alpha_1)=\beta_1$, $h_4\,\sigma\,h_3\,\sigma\,h_2\,\sigma\,h_1\,\sigma(\beta_2)=\beta_1$ and*

$$h_5\,\sigma\,h_4\,\sigma\,h_3\,\sigma\,h_2\,\sigma\,h_1\,\sigma(\alpha_2)=\beta_1.$$

The choice of one 5*-tuple $(h_1,\dots,h_5)\in H^5$ suffices.*

Remark. Relations (C) and (E) are slightly different from [H-T]. Moreover, Hatcher and Thurston worked only with closed surfaces. The result above is due to Harer ([H]).

Corollary 2.2. *Let Σ be a surface with at least one boundary component. Then $\mathcal{M}_\Sigma$ admits a presentation with generators α_i, $\delta_{i,j}$, ξ, θ_i, σ and relations (i)–(iii) above added to the following:*
(iv) *σ commutes with $(\alpha_i)_{2\le i\le g+r-1}$, $(\theta_i)_{2\le i\le g-1}$, $\xi'=\overline{\theta_1}\,\delta_{2,4}\,\overline{\alpha_1}(\xi)$, $\delta_{1,2}$, $(\delta_{ij})_{3\le i<j\le 2g+r-1}$ and $(\varphi_i)_{3\le i\le 2g+r-1}$, where $\varphi_i=\alpha_1^2(\xi\,\delta_{2,i})^2\overline{\alpha_{k_i}}$ with $k_i=[(i+1)/2]$ if $i\le 2g$ and $k_i=i-g$ if $i\ge 2g+1$.*
(v) *$\sigma^2\in H$.*
(vi) *$\sigma h\sigma h\sigma\in H$ for all $h\in\mathcal{H}=\{\alpha_1\ ;\ \delta_{1,3}\ ;\ \delta_{1,4}\,\delta_{1,3}\,\overline{\alpha_1}\ ;\ \delta_{1,5}\,\delta_{1,3}\,\overline{\alpha_1}\ ;\ \delta_{1,2g+1},\dots,\delta_{1,2g+r-1}\ ;\ \delta_{1,2g+1}\,\delta_{1,3}\,\overline{\alpha_1},\dots,\delta_{1,2g+r-1}\,\delta_{1,3}\,\overline{\alpha_1}\ ;\ \delta_{1,k}\,\delta_{1,j}\,\overline{\alpha_1}$ for $2g+1\le j<k\le 2g+r-1\ \}$.*

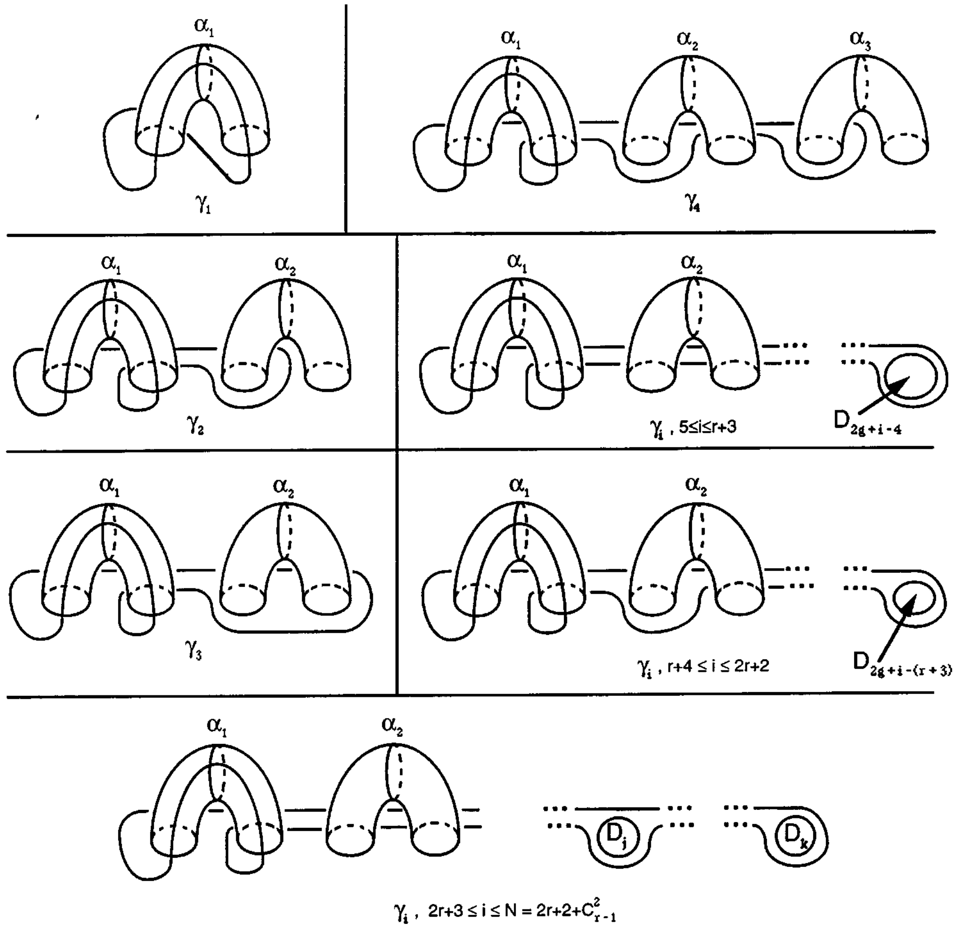

FIGURE 18

(*vii*) σ *commutes with* $\overline{h}\sigma h$, *where* $h=\alpha_1\,\overline{\delta_{2,4}}\,\theta_1$.

(*viii*) $\sigma\, h_5\, \sigma\, h_4\, \sigma\, h_3\, \sigma\, h_2\, \sigma\, h_1\, \sigma \in H$, *where* $h_1=h_3=h_5=\overline{\alpha_1}\,\delta_{2,3}$, $h_2=\overline{\delta_{2,3}}\,\theta$ *and* $h_4=\delta_{2,3}\,\overline{\alpha_1}\,\overline{\theta}$; *here* $\theta=\overline{\varphi_3}\,\xi\,\alpha_1^2\alpha_2(\theta_1)$.

Remarks. (i) φ_i is the curve shown in Figure 17. The chain $(\alpha_1, \beta_1, \delta_{2,i})$ and the computations of Lemma 1.2 prove that

$$\varphi_i=(\alpha_1\beta_1\delta_{2,i})^4\,\overline{\alpha_{k_i}}=\alpha_1^2(\xi\,\delta_{2,i})^2\overline{\alpha_{k_i}}.$$

(ii) As diffeomorphisms of Σ, ξ and θ can be seen to be equal to $\beta_2\,\alpha_2\,\alpha_2\,\beta_2$ and $\varepsilon\,\alpha_1\,\alpha_2\,\varepsilon$ respectively, where the curves are whose which are shown on Figures 15 and 16.

Proof. We have to prove that the relations (*iv*)–(*viii*) are equivalent to the relations (A)–(E).

(*iv*) Let $H_0(\alpha_1,\beta_1)$ be the subgroup of elements of $H(\alpha_1,\beta_1)$ which leave each curve α_i fixed. Then, $H(\alpha_1,\beta_1)$ is generated by $H_0(\alpha_1,\beta_1)$ and elements permuting the α_i's for $2\le i\le g$ or reversing their orientation. We have seen that θ_i permutes α_i and α_{i+1}. The diffeomorphism ξ' preserves α_1, β_1 and α_i for

$i \geq 3$ and reverses the orientation of α_2 (using braid relations, one can see that $\xi' = \beta_2\,\alpha_2\,\alpha_2\,\beta_2$). So, $H(\alpha_1, \beta_1)$ is generated by $H_0(\alpha_1, \beta_1)$, ξ' and $(\theta_i)_{2 \leq i \leq g-1}$. Furthermore, following the proof of Hatcher-Thurston for H_0, one can prove that $H_0(\alpha_1, \beta_1)$ is isomorphic to $P_{2g+r-2} \times \mathbb{Z}^{g+r-2}$ and deduce that it is generated by $\delta_{1,2}$, $(\alpha_i)_{2 \leq i \leq g+r-1}$, $(\delta_{ij})_{3 \leq i < j \leq 2g+r-1}$ and $(\varphi_i)_{3 \leq i \leq 2g+r-1}$. So, we have found the required generators for $H(\alpha_1, \beta_1)$, and this proves that (iv) is equivalent to (A).

(vi) For any s.c.c. γ in $\{\gamma_1, \dots, \gamma_N\}$, we have to find an element h of H such that $h\sigma(\alpha_1) = \gamma$, $h\sigma(\gamma) = \beta_1$ and $h\sigma(\beta_1) = \alpha_1$. $\mathcal{H}$ is exactly the set of the elements that we have found:

- $\gamma = \gamma_1 \;\Rightarrow\; h = \alpha_1$,
- $\gamma = \gamma_2 \;\Rightarrow\; h = \delta_{1,3}$,
- $\gamma = \gamma_3 \;\Rightarrow\; h = \delta_{1,4}\,\delta_{1,3}\,\overline{\alpha_1}$,
- $\gamma = \gamma_4 \;\Rightarrow\; h = \delta_{1,5}\,\delta_{1,3}\,\overline{\alpha_1}$,
- $\gamma = \gamma_i$, $i = 5, \dots, r+3 \;\Rightarrow\; h = \delta_{1,2g+i-4}$,
- $\gamma = \gamma_i$, $i = r+4, \dots, 2r+2 \;\Rightarrow\; h = \delta_{1,2g+i-r-3}\,\delta_{1,3}\,\overline{\alpha_1}$,
- $\gamma = \gamma_i$, $i = 2r+3, \dots, N \;\Rightarrow\; h = \delta_{1,k}\,\delta_{1,j}\,\overline{\alpha_1}$ if γ_i encircles D_j and D_k.

(vii) $h = \alpha_1\,\overline{\delta_{2,4}}\,\theta_1$ is an element of H satisfying $h(\beta_2) = \beta_1$, so (vii) is equivalent to (D).

(viii) If we choose $h_1 = h_3 = h_5 = \overline{\alpha_1}\,\delta_{2,3}$, $h_2 = \overline{\delta_{2,3}}\,\theta$ and $h_4 = \delta_{2,3}\,\overline{\alpha_1}\,\overline{\theta}$, then $h_i \in H$ for all i and the 5-tuple $(h_1, h_2, h_3, h_4, h_5)$ satisfies the condition of the relation (E). □

2.1. **Proof of Theorem A.** Let G be a group with generators $\{a \equiv a_\alpha \,/\, \alpha$ s.c.c. in $\Sigma\}$ and relations (I), (II) and (III) (Theorem A). We shall denote by H'_0 the subgroup of G generated by the a_i's and the d_{ij}'s, and by H' the subgroup generated by H'_0, $x = b_1\,a_1\,a_1\,b_1$, and the t_i's, where $t_i = e_i\,a_i\,a_{i+1}\,e_i$ $(i = 1, \dots, g-1)$.

Remark. Since G is a quotient of R, the relations which have been shown in Section 1 are also true in G.

Let $\varphi : G \to \mathcal{M}$ be the epimorphism defined by $\varphi(a) = \alpha$. In order to prove that φ is an isomorphism, we shall construct an inverse morphism $\psi : \mathcal{M} \to G$. So, let us define ψ on the generators of $\mathcal{M}$ as follows:

$$\psi(\alpha) = a \quad \forall \alpha \in \Big\{ (\alpha_i)_{1 \leq i \leq g+r-1}\,;\, (\delta_{ij})_{1 \leq i < j \leq 2g+r-1} \Big\},$$

$$\psi(\xi) = x\,, \quad \psi(\theta_i) = t_i\,, \quad \psi(\sigma) = a_1\,b_1\,a_1 = s\,.$$

First of all, let us prove that ψ is a homomorphism defined on $\mathcal{M}$; to do this, we will show that the relations (i)–$(viii)$ are mapped by ψ into true relations in G.

Relations *(i)*. $(i)_a$ and $(i)_b$ are consequences of (0), and since $\lambda_{ijkl}(\delta_{ij}) = \delta_{ij}$, $(i)_c$ is a consequence of Corollary 1.4.

