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PACKING MEASURE OF THE SAMPLE PATHS OF FRACTIONAL BROWNIAN MOTION

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ABSTRACT. Let $X(t)$ $(t \in \mathbf{R})$ be a fractional Brownian motion of index α in $\mathbf{R}^d$. If $1 < \alpha d$, then there exists a positive finite constant K such that with probability 1,

$$\phi\text{-}p(X([0,t])) = Kt \quad \text{for any} \quad t > 0 \ ,$$

where $\phi(s) = s^{\frac{1}{\alpha}}/(\log\log \frac{1}{s})^{\frac{1}{2\alpha}}$ and $\phi\text{-}p(X([0,t]))$ is the ϕ-packing measure of $X([0,t])$.

1. Introduction

Packing measure was introduced by Taylor and Tricot ([TT]) as a dual concept to Hausdorff measure. Like Hausdorff measure and Hausdorff dimension (cf. [F]), it is a very useful tool in analyzing fractal sets and in studying sample path properties of stochastic processes. There has been a lot of work on the packing measure of the image and graph of processes with stationary independent increments (see [FT], [LeT], [RT], [TT], [Tay1], and references therein). The objective of this paper is to study the packing measure of the sample paths of fractional Brownian motion. For processes with stationary independent increments, the strong Markov property plays a crucial role. For this reason some of the techniques used previously are not available in the present situation.

Fix $\alpha \in (0,1)$ and consider the centered, real gaussian process $Y(t)$ $(t \in \mathbf{R})$ with covariance

$$E\big(Y(t)Y(s)\big) = \frac{1}{2}\big(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\big) \ .$$

We will use the fact that $Y(t)$ $(t \in \mathbf{R})$ can be represented as a stochastic integral

$$Y(t) = \int_{\mathbf{R}} G(t,x)dW(x) \ , \tag{1.1}$$

where $dW(x)$ is a scattered gaussian random measure with Lebesgue measure as control measure and

$$G(t,x) = c_\alpha\left(\max(t-x,0)^{\alpha-\frac{1}{2}} - \max(-x,0)^{\alpha-\frac{1}{2}}\right) \ , \tag{1.2}$$

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where c_α is a normalizing constant depending on α only. We associate with Y a gaussian process in $\mathbf{R}^d$ by

$$X(t) = \big(X_1(t), \cdots, X_d(t)\big) ,$$

where $X_1, \cdots, X_d$ are independent copies of Y. Using the terminology of Kahane ([K], Chapter 18), we call X the $(1, d, \alpha)$ process or the d-dimensional fractional Brownian motion of index α . When $\alpha = \frac{1}{2}$, $X(t)$ is ordinary d-dimensional Brownian motion. It is easy to see that $X(t)$ $(t \in \mathbf{R})$ is a self-similar process with exponent α, i.e. for any $a > 0$,

$$X(a\ \cdot) \overset{\mathrm{d}}{=} a^\alpha X(\cdot) , \tag{1.3}$$

where $X \overset{\mathrm{d}}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions.

It is well known that with probability 1,

$$\mathrm{Dim} X([0,1]) = \min\{d;\ \frac{1}{\alpha}\} ,$$

where $\mathrm{Dim} E$ is the packing dimension of E (cf. [Tay2]). We shall prove that, in the transient case (that is, $1 < \alpha d$), there exist positive constants C_1 and C_2, such that

$$C_1 \le \phi\text{-}p(X([0,1])) \le C_2 , \quad \text{a.s.} \tag{1.4}$$

where $\phi(s) = s^{\frac{1}{\alpha}}/(\log\log\frac{1}{s})^{\frac{1}{2\alpha}}$ and ϕ-p is the ϕ-packing measure. If $1 > \alpha d$, then by a result of Pitt [P] (see also [K], Ch.18, Theorem 2), $X([0,1])$ a.s. contains interior points and hence $0 < L_d(X([0,1])) < \infty$, a.s., where L_d is the Lebesgue measure in $\mathbf{R}^d$. In the critical case of $1 = \alpha d$, the problem of finding the packing measure of $X([0,1])$ is much deeper. Except in the special case of planar Brownian motion, for which LeGall and Taylor ([LeT]) proved that ϕ-$p(X([0,1]))$ is either zero or infinite, this problem is still open.

The paper is organized as follows. In Section 2, we collect definitions and lemmas which are essential to our calculations. In Section 3, we obtain upper bounds for the probability that a fractional Brownian motion hits a ball, generalizing classical results for ordinary Brownian motion in $\mathbf{R}^d$ $(d \ge 3)$. In Section 4, we prove a lim inf theorem for the sojourn time of fractional Brownian motion. In Section 5, we prove (1.4). We have to use methods quite different from those for Brownian motion to prove our results. The reason for this is not only that for $\alpha \ne \frac{1}{2}$ fractional Brownian motion of index α does not have independent increments; there are also other technical difficulties. For example, the distribution of the sojourn time of d-dimensional Brownian motion in the unit ball $\mathbf{R}^d$ is explicitly known ([CT], [TT]) but for fractional Brownian motion of index $\alpha \ne \frac{1}{2}$, we can not even prove that the sojourn time has a bounded density.

