TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 8, August 1996

WHICH FAMILIES OF l-MODAL MAPS ARE FULL?

R. GALEEVA AND S. VAN STRIEN

Abstract. In this paper we shall show that certain conditions which are sufficient for a family of one-dimensional maps to be full cannot be dispensed with.

We say that a continuous map $f\colon [0,1] \to [0,1]$ is l-modal if it has local extrema at $0 = c_0 < c_1 < c_2 < \cdots < c_l < c_{l+1} = 1$ and if $f(\{0,1\}) \subset \{0,1\}$. Let $f_\mu\colon [0,1] \to [0,1]$, $\mu \in \Delta$, be a l-modal family where Δ is some connected subset of a Euclidean space.

In this paper we want to discuss which conditions are necessary and sufficient for such a family to be full. Let us first give one definition of this notion. As usual we define the *kneading invariant* of f as follows. Let $I_i = (c_i, c_{i+1})$ and $\Sigma = \{I_1, c_1, I_2, c_2, \dots, I_{l+1}\}^{\mathbb{N}}$ endowed with the metric $d(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} d(x_i, y_i)$ where $d(x_i, y_i) = 1$ if $x_i \neq y_i$ and $d(x_i, x_i) = 0$. Let $\underline{i}\colon [0,1] \to \Sigma$ be defined by $\underline{i}(x) = (i_0(x), i_1(x), \dots, i_n(x), \dots)$ where $i_n(x) = I_k$ if $f^n(x) \in I_k$ and $i_n(x) = c_k$ if $f^n(x) = c_k$. The sequence $\underline{i}(x)$ is called the *itinerary* of x under f. Given $n \in \mathbb{N}$ and $x \in I$, there exists $\delta > 0$ such that $i_n(y)$ is constant on the interval $(x, x+\delta)$. It follows that the one-sided limit $\underline{i}(x^+) = \lim_{y \downarrow x} \underline{i}(y)$ always exists. The sequences $\nu_1, \dots, \nu_l$ defined by

$$\nu_i = \underline{i}(c_i^+)$$

are called the *kneading invariants* of f.

For the purposes of this paper (see the remark below the next theorem), we say that a l-modal family $f_\mu\colon [0,1] \to [0,1]$, $\mu \in \Delta$, is *full* if the following property holds. Given a l-modal map $g\colon [0,1] \to [0,1]$ such that

1. no lap of g is a homterval (an interval J is called a *homterval* for g if $g^n|J$ is a homeomorphism for each n);
2. $g(0) = f_\mu(0)$ and $g(1) = f_\mu(1)$ for $\mu \in \Delta$,

there exists $\mu' \in \Delta$ such that g and $f_{\mu'}$ have the same kneading invariants.

Let us first cite a theorem which gives a sufficient condition for a family to be full. In order to state this theorem we need to introduce the map $F\colon \Delta \to [0,1]^l$ defined by

$$F(\mu) = (f_\mu(c_1), \dots, f_\mu(c_l)).$$

Since successive turning points of f_μ are alternating local maxima and local minima, the map F has values in

$$V = \{(v_1, \dots, v_l) \in I^l;\ s(-1)^i(v_{i+1} - v_i) < 0 \text{ for } i = 0, \dots, l\}$$

Received by the editors June 6, 1994 and, in revised form, September 21, 1995.
1991 *Mathematics Subject Classification.* Primary 58Fxx, 34Cxx, 30-xx.
Key words and phrases. One-dimensional dynamics.

©1996 American Mathematical Society

where $s \in \{-,+\}$ depending on the orientation of f_μ and where v_0 and v_{l+1} are respectively the images of the left and right endpoint of I. (By the definition of an l-modal map, $v_0, v_{l+1} \in \{0,1\}$.) Here, and below, we have written $f_\mu(c_i)$ instead of $f_\mu(c_i(\mu))$.

In the unimodal case, there is a well-known condition which suffices for a family to be full: for this it is sufficient that the map

$$\Delta \ni \mu \mapsto F(\mu) \in \Delta = [0,1] \setminus \{F(0) = F(1)\}$$

is surjective, see [M.T.] and [C.E.]. In [M.S.] this result is generalized:

Theorem 1 (See [M.S.]). *Assume that f_μ, $\mu \in \Delta$, is a family of l-modal C^1 maps such that*

1. *$\Delta \ni \mu \mapsto f_\mu \in C^1$ extends continuously to a map $\bar{\Delta} \ni \mu \mapsto f_\mu \in C^1$ and that the turning points $c_1(\mu), c_2(\mu), \dots, c_l(\mu)$ depend continuously on $\mu \in \Delta$;*
2. *$F\colon \Delta \to V$ is a homeomorphism*

then f_μ is a full family.