Relation *(ii)*$_{\boldsymbol{a}}$. (0) implies $t_i\,t_j = t_j\,t_i$ when $|i-j| \geq 2$. For $1 \leq i \leq g-1$, one has

$$\begin{aligned}
t_{i+1}(t_i) &= (e_{i+1}\, a_{i+1}\, a_{i+2}\, e_{i+1})(e_i\, a_i\, a_{i+1}\, e_i)(\overline{e_{i+1}}\,\overline{a_{i+1}}\,\overline{a_{i+2}}\,\overline{e_{i+1}}) \\
&= e_{i+1}\, a_{i+1}\, a_{i+2}\, e_i\, a_i\, e_{i+1}\, a_{i+1}\, \overline{e_{i+1}}\, e_i\, \overline{a_{i+1}}\,\overline{a_{i+2}}\,\overline{e_{i+1}} \quad \text{by } (0) \\
&= e_{i+1}\, a_{i+1}\, a_{i+2}\, e_i\, a_i\, \overline{a_{i+1}}\, e_{i+1}\, a_{i+1}\, e_i\, \overline{a_{i+1}}\,\overline{a_{i+2}}\,\overline{e_{i+1}} \quad \text{by } (1) \\
&= e_{i+1}\, a_{i+2}\, a_{i+1}\, e_i\, \overline{a_{i+1}}\, a_i e_{i+1}\, a_{i+1}\, e_i\, \overline{a_{i+1}}\,\overline{a_{i+2}}\,\overline{e_{i+1}} \quad \text{by } (0) \\
&= e_{i+1}\, a_{i+2}\, \overline{e_i}\, a_{i+1}\, e_i\, a_i\, e_{i+1}\, \overline{e_i}\, a_{i+1}\, e_i\, \overline{a_{i+2}}\,\overline{e_{i+1}} \quad \text{by } (1) \\
&= \overline{e_i}\, e_{i+1}\, a_{i+2}\, a_{i+1}\, e_i\, a_i\, \overline{e_i}\, e_{i+1}\, a_{i+1} \overline{a_{i+2}}\,\overline{e_{i+1}}\, e_i \quad \text{by } (0) \\
&= \overline{e_i}\, e_{i+1}\, a_{i+2}\, a_{i+1}\, \overline{a_i}\, e_i\, a_i\, e_{i+1}\, a_{i+1} \overline{a_{i+2}}\,\overline{e_{i+1}}\, e_i \quad \text{by } (1) \\
&= \overline{e_i}\,\overline{a_i}\, e_{i+1}\, a_{i+1}\, e_i\, a_{i+2}\, e_{i+1}\, \overline{a_{i+2}}\, a_{i+1} \overline{e_{i+1}}\, a_i\, e_i \quad \text{by } (0) \\
&= \overline{e_i}\,\overline{a_i}\, e_{i+1}\, a_{i+1}\, e_i\, \overline{e_{i+1}}\, a_{i+2}\, e_{i+1}\, a_{i+1}\, \overline{e_{i+1}}\, a_i\, e_i \quad \text{by } (1) \\
&= \overline{e_i}\,\overline{a_i}\, e_{i+1}\, a_{i+1}\, \overline{e_{i+1}}\, e_i\, a_{i+2}\, e_{i+1} a_{i+1}\, \overline{e_{i+1}}\, a_i\, e_i \quad \text{by } (0) \\
&= \overline{e_i}\,\overline{a_i}\,\overline{a_{i+1}}\, e_{i+1}\, a_{i+1}\, e_i\, a_{i+2}\, \overline{a_{i+1}}\, e_{i+1}\, a_{i+1}\, a_i\, e_i \quad \text{by } (1) \\
&= \overline{e_i}\,\overline{a_i}\,\overline{a_{i+1}}\, e_{i+1}\, a_{i+1}\, e_i\, \overline{a_{i+1}}\, a_{i+2}\, e_{i+1}\, a_{i+1}\, a_i\, e_i \quad \text{by } (0) \\
&= \overline{e_i}\,\overline{a_i}\,\overline{a_{i+1}}\, e_{i+1}\, \overline{e_i}\, a_{i+1}\, e_i\, a_{i+2}\, e_{i+1}\, a_{i+1}\, a_i\, e_i \quad \text{by } (1) \\
&= \overline{e_i}\,\overline{a_i}\,\overline{a_{i+1}}\,\overline{e_i}\, e_{i+1}\, a_{i+1}\, a_{i+2}\, e_{i+1}\, e_i\, a_{i+1}\, a_i\, e_i \quad \text{by } (0) \\
&= \overline{t_i}(t_{i+1}),
\end{aligned}$$

which implies $t_{i+1}\, t_i\, t_{1+1} = t_i\, t_{i+1}\, t_i$.

Relation $(\boldsymbol{ii})_{\boldsymbol{b}}$. Let δ'_{ij} be the curves shown in Figure 17.

Lemma 2.3. *$d'_{ij} \in H'_0$ for all $1 \le i < j \le 2g+r-1$, and H'_0 is generated by a_i's and d'_{ij}'s.*

Proof. Since $\delta_{i,j} = \tau_{\delta_{i,j-1}} \ldots \tau_{\delta_{i,i+1}}(\delta'_{i,j})$, one has

$$d'_{i,j} = \overline{d_{i,i+1}} \ldots \overline{d_{i,j-1}}(d_{i,j}) \in H'_0$$

(use Corollary 1.4). Symmetrically, we obtain $d_{i,j} = d'_{i,i+1} \ldots d'_{i,j-1}(d'_{i,j})$, which proves the lemma. □

By Lemma 1.2, one has $(a_i e_i a_{i+1})^4 = a_i^2 a_{i+1}^2 t_i^2$. But the chain $(\alpha_i, \varepsilon_i, \alpha_{i+1})$ gives:

$$(a_i e_i a_{i+1})^4 = d'_{2i-1,2i+1} d_{2i,2i+2}\,.$$

Thus, by Lemma 2.3, t_i^2 is an element of H'_0.

Relation $(\boldsymbol{ii})_{\boldsymbol{c}}$. The chain $(\alpha_1, \beta_1, \alpha_1)$ yields the relation $a_1^4 (b_1 a_1 a_1 b_1)^2 = d_{1,2}$ (using Lemma 1.2). Thus we have $x^2 = d_{1,2} a_1^{-4} \in H'_0$.

Relation $(\boldsymbol{iii})_{\boldsymbol{a}}$. For $i \ge 2$, $x(a_i) = a_i \in H'_0$ by (0). One has

$$x(a_1) = b_1\, a_1\, a_1\, b_1(a_1) = b_1\, a_1\, a_1\, \overline{a_1}(b_1) = b_1 \overline{b_1}(a_1) = a_1 \in H'_0 \quad \text{by (1).}$$

Relation $(\boldsymbol{iii})_{\boldsymbol{b}}$. When $3 \le i < j \le 2g+r-1$ or $(i,j) = (1,2)$, one has $x(d_{i,j}) = d_{i,j} \in H'_0$ by (0).

When $3 \le j \le 2g+r-1$, one has $\tau^{-1}_{\delta_{1,2}}\, \xi(\delta_{1,j}) = \delta_{2,j}$ so, according to Corollary 1.4, $x(d_{1,j}) = d_{1,2}(d_{2,j}) \in H'_0$.

Similarly, when $3 \le j \le 2g+r-1$, we have $x(d_{2,j}) = d_{1,j} \in H'_0$.

Relation $(\boldsymbol{iii})_{\boldsymbol{c}}$. We have seen that θ_k permutes α_k and α_{k+1} and leaves α_i fixed when $i \ne k,\, k+1$. So, by Corollary 1.4, one has $t_k(a_i) \in H'_0$.

Relation $(\boldsymbol{iii})_{\boldsymbol{d}}$. For all i, j, k, we shall find $h \in H_0$ such that $\theta_k(\delta_{ij}) = h(\delta_{i',j'})$ for some i', j'. Then, if h is given as a word on the generators a_i's and $\delta_{i,j}$'s, and m is the associated word in H'_0, one has by Corollary 1.4

$$t_k(d_{ij}) = m(d_{i',j'}) \in H'_0.$$

The cases of $\delta_{2k-1,2k+1}$, $\delta_{2k+1,j}$ and $\delta_{i,2k+1}$ will be proved in a slightly different way:

* $i, j \neq 2k-1,\ 2k,\ 2k+1,\ 2k+2 \ \Rightarrow\ \theta_k(\delta_{i,j}) = \delta_{i,j}$.
* $j \geq 2k+3 \ \Rightarrow\ \theta_k(\delta_{2k-1,j}) = \delta'_{2k-1,2k+1}\delta'_{2k,2k+1}(\delta_{2k+1,j})$.
  $i \leq 2k-2 \ \Rightarrow\ \theta_k(\delta_{i,2k-1}) = \delta'_{2k-1,2k+1}\delta'_{2k,2k+1}(\delta_{i,2k+1})$.
* $j \geq 2k+3 \ \Rightarrow\ \theta_k(\delta_{2k,j}) = \delta_{2k,2k+2}(\delta_{2k+2,j})$.
  $i \leq 2k-2 \ \Rightarrow\ \theta_k(\delta_{i,2k}) = \delta_{2k,2k+2}(\delta_{i,2k+2})$.
* $j \geq 2k+3 \ \Rightarrow\ \theta_k(\delta_{2k+2,j}) = \delta_{2k+1,j}$.
  $i \leq 2k-2 \ \Rightarrow\ \theta_k(\delta_{i,2k+2}) = \delta_{i,2k+1}$.
* $\theta_k(\delta_{2k-1,2k}) = \delta'_{2k-1,2k+1}(\delta_{2k+1,2k+2})$.
* $\theta_k(\delta_{2k-1,2k+2}) = \delta^{-1}_{2k-1,2k}(\delta_{2k,2k+1})$.
* $\theta_k(\delta_{2k,2k+1}) = \delta^{-1}_{2k+1,2k+2}(\delta'_{2k-1,2k+2})$.
* $\theta_k(\delta_{2k,2k+2}) = \delta_{2k,2k+2}$.
* $\theta_k(\delta_{2k+1,2k+2}) = \delta_{2k,2k+2}(\delta_{2k-1,2k})$.
* Since $\delta_{2k-1,2k+1} = \delta_{2k-1,2k}(\delta'_{2k-1,2k+1})$, one has

$$\theta_k(\delta_{2k-1,2k+1}) = \theta_k\, \delta_{2k-1,2k}(\delta'_{2k-1,2k+1}) = \theta_k\, \delta_{2k-1,2k}\, \overline{\theta_k}\, \theta_k(\delta'_{2k-1,2k+1})\,.$$

  But since

$$\theta_k(\delta'_{2k-1,2k+1}) = \delta'_{2k-1,2k+1},$$

  one has using Corollary 1.4

$$t_k(d_{2k-1,2k+1}) = t_k\, d_{2k-1,2k}\, \overline{t_k}(d'_{2k-1,2k+1})\,.$$

  We have seen that $t_k(d_{2k-1,2k}) \in H'_0$; therefore $t_k(d_{2k-1,2k+1}) \in H'_0$ by Lemma 2.3.
* We saw in Lemma 2.3 that

$$d_{2k+1,j} = d_{2k+1,j-1} \dots d_{2k+1,2k+2}(d'_{2k+1,j}).$$

  Since, for $j \geq 2k+3$, $\theta_k(\delta'_{2k+1,j}) = \delta'_{2k-1,2k+1}(\delta'_{2k-1,j})$, one has $t_k(d'_{2k+1,j}) \in H'_0$ by Corollary 1.4. Furthermore, we already have $t_k(d_{2k+1,2k+2}) \in H'_0$, thus, $t_k(d_{2k+1,j}) \in H'_0$ by induction on j.
* Similarly, one has $d_{i,2k+1} = d_{i,2k} \dots d_{i,i+1}(d'_{i,2k+1})$ and, for i less than $2k-1$, $t_k(d'_{i,2k+1}) = d'_{2k-1,2k+1}(d'_{i,2k-1}) \in H'_0$. Since $t_k(d_{i,l}) \in H'_0$ for $l = i+1, \dots, 2k$, we have $t_k(d_{i,2k+1}) \in H'_0$.

Relation $(\boldsymbol{ii})_{\boldsymbol{d}}$.