We will use $K, K_1, K_2, \cdots,\ c_1, c_2, \cdots$ to denote unspecified positive finite constants which may take different values from line to line.

2. Preliminaries

First we recall briefly the definitions of packing measure and packing dimension. Let Φ be the class of functions $\phi : (0, \delta) \to (0, 1)$ which are right continuous, monotone increasing with $\phi(0+) = 0$, and such that there exists a finite constant $K_1 > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \le K_1 \qquad \text{for } 0 < s < \frac{1}{2}\delta\ . \tag{2.1}$$

For $\phi \in \Phi$, Taylor and Tricot ([TT]) defined the set function ϕ-$P(E)$ on $\mathbf{R}^N$ by

$$\phi\text{-}P(E) = \lim_{\epsilon\to 0}\ \sup\left\{\sum_i \phi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E,\ r_i < \epsilon\right\}\ , \tag{2.2}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. A sequence of closed balls satisfying the conditions in the right hand side of (2.2) is called an ϵ-packing of E. Observe that ϕ-P is not an outer measure because it fails to be countably subadditive. However, ϕ-P is a premeasure, so one can obtain a metric outer measure ϕ-p on $\mathbf{R}^N$ by defining

$$\phi\text{-}p(E) = \inf\left\{\sum_n \phi\text{-}P(E_n) :\quad E \subseteq \bigcup_n E_n\right\}\ . \tag{2.3}$$

ϕ-$p(E)$ is called the ϕ-packing measure of E. Every Borel set in $\mathbf{R}^N$ is ϕ-p measurable. If $\phi(s) = s^\alpha$, s^α-$p(E)$ is called the α-dimensional packing measure of E. The packing dimension of E is defined by

$$\begin{aligned}\mathrm{Dim}E &= \inf\{\alpha > 0 :\ \ s^\alpha\text{-}p(E) = 0\}\\ &= \sup\{\alpha > 0 :\ \ s^\alpha\text{-}p(E) = +\infty\}\ .\end{aligned}$$

By (2.3), we see that, for any $E \subset \mathbf{R}^N$,

$$\phi\text{-}p(E) \le \phi\text{-}P(E)\ . \tag{2.4}$$

The upper bound for ϕ-$P(E)$ is usually not easy to determine, because we need to consider all the possible packings in (2.2). A lower bound for ϕ-$p(E)$ can be obtained by using the following density theorem for packing measures (see [TT] and [RT] for a proof).

Lemma 2.1. *Let μ be a Borel measure on $\mathbf{R}^N$ and $\phi \in \Phi$. Then for any Borel set $E \subseteq \mathbf{R}^N$,*

$$\phi\text{-}p(E) \ge K_1^{-3}\mu(E) \inf_{x\in E}\left\{\underline{D}^\phi_\mu(x)\right\}^{-1}\ ,$$

where K_1 is the constant in (2.1) *and*

$$\underline{D}^\phi_\mu(x) = \liminf_{r\to 0} \frac{\mu(B(x, r))}{\phi(2r)}$$

is the lower ϕ-density of μ at x.

Now we recall some facts about gaussian processes. Let $Y(t)$ $(t \in S)$ be a gaussian process. We define a metric d on S by

$$d(s, t) = \left(E(Y(s) - Y(t))^2\right)^{\frac{1}{2}}\ .$$

Denote by $N_d(S, \epsilon)$ the smallest number of open d-balls of radius ϵ needed to cover S, and write $D = \sup\{d(s, t) : s,\ t \in S\}$.

The following lemma is well known. It is a consequence of the gaussian isoperimetric inequality and Dudley's entropy bound ([LT], see also [Ta1]). We will use it when S is an interval.

Lemma 2.2. *There exists an absolute constant $K > 0$ such that for any $u > 0$, we have*

$$P\left\{\sup_{s,\ t\in S}|Y(s)-Y(t)| \geq K\left(u+\int_0^D\sqrt{\log N_d(S,\epsilon)}d\epsilon\right)\right\} \leq exp\left(-\frac{u^2}{D^2}\right) .$$

The following lemma is a corollary of Lemma 2.2, and (ii) is proved in [M].

Lemma 2.3. *Let $Y(t)$ $(t \in \mathbf{R})$ be a centered gaussian process with values in* $\mathbf{R}$ *such that $Y(0) = 0$,*

$$E\big(Y(t)-Y(s)\big)^2 \leq \theta^2|t-s|^{2\alpha} \quad (0<\alpha<1) .$$

Then:

(i) *For any $r > 0, u \geq Kr^\alpha$, we have*

$$P\left\{\sup_{|t|\leq r}|Y(t)| \geq \theta u\right\} \leq exp(-\frac{u^2}{Kr^{2\alpha}}) .$$

(ii) *Let*

$$\omega_Y(h) = \sup_{t,\ t+s\in[0,1],\ |s|\leq h}|Y(t+s)-Y(t)|$$

be the uniform modulus of continuity of $Y(t)$ on $[0,1]$. Then

$$\limsup_{h\to 0}\frac{\omega_Y(h)}{\theta h^\alpha(\log\frac{1}{h})^{\frac{1}{2}}} \leq K_2 , \quad a.s.$$

Let $\{A_k\}$ be a sequence of events in a probability space. In the following lemma, (i) is well known and (ii) is proved implicitly in [M] (see also [Ta2]).