Proof. In the unimodal case one can use the order structure on the space of kneadings. In the general case one can proceed as in [M.S.].

The purpose of this paper is to show that neither of the two conditions in the above theorem can be dropped.

Remark 1. Let us discuss the notion of full family in a little more detail. For this notion to be useful one might hope that polynomial families are full. However, maps from such families do not have wandering intervals, see [M.S.]. So we would like to make sure that even if g has homtervals, the fact that f_μ and g have the same kneading invariants does not imply that f_μ also has wandering intervals. So let us check that we have defined everything so that we indeed do not force wandering intervals! Let us define $x \sim y$ if the interval connecting x and y is a union of homtervals of g. Note that $g/\sim\colon [0,1]/\sim \to [0,1]/\sim$ is a continuous map (since the image of a homterval is again a homterval). Because of condition 1 above, $g/\sim$ has the same modality as g. (In fact, if we did not impose condition 1 then g could be of the form $g(x) = x$ for $x \in [0,1/2]$ and $g(x) = 1-x$ for $x \in [1/2,1]$ in which case $[0,1]/\sim$ would be a single point.) Clearly, the kneading invariants of g and $g/\sim$ are the same: $\nu_i(g) = \nu_i(g/\sim)$. (This is the reason to choose the kneading invariants ν_i from above, rather than the alternative kneading invariants $\hat{\nu}_i(g) = \underline{i}(g(c_i))$, because if the interval connecting $g^n(c_i)$ and c_j is a homterval then $\hat{\nu}_i(g) \neq \hat{\nu}_i(g/\sim)$. So we cannot take the quotient $g/\sim$ of g without changing these alternative kneading invariants.)

Remark 2. In [M.S.], maps were defined to be *essentially combinatorially equivalent* if 'the dynamics inside periodic attractors' also matched. If two such maps are equivalent then they certainly have the same kneading invariants. In [M.S.], a family f_μ was called full, if for any g as above – but satisfying some additional hypothesis – there exists μ such that f_μ and g are essentially combinatorially equivalent. The additional conditions on g in the theorem in [M.S.] become irrelevant if we only care about the kneading invariant of g. This is the reason why Theorem 1 follows from Theorem II.4.1 from [M.S.].

Remark 3. As is shown in [M.S.], the two assumptions in the statement of Theorem 1 can be weakened.

In Theorem 2 we shall show that condition 1 in Theorem 1 is necessary. That is, we shall show that if for each μ the map f_μ has a non-zero derivative in one of its turning points then f_μ cannot be a full family. Since Δ is not compact, we shall need to assume in Theorem 2 that f_μ does not degenerate too much as $\mu \to \partial\Delta$. To be specific, we shall assume in Theorem 2 that there exists $K < \infty$ such that for the family of ℓ-modal maps f_μ one has

1. for each $i = 1, 2, \ldots, \ell + 1$ the restriction $f_\mu | I_i(\mu)$ is C^2 and $|D^2 f_\mu(x)| \le K$ for each $\mu \in \Delta$ and each $x \in I_i(\mu) = (c_i(\mu), c_{i+1}(\mu))$;
2. small derivatives only occur at the boundary of intervals of monotonicity or when the image of the whole interval is small: for each $\epsilon > 0$ there exists $\delta > 0$ such that $|Df_\mu(x)| \le \delta$ for some $\mu \in \Delta$ and $x \in I_i(\mu)$ then either $d(x_n, \{c_1, \ldots, c_l\}) \le \epsilon$ or $|f_\mu(I_i(\mu))| \le \epsilon$ (here $|U|$ stands for the length of an interval U);
3. the intervals of monotonicity do not shrink to zero: $|c_i(\mu) - c_{i+1}(\mu)| \ge 1/K$ for all $\mu \in \Delta$ and each $i = 0, 1, 2, \ldots, \ell$;
4. one of the following two conditions is satisfied:
   - 4(a) if 0 (resp. 1) is a fixed point then for each sequence of points x_n and parameters μ_n such that $x_n, f_{\mu_n}(x_n), \ldots, f^n_{\mu_n}(x_n) \in I_1$ (resp. $\in I_{l+1}$) for which x_n is not in the basin of an attracting fixed point of f_{μ_n} we have $x_n \to 0$ (resp. $x_n \to 1$);
   - 4(b) if 0 and 1 are both not fixed points then the endpoints are exchanged by f (i.e., $f_\mu(0) = 1$, $f_\mu(1) = 0$) and then we assume that for each sequence of points x_n and parameters μ_n such that $x_n, f^2_{\mu_n}(x_n), \ldots, f^{2n}_{\mu_n}(x_n) \in I_1$ for which x_n is not in the basin of an attracting point of period 2 of f_{μ_n} we have $x_n \to 0$.