Lemma 2.4. *t_1 commutes with $x\, t_1\, x$ in G.*

Proof. We have

$$
\begin{aligned}
x\,t_1\,x\,t_1 &= (b_1\,a_1\,a_1\,b_1)(e_1\,a_1\,a_2\,e_1)(b_1\,a_1\,a_1\,b_1)(e_1\,a_1\,a_2\,e_1)\\
&= b_1\,a_1\,a_1\,e_1\,b_1\,a_1\,b_1\,a_2\,e_1\,a_1\,a_1\,b_1\,e_1\,a_1\,a_2\,e_1 \quad \text{by } (0)\\
&= b_1\,a_1\,a_1\,e_1\,a_1\,b_1\,a_1\,a_2\,e_1\,a_1\,a_1\,b_1\,e_1\,a_1\,a_2\,e_1 \quad \text{by } (1)\\
&= b_1\,a_1\,e_1\,a_1\,e_1\,b_1\,a_2\,a_1\,e_1\,a_1\,a_1\,e_1\,b_1\,a_1\,a_2\,e_1 \quad \text{by } (0) \text{ and } (1)\\
&= b_1\,e_1\,a_1\,e_1\,b_1\,e_1\,a_2\,e_1\,a_1\,e_1\,a_1\,e_1\,b_1\,a_1\,a_2\,e_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,b_1\,a_1\,b_1\,e_1\,e_1\,a_2\,e_1\,a_1\,a_1\,e_1\,a_1\,b_1\,a_1\,a_2\,e_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,b_1\,a_1\,e_1\,a_2\,e_1\,a_2\,a_1\,a_1\,e_1\,b_1\,a_1\,b_1\,a_2\,e_1 \quad \text{by } (1)\\
&= e_1\,a_1\,b_1\,a_1\,a_2\,e_1\,a_2\,a_1\,a_1\,a_2\,e_1\,a_2\,b_1\,a_1\,e_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,b_1\,a_1\,e_1\,a_1\,a_2\,a_1\,b_1\,e_1\,a_2\,e_1\,a_1\,e_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,b_1\,e_1\,a_1\,e_1\,a_1\,b_1\,a_2\,e_1\,a_2\,a_1\,e_1\,a_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,e_1\,a_1\,b_1\,e_1\,a_2\,e_1\,a_1\,e_1\,a_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,e_1\,a_1\,e_1\,b_1\,a_2\,a_1\,e_1\,a_1\,a_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,a_1\,e_1\,a_1\,b_1\,a_1\,a_2\,e_1\,a_1\,a_1\,b_1 \quad \text{by } (0) \text{ and } (1)\\
&= e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,a_1\,e_1\,b_1\,a_1\,b_1\,a_2\,e_1\,a_1\,a_1\,b_1 \quad \text{by } (1)\\
&= e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,a_1\,b_1\,e_1\,a_1\,a_2\,e_1\,b_1\,a_1\,a_1\,b_1 \quad \text{by } (0)\\
&= t_1\,x\,t_1\,x. \quad \square
\end{aligned}
$$

Denote $\psi(\xi_i)=t_{i-1}\,t_{i-2}\ldots t_1(x)$ by x_i; we have to prove that $[x_i,x_j]\in H_0'$ $(i<j)$. Relations $(\mathrm{iii})_{c-d}$ show that H_0' is stable under conjugation by t_i. So, one can conjugate by $g=\overline{t_2}\ldots\overline{t_i}\,\overline{t_1}\ldots\overline{t_{j-1}}$:

$$g\,t_{j-1}\ldots t_1(x)=\overline{t_2}\ldots\overline{t_i}(x)=x \quad \text{by } (0),$$

and

$$
\begin{aligned}
g\,t_{i-1}\ldots t_1(x) &= \overline{t_2}\ldots\overline{t_i}\,\overline{t_1}\ldots\overline{t_{i-1}}\,\overline{t_i}\ldots\overline{t_{j-1}}\,t_{i-1}\ldots t_1(x)\\
&= \overline{t_2}\ldots\overline{t_i}\,\overline{t_1}\ldots\overline{t_{i-1}}\,\overline{t_i}\,t_{i-1}\ldots t_1(x) \quad \text{by } (ii)_a \text{ and } (0)\\
&= \overline{t_2}\ldots\overline{t_i}\,\overline{t_1}\ldots\overline{t_{i-2}}\,t_i\,\overline{t_{i-1}}\,\overline{t_i}\,t_{i-2}\ldots t_1(x) \quad \text{by } (ii)_a\\
&= \overline{t_2}\ldots\overline{t_i}\,t_i\,\overline{t_1}\ldots\overline{t_{i-2}}\,\overline{t_{i-1}}\,t_{i-2}\ldots t_1(x) \quad \text{by } (ii)_a \text{ and } (0)\\
&= \overline{t_2}\ldots\overline{t_{i-1}}\,\overline{t_1}\ldots\overline{t_{i-3}}\,t_{i-1}\overline{t_{i-2}}\,\overline{t_{i-1}}\,t_{i-3}\ldots t_1(x) \quad \text{by } (ii)_a\\
&\;\vdots\\
&= \overline{t_2}\,\overline{t_1}\,\overline{t_2}\,t_1(x)\\
&= \overline{t_2}\,t_2\,\overline{t_1}\,\overline{t_2}(x) \quad \text{by } (ii)_a\\
&= \overline{t_1}(x) \quad \text{by } (0).
\end{aligned}
$$

Thus, $(ii)_d$ is equivalent to $[\overline{t_1}(x),x]\in H_0'$. But, by Lemma 2.4, we have

$$[\overline{t_1}(x),x]=\overline{t_1}\,x\,t_1\,x\,\overline{t_1}\,\overline{x}\,t_1\overline{x}=\overline{t_1}^2\,x\,t_1^2\,\overline{x}.$$

Then, relations $(ii)_b$ and $(iii)_{a-b}$ yield $(ii)_d$.

Relation $(ii)_e$. For $i\geq j+1$, relations $(ii)_a$ and (0) give

$$t_i(x_j)=t_i\,t_{j-1}\ldots t_1(x)=t_{j-1}\ldots t_1\,t_i(x)=x_j$$

i.e.

$$t_i,x_j]=1\in H_0'.$$

For $i \leq j-2$, the same relations give

$$\begin{aligned} t_i(x_j) &= t_i\, t_{j-1} \dots t_1(x) \\ &= t_{j-1} \dots t_i\, t_{i+1}\, t_i \dots t_1(x) \\ &= t_{j-1} \dots t_{i+1}\, t_i\, t_{i+1} \dots t_1(x) \\ &= t_{j-1} \dots t_{i+1}\, t_i \dots t_1\, t_{i+1}(x) \\ &= t_{j-1} \dots t_{i+1}\, t_i \dots t_1(x) = x_j. \end{aligned}$$

Thus $[t_i, x_j] = 1 \in H'_0$.

Relation (*iv*). The relation (0) shows that s commutes with $(a_i)_{2\leq i\leq g+r-1}$, $(t_i)_{2\leq i\leq g-1}$, $d_{1,2}$, $(d_{ij})_{3\leq i<j\leq 2g+r-1}$.

By Corollary 1.4, one has $\psi(\xi') = \overline{t_1}\, d_{2,4}\, \overline{a_1}(x) = b_2\, a_2\, a_2\, b_2$ and so (0) implies that s commutes with $\psi(\xi')$.

Chain relations and Lemma 1.2 show that $\psi(\varphi_i) = a_1^2(\, x\, d_{2,i})^2\overline{a_{k_i}}$ is equal to f_i. Thus, s commutes with $\psi(\varphi_i)$ by (0).

Relation (*v*). $s^2 = a_1\, x\, a_1 \in H'$.

Relation (*vi*). Suppose first that $m = \psi(h) \in \{a_1;\ d_{1,3};\ d_{1,2g+1}, \dots, d_{1,2g+r-1}\}$. Then

$$m\, a_1 = a_1\, m \ \text{ by (0)} \qquad \text{and} \qquad b_1\, m\, b_1 = m\, b_1\, m \ \text{ by (1)}.$$

So,

$$\begin{aligned} s\, m\, s\, m\, s &= a_1\, b_1\, a_1\, m\, a_1\, b_1\, a_1\, m\, a_1\, b_1\, a_1 = a_1\, b_1\, a_1\, a_1\, m\, b_1\, m\, a_1\, a_1\, b_1\, a_1 \\ &= a_1\, x\, m\, x\, a_1 \in H'. \end{aligned}$$

Suppose next that

$$m \in \{d_{1,4}\, d_{1,3}\, \overline{a_1}\ ;\ d_{1,5}\, d_{1,3}\, \overline{a_1}\ ;\ d_{1,2g+1}\, d_{1,3}\, \overline{a_1}, \dots,\ d_{1,2g+r-1}\, d_{1,3}\, \overline{a_1}\ ; d_{1,k}\, d_{1,j}\, \overline{a_1} \text{ for } 2g+1 \leq j < k \leq 2g+r-1\};$$

m can be seen as a product $d_{1,k}\, d_{1,j}\, \overline{a_1}$ with $j = 3$ and $k \in \{4, 5, 2g+1, \dots, 2g+r-1\}$ or $2g+1 \leq j < k \leq 2g+r-1$.

If μ_{jk} is a curve in $\Sigma \setminus \{\alpha_i\} \subset \Sigma$ which encircles D_1, D_j and D_k, the lantern $L(\delta_{j,k}, \delta_{1,k})$ yields

$$a_1\, a_{i_j}\, a_{i_k}\, m_{jk} = d_{1,k}\, d_{1,j}\, d_{jk}. \tag{4}$$

Thus, $m = a_{i_j}\, a_{i_k}\, m_{jk}\, \overline{d_{jk}}$ and

$$\begin{aligned} s\, m\, s\, m\, s &= (a_1\, b_1\, a_1)(\overline{d_{jk}}\, a_{i_j}\, a_{i_k}\, m_{jk})(a_1\, b_1\, a_1)(\overline{d_{jk}}\, a_{i_j}\, a_{i_k}\, m_{jk})(a_1\, b_1\, a_1) \\ &= \overline{d_{jk}}^2\, a_{i_j}^2\, a_{i_k}^2\, a_1\, b_1\, a_1\, a_1\, m_{jk}\, b_1\, m_{jk}\, a_1\, a_1\, b_1\, a_1 \quad \text{by (0)} \\ &= \overline{d_{jk}}^2\, a_{i_j}^2\, a_{i_k}^2\, a_1\, x\, m_{jk}\, x\, a_1 \quad \text{by (1)}. \end{aligned}$$

It is already known that a_1, a_{i_j}, a_{i_k}, x and d_{jk} are elements of H'; (4) shows that $m_{jk} \in H'$. Therefore, $s\, m\, s\, m\, s \in H'$.

Relation (*vii*). If $h = \tau_{\alpha_1} \tau_{\delta_{2,4}}^{-1} \tau_{\varepsilon_1} \tau_{\alpha_1} \tau_{\alpha_2} \tau_{\varepsilon_1}$, then $h^{-1}(\alpha_1) = \alpha_2$ and $h^{-1}(\beta_1) = \beta_2$. Thus, by Corollary 1.4, one has $\overline{m}(a_1) = a_2$ and $\overline{m}(b_1) = b_2$, where $m = \psi(h) = a_1\, d_{2,4}\, t_1$, so $\overline{m}(s) = a_2\, b_2\, a_2$. Then, s commutes with $\overline{m}(s)$ by (0).

Relation (*viii*). Set $m_i = \psi(h_i)$ and $t = \psi(\theta) = \psi(\varphi_3)^{-1}\, x\, a_1^2\, a_2\Big(t_1\Big)$. We have seen that $\psi(\varphi_3) = f_3$ and so, using Corollary 1.4, we can see that $t = e\, a_1\, a_2\, e$ where ε is the curve shown in Figure 16.