Lemma 2.4. (i) *If $\sum_{k=1}^\infty P(A_k) < \infty$, then $P(\limsup_{k\to\infty} A_k) = 0$.*

(ii) *If there exist positive constants K, ϵ and positive integers k_0, J such that for $k_0 \leq k < J$,*

$$\sum_{j=k+1}^J P(A_k\cap A_j) \leq P(A_k)\left(K+(1+\epsilon)\sum_{j=k+1}^J P(A_j)\right) \tag{2.5}$$

and

$$\sum_{k=k_0}^J P(A_k) \geq \frac{1+2K}{\epsilon}. \tag{2.6}$$

then

$$P\left(\bigcup_{k=k_0}^J A_k\right) \geq \frac{1}{1+2\epsilon} .$$

Remark. If $\sum_k P(A_k) = \infty$, then for each fixed k_0, we can take J large enough so that (2.6) holds.

3. Hitting Probabilities

In this section, we consider the probability that a d-dimensional fractional Brownian motion of index α hits a ball $B(y,r)$ in $\mathbf{R}^d$. The main results are Theorems 3.1 and 3.2, which generalize the classical results about the hitting probability and delayed hitting probability of d-dimensional Brownian motion ($d \geq 3$).

Let $Z(t)$ $(t \in \mathbf{R})$ be a centered, real valued gaussian process. We write

$$\sigma^2(t,s) = E(Z(t)-Z(s))^2 \ , \quad \sigma^2(t) = E(Z(t))^2 \ .$$

Suppose that

$$\sigma^2(t,s) \leq \theta^2 |t-s|^{2\alpha} \quad (0 < \alpha < 1) \tag{3.1}$$

and there exist positive constants $c_1,\ c_2$ such that

$$c_1 \theta^2 t^{2\alpha} \leq \sigma^2(t) \leq c_2 \theta^2 t^{2\alpha} \ . \tag{3.2}$$

Let $Y_1, \cdots, Y_d$ be independent copies of Z and let

$$Y(t) = (Y_1(t), \cdots, Y_d(t)) \ .$$

Lemma 3.1. *For any $a > 0$, let $S = [a, 2a]$. Consider a centered real gaussian process $Z(t)$ $(t \in \mathbf{R})$ that satisfies* (3.1) *and* (3.2) *on S, where θ may depend on a. Let $Y(t)$ $(t \in \mathbf{R})$ be the associated gaussian process in $\mathbf{R}^d$. Then there exist positive constants K_3 and K_4, depending only on α and d, such that for any $r > 0$ and any $y \in \mathbf{R}^d$ with $|y| > K_3 r$, we have*

$$P\left\{\inf_{t \in S} |Y(t) - y| < r\right\} \leq K_4 exp\left(-\frac{|y|^2}{K_4 \theta^2 a^{2\alpha}}\right) \cdot a^{1-\alpha d} \cdot \left(\frac{r}{\theta}\right)^{d - \frac{1}{\alpha}} \ . \tag{3.3}$$

Proof. Denote by $N(S, (\frac{r}{\theta})^{1/\alpha})$ the smallest number of open balls of radius $(\frac{r}{\theta})^{1/\alpha}$ that are needed to cover S. Then

$$N(S, (\frac{r}{\theta})^{1/\alpha}) \leq a \cdot (\frac{r}{\theta})^{-\frac{1}{\alpha}} \ , \tag{3.4}$$

Let $\{S_p\}$ $(1 \leq p \leq N(S, (\frac{r}{\theta})^{1/\alpha}))$ be a family of balls of radius $(\frac{r}{\theta})^{1/\alpha}$ that cover S and let

$$A = \left\{\inf_{t \in S} |Y(t) - y| < r\right\} ,$$

$$A_p = \left\{\inf_{t \in S_p} |Y(t) - y| < r\right\} .$$

Then

$$A \subseteq \bigcup_{p=1}^{N(S, (\frac{r}{\theta})^{1/\alpha})} A_p \ . \tag{3.5}$$

Now fix a $1 \leq p \leq N(S, (\frac{r}{\theta})^{1/\alpha})$. Let

$$\epsilon_n = \left(\frac{r}{\theta}\right)^{\frac{1}{\alpha}} exp(-2^{n+1})$$

and let $\{t_i^{(n)},\ 1 \leq i \leq N(S_p, \epsilon_n)\}$ be a set of the centers of open balls with radius ϵ_n that cover S_p. Denote

$$r_n = \beta \theta d \epsilon_n^{\alpha} 2^{\frac{n+1}{2}} \ ,$$

where $\beta \geq K_2$ is a constant to be determined later, and

$$
\begin{aligned}
A_i^{(j)} &= \left\{ |Y(t_i^{(j)}) - y| \leq r + \sum_{k=j}^{\infty} r_k \right\} , \\
A^{(n)} &= \bigcup_{j=1}^{n} \bigcup_{i=1}^{N(S_p,\epsilon_j)} A_i^{(j)} \\
&= A^{(n-1)} \cup \bigcup_{i=1}^{N(S_p,\epsilon_n)} A_i^{(n)} \ .
\end{aligned} \tag{3.6}
$$