The last assumption prohibits that there exists a sequence of maps in the family f_μ which more and more 'almost' have a fixed point. This would happen, for instance, if a limit map has a saddle-node fixed point. If, for example, the left endpoint of the interval is a fixed point, then it is prohibited that an orbit remains longer and longer in I_1 without converging to an attracting fixed point in I_1. For all reasonable families of maps this condition is satisfied because of some type of convexity. Certainly families of piecewise affine maps satisfy the above assumptions.

Theorem 2 (Condition 1 from Theorem 1 is necessary)**.** *Suppose that f_μ is as above and satisfies the conditions from Theorem 1, except that f_μ is not differentiable in at least one of the turning points $c_i(\mu)$ in the sense that there exists $k' > 0$ and $\delta > 0$ such that $|Df_\mu(x)| \ge k' > 0$ for all $\mu \in \Delta$ and all $x \in (c_i(\mu) - \delta, c_i(\mu) + \delta)$ with $x \ne c_i(\mu)$. Then f_μ is not full.*

We should emphasize that the above theorem applies to any family of piecewise linear maps or piecewise polynomial maps where the slope at one of the critical points is bounded away from zero. Such families are therefore not full.

Proof. In order to be definite, assume that $f_\mu(0) = f_\mu(1) = 0$ and that c_i is a local maximum. The other case can be dealt with similarly. To show that f_μ is not a full family we choose a sequence of smooth ℓ-modal maps $g_n\colon [0,1] \to [0,1]$ and show that for sufficiently large n, there exists no $\mu \in \Delta$ for which f_μ and g_n have the same kneading sequence. So choose smooth ℓ-modal maps g_n with $g_n(0) = g_n(1) = 0$ with the following properties. Take g_n so that $g_n(c_k)$ is equal to 0 or 1 except when

$k = i$. So the image of each lap, except that of I_{i-1} and I_i, is equal to the entire interval $[0,1]$. Furthermore, we choose $g_n(c_i) \in I_{l+1}$ so that it is 'close' to 1 in the following precise fashion:
i) $g_n^2(c_i), \dots, g_n^{n+1}(c_i) \in I_1$,
ii) $g_n^{n+2}(c_i) \in I_{i-1} = (c_{i-1}, c_i)$,
iii) $g_n^{n+3}(c_i) \in I_{l+1}$,
iv) $g_n^{n+4}(c_i), \dots, g_n^{2n+3}(c_i) \in I_1$, and
v) $g_n^{2n+4}(c_i) \in I_i = (c_i, c_{i+1})$ is a fixed point of the map g_n^{n+2}.
Clearly g_n can be chosen so that i)-iv) hold. Let us explain why we can choose g_n so that property v) also holds. Since $g_n|I_{i-1}$ is increasing ii) implies that $g_n^{n+3}(c_i)$ lies to the left of $g_n(c_i)$ in I_{l+1}. Since $g_n|I_{l+1}$ is decreasing this implies that $g_n^{n+4}(c_i)$ lies to the right of $g_n^2(c_i)$ in I_1. Hence g_n^{n+4+j} lies to the right of g_n^{2+j} for $j = 0, 1, \dots, n$. This means that we can modify g_n so that $g_n^{2n+4}(c_i)$ is an arbitrary point in I_i. In particular, taking $p \in I_i$ so that $g_n(p) = g_n(g_n^{n+2}(c_i))$, we can make sure that $g_n^{2n+4}(c_i) = p$. This implies that p is a fixed point of g_n^{n+2}. In terms of kneading sequences, i)-v) read as

$$\underline{i}(c_i) = c_i I_{l+1} I_1^n I_{i-1} \left(I_{l+1} I_1^n I_i\right)^\infty .$$

The above choice implies that the restriction of g_n^{n+2} to some interval containing the point c_i is a full unimodal map. More precisely, $T_n = [g_n^{n+2}(c_i), g_n^{2n+4}(c_i)]$ contains c_i, the intervals $T_n, \dots, g_n^{n+1}(T_n)$ are disjoint and g_n^{n+2} is a unimodal map from T_n onto T_n. Moreover, $g_n(T_n) \subset I_{l+1}$ and $g_n^i(T_n) \subset I_1$ for $i = 1, 2, \dots, n+1$. (In fact, this also implies that $g_n^{n+2}|T$ has a local minimum at c_i because g_n has a local maximum at c_i, $g|I_{l+1}$ is orientation reversing and $g|I_1$ is orientation preserving.) This choice implies that no lap of g_n is a homterval and so the ℓ-modal map g_n satisfies all the required conditions.