Recall that by relation (2) (see the proof of Lemma 1.2), one has $a_1\,t_1 = t_1\,a_2$ and $a_2\,t_1 = t_1\,a_1$. Then one can compute:

$$
\begin{aligned}
& s\,m_5\,s\,m_4\,s\,m_3\,s\,m_2\,s\,m_1\,s \\
&\quad = (a_1\,b_1\,a_1)(\overline{a_1}\,d_{2,3})(a_1\,b_1\,a_1)(d_{2,3}\,\overline{a_1}\,\overline{t})(a_1\,b_1\,a_1)(\overline{a_1}\,d_{2,3}) \\
&\qquad \times (a_1\,b_1\,a_1)(\overline{d_{2,3}}\,t)(a_1\,b_1\,a_1)(\overline{a_1}\,d_{2,3})(a_1\,b_1\,a_1) \\
&\quad = a_1\,b_1\,a_1\,d_{2,3}\,b_1\,d_{2,3}\,\overline{t}\,a_1\,b_1\,d_{2,3}\,a_1\,b_1\,a_1\,\overline{d_{2,3}}\,t\,a_1\,b_1\,d_{2,3}\,a_1\,b_1\,a_1 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,\overline{t}\,b_1\,a_1\,b_1\,d_{2,3}\,b_1\,\overline{d_{2,3}}\,a_1\,t\,a_1\,b_1\,d_{2,3}\,a_1\,b_1\,a_1 \quad \text{by } (0) \text{ and } (1) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,\overline{t}\,b_1\,a_1\,d_{2,3}\,b_1\,d_{2,3}\,\overline{d_{2,3}}\,a_2\,t\,a_2\,b_1\,d_{2,3}\,a_1\,b_1\,a_1 \quad \text{by } (1) \text{ and } (2) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,\overline{t}\,a_2\,b_1\,d_{2,3}\,a_1\,b_1\,e\,a_1\,a_2\,e\,b_1\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{t}\,b_1\,d_{2,3}\,a_1\,e\,b_1\,a_1\,b_1\,a_2\,e\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (0) \text{ and } (2) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{t}\,b_1\,d_{2,3}\,a_1\,e\,a_1\,b_1\,a_1\,a_2\,e\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (1) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{t}\,b_1\,d_{2,3}\,e\,a_1\,e\,b_1\,a_1\,a_2\,e\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (1) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{e}\,\overline{a_1}\,\overline{a_2}\,\overline{e}\,b_1\,d_{2,3}\,e\,a_1\,b_1\,t\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{e}\,\overline{a_1}\,\overline{a_2}\,b_1\,d_{2,3}\,a_1\,b_1\,t\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{e}\,\overline{a_1}\,b_1\,a_1\,d_{2,3}\,b_1\,\overline{a_2}\,t\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,\overline{e}\,b_1\,a_1\,\overline{b_1}\,d_{2,3}\,b_1\,t\,\overline{a_1}\,d_{2,3}\,a_1\,b_1\,a_1\,a_2 \quad \text{by } (1) \text{ and } (2) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,b_1\,\overline{e}\,a_1\,d_{2,3}\,b_1\,\overline{d_{2,3}}\,t\,d_{2,3}\,b_1\,a_1\,a_2 \quad \text{by } (0) \text{ and } (1) \\
&\quad = (a_1\,b_1\,d_{2,3})^2\,a_1\,b_1\,d_{2,3}\,\overline{e}\,a_1\,b_1\,t\,b_1\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^3\,\overline{e}\,a_1\,b_1\,e\,a_1\,a_2\,e\,b_1\,a_1\,a_2 \\
&\quad = (a_1\,b_1\,d_{2,3})^3\,\overline{e}\,a_1\,e\,b_1\,a_1\,b_1\,a_2\,e\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^3\,a_1\,e\,\overline{a_1}\,a_1\,b_1\,a_1\,a_2\,e\,a_1\,a_2 \quad \text{by } (1) \\
&\quad = (a_1\,b_1\,d_{2,3})^3\,a_1\,b_1\,e\,a_1\,a_2\,e\,a_1\,a_2 \quad \text{by } (0) \\
&\quad = (a_1\,b_1\,d_{2,3})^4\,\overline{d_{2,3}}\,t\,a_1\,a_2 \\
&\quad = a_1^2(x\,d_{2,3})^2\overline{d_{2,3}}\,t\,a_1\,a_2 \quad \text{by Lemma 1.2.}
\end{aligned}
$$

It is known that a_i, x and $d_{2,3}$ are elements of H'. Since $\psi(\varphi_3) = a_1^2(x\,d_{2,3})^2\,\overline{a_{k_i}} \in H'$, one has $t = \psi(\theta) = \psi(\varphi_3)\,x\,a_1^2\,a_2(t_1) \in H'$. This concludes the proof of $(viii)$.

This ends the proof that $\psi : \mathcal{M} \to G$ is a homomorphism. One has $\varphi\psi = Id_{\mathcal{M}}$. So, to prove Theorem A, it is enough to show that ψ is onto. Let α be a s.c.c. in Σ. We have to show that a is in K, the subgroup of G generated by $\{a_i{}'\text{s},\ d_{ij}{}'\text{s},\ x,\ t_i{}'\text{s},\ s\}$ (one has $\psi(\mathcal{M}) = K$).

First case: α does not separate Σ. There exists a diffeomorphism h of Σ such that $h(\alpha_1) = \alpha$. If we decompose h into a product of elements of $\{\alpha_i{}'\text{s},\ \delta_{ij}{}'\text{s},\ \xi,\ \theta_i{}'\text{s},\ \sigma\}$ (these elements generate $\mathcal{M}$ by Corollary 2.2) and set $m = \psi(h)$, then Corollary 1.4 implies $a = m(a_1) \in K$.

Second case: α separates in Σ. Denote by g_α the genus of the component of $\Sigma \backslash \alpha$ which does not contain δ, and by r_α its number of boundary components. Up to diffeomorphism, α is the curve shown in Figure 19.

If, in $\Sigma \backslash (\bigcup \alpha_i)$, α encircles only one hole, then $\alpha = \alpha_{g+1}$ and $a \in K$. If α encircles two holes, then α is one of the curves $\delta_{2g-1,2g}$ or $\delta_{2g+1,2g+2}$ and $a \in$

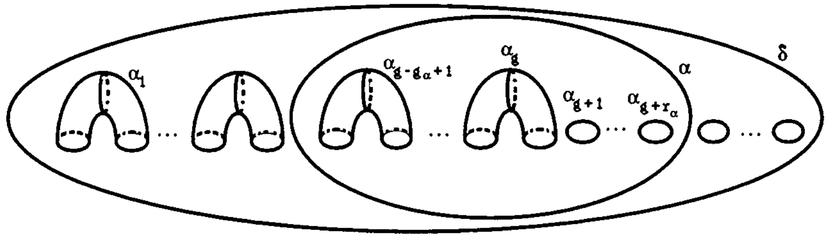

FIGURE 19

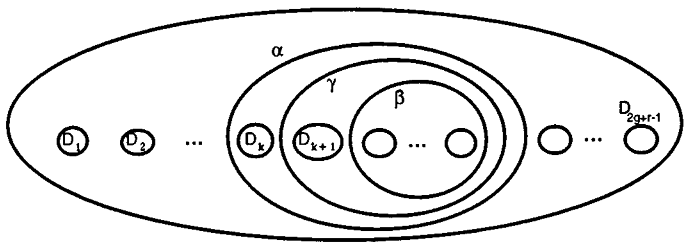

FIGURE 20

K. Otherwise, one can find three curves β, γ, λ verifying the lantern relation $a\, a_{i_k}\, a_{i_{k+1}}\, b = d_{k,k+1}\, c\, l$ in G and such that if one of them separates, it encircles fewer holes than α does (see Figure 20). Then we conclude, using the first case and an inductive argument.

We have proved the first part of Theorem A when $r \geq 1$. If $r = 0$, it is clear that the presentation given by Wajnryb in [W] together with Theorem 1.3 implies Theorem A. The last statement of the theorem was proven in Theorem 1.6.

2.2. **Proof of Theorem B.** Harer proved in [H] that, when $g \geq 3$, $\mathcal{M}_{g,r}$ is generated by twists along non-separating curves . The following two lemmas tell us what happens in genus one and two.

Lemma 2.5. *For any $r \geq 0$, the twists on non-separating curves generate $\mathcal{M}_{2,r}$.*

Proof. We only need to prove that a twist on a separating curve can be decomposed into a product of twists on non-separating curves. So, let α be a separating curve in $\Sigma_{2,r}$.

First case. One of the two components of $\Sigma\backslash\alpha$ has genus zero. Then the star $(\alpha_1, \alpha_2, \alpha_3, \beta)$ (see Figure 21) gives $(\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_\beta)^3 = \tau_\alpha\tau_{\delta_1}\tau_{\delta_2}$.

Second case. The two components of $\Sigma\backslash\alpha$ have genus one. The chain $(\alpha_1, \beta, \alpha_2)$ (see Figure 22) gives $(\tau_{\alpha_1}\tau_\beta\tau_{\alpha_2})^4 = \tau_\alpha\tau_\delta$ and we conclude using the first case. □

Lemma 2.6. *For any $r \geq 1$, $\mathcal{M}_{1,r}$ is generated by twists on $r-1$ boundary curves together with those on non-separating curves.*

This lemma will be proved in Section 5.

We shall now prove Theorem B. From now on, Σ will denote a surface of genus $g \geq 2$ with r boundary components $(r \geq 0)$. Let G' be a group with generators

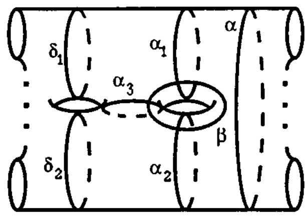

Figure 21

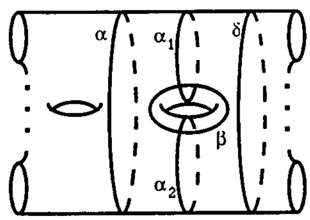

Figure 22

$\{a \equiv a_\alpha \,/\, \alpha$ non-separating s.c.c. in $\Sigma\}$ and relations (I)', (II)' and (III)'. Since G' is a quotient of S, the relations which have been shown in Section 1 will be true in G'. In particular, all lanterns, stars and chains are equal to one if the curves involved do not separate (Theorem 1.6).

Define $\varphi\colon G' \to \mathcal{M}$ by $\varphi(a)=\alpha$. φ is clearly a homomorphism. Harer's result and Lemma 2.5 show that φ is onto. We shall construct an inverse morphism $\psi\colon \mathcal{M} \to G'$. To do this, we consider $\mathcal{M}$ given by the presentation of Theorem A. So, let α be a s.c.c. in Σ. If α does not separate, we set $\psi(\alpha)=a$. Otherwise, there are the two following cases.

First case. One of the two components of $\Sigma\setminus\alpha$ has genus greater than or equal to two. Consider the curves shown in Figure 21 and define $\psi(\alpha)$ by $\psi(\alpha)= (a_1a_2a_3b)^3\,\overline{d_1}\,\overline{d_2}$. We have to show that $\psi(\alpha)$ is independent of the choice of the star. So, let $(\alpha_1', \alpha_2', \alpha_3', \beta')$ be a second star and denote $(a_1a_2a_3b)^3\,\overline{d_1}\,\overline{d_2}$ by a and $(a_1'a_2'a_3'b')^3\,\overline{d_1'}\,\overline{d_2'}$ by a'. We want to show that $a=a'$.

If all the curves involved lie in the same component Σ_1 of $\Sigma\setminus\alpha$, there is a diffeomorphism h of Σ_1 which sends the first star onto the second one fixing α (this is a consequence of the classification of surfaces). Decompose h into a product of twists on non-separating curves in Σ_1 (it can be done since Σ_1 has genus greater than or equal to two) and extend it to Σ by the identity. Then, if m denotes the word of G' associated to h, one has $a'=m(a)$ using Corollary 1.4. But if γ is a curve in the decomposition of h, then γ and α are disjoint. So, one has

$$\gamma = \tau_\alpha(\gamma) = (\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_\beta)^4\tau_{\delta_1}^{-1}\tau_{\delta_2}^{-2}(\gamma)$$

and by Corollary 1.4, $a(c)=a$. This proves that m commutes with a and so $a'=a$.

We suppose now that the two stars lie in distinct components of $\Sigma\setminus\alpha$ (see Figure 23).