Then by Lemma 2.3 (ii), we have

$$
P(A_p) \leq \lim_{n\to\infty} P(A^{(n)}) \ . \tag{3.7}
$$

By (3.6), we have

$$
P(A^{(n)}) \leq P(A^{(n-1)}) + P(A^{(n)} \backslash A^{(n-1)}) \tag{3.8}
$$

and

$$
P(A^{(n)} \backslash A^{(n-1)}) \leq \sum_{i=1}^{N(S_p,\epsilon_n)} P(A_i^{(n)} \backslash A_{i'}^{(n-1)}) \ , \tag{3.9}
$$

where i' is chosen so that $|t_i^{(n)} - t_{i'}^{(n-1)}| < \epsilon_{n-1}$. Consider

$$
\begin{aligned}
&P\Big(A_i^{(n)} \backslash A_{i'}^{(n-1)} \Big) \\
&= P\Big\{ |Y(t_i^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k \ , \ |Y(t_{i'}^{(n-1)}) - y| > r + \sum_{k=n-1}^{\infty} r_k \Big\} \\
&\leq P\Big\{ |Y(t_i^{(n)}) - y| < r + \sum_{k=n}^{\infty} r_k \ , \ |Y(t_i^{(n)}) - Y(t_{i'}^{(n-1)})| \geq r_{n-1} \Big\} \ .
\end{aligned} \tag{3.10}
$$

By elementary properties of gaussian variables, we can write

$$
\frac{Z(t_i^{(n)}) - Z(t_{i'}^{(n-1)})}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)})} = \rho \ \frac{Z(t_i^{(n)})}{\sigma(t_i^{(n)})} + \sqrt{1-\rho^2} \ \xi \ ,
$$

where

$$
\rho = \frac{E\Big[\Big(Z(t_i^{(n)}) - Z(t_{i'}^{(n-1)}) \Big) Z(t_i^{(n)}) \Big]}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)}) \sigma(t_i^{(n)})}
$$

and where ξ is a centered gaussian variable with variance 1 and is independent of $Z(t_i^{(n)})$. Hence there exists a centered gaussian vector Ξ with the identity matrix as its covariance matrix such that Ξ is independent of $Y(t_i^{(n)})$ and

$$
\frac{Y(t_i^{(n)}) - Y(t_{i'}^{(n-1)})}{\sigma(t_i^{(n)}, t_{i'}^{(n-1)})} = \rho \ \frac{Y(t_i^{(n)})}{\sigma(t_i^{(n)})} + \sqrt{1-\rho^2} \ \Xi.
$$

We observe that

$$\begin{aligned} r+\sum_{k=n}^{\infty} r_k &\le r+\sum_{k=0}^{\infty} r_k \\ &\le \Big(1+c_3\int_0^{\infty} exp(-\alpha x^2)dx\Big)r \\ &= c_4 r\ . \end{aligned}$$

and by (3.1),

$$\sigma(t_i^{(n)},\ t_{i'}^{(n-1)}) \le \theta|t_i^{(n)}-t_{i'}^{(n-1)}|^{\alpha}\ .$$

It follows that (3.10) is less than

$$\begin{aligned} &P\Big\{|Y(t_i^{(n)})-y|\le c_4 r\ ,\ |\Xi| \ge \frac{1}{\sqrt{1-\rho^2}}\Big[\frac{r_{n-1}}{\theta|t_i^{(n)}-t_{i'}^{(n-1)}|^{\alpha}} - \rho\ \frac{Y(t_i^{(n)})}{\sigma(t_i^{(n)})}\Big]\Big\} \\ &\le P\Big\{|Y(t_i^{(n)})-y|\le c_4 r\ ,\ |\Xi|\ge \frac{\beta d}{2}\ 2^{\frac{n}{2}}\Big\} \\ &\quad + P\Big\{|Y(t_i^{(n)})-y|\le c_4 r\ ,\ \rho\ \frac{|Y(t_i^{(n)})|}{\sigma(t_i^{(n)})} \ge \frac{\beta d}{2}\ 2^{\frac{n}{2}}\Big\} \\ &\hat{=}\ I_1+I_2\ . \end{aligned} \tag{3.11}$$

By the independence of Ξ and $Y(t_i^{(n)})$, we have

$$\begin{aligned} I_1 &= P\Big\{|Y(t_i^{(n)})-y|\le c_4 r\ \Big\}\cdot P\Big\{|\Xi|\ge \frac{\beta d}{2}\ 2^{\frac{n}{2}}\Big\} \\ &= \int_{\{|u-y|\le c_4 r\}} \frac{1}{(2\pi)^{d/2}\sigma^d(t_i^{(n)})}\ exp\Big(-\frac{|u|^2}{2\sigma^2(t_i^{(n)})}\Big)du\cdot P\Big\{|\Xi|\ge\frac{\beta d}{2}\ 2^{\frac{n}{2}}\Big\} \\ &\le c_5\ exp\Big(-\frac{|y|^2}{4\sigma^2(t_i^{(n)})}\Big)\Big(\frac{r}{\sigma(t_i^{(n)})}\Big)^d\cdot P\Big\{|\Xi|\ge\frac{\beta d}{2}\ 2^{\frac{n}{2}}\Big\}\quad (\text{ if } |y|\ge K_3 r) \\ &\le c_6\ exp\Big(-\frac{|y|^2}{4c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{c_1\theta a^{\alpha}}\Big)^d\cdot exp\Big(-\frac{(\beta d)^2}{16}2^n\Big)\ , \end{aligned} \tag{3.12}$$