Let us assume by contradiction that for each n there exists $\mu_n \in \Delta$ for which f_{μ_n} and g_n have the same kneading invariants. First notice that the turning points of g_n are eventually periodic. Moreover, by construction of g_n no two distinct iterates of turning points are in a homterval. It follows that the turning points of f_{μ_n} are also eventually periodic and that if we take $U_n = [f_{\mu_n}^{n+2}(c_i), f_{\mu_n}^{2n+4}(c_i)]$ then $U_n \ni c_i$ and

- $U_n, \dots, f_{\mu_n}^{n+1}(U_n)$ are disjoint;
- $f_{\mu_n}(U_n) \subset I_{l+1}$;
- $f_{\mu_n}^2(U_n), \dots, f_{\mu_n}^{n+1}(U_n) \subset I_1$;
- $f_{\mu_n}^{n+2}$ is a unimodal map from U_n onto U_n;
- $f_{\mu_n}(I_j) = [0,1]$ except for $j = i-1, i$.

First notice that $|f_{\mu_n}(U_n)|$ is at least $(k'/2)|U_n|$ for n sufficiently large, because of the assumption on Df_{μ_n} near c_i. Moreover, because of assumptions 2 and 3, there exists a uniform constant $C > 0$ such that $|f_{\mu_n}^2(U_n)| \geq C|U_n|$. Now let a_n be the left endpoint of $U_n' = f_{\mu_n}^2(U_n)$. Then $U_n', \dots, f_{\mu(n)}^{n-1}(U_n') \subset I_1$, $f_{\mu_n}^n(U_n') \ni c_i$ are disjoint (and lie in increasing order). Let W_n be the 'fundamental neighbourhood' $W_n = [a_n, f(a_n)]$ (it strictly contains U_n'). Since $W_n, \dots, f_{\mu_n}^{n-2}(W_n)$ are all contained in I_1, assumption 4 implies that $W_n \to 0$. Because of assumption 3 we get that $|f_{\mu_n}^{n-2}(W_n)|$, and therefore $|f_{\mu_n}^n(W_n)|$ is bounded away from zero. Hence

$$|f_{\mu_n}^n(W_n)|/|W_n| \to \infty \text{ as } n \to \infty. \tag{1}$$

Notice that $x \mapsto \log|Df_{\mu_n}(x)|$ is uniformly Lipschitz on the interval J_{μ_n} connecting 0 to $I_1 \cap f_{\mu_n}^{-1}(c_i)$ (here we use that this last point is not too close to c_1 and assumption 2. Therefore

$$\begin{aligned}\log\frac{|Df_{\mu_n}^n(u)|}{|Df_{\mu_n}^n(v)|} &= \sum_{i=0}^{n-1}\log|Df_{\mu_n}(f_{\mu_n}^i(u))| - \log|Df_{\mu_n}(f_{\mu_n}^i(v))| \\ &\le K\sum_{i=0}^{n}|f_{\mu_n}^i(u) - f_{\mu_n}^i(v)| \le K\end{aligned}$$

for $u, v \in W_n$. Combining this with (1) we get that there exists a function $C(n)$ with $C(n) \to \infty$ as $n \to \infty$ such that

$$\frac{|f_{\mu_n}^n(U_n')|}{|U_n'|} \ge C(n).$$

In particular for large n, there exists no $\mu_n \in \Delta$ for which f_{μ_n} maps some interval U_n in $n+2$ steps into itself (in the required way). Q.E.D.

Naively, one might think that condition 2 in Theorem 1 can be replaced by the condition that F is surjective. This is false!