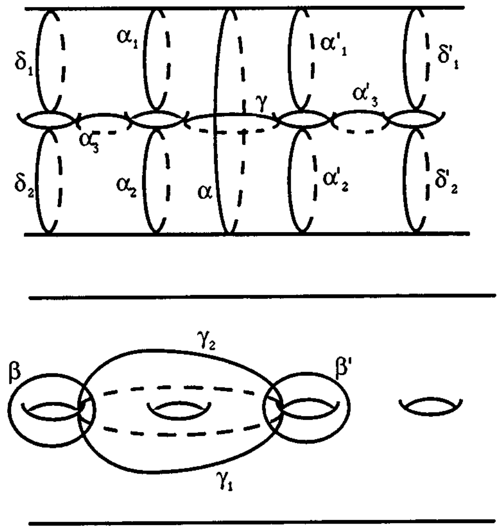

FIGURE 23

By Lemma 1.7 and stars and chains relations E_0 and C_0, one has

$$(c\,b\,a_1\,a_2\,b\,c)(a_3\,b\,a_1\,a_2\,b\,a_3) = a_1'\,a_2'\,d_1\,d_2,$$

$$(c\,b'\,a_1'\,a_2'\,b'\,c)(a_3'\,b'\,a_1'\,a_2'\,b'\,a_3') = a_1\,a_2\,d_1'\,d_2'\,.$$

Thus, we obtain, using Lemma 1.2 and (0),

$$\begin{aligned} a &= (a_1\,a_2\,a_3\,b)^3\overline{d_1}\,\overline{d_2} \\ &= [(a_1\,a_2)(b\,a_1\,a_2\,b)(a_3\,b\,a_1\,a_2\,b\,a_3)][(a_1'\,a_2')(\overline{a_3}\,\overline{b}\,\overline{a_1}\,\overline{a_2}\,\overline{b}\,\overline{a_3})(\overline{c}\,\overline{b}\,\overline{a_1}\,\overline{a_2}\,\overline{b}\,\overline{c})] \\ &= a_1\,a_2\,a_1'\,a_2'\,[b\,a_1\,a_2\,b\,\overline{c}\,\overline{b}\,\overline{a_1}\,\overline{a_2}\,\overline{b}]\,\overline{c} \end{aligned}$$

and, similarly,

$$a' = (a_1'\,a_2'\,a_3'\,b')^3\overline{d_1'}\,\overline{d_2'} = a_1'\,a_2'\,a_1\,a_2\,[b'\,a_1'\,a_2'\,b'\,\overline{c}\,\overline{b'}\,\overline{a_1'}\,\overline{a_2'}\,\overline{b'}]\,\overline{c}\,.$$

But one can check that $\tau_\beta\tau_{\alpha_1}\tau_{\alpha_2}\tau_\beta(\gamma) = \tau_{\beta'}\tau_{\alpha_1'}\tau_{\alpha_2'}\tau_{\beta'}(\gamma)$. So, if γ' denotes this curve, one has (by Corollary 1.4)

$$a = a_1\,a_2\,a_1'\,a_2'\,\overline{c'}\,\overline{c} = a'\,.$$

Lemma 2.7. *Let δ be a separating curve in Σ such that $\Sigma\backslash\delta$ has a component of genus greater than two. Let h be a diffeomorphism of Σ which is decomposed into a product of twists along non-separating curves. Then, if $\delta' = h(\delta)$ and m is the word associated to h in G', one has $\psi(\delta') = m\psi(\delta)\overline{m}$.*

Proof. Let $E = (\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star such that $\psi(\delta) = (a_1a_2a_3b)^3\overline{d_1}\,\overline{d_2}$ (Figure 21). Then, if $E' = (\alpha_1', \alpha_2', \alpha_3', \beta')$ is the image of E by h, one has $\psi(\delta') = (a_1'a_2'a_3'b')^3\overline{d_1'}\,\overline{d_2'}$. Furthermore, Corollary 1.4 gives $a_k' = m(a_k)$, $b' = m(b)$ and $d_k' = m(d_k)$, which imply the required result. □

Second case. The two components of $\Sigma\backslash\alpha$ have genus one. Consider the curves shown in Figure 22 and define $\psi(\alpha)$ by $\psi(\alpha) = (a_1ba_2)^4\,\overline{\psi(\delta)}$ (according to the first case, $\psi(\delta)$ is well defined). We have to show that $\psi(\alpha)$ is independent of the choice

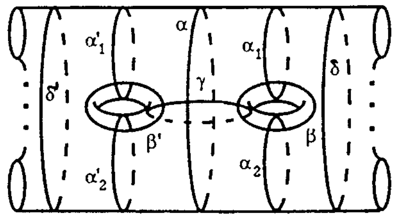

FIGURE 24

of the chain. So, let $(\alpha_1',\beta',\alpha_2')$ be a second chain and denote $(a_1ba_2)^4\,\overline{\psi(\delta)}$ by a and $(a_1'b'a_2')^4\,\overline{\psi(\delta')}$ by a'.

If the two chains lie in the same component Σ_1 of $\Sigma\setminus\alpha$, there is a diffeomorphism h of Σ_1 which sends one of the two chains to the other one fixing α. By Lemma 2.6, one can decompose h into a product of twists along non-separating curves or boundary curves. But since a twist on a boundary curve is central, one can suppose that h is a product of twists along only non-separating curves. Extend h to Σ by identity. Then, if m is the word in G' corresponding to the decomposition of h, one has by Corollary 1.4 and Lemma 2.7 $a'=m(a)$. At last, since all the curves involved in the decomposition of m are disjoint from α, m commutes with a by Corollary 1.4.

We now suppose that the chains lie in distinct components of $\Sigma\setminus\alpha$ (Figure 24). We set

$$X_1 = a_1\,a_2\,,\quad X_2 = b\,X_1\,b\,,\quad X_3 = c\,X_2\,c\,,\quad X_1' = a_1'\,a_2'\,,$$
$$X_2' = b'\,X_1'\,b'\,,\quad\text{and}\quad X_3' = c\,X_2'\,c.$$

Then, one has, using Lemma 1.2, $a=X_1^2X_2^2\,\overline{\psi(\delta)}$ with $\psi(\delta)=X_1X_2X_3\,\overline{a_1'}\,\overline{a_2'}$ and $a'={X'}_1^2{X'}_2^2\,\overline{\psi(\delta')}$ with $\psi(\delta')=X_1'X_2'X_3'\,\overline{a_1}\,\overline{a_2}$. Thus,

$$a = (X_1\,X_2)^2\,\overline{X_1}\,\overline{X_2}\,\overline{X_3}\,X_1' = X_1\,X_2\,\overline{c}\,\overline{X_2}\,\overline{c}\,X_1' \qquad\text{and}\qquad a' = X_1'\,X_2'\,\overline{c}\,\overline{X_2'}\,\overline{c}\,X_1\,.$$

But since $\tau_\beta\tau_{\alpha_1}\tau_{\alpha_2}\tau_\beta(\gamma) = \tau_{\beta'}\tau_{\alpha_1'}\tau_{\alpha_2'}\tau_{\beta'}(\gamma)$, by Corollary 1.4 we have $X_2(c) = X_2'(c)$. Thus, we obtain $a=X_1\,X_2'\,\overline{c}\,\overline{X_2'}\,\overline{c}\,X_1'=a'$.

We have proved that $\psi:\mathcal{M}\to G'$ is well defined. It remains to prove that ψ is a homomorphism. To do this, it is enough to show that the relations (I) and (IV) are mapped by ψ into true relations in G'.

The braids. Let λ and μ be two curves in Σ, $\nu=\tau_\mu(\lambda)$. We have to prove that $\psi(\nu)=\psi(\mu)\psi(\lambda)\psi(\mu)^{-1}$. This is shown by Corollary 1.4 if λ does not separate. Otherwise, if a component of $\Sigma\setminus\lambda$ has genus greater than two, the relation is still true by Lemma 2.7.

Finally, let us suppose that λ separates and that the two components of $\Sigma\setminus\lambda$ have genus one. Let $C(\alpha_1,\beta,\alpha_2)$ be a chain such that $\psi(\lambda)=(a_1\,b\,a_2)^4\,\overline{\psi(\delta)}$ (see Figure 22) and denote by α_1', β', α_2' and δ' the respective image of α_1, β, α_2 and δ under τ_μ. Then, one has $\psi(\nu)=(a_1'\,b'\,a_2')^4\,\overline{\psi(\delta')}$.

But there are some non-separating curves $\varepsilon_1,\dots,\varepsilon_p$ such that $\psi(\mu)$ is equal to the product $e_1\dots e_p$. Thus, by Corollary 1.4, one has $a_k'=\psi(\mu)(a_k)$, $b'=\psi(\mu)(b)$ and, by Lemma 2.7, $\psi(\delta')=\psi(\mu)(\psi(\delta))$. So, we obtain $\psi(\nu)=\psi(\mu)\psi(\lambda)\psi(\mu)^{-1}$.

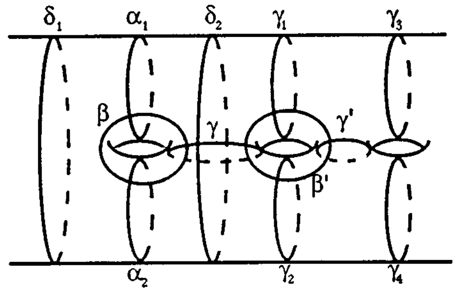

FIGURE 25

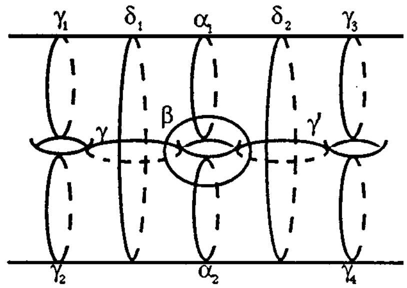

FIGURE 26

Remark. Since the braids relations are mapped by ψ into true relations, one has in G', for any disjoint s.c.c. α and β in Σ, the relation

$$(\mathcal{B}) \qquad \psi(\alpha)\psi(\beta) = \psi(\beta)\psi(\alpha).$$

Lemma 2.8. *The chain relations are satisfied in G'.*

Proof. Let $(\alpha_1, \beta, \alpha_2)$ be a chain C, and δ_1 and δ_2 its boundary components. If δ_1 and δ_2 do not separate, the relation is satisfied by (II)' and Theorem 1.6. So, we suppose that δ_1 and δ_2 separate in Σ. If g is equal to two, the relation is satisfied by definition of $\psi(\delta_k)$. And when $g \geq 3$, two different cases occur, which are described by Figures 25 and 26 (we have to consider the second case only if $g=3$).

First case (*Figure* 25). The definition of $\psi(\delta_k)$ gives $\psi(\delta_1) = X_1X_2X_3\overline{Y_1}$ and $\psi(\delta_2) = Y_1Y_2Y_3'\,\overline{c_3}\,\overline{c_4}$, where

$$X_1 = a_1\,a_2, \qquad X_2 = b\,X_1\,b, \qquad X_3 = c\,X_2\,c\ ,$$
$$Y_1 = c_1\,c_2, \qquad Y_2 = b'\,Y_1\,b', \qquad Y_3' = c'\,Y_2\,c'\ .$$

Furthermore, if $Y_3 = cY_2c$, one has by Lemma 1.7 $Y_3Y_3' = a_1a_2c_3c_4$. Thus, we obtain

$$\begin{aligned}\psi(\delta_1)\psi(\delta_2) &= X_1\,X_2\,X_3\,Y_2\,Y_3'\,\overline{c_3}\,\overline{c_4} = (X_1\,X_2\,X_3\,Y_2)(\overline{Y_3}\,X_1)\\ &= X_1^2\,X_2^2\,\overline{X_2}\,c\,X_2\,c\,Y_2\,\overline{c}\,\overline{Y_2}\,\overline{c}\,.\end{aligned}$$

At last, since $\overline{X_2}(c) = cY_2(c)$ (Corollary 1.4), we have $X_1^2X_2^2 = \psi(\delta_1)\psi(\delta_2)$.

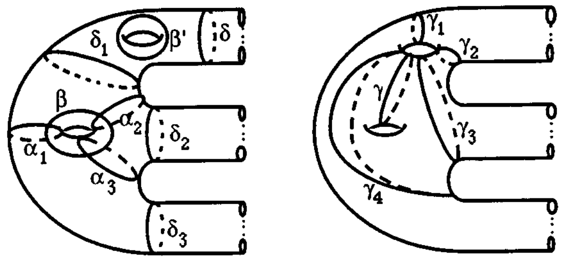

FIGURE 27

Second case (*Figure* 26). Set $X_1 = a_1\, a_2$, $X_2 = b\, X_1\, b$, $X_3 = c\, X_2\, c$ and $X'_3 = c' X_2 c'$. Then, by definition, we have

$$\psi(\delta_1) = X_1 X_2 X'_3\, \overline{c_3}\, \overline{c_4} \quad \text{and} \quad \psi(\delta_2) = X_1 X_2 X_3 \overline{c_1}\, \overline{c_2}$$

and so

$$\psi(\delta_1)\psi(\delta_2) = X_1^2 X_2^2 X'_3 X_3 \overline{c_1}\, \overline{c_2}\, \overline{c_3}\, \overline{c_4}.$$

Then, using the equality $X'_3 X_3 = c_1\, c_2\, c_3\, c_4$ (given by Lemma 1.7), one obtains the required result. □