where the last inequality follows from (3.2) and the tail probability of the gaussian vector. On the other hand,

$$\begin{aligned} I_2 &\le \int_{\{|u-y|\le c_4 r,\ |u|\ge \frac{\beta d}{2}2^{n/2}\sigma(t_i^{(n)})\}} \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \frac{1}{\sigma^d(t_i^{(n)})} exp\Big(-\frac{|u|^2}{2\sigma^2(t_i^{(n)})}\Big)du \\ &\le c_7\int_{\{|u-y|\le c_4 r\}}\frac{1}{\sigma^d(t_i^{(n)})} exp\Big(-\frac{|u|^2}{4\sigma^2(t_i^{(n)})}\Big)du\cdot exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big) \\ &\le c_8\ exp\Big(-\frac{|y|^2}{8\sigma^2(t_i^{(n)})}\Big)\Big(\frac{r}{\sigma(t_i^{(n)})}\Big)^d\cdot exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big)\quad (\text{if } |y|\ge K_3 r) \\ &\le c_9\ exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\ \Big(\frac{r}{c_1\theta a^{\alpha}}\Big)^d\cdot exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big)\ . \end{aligned} \tag{3.13}$$

Combining (3.8) through (3.13) and choosing $\beta\ge K_2$ satisfying

$$\frac{(\beta d)^2}{16} > 2\ ,$$

we obtain

$$\begin{aligned} P(A^{(n)}) &\le P(A^{(n-1)}) + c_{10}\ N(S_p,\epsilon_n) exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d \\ &\quad \cdot exp\Big(-\frac{(\beta d)^2}{16}\ 2^n\Big) \\ &\le c_{11}\Big[N(S_p,\epsilon_0) + \sum_{k=1}^{\infty} N(S_p,\epsilon_k) exp\Big(-\frac{(\beta d)^2}{16}\ 2^k\Big)\Big] \\ &\quad \cdot exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d \\ &\le c_{12}\ exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\Big(\frac{r}{\theta a^\alpha}\Big)^d . \end{aligned}$$

Therefore, by (3.4), (3.5) and (3.7), we have

$$P\Big\{\inf_{t\in S}|Y(t)-y| \le r\Big\} \le c_{13} exp\Big(-\frac{|y|^2}{8c_2\theta^2(2a)^{2\alpha}}\Big)\cdot\frac{1}{a^{\alpha d-1}}\cdot\Big(\frac{r}{\theta}\Big)^{d-\frac{1}{\alpha}} .$$

This proves (3.3) with $K_4 = \max(8c_2 2^{2\alpha};\ c_{13})$. □

Theorem 3.1. *Let $X(t)$ $(t \in \mathbf{R})$ be a d-dimensional fractional Brownian motion of index α. If $1 < \alpha d$, then there exists a positive constant K, depending only on α and d, such that for any $r > 0$ and any $y \in \mathbf{R}^d$ with $|y| \ge K_3 r$, we have*

$$P\Big\{\exists\ t > 0\ \ \text{such that}\ |X(t)-y| < r\Big\} \le K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}} . \tag{3.14}$$

Proof. Clearly, $X(t)$ $(t \in \mathbf{R})$ satisfies (3.1) and (3.2) with $\theta = 1$. For any $n \in \mathbf{Z}$, let $S_n = [2^n, 2^{n+1}]$. Then

$$(0,\infty) = \bigcup_{n\in\mathbf{Z}} S_n .$$

By Lemma 3.1, we have

$$\begin{aligned} &P\Big\{\exists\ t > 0 \text{ such that } |X(t)-y| < r\Big\} \\ &\le \sum_{n\in\mathbf{Z}} P\Big\{\inf_{t\in S_n}|X(t)-y| < r\Big\} \\ &\le K_4 \sum_{n\in\mathbf{Z}} exp\Big(-\frac{|y|^2}{K_4 2^{2\alpha n}}\Big)\cdot 2^{n(1-\alpha d)}\cdot r^{d-\frac{1}{\alpha}} \\ &\le c_{14}\int_0^\infty \frac{1}{x^{\alpha d}} exp\Big(-\frac{|y|^2}{K_4 x^{2\alpha}}\Big) dx \cdot r^{d-\frac{1}{\alpha}} \\ &\le K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}} . \end{aligned}$$

This proves (3.14). □

Remark. If $X(t)$ $(t \in \mathbf{R}_+)$ is a Brownian motion in $\mathbf{R}^d$ with $d \ge 3$, then it is well known that (3.14) holds with equality and $K = 1$. This result was stated by Kakutani in 1944 ([Ka]) for $d = 3$. For a proof of the general result, see [PS].

The following theorem is a generalization of the delayed hitting probability of Brownian motion in $\mathbf{R}^d$ $(d \geq 3)$ in [DE].