Theorem 3 (Condition 2 in Theorem 1 is necessary)**.** *There exists a family $f_\mu\colon [0,1] \to [0,1]$ of bimodal maps which satisfies condition 1 from Theorem 1, for which $F\colon \Delta \to V$ is surjective, but which is not full.*

Proof. We will give the proof of Theorem 3 in two steps. The idea of the construction is to take a cubic family P_μ which is full, remove a point μ_0 from the parameter space Δ, excluding in this way some type of combinatorial behaviour (kneading invariant). As a consequence the family P_μ, $\mu \in \Delta \setminus \{\mu_0\}$ is nonfull. Next we choose a parameter-dependent coordinate change h_μ, so that the extremal value map $\hat{F}\colon \Delta \to V$ corresponding to $h_\mu \circ P_\mu \circ h_\mu^{-1}$, $\mu \in \Delta \setminus \{\mu_0\}$ becomes surjective again.

Step 1. Let us consider the cubic family:

$$P_\mu(x) = (1-a-b)x^3 + bx^2 + ax$$

where

$$\mu = (a,b) \in \Delta = \{\mu = (a,b);\ P_\mu\colon [0,1] \to [0,1] \text{ is bimodal}\}.$$

Let V be the set of extremal values, i.e., let $v_1 = P_\mu(c_1)$ and $v_2 = P_\mu(c_2)$ and define

$$V = \{(v_1, v_2);\ 1 \ge v_1 > v_2 \ge 0\}.$$

According to the Corollary to Theorem II.4.1 in [M.S.], the associated map $F\colon \Delta \to V$ is a homeomorphism. In particular, it is a full family and therefore there exists a parameter $\mu_0 = (a_0, b_0) \in \Delta$ for which $P_{\mu_0}(c_1) = c_2$, $P_{\mu_0}(c_2) = c_1$, i.e., P_{μ_0} has a superstable cycle, containing both turning points. Since there exists no parameter $\mu \ne \mu_0$ such that P_μ and P_{μ_0} have the same kneading invariants (see [M.T.] or Section II.10 in [M.S.]), the family $P_\mu(x)$, $\mu \in \Delta \setminus \{\mu_0\}$ is not full and also the map $F\colon \Delta \setminus \{\mu_0\} \to V$ is not surjective.

Step 2. Now we will choose a parameter dependent diffeomorphism $h_\mu(x)$ such that

$$\Delta \setminus \{\mu_0\} \ni \mu \mapsto \hat{F}(\mu) \in V$$

is surjective. Here $\hat{F}$ is the map which associates to μ the extremal values of $h_\mu \circ f_\mu \circ h_\mu^{-1}$. This means that

$$\hat{F}(\mu) = (h_\mu(f_\mu(c_1(\mu))), h_\mu(f_\mu(c_2(\mu)))) .$$

To construct h_μ we fix $\epsilon_0 > 0$ so small that the $2\epsilon_0$ neighbourhoods of $c_i(\mu_0)$ are contained in $(0,1)$ and are disjoint. Moreover let $p = (c_1(\mu_0) + c_2(\mu_0))/2$ and define two smooth functions $\rho_i \colon \mathbb{R} \to [0,1]$ such that $\text{supp}(\rho_1) \subset [0,p]$ and $\text{supp}(\rho_2) \subset [p,1]$ and such that $\rho_i = 1$ for $x \in [c_i - \epsilon_0, c_i + \epsilon_0]$. Define $A = \sup_{x\in[0,1]}\{|D\rho_1(x)|, |D\rho_2(x)|\}$ and choose $\epsilon \in (0,\epsilon_0)$ so that $\epsilon A < 1$ and $B_\epsilon(v_1(\mu_0), v_2(\mu_0)) \subset V$ (here $B_t(z)$ is the ball with radius t centered in $z \in \mathbb{R}^2$). Choose $\delta > 0$ so small that $F(B_\delta(\mu_0)) \subset B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0))$. Next we consider a surjective smooth function $\theta \colon \mathbb{R} \to [-\epsilon, +\epsilon]$ so that $\text{supp}(\theta) \subset (0, \delta^2]$. Define $\Psi \colon \Delta \setminus \{\mu_0\} \to \mathbb{R}^2$ by

$$\Psi(\mu) = (\Psi_1(\mu), \Psi_2(\mu)) := \theta(||\mu - \mu_0||^2) \cdot \frac{(\mu - \mu_0)}{||mu - \mu_0||}.$$

Since $\text{supp}(\theta) \subset (0, \delta^2]$, the map Ψ is smooth. Furthermore, define

$$h_\mu(x) = x + \Psi_1(\mu)\rho_1(x) + \Psi_2(\mu)\rho_2(x).$$

Since $\text{supp}(\rho_1)$ and $\text{supp}(\rho_2)$ are disjoint sets and since $|\Psi_i||D\rho_i| \le \epsilon A < 1$, it follows that $Dh_\mu(x) > 0$ for $x \in [0,1]$ and that $[0,1] \ni x \mapsto h_\mu(x) \in [0,1]$ is a diffeomorphism. Moreover, $(\mu, x) \mapsto h_\mu(x)$ is smooth.