The stars. Let $(\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star E in Σ, and let δ_1, δ_2, and δ_3 be the boundary components. If all these curves do not separate Σ, E is satisfied in G' by (II)', (III)' and Theorem 1.6. If only one curve δ_i separates, E is satisfied by definition of $\psi(\delta_i)$. So, let us suppose that δ_1, δ_2 and δ_3 separate Σ. Since $g \geq 2$, one can suppose that the component of $\Sigma \backslash \delta_1$ which does not contain δ_2 is of genus greater than or equal to one. Consider the curves shown on Figure 27, and set

$$X_1 = a_3\, c\ , \quad X_2 = b\, X_1\, b\ , \quad X_3 = a_1\, X_2\, a_1\ , \quad X'_3 = a_2\, X_2\, a_2\ ,$$

$$Y_1 = c_1\, c_2\ , \quad Y_2 = b'\, Y_1\, b'\ , \quad Y_3 = c\, Y_2\, c\ .$$

By definition of $\psi(\delta_2)$, $\psi(\delta_3)$ and $\psi(\delta)$, one has

$$X_1\, X_2\, X_3 = c_1\, c_3\, \psi(\delta_3)\ ,\ X_1\, X_2\, X'_3 = c_2\, c_4\, \psi(\delta_2)\ ,\ Y_1\, Y_2\, Y_3 = a_1\, a_2\, \psi(\delta)$$

and Lemma 2.8 gives $X_1^2\, X_2^2 = c_3\, c_4$ and $Y_1^2\, Y_2^2 = \psi(\delta_1)\psi(\delta)$. From these relations we obtain

$$\begin{aligned}
\psi(\delta_2)\psi(\delta_3) &= X_1\, X_2\, X'_3\, \overline{c_2}\, \overline{c_4}\, X_1\, X_2\, X_3\, \overline{c_1}\, \overline{c_3} \\
&= X_1^2 X_2^2\, \overline{c_1}\, \overline{c_2}\, \overline{c_3}\, \overline{c_4}\, X_3\, X'_3 \quad \text{by } (\mathcal{B}) \text{ and Lemma 1.2} \\
&= \overline{c_1}\, \overline{c_2}\, (a_1\, b\, a_3\, c\, b\, a_1)(a_2\, b\, a_3\, c\, b\, a_2) \\
&= \overline{c_1}\, \overline{c_2}\, (c\, b\, a_1\, a_2\, b\, c)(a_3\, b\, a_1\, a_2\, b\, a_3) \quad \text{by } (\mathcal{B}), \\
\psi(\delta_1)\psi(\delta_2)\psi(\delta_3) &= (Y_1^2 Y_2^2\, \psi(\delta)^{-1})\overline{c_1}\, \overline{c_2}\, (c\, b\, a_1\, a_2\, b\, c)(a_3\, b\, a_1\, a_2\, b\, a_3) \\
&= Y_1^2 Y_2^2\, (\overline{Y_1}\, \overline{Y_2}\, \overline{Y_3}\, a_1\, a_2)\, \overline{Y_1}\, (c\, b\, a_1\, a_2\, b\, c)(a_3\, b\, a_1\, a_2\, b\, a_3) \\
&= a_1\, a_2\, Y_2\, \overline{c}\, \overline{Y_2}\, \overline{c}\, (c\, b\, a_1\, a_2\, b\, c)(a_3\, b\, a_1\, a_2\, b\, a_3) \quad \text{by } (0).
\end{aligned}$$

Now, set $Z_1 = a_1 a_2$, $Z_2 = b Z_1 b$ and $Z_3 = a_3 Z_2 a_3$. Then, using Corollary 1.4 one can check that $Y_2(c) = Z_2(c)$. Thus, we obtain

$$\psi(\delta_1)\psi(\delta_2)\psi(\delta_3) = Z_1\, [Z_2\, \overline{c}\, \overline{Z_2}\, \overline{c}][c\, Z_2\, c] Z_3 = Z_1\, Z_2\, Z_3\, .$$

Lemma 1.2 shows that this last relation is the star E.

This proves that ψ is a homomorphism. One has clearly $\psi \circ \varphi = \mathrm{Id}_{G'}$ so φ is a monomorphism from G' to $\mathcal{M}$. Since φ is onto, this concludes the proof of Theorem B.

3. Mapping class group with p_1-structure

A Topological Quantum Field Theory is a functor from a cobordism category to a category of modules with suitable properties (see for instance [A1]). When we consider the cobordism category of 2-dimensional surfaces and 3-dimensional cobordisms, this functor produces projective representations of mapping class groups. One can linearize this representation by considering a certain central extension (G. Masbaum and J. Roberts describe some of these extensions in [M-R]).

In [BHMV2], the authors construct a TQFT V_p from the Kauffman bracket. In order to solve the framing anomaly, all the manifolds are equipped with p_1-structure. The central extension obtained this way is the mapping class group with p_1-structure and the goal of this section is to give a presentation of this group.

3.1. **Definitions.**

Definition 3.1. Let X be the homotopy fiber of the map $p_1 : BO \to K(\mathbb{Z}, 4)$ corresponding to the first Pontryagin class of the universal stable bundle γ over BO. A p_1-*structure* on a manifold M is a lifting of a classifying map of the stable tangent bundle of M to X.

Definition 3.2. A *homotopy* between two p_1-structures Φ_0 and Φ_1 on a manifold M is a p_1-structure Φ on $M \times I$ such that $\Phi_{|M\times 0}$ and $\Phi_{|M\times 1}$ coincide respectively with Φ_0 and Φ_1.

If N is a submanifold of M, one can define a homotopy $\mathrm{rel}\, N$ in the obvious way.

Remarks. (i) A p_1-structure up to homotopy is the analogue of a spin structure, where the second Stiefel-Whitney class w_2 is replaced by the first Pontryagin class p_1.

(ii) We consider actual p_1-structure rather than the homotopy class of such a structure. This allows manifolds with p_1-structure to be glued along parts of their boundary.

For the relevant classical algebraic topology or structure on manifolds, see for instance [S] or [Sw]. The following fact is easily proven using obstruction theory.

Proposition 3.3. *If M is a manifold and N a submanifold of M, then, the set of homotopy class* $\mathrm{rel}\, N$ *of p_1-structure on M is affinely isomorphic to* $H^3(M, N; \mathbb{Z})$.

Thus, manifolds of dimension less than or equal to two admit a unique p_1-structure up to homotopy, and the set of homotopy class rel boundary of p_1-structure on an oriented, compact, connected 3-manifold is affinely isomorphic to $\mathbb{Z}$.

Now, let Σ be an oriented, connected surface and φ a given p_1-structure on Σ. We shall define $\widetilde{\mathcal{M}}_\Sigma$, the mapping class group with p_1-structure. For $f \in \mathcal{M}_\Sigma$, we provide $\partial(\Sigma \times I)$ with the following p_1-structure: $\varphi \circ f$ on $\Sigma \times 0$ and φ on $(\Sigma \times 1 \cup \partial\Sigma \times I)$. This can be extended to $\Sigma \times I$, and P_f, the set of homotopy class of such extensions, is affinely isomorphic to $\mathbb{Z}$.

Definition 3.4. The *mapping class group with p_1-structure* is the set of all pairs (f, F) where $f \in \mathcal{M}_\Sigma$ and $F \in P_f$, together with the obvious composition.

Remark. Atiyah ([A2]) has previously defined this group in a different way.

The forgetful map μ is an epimorphism from $\widetilde{\mathcal{M}}_\Sigma$ to $\mathcal{M}_\Sigma$, with kernel $\mathbb{Z}$. Let us denote by u a generator of $Ker\,\mu$. Since to modify a given p_1-structure on $\Sigma \times I$, it is enough to change it in a small 3-ball, u is in the center of $\widetilde{\mathcal{M}}_\Sigma$. So, we obtain the following central extension of $\mathcal{M}_\Sigma$:

$$(**) \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathcal{M}} \xrightarrow{\mu} \mathcal{M} \longrightarrow 1.$$

Note that this extension is natural in the following sense: if Σ' is a subsurface of Σ, then we have the following commutative diagram, where i_* and $i_\sharp$ are the obvious morphisms induced by the inclusion i of Σ' in Σ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathcal{M}}_{\Sigma'} & \xrightarrow{\mu'} & \mathcal{M}_{\Sigma'} & \longrightarrow & 1 \\ & & || & & \downarrow i_* & & \downarrow i_\sharp & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathcal{M}}_{\Sigma} & \xrightarrow{\mu} & \mathcal{M}_{\Sigma} & \longrightarrow & 1. \end{array}$$

3.2. **Presentation of $\widetilde{\mathcal{M}}$.** Let τ_α be a Dehn twist. We shall construct a canonical lifting $\tilde{\tau}_\alpha = (\tau_\alpha, A)$. To do this, consider a neighborhood $V \approx S^1 \times I_1$ $(I_1 = [0,1])$ of α in Σ and define A on $(\Sigma \backslash V) \times I$ by $A(\sigma, t) = \varphi(\sigma)$. We have to extend to $V \times I$ the p_1-structure on $\partial(V \times I)$ which is equal to $\varphi \circ \tau_\alpha$ on $V \times 0$ and to φ on $(V \times 1 \cup (\partial V) \times I)$.

Now, we paste a copy of $D^2 \times I_1$ to V identifying $S^1 \times I_1$ with V.

* τ_α is a diffeomorphism of $S^1 \times I_1$ which can be extended to $D^2 \times I_1$.
* Since $H^p(D^2 \times I_1, S^1 \times I_1; \mathbb{Z}) = 0$ for $p = 3$ or 4, there is one and only one way to extend $\varphi_{|V}$ to $D^2 \times I_1$ (up to homotopy).

So, we can define a p_1-structure Φ on $D^2 \times \partial(I_1 \times I)$ which extends the given one on $\partial(V \times I) = S^1 \times \partial(I_1 \times I)$ as follow:

$$\begin{array}{ll} \Phi(x, \tau, t) = \varphi(x, \tau) & \text{for } (x, \tau, t) \in D^2 \times \partial I_1 \times I, \\ \Phi(x, \tau, 0) = \varphi \circ \tau_\alpha(x, \tau) & \text{for } (x, \tau) \in D^2 \times I_1, \\ \Phi(x, \tau, 1) = \varphi(x, \tau) & \text{for } (x, \tau) \in D^2 \times I_1. \end{array}$$

Since $H^p(D^2 \times I_1 \times I, D^2 \times \partial(I_1 \times I); \mathbb{Z}) = 0$ when p is equal to four or three, there is one and only one way to extend Φ to $D^2 \times I_1 \times I$ up to homotopy. We will denote this extension by π_α, and we define A on $V \times I$ by $A(\sigma, t) = \pi_\alpha(\sigma, t)$. This gives a well defined p_1-structure A on $\Sigma \times I$, and we will denote by $\tilde{\tau}_\alpha$ the element (τ_α, A) of $\widetilde{\mathcal{M}}$.

Remark. It is easy to see that when α bounds a disc in Σ, then $\tilde{\tau}_\alpha = 1_{\widetilde{\mathcal{M}}}$.

Since Dehn twists generate $\mathcal{M}$, the set of all $\tilde{\tau}_\alpha$ together with the generator u of $\mathbb{Z}$ generate $\widetilde{\mathcal{M}}$. Because u is central in $\widetilde{\mathcal{M}}$, one has the relation $u\tilde{\tau}_\alpha = \tilde{\tau}_\alpha u$ for any s.c.c. α in Σ. Theorems 3.7, 3.8 and 3.9 below tell us how braids, lanterns, stars and chains are lifted to $\widetilde{\mathcal{M}}$ by the section $s : \mathcal{M} \to \widetilde{\mathcal{M}}$, $s(\tau_\alpha) = \tilde{\tau}_\alpha$. Using these results, we obtain from Theorems A and B the two following presentations of $\widetilde{\mathcal{M}}$.

Notation. If $m\in\mathcal{M}$ is a word $\tau_{\alpha_1}\dots\tau_{\alpha_p}$, we will denote by $\widetilde{m}$ the corresponding word $\tilde{\tau}_{\alpha_1}\dots\tilde{\tau}_{\alpha_p}\in\widetilde{\mathcal{M}}$.

Theorem 3.5. *For any oriented surface Σ, $\widetilde{\mathcal{M}}_\Sigma$ has the following presentation:*
generators: $\{\tilde{\tau}_\alpha \,/\, \alpha$ *s.c.c. in* $\Sigma\}\cup\{u\}$ *(if α bounds a disc in Σ, then $\tilde{\tau}_\alpha=1$) .*
relations: *(1) For any s.c.c. α in Σ, $\tilde{\tau}_\alpha u=u\tilde{\tau}_\alpha$.*
(2) $\widetilde{T}=1$ for any braids T_0, T_1 and T_{2_0} .
(3) $\widetilde{C}=u^{12}$ for any chain C.
(4) $\widetilde{L}=1$ for any lantern L.
If $g\geq 1$, one can replace (3) and (4) by
(5) $\widetilde{E}=u^{12}$ for any star E.

With the non-separating curves, we obtain

Theorem 3.6. *For any oriented surface Σ of genus greater than or equal to two, $\widetilde{\mathcal{M}}_\Sigma$ has the following presentation:*
generators: $\{\tilde{\tau}_\alpha \,/\, \alpha$ *non-separating s.c.c. in* $\Sigma\}\cup\{u\}$
relations: *(1)' For any non-separating s.c.c. α in Σ, $\tilde{\tau}_\alpha u=u\tilde{\tau}_\alpha$.*
(2)' $\widetilde{T}=1$ for any braids T_0 and T_1 between only non-separating curves.
(3)' $\widetilde{C_0}=u^{12}$ for one chain C_0 with non-separating curves.
(4)' $\widetilde{L_0}=1$ for one lantern L_0 with non-separating curves.
When $g\geq 3$, one can replace (3)' or (4)' by
(5)' $\widetilde{E_0}=u^{12}$ for one star E_0 (with non-separating curves).