Theorem 3.2. *Let $X(t)$ $(t \in \mathbf{R})$ be a d-dimensional fractional Brownian motion of index α $(0 < \alpha < 1)$ with $1 < \alpha d$. Then for any $T > 0$ and any $0 < r < T^\alpha$, we have*

$$P\left\{\exists\, t \in \mathbf{R} \text{ such that } |t| > T \text{ and } |X(t)| < r\right\} \leq K\left(\frac{r}{T^\alpha}\right)^{d-\frac{1}{\alpha}}, \tag{3.15}$$

where $K > 0$ is a constant depending only on α and d.

Proof. Observing that the distribution of $X(T)$ and $X(-T)$ has the density

$$p(x,T) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \frac{1}{T^{\alpha d}} \cdot exp\left(-\frac{|x|^2}{2T^{2\alpha}}\right),$$

we have

$$\begin{aligned} &P\left\{\exists\, t \in \mathbf{R} \text{ such that } |t| > T \text{ and } |X(t)| < r\right\} \\ &\leq 2P\left\{\exists\, t > T \text{ such that } \ |X(t)| < r\right\} \\ &= 2\int_{\mathbf{R}^d} P\left\{\exists\, t > T \text{ such that } |X(t)| < r\,|X(T) = y\right\} p(y,T)dy \end{aligned} \tag{3.16}$$

Using conditional expectation, for any $t > 0$, we can write

$$X(tT + T) = X_1(t) + c(t)X(T), \tag{3.17}$$

where X_1 is independent of $X(T)$ and

$$c(t) = \frac{(1+t)^{2\alpha} + 1 - t^{2\alpha}}{2}.$$

It is clear that $c(t) \geq \frac{1}{2}$ and $c(t) \equiv 1$ if $\alpha = \frac{1}{2}$; $c(t)$ is decreasing if $\alpha < \frac{1}{2}$ and $c(t)$ is increasing if $\alpha > \frac{1}{2}$. Moreover, there exists a constant $c_{15} > 0$ such that

$$|c(t) - c(s)| \leq c_{15}\ |t-s|^\gamma, \tag{3.18}$$

where $\gamma = 2\alpha$ if $\alpha \leq \frac{1}{2}$; $\gamma = 1$ if $\alpha > \frac{1}{2}$. In order to estimate the conditional probability in (3.16), let

$$Y(t) = \frac{X_1(t)}{c(t)}.$$

Now we verify that $Y(t)$ satisfies the conditions in Lemma 3.1. By (3.17), (3.18) and elementary calculations, we have

$$\begin{aligned} \sigma^2(t,s) &= T^{2\alpha}\ \frac{|t-s|^{2\alpha}}{c(t)c(s)} + T^{2\alpha}\ \frac{c(t)-c(s)}{c(t)c(s)}\ \left(\frac{(1+s)^{2\alpha}}{c(s)} - \frac{(1+t)^{2\alpha}}{c(t)}\right) \\ &\leq \frac{c_{16}T^{2\alpha}}{c(t)c(s)}\ |t-s|^{2\alpha}, \end{aligned}$$

$$\sigma^2(t) = \frac{T^{2\alpha}}{c^2(t)}\left[(1+t)^{2\alpha} - \frac{\left((1+t)^{2\alpha} + 1 - t^{2\alpha}\right)^2}{4}\right],$$

and (3.2) holds on any $S = [a, 2a]$ with

$$\theta = \frac{T^\alpha}{m}\ , \qquad \text{where } m = \min\{c(a),\ c(2a)\}\ .$$

Hence by Lemma 3.1, for any $r > 0$ and for any $y \in \mathbf{R}^d$ with $|y| \geq K_3 r$, we have

$$\begin{aligned} &P\Big\{\inf_{t\in S}|X_1(t) - c(t)y| < r\Big\} \\ &\leq P\Big\{\inf_{t\in S}|Y(t) - y| < \frac{r}{m}\Big\} \\ &\leq K_4 exp\Big(-\frac{|y|^2 m^2}{K_4 T^{2\alpha} a^{2\alpha}}\Big) a^{1-\alpha d}\cdot\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}}\ . \end{aligned} \tag{3.19}$$

By using the same argument as in the proof of Theorem 3.1, we obtain that for $|y| \geq K_3 r$,

$$P\Big\{\exists\ t > 0 \text{ such that } |X_1(t) + c(t)y| < r\Big\} \leq K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}}\ . \tag{3.20}$$

Putting (3.20) into (3.16), we have

$$\begin{aligned} (3.16) &\leq 2\int_{\mathbf{R}^d} \min\{1,\ K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}}\}\cdot p(y,T)dy \\ &\leq 2\int_{\{|y|\leq c_{17}r\}} p(y,T)dy + 2\int_{\{|y|\geq c_{17}r\}} K\Big(\frac{r}{|y|}\Big)^{d-\frac{1}{\alpha}} p(y,T)dy \\ &\leq c_{18}\Big(\frac{r}{T^\alpha}\Big)^d + c_{19}\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}} \\ &\leq K\Big(\frac{r}{T^\alpha}\Big)^{d-\frac{1}{\alpha}}\ . \end{aligned}$$

This completes the proof of (3.15). □

Dvoretzky and Erdös ([DE]) proved that if $X(t)$ is d-dimensional Brownian motion with $d \geq 3$, then the upper bound in (3.15) also serves as a lower bound (with a different constant). We have not been able to prove an analogous result for fractional Brownian motion, so we pose the following

Question. Are the upper bounds in Theorems 3.1 and 3.2 the best possible?