Next we show that $\hat{F}$ is surjective. By the choice of ρ_i,

$$h_\mu(f_\mu(c_i(\mu))) = f_\mu(c_i(\mu)) + \Psi_i(\mu)$$

and therefore

$$\hat{F}(\mu) = F(\mu) + \Psi(\mu). \tag{2}$$

Because $\text{supp}(\theta) \subset [0, \delta^2]$ we have that $\Psi(\mu) = 0$ for $\mu \notin B_\delta(\mu_0)$. Hence

$$\hat{F}(\Delta \setminus B_\delta(\mu_0)) = F(\Delta \setminus B_\delta(\mu_0)) \supset V \setminus B_\epsilon(v_1(\mu_0), v_2(\mu_0)).$$

Since θ is surjective there exists $\delta_1, \delta_2 \in (0,\delta)$ such that $\theta(\delta_1^2) = -\epsilon$ and $\theta(\delta_2^2) = \epsilon$. Hence

- $\Psi(\partial B_r(\mu_0)) = \partial B_\epsilon(0)$ for $r = \delta_i$;
- the winding number of the parameterized closed loop

  $$S^1 \ni \phi \mapsto \Psi\left(\mu_0 + (r\cos\phi, r\sin\phi)\right)$$

  with respect to a point z in $B_\epsilon(0)$ depends on $r \in [0,\delta]$: it is equal to $(-1)^i$ if $r = \delta_i$. (In fact, for some r it is not defined because the loop goes through z.)

From $\hat{F}(\mu) = F(\mu) + \Psi(\mu)$, since $\Psi(\partial B_r(\mu_0)) = \partial B_\epsilon(0)$ for $r = \delta_i$ and because $F(B_\delta(\mu_0)) \subset B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0))$ it follows that for $r = \delta_i$ the loop

$$\gamma\colon \quad S^1 \ni \phi \mapsto \hat{F}\left(\mu_0 + (r\cos\phi, r\sin\phi)\right) \tag{3}$$

lies outside $B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0))$ and that its winding number w.r.t. a point z in the disc $B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0))$ is still equal to $(-1)^i$ when $r = \delta_i$. Hence

$$\hat{F}(B_\delta(\mu_0) \setminus \{\mu_0\}) \supset B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0)),$$

because otherwise there exists a point $z \in B_{\epsilon/2}(v_1(\mu_0), v_2(\mu_0))$ such that $z \notin$ the loop γ for each $r \in [0, \delta]$. This would imply that the winding number of γ with respect to z would be independent of r, which contradicts our former assertion. It follows that $\hat{F}\colon \Delta \setminus \{\mu_0\} \to V$ is surjective even though P_μ, $\mu \in \Delta \setminus \{\mu_0\}$, is not a full family. Q.E.D.

Acknowledgement

We would like to thank the referee for his very useful and careful comments.

Addition: After this paper was submitted we learned about related preprints by Jimenez López and Snoha, *There are no Piecewise Linear Maps of type* 2^∞, and by Martens and Tresser, *Forcing of periodic orbits and renormalization of piecewise affine maps*. From these papers it follows that piecewise affine maps are never infinitely renormalizable. The maps from our families in Theorem 1 are more general, but as a consequence we can only show that these families cannot contain infinite "kneadings" of arbitrary maps (and thus show that the families cannot be full).

References

[C.E.] P. Collet and J.-P. Eckmann: *Iterated Maps of the Interval as Dynamical Systems*. Birkhäuser, Boston (1980). MR **82j:**58078

[M.T.] J. Milnor and W. Thurston: *On iterated maps of the interval: I,II*. In: "Dynamical Systems: Proc. Univ, of Maryland 1986-87", Lecture Notes in Mathematics **1342**, (1988), 465–563. MR **90a:**98083

[M.S.] W. de Melo and S. van Strien, *One-dimensional dynamics*. Ergebnisse Series **25**, Springer Verlag, Berlin (1993). MR **95a:**58035

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

Current address: UMR 129 CNRS UNSA, 1361 Route des Lucioles Sophia Antipolis, 06560 Valbonne, France

E-mail address: `galeeva@doublon.unice.fr`

Department of Mathematics, University of Amsterdam, Amsterdam, The Netherlands

E-mail address: `strien@fwi.uva.nl`