3.3. **Lifting of the braid relations.** The following theorem proves that the braid relations are still true in $\widetilde{\mathcal{M}}_{g,r}$, where g and r are non-negative.

Theorem 3.7. *Let α be a s.c.c. in Σ, $m=(f,F)\in\widetilde{\mathcal{M}}$ and $\gamma=f(\alpha)$. Then, in $\widetilde{\mathcal{M}}$ we have the relation*

$$\tilde{\tau}_\gamma = m\tilde{\tau}_\alpha\overline{m}\,.$$

Proof. Denote the element $m\tilde{\tau}_\alpha\overline{m}$ by (τ_γ,Φ) (see Figure 28) and consider a neighborhood V of γ in Σ. $V'=f^{-1}(V)$ is a neighborhood of α.

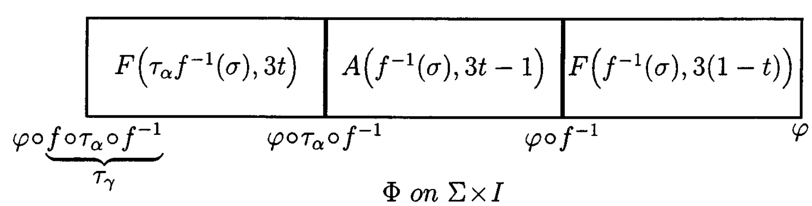

FIGURE 28

First step. We shall homotope Φ rel $\partial(\Sigma\times I)$ to a p_1-structure Ψ on $\Sigma\times I$ such that $\Psi(\sigma,t)=\varphi(\sigma)$ for $\sigma\notin V$. To do this, we have to define a p_1-structure H on $\Sigma\times I\times I$ such that:

$$H_{|\Sigma\times I\times 0}(\sigma,t,0)=\Phi(\sigma,t)\ ,\quad H_{|(\Sigma\setminus V)\times I\times 1}(\sigma,t,1)=\varphi(\sigma)\ ,$$
$$H_{|\Sigma\times 0\times I}(\sigma,0,\theta)=\varphi\circ\tau_\gamma(\sigma)\ ,\quad H_{|\Sigma\times 1\times I}(\sigma,1,\theta)=\varphi(\sigma)=H_{|\partial\Sigma\times I\times I}(\sigma,t,\theta)\ .$$

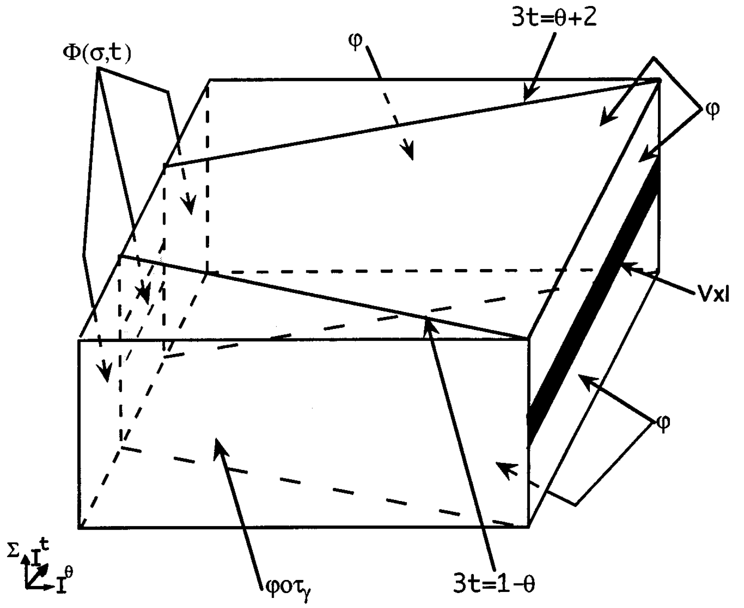

FIGURE 29

First, we define

$$H(\sigma,t,\theta)=\left|\begin{array}{ll} F(\tau_\alpha f^{-1}(\sigma),3t) & \text{if } 3t\le 1-\theta\,,\\ F(f^{-1}(\sigma),3(1-t)) & \text{if } 3t\ge 2+\theta\,,\\ F(f^{-1}(\sigma),1-\theta) & \text{if } 1-\theta\le 3t\le 2+\theta \text{ and } \sigma\notin V\,.\end{array}\right.$$

Then, it remains us to extend H to $W=\{(\sigma,t,\theta)\in V\times I\times I$ such that $1-\theta\le 3t\le 2+\theta\}$, which is diffeomorphic to $V\times I\times I$ (see Figure 29; we will identify W with $V\times I\times I$). H is given on $M=(\partial(V\times I)\times I)\cup(V\times I\times 0)$. Since $H^4(W,M)=H^3(W,M)=0$, there is one and only one way to extend H to W up to homotopy. We define Ψ by $\Psi(\sigma,t)=H(\sigma,t,1)$.

Second step. To show that $(\tau_\gamma,\Psi)=\tilde\tau_\gamma$, we only need to look at the p_1-structure on $V\times I$. So, from now on, φ is considered as a p_1-structure on V and V', and f is considered as a diffeomorphism from V' to V. Consider two copies of $D^2\times I_1$ (one is denoted $D'^2\times I_1$) and paste them on V and V'. Extend f to a diffeomorphism from $D'^2\times I_1$ to $D^2\times I_1$, and φ to a p_1-structure on $D^2\times I_1$ and $D'^2\times I_1$. Then, we define a p_1-structure Ξ on $\partial(D^2\times I_1\times I)$ by

$$\Xi(x,\tau,t)=\left\{\begin{array}{ll} \Psi(x,\tau,t) & \text{if } x\in S^1\,,\\ \varphi\circ\tau_\gamma(x,\tau) & \text{if } t=0\,,\\ \varphi(x,\tau) & \text{if } t=1 \text{ or } \tau\in\{0,1\}\,.\end{array}\right.$$

To prove that $(\tau_\gamma,\Psi)=\tilde\tau_\gamma$, it is enough to show that Ξ can be extended to $D^2\times I_1\times I$ (see definition of $\tilde\tau_\gamma$ in 3.2). To do this, we shall find a p_1-structure K on $D^2\times I_1\times I\times I$ such that $K_{|\partial(D^2\times I\times I)\times 1}$ coincides with Ξ. Then, $K_{|D^2\times I\times I\times 1}$ will be the extension of Ξ.

We recall that F is a p_1-structure on $\Sigma\times I$. We shall extend it to $D'^2\times I_1\times I$. We consider its restriction to $V'\times I=S^1\times I_1\times I$, which we will also denote F.

Since $H^4(D^2\times I_1\times I,(S^1\times I_1\times I)\cup(D^2\times I_1\times\partial I))=0$, there exists a p_1-structure $\hat{F}$ on $D'^2\times I_1\times I$ such that $\hat{F}_{|S^1\times I_1\times I}=F$, $\hat{F}_{|D'^2\times I_1\times 0}=\varphi\circ f$ and $\hat{F}_{|D'^2\times I_1\times 1}=\varphi$.

Now, we can define K on $M'=\overline{\partial(D^2\times I_1\times I\times I)\setminus D^2\times I\times I\times 1}$ by

$$K(x,\tau,t,\theta)=\begin{cases}\pi_\alpha(f^{-1}(x,\tau),t) & \text{if } \theta=0\,,\\ H(x,\tau,t,\theta) & \text{if } x\in S^1 \ \ (\text{with } W\equiv S^1\times I_1\times I\times I)\,,\\ \hat{F}(f^{-1}(x,\tau),1-\theta) & \text{if } t=1 \ \text{ or } \ \tau\in\{0,1\}\,,\\ \hat{F}(\tau_\alpha f^{-1}(x,\tau),1-\theta) & \text{if } t=0\,.\end{cases}$$

One can check that K coincides with Ξ on $\partial(D^2\times I_1\times I)\times 1$; since when $p=3$ or 4 we have $H^p(D^2\times I_1\times I\times I,M')=0$ K can be extended in a unique way up to homotopy to $D^2\times I_1\times I\times I$. □

3.4. **Lifting of the lantern relations.**

Theorem 3.8. *Let Σ be a sphere with four holes. If $\alpha_1,\dots,\alpha_4,\beta_1,\beta_2$ and β_3 are the curves shown in Figure 1, then one has in $\widetilde{\mathcal{M}}_\Sigma$:*

$$\tilde\tau_{\alpha_1}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_3}\tilde\tau_{\alpha_4}=\tilde\tau_{\beta_3}\tilde\tau_{\beta_2}\tilde\tau_{\beta_1}.$$

Consequence. Since the extension (∗∗) is natural, the lantern relations will be true in $\widetilde{\mathcal{M}}_\Sigma$ for any surface Σ.

Proof. Since $\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4}=\tau_{\beta_3}\tau_{\beta_2}\tau_{\beta_1}$, there exists an integer n such that

$$\tilde\tau_{\alpha_1}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_3}\tilde\tau_{\alpha_4}=\tilde\tau_{\beta_3}\tilde\tau_{\beta_2}\tilde\tau_{\beta_1}u^n.$$

Consider an embedding of Σ in a surface Σ' such that α_4 bounds a disc in Σ'. Then, we have $\tilde\tau_{\alpha_4}=1$ and β_k is isotopic to α_k, thus $\tilde\tau_{\beta_k}=\tilde\tau_{\alpha_k}$ and $\tilde\tau_{\alpha_1}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_3}=\tilde\tau_{\alpha_3}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_1}u^n$, which implies that $n=0$. □

3.5. **Lifting of the star and chain relations.**

Theorem 3.9. *Let Σ be a surface of genus one with three holes. If α_1, α_2, α_3, β, δ_1, δ_2 and δ_3 are the curves shown in Figure 2, then, in $\widetilde{\mathcal{M}}_\Sigma$, we have the relation:*

$$(\tilde\tau_{\alpha_1}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_3}\tilde\tau_\beta)^3=\tilde\tau_{\delta_1}\,\tilde\tau_{\delta_2}\,\tilde\tau_{\delta_3}\,u^{12}\,.$$

Consequence. For any surface Σ and any stars E or chain C in Σ, in $\widetilde{\mathcal{M}}_\Sigma$ we have the relation $\widetilde{E}=\widetilde{C}=u^{12}$.

Proof. Since the star relations are verified in $\mathcal{M}$, there exists an integer n such that $(\tilde\tau_{\alpha_1}\tilde\tau_{\alpha_2}\tilde\tau_{\alpha_3}\tilde\tau_\beta)^3=\tilde\tau_{\delta_1}\,\tilde\tau_{\delta_2}\,\tilde\tau_{\delta_3}\,u^n$. Consider an embedding of Σ in $S^1\times S^1$ and denote by α the common image of α_1, α_2 and α_3. Since each of the three curves δ_1, δ_2 and δ_3 bounds a disc in $S^1\times S^1$, the relation becomes

$$(\star)\qquad (\tilde\tau_\alpha\tilde\tau_\beta\tilde\tau_\alpha)^4=u^n.$$

In order to prove that $n=12$, we shall use the action of $\widetilde{\mathcal{M}}_{S^1\times S^1}$ on $V_p=V_p(S^1\times S^1)$ induced by the TQFT developped in [BHMV2].

First, let us describe V_p. We suppose that $p\geq 4$ and we consider the ring k_p define by

$$k_p=\mathbb{Z}[\tfrac{1}{p},A,\kappa]/(\varphi_{2p}(A),\kappa^6-u)$$

where $\varphi_{2p}(A)$ is the $2p$-th cyclotomic polynomial and $u = A^{-6-\frac{p(p+1)}{2}}$. Let us denote by $K(D^2\times S^1)$ (resp. $K(S^1\times D^2)$) the Jones-Kauffman module of $D^2\times S^1$ (resp. $S^1\times D^2$) over k_p. There is a bilinear form

$$\langle\ ,\ \rangle : K(D^2\times S^1)\times K(-S^1\times D^2)\longrightarrow k_p$$

which associates to a pair of banded links L, L' in $D^2\times S^1$ and $S^1\times D^2$ the Kauffman bracket of $L\cup L'$ in S^3, using the standard decomposition

$$D^2\times S^1 \underset{S^1\times S^1}{\cup} S^1\times(-D^2)$$

of S^3. Then, V_p is the quotient of $K(D^2\times S^1)$ by the left kernel. It is shown in [BHMV1] that V_p is a k_p-module of dimension $[(p-1)/2]$.