4. A Limit Theorem For the Sojourn Time

In this section, we prove a lim inf theorem for the sojourn time of fractional Brownian motion in the ball $B(0, r)$. In next section, we will use the result to calculate the packing measure of the sample paths of a transient fractional Brownian motion.

For any $r > 0$ and any $y \in \mathbf{R}^d$, let

$$T_y(r) = \int_{\mathbf{R}} 1_{B(y,r)}(X(t))dt$$

be the sojourn time of $X(t)$ ($t \in \mathbf{R}$) in $B(y,r)$. If $y = 0$, we denote $T_y(r)$ by $T(r)$. By the self-similarity of $X(t)$, we have for any $a > 0$,

$$T(a\ \cdot) \overset{\mathrm{d}}{=} a^{\frac{1}{\alpha}} T(\cdot)\ . \tag{4.1}$$

Lemma 4.1. *For any* $0 < u < 1$,

$$exp\left(-\frac{K}{u^{2\alpha}}\right) \le P\{T(1) \le u\} \le exp\left(-\frac{1}{Ku^{2\alpha}}\right)\ , \tag{4.2}$$

where $K > 0$ *is a constant depending only on* α *and* d.

Proof. The right inequality in (4.2) is easy:

$$\begin{aligned} P\{T(1) \le u\} &\le P\left\{\max_{t\in[0,u]} |X(t)| \ge 1\right\} \\ &= P\left\{\max_{t\in[0,1]} |X(t)| \ge u^{-\alpha}\right\} \\ &\le exp\left(-\frac{1}{Ku^{2\alpha}}\right)\ , \end{aligned}$$

where the equality in the second line follows from (1.3) and the last inequality follows from Lemma 2.3 (i). To prove the left inequality in (4.2), we use the same argument as in the proof of Theorem 3.2,

$$\begin{aligned} P\{T(1) \le u\} &\ge P\left\{\text{for any } t \text{ with } |t| > \frac{u}{2},\ |X(t)| \ge 1\right\} \\ &= 1 - P\left\{\exists\ t \text{ with } |t| > \frac{u}{2} \text{ such that } |X(t)| < 1\right\} \\ &\ge 1 - \int_{\mathbf{R}^d} \min\left\{1,\ K\left(\frac{1}{|y|}\right)^{d-\frac{1}{\alpha}}\right\} \cdot p(y, \frac{u}{2})dy \\ &\ge \int_{\{|y|\ge c_{20}\}} \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \cdot \left(\frac{2}{u}\right)^{\alpha d} exp\left(-\frac{|y|^2}{2^{1-2\alpha}u^{2\alpha}}\right) \cdot \left(1 - \frac{K}{|y|^{d-\frac{1}{\alpha}}}\right)dy \\ &\ge c_{21} \int_{\{|z|\ge c_{22}/u^{\alpha}\}} exp(-\frac{|z|^2}{2})dz \\ &\ge exp\left(-\frac{K}{u^{2\alpha}}\right)\ . \end{aligned}$$

This proves (4.2). ∎

Since it is not known whether $T(1)$ has a bounded density, we will use the following lemma, which gives some information about the local density of a probability measure.

Lemma 4.2. *Given* $\lambda_0 > 0$, *there exists a positive constant* $K = K(\lambda_0)$ *with the following property: for any Borel probability measure* μ *on* $\mathbf{R}$, *there exists* $x \in [\lambda_0, 2\lambda_0]$ *such that*

$$\mu\big((x-\delta, x+\delta)\big) \le K\delta^{\frac{1}{2}} \quad \text{for every } 0 < \delta < \frac{1}{8}\ . \tag{4.3}$$

Proof. For $n = 1, 2, \cdots$, let $\delta_n = \frac{1}{n^3}$ and let

$$A_n = \left\{ x \in [\lambda_0, 2\lambda_0] : \ \mu\big((x - \delta_n, x + \delta_n)\big) \geq K\delta_n^{\frac{1}{2}} \right\} ,$$

where $K = K(\lambda_0) > 0$ will be chosen later. Let

$$A = \left\{ x \in [\lambda_0, 2\lambda_0] : \ \mu((x - \delta_n, x + \delta_n)) \geq K\delta_n^{\frac{1}{2}} \quad \text{for some} \ \ n = 1, \ 2, \cdots \right\}.$$

Denote the Lebesgue measure on $\mathbf{R}$ by L_1. Then by Fubini's theorem, we have

$$\begin{aligned} L_1(A_n) &= L_1\big\{ x \in [\lambda_0, 2\lambda_0] : \ \mu\big((x - \delta_n, x + \delta_n)\big) \geq K\delta_n^{\frac{1}{2}} \big\} \\ &\leq \frac{\int_{\lambda_0}^{2\lambda_0} \ \mu\big((x - \delta_n, x + \delta_n)\big) dx}{K \ \delta_n^{\frac{1}{2}}} \\ &= \frac{\int_{\lambda_0}^{2\lambda_0} dx \int 1_{(x-\delta_n, x+\delta_n)}(t) d\mu(t)}{K \ \delta_n^{\frac{1}{2}}} \\ &= \frac{\int d\mu(t) \int_{\lambda_0}^{2\lambda_0} 1_{(t-\delta_n, t+\delta_n)}(x) dx}{K\delta_n^{\frac{1}{2}}} \\ &\leq \frac{2}{K} \ \delta_n^{\frac{1}{2}} = \frac{2}{K} \ n^{-\frac{3}{2}} \ . \end{aligned}$$

Hence

$$L_1(A) \leq \sum_{n=1}^{\infty} \ \frac{2}{K} \ n^{-\frac{3}{2}} \ .$$

Take $K = K(\lambda_0) > 0$ such that $L_1(A) \leq \frac{1}{2}\lambda_0$. Then there exist $x \in [\lambda_0, 2\lambda_0]$ such that

$$\mu((x - n^{-3}, x + n^{-3})) \leq K \ n^{-\frac{3}{2}} \qquad \text{for every} \ \ n = 1, \ 2, \cdots . \tag{4.4}$$

For any $0 < \delta < \frac{1}{8}$, there exists an integer n such that

$$\frac{1}{(n+1)^3} \leq \delta < \frac{1}{n^3} \ .$$

By (4.4) we have

$$\begin{aligned} \mu((x - \delta, x + \delta)) &\leq \mu((x - n^{-3}, x + n^{-3})) \\ &\leq K \ n^{-\frac{3}{2}} \\ &\leq K \ \delta^{\frac{1}{2}} \ . \end{aligned}$$

This proves (4.3). □

Now we prove the main result of this section.

Theorem 4.1. *Let $X(t)$ $(t \in \mathbf{R})$ be a fractional Brownian motion of index α in $\mathbf{R}^d$ and $1 < \alpha d$. Then with probability 1,*

$$\liminf_{r \to 0} \ \frac{T(r)}{\phi(r)} = \gamma \ , \tag{4.5}$$

where $\phi(s) = s^{\frac{1}{\alpha}} / (\log\log \frac{1}{s})^{\frac{1}{2\alpha}}$ and $0 < \gamma < \infty$ is a constant depending on α and d only.

Proof. We start with the easy part and prove that there exists a constant $\gamma_1 > 0$ such that

$$\liminf_{r\to 0} \frac{T(r)}{\phi(r)} \geq \gamma_1 \quad \text{a.s.} \tag{4.6}$$

For $k = 1, 2, \cdots$, let $a_k = exp(-\frac{k}{\log k})$ and

$$A_k = \{\omega : T(a_k) \leq \lambda\phi(a_k)\} ,$$

where $\lambda > 0$ is a constant to be determined later. Then by the scaling property (4.1) of $T(r)$ and Lemma 4.1, we have, for k large enough,

$$\begin{aligned} P(A_k) &= P\Big\{T(1) \leq \frac{\lambda}{(\log\log\frac{1}{a_k})^{\frac{1}{2\alpha}}}\Big\} \\ &\leq exp\Big(-\frac{1}{K\lambda^{2\alpha}}\log\log\frac{1}{a_k}\Big) \\ &\leq k^{-\frac{1}{K\lambda^{2\alpha}}} . \end{aligned}$$

Take $\lambda = \gamma_1 > 0$ such that $K\gamma_1^{2\alpha} < 1$. Then

$$\sum_{k=1}^{\infty} P(A_k) < \infty .$$

By Lemma 2.4 (i), with probability 1, there exists $k_0 = k_0(\omega)$ such that $k \geq k_0$ implies

$$T(a_k) \geq \gamma_1\phi(a_k) .$$

Thus

$$\liminf_{k\to\infty} \frac{T(a_k)}{\phi(a_k)} \geq \gamma_1 \quad \text{a.s.}$$

Since $T(r)$ and $\phi(r)$ are monotone increasing for small r, we have that, for $r > 0$ small enough and $a_{k+1} \leq r < a_k$,

$$\frac{T(r)}{\phi(r)} \geq \frac{T(a_{k+1})}{\phi(a_{k+1})} \cdot \frac{\phi(a_{k+1})}{\phi(a_k)} .$$

Observing that

$$\lim_{k\to\infty} \frac{\phi(a_{k+1})}{\phi(a_k)} = 1 ,$$

we conclude that

$$\liminf_{r\to 0} \frac{T(r)}{\phi(r)} \geq \gamma_1 \quad \text{a.s.}$$

This proves (4.6).

In order to prove that there exists a positive constant γ_2 such that

$$\liminf_{r\to 0} \frac{T(r)}{\phi(r)} \leq \gamma_2 , \tag{4.7}$$

we let $b_k = exp(-k^2)$, $\tau_k = k^\theta b_k^{\frac{1}{\alpha}}$, where $\theta = \frac{3}{\alpha d - 1}$, and let $\lambda > 0$ be a constant to be determined later. Denote

$$T(b_k, \tau_k) = \int_{|t|\leq\tau_k} 1_{B(0,b_k)}\big(X(t)\big)dt .