The Dehn twist τ_α induces an obvious endomorphism t on $K(D^2\times S^1)$ which descends to V_p. Similarly, τ_β induces an endomorphism on $K(-S^1\times D^2)$. Considering its adjoint, we obtain a new endomorphism t' on V_p. Then, the actions of $\tilde\tau_\alpha$ and $\tilde\tau_\beta$ on V_p coincide respectively with t and t'. By using methods and results of [BHMV1], one can check that $(tt't)^2 = -A^{-6}A^{-p(p-1)/2}\,\mathrm{Id}_{V_p}$ when p is odd. So, since u acts as the multiplication by the scalar κ, $(\star)$ implies $A^{-12}A^{-p(p-1)} = \kappa^n$ in k_p. Suppose now that $p\equiv -1 \mod 4$. Then, $\kappa^n = \kappa^{12}$ and, since $\kappa^6 = A^{-6}$, we obtain $1 = \kappa^{6(n-12)} = A^{-6(n-12)}$. Moreover, since A is a primitive $2p$-th root of unity, one has $-6(n-12)\equiv 0\,(2p)$ for all integers $p\geq 4$ such that $p\equiv -1 \mod 4$. This proves the required equality $n=12$. □

4. On central extensions of $\mathcal{M}_\Sigma$

Let Σ be a surface of genus $g\geq 0$ with r boundary components ($r\geq 0$). Recall that S_Σ is a group generated by $\{a_\alpha\,/\,\alpha$ a non-separating s.c.c. in $\Sigma\}$ and defined by the relations T_0 and T_1. Let $\mathcal{T}_\Sigma$ be the subgroup of $\widetilde{\mathcal{M}}_\Sigma$ generated by $\{\tilde\tau_\alpha\,/\,\alpha$ s.c.c. in $\Sigma\}\cup\{u^{12}\}$. In this section, we shall prove the following theorem:

Theorem 4.1. ***(i)*** *Considering the restriction of μ to $\mathcal{T}_\Sigma$, we have the natural central extension*

$$(*)\qquad 0\longrightarrow \mathbb{Z}\longrightarrow \mathcal{T}_\Sigma \xrightarrow{\ \mu\ } \mathcal{M}_\Sigma\longrightarrow 1.$$

(ii) *If $g\geq 2$, $\mathcal{T}_\Sigma$ is generated by $\{\tilde\tau_\alpha\,/\,\alpha$ non-separating s.c.c. in $\Sigma\}$.*

(iii) *One can define natural homorphisms $\varphi : S_\Sigma\to\mathcal{T}_\Sigma$ and $\psi : S_\Sigma\to\mathbb{Z}$ by $\varphi(a_\alpha)=\tilde\tau_\alpha$ and $\psi(a_\alpha)=1$.*

When $g\geq 3$, φ and ψ induce an isomorphism between S_Σ and $\mathcal{T}_\Sigma\times\mathbb{Z}$.

When $g=2$, φ is an isomorphism from S_Σ to $\mathcal{T}_\Sigma$.

(iv) *When $g\geq 3$, the extension (*) is the universal central extension of $\mathcal{M}_\Sigma$.*

Clearly, this theorem proves Theorem C (the functorial properties are obvious) and implies the following two corollaries:

Corollary 4.2. *The kernel of the quotient map $S_\Sigma\to\mathcal{M}_\Sigma$ is isomorphic to $\mathbb{Z}$ (generated by C_0) when $g=2$ and to $\mathbb{Z}\oplus\mathbb{Z}$ (generated by L_0 and C_0) when $g\geq 3$.*

Corollary 4.3. *Suppose that $g\geq 3$. Then, $\mathcal{T}_\Sigma$ admits a presentation with generators $\{a_\alpha\,/\,\alpha$ non-separating s.c.c. in $\Sigma\}$ and the relations* (I)' *and* (II)' *(braids T_0 and T_1 and a lantern L with only non-separating curves).*

Proof of Theorem 4.1. (i) is clear since the kernel of $\mu_{|\mathcal{T}_\Sigma}$ is generated by u^{12}.

To prove (ii), first one can see, by considering a chain C without separating curves and using Theorem 3.9, that u^{12} is in the subgroup of $\widetilde{\mathcal{M}}_\Sigma$ generated by $\{\tilde{\tau}_\alpha \,/\, \alpha$ a non-separating s.c.c. in $\Sigma\}$. Then, we can conclude exactly as in the proof of Lemma 2.5.

Since braid relations are verified in $\widetilde{\mathcal{M}}_\Sigma$, the map φ is a morphism. Furthermore, (ii) implies that φ is onto when $g \geq 2$. Now, we shall determine $\operatorname{Ker}\varphi$. Denote by $q : S_\Sigma \to \mathcal{M}_\Sigma$ the quotient map given by Theorem B. First, we suppose that $g \geq 3$. Then, $\operatorname{Ker} q$ is generated by two elements C_0 and L_0 coming from chains and lanterns. So, considering the commutative diagram

$$\begin{array}{ccc} S_\Sigma & \xrightarrow{\quad q \quad} & \mathcal{M}_\Sigma \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\mu} & \\ & \mathcal{T}_\Sigma & \end{array}$$

we see that $\operatorname{Ker}\varphi \subset \langle C_0, L_0\rangle$. Since this group is abelian (Theorem 1.6) and $\varphi(L_0)=1$ and $\varphi(C_0)=u^{12}$, we have $\operatorname{Ker}\varphi = \langle L_0\rangle$. But since ψ is the abelianized morphism and $\psi(L_0) = 1$, we have $\langle L_0\rangle = \mathbb{Z}$ and we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow S_\Sigma \xrightarrow{\varphi} \mathcal{T}_\Sigma \longrightarrow 1 .$$

Furthermore, since $\psi : S_\Sigma \to \mathbb{Z}$ is a section, this sequence splits. When $g=2$, $\operatorname{Ker} q$ is cyclic generated by C_0. Thus, φ is an isomorphism.

(iv) was shown by Harer in [H]. □

5. The genus one case

The goal of this section is to prove Lemma 2.6 and the fact that there is no epimorphism from $S_{1,r}$ to $\mathcal{M}_{1,r}$ if $r \geq 2$. In fact, we shall prove stronger results.

Lemma 5.1. *Considering the curves shown on Figure* 30, $\mathcal{M}_{1,r}$ *is generated by the twists* τ_{α_i}, τ_β *and* τ_{δ_j} *for* $j \geq 2$.

Proof. We shall use an inductive argument. Capping off δ_r by a disc with center p, one can obtain the following two exact sequences (see [B] and [H]):

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_1} \mathcal{M}_{1,r} \xrightarrow{f_2} \mathcal{N}_{1,r-1} \longrightarrow 1,$$

$$0 \longrightarrow \pi_1(\Sigma_{1,r-1}, p) \xrightarrow{f_3} \mathcal{N}_{1,r-1} \xrightarrow{f_4} \mathcal{M}_{1,r-1} \longrightarrow 1,$$

where $\mathcal{N}_{1,r-1}$ is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{1,r-1}$ which are the identity on the boundary and which leave fixed the base point p. The four morphisms f_k are defined as follows:

- → $f_1(1) = \delta_r$.
- → f_2 is obtained by extending each map over the disc by the identity.
- → If α is a loop based on p and α' and α'' the two boundary components of a neighborhood of α, then $f_3(\alpha) = \tau_{\alpha'}\tau_{\alpha''}^{-1}$ (here and below, we will denote by the same letter a curve and its homotopy class).
- → f_4 is defined by forgetting the base point p.

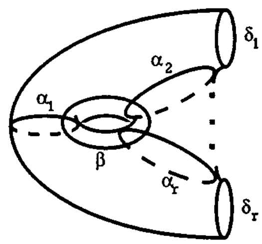

FIGURE 30

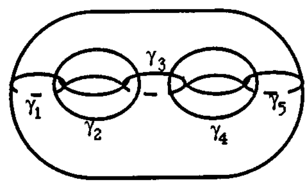

FIGURE 31

It is known that $\mathcal{M}_{1,1}$ is generated by τ_{α_1} and τ_β. Suppose inductively that $\mathcal{M}_{1,r-1}$ is generated by $\tau_{\alpha_1},\dots,\tau_{\alpha_{r-1}},\tau_\beta,\tau_{\delta_2},\dots,\tau_{\delta_{r-1}}$. Then, $\mathcal{N}_{1,r-1}$ is generated by these twists and $\operatorname{Im} f_3$. Furthermore, $\pi_1(\Sigma_{1,r-1},p)$ is generated by the homotopy classes of the curves $\alpha_1,\dots,\alpha_{r-1}$ and β. If we denote by α_i' the curve $\tau_\beta\tau_{\alpha_1}\tau_{\alpha_r}^{-1}\tau_\beta^{-1}(\alpha_i)$, then one can check that f_3 maps α_i into $\tau_{\alpha_i}\tau_{\alpha_i'}^{-1}$, and if $\beta'=\tau_{\alpha_1}\tau_\beta(\alpha_r)$, then $f_3(\beta)=\tau_\beta\tau_{\beta'}^{-1}$. Thus, since $\tau_{\alpha_i'}=\tau_\beta\tau_{\alpha_1}\tau_{\alpha_r}^{-1}\tau_\beta^{-1}(\tau_{\alpha_i})$ and $\tau_{\beta'}=\tau_{\alpha_1}\tau_\beta(\tau_{\alpha_r})$, the group $\mathcal{N}_{1,r-1}$ is generated by $\tau_{\alpha_1},\dots,\tau_{\alpha_{r-1}},\tau_\beta,\tau_{\delta_2},\dots,\tau_{\delta_{r-1}}$ and τ_{α_r}.

Finally, we obtain the required result for $\mathcal{M}_{1,r}$ using the first exact sequence. □

Proposition 5.2. *Suppose that $r\geq 2$. Then, the twists along non-separating curves and $r-2$ boundary curves do not generate $\mathcal{M}_{1,r}$.*

Proof. Referring to Figure 30, suppose that τ_{δ_1} and τ_{δ_2} can be decomposed into a product of twists along non-separating curves or the curves $\delta_3,\dots,\delta_r$. First, let α be a non-separating curve in $\Sigma=\Sigma_{1,r}$. There is a diffeomorphism h of Σ which sends α_1 on α. Lemma 5.1 tells us that h can be decomposed into a product of twists along the curves $\alpha_1,\dots,\alpha_r,\beta,\delta_2,\dots,\delta_r$. One has $\tau_\alpha=h\,\tau_{\alpha_1}\,\overline{h}$; since Dehn twists along boundary curves are central in $\mathcal{M}_{1,r}$, one still has $\tau_\alpha=h'\,\tau_{\alpha_1}\,\overline{h'}$, where h' is obtain from h by removing the boundary twits in its decomposition. Thus, one can suppose that τ_{δ_1} and τ_{δ_2} are decomposed into a product of twists along the curves $\alpha_1,\dots,\alpha_r,\beta,\delta_3,\dots,\delta_r$.

Now, let $\varphi:\mathcal{M}_{1,r}\to\mathcal{M}_{2,0}$ be the morphism obtained by capping off each of the boundary components δ_j for $j\geq 3$ with a disc and gluing a cylinder onto δ_1 and δ_2. Then, if $\gamma_1,\dots,\gamma_5$ are the curves shown in Figure 31, one has $\varphi(\tau_\beta)=\tau_{\gamma_2}$, $\varphi(\tau_{\alpha_i})=\tau_{\gamma_1}$ for $i\neq 2$, $\varphi(\tau_{\alpha_2})=\tau_{\gamma_3}$, $\varphi(\tau_{\delta_1})=\varphi(\tau_{\delta_2})=\tau_{\gamma_5}$ and $\varphi(\tau_{\delta_j})=1$ for $j\geq 3$. Thus, our hypothesis implies that τ_{γ_5} can be decomposed into a product of twists along the three curves γ_1, γ_2 and γ_3. Then, since $\tau_{\gamma_1},\dots,\tau_{\gamma_5}$ generate

$\mathcal{M}_{2,0}$, one only needs $\tau_{\gamma_1}, \dots, \tau_{\gamma_4}$ to generate $\mathcal{M}_{2,0}$. This is false according to Humphries ([Hu]). □

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URA n° 758 du C.N.R.S., Université de Nantes, Département de Mathématiques, 2 Rue de la Houssinière, F-44072 Nantes Cedex 03, France

E-mail address: gervais@math.univ-nantes.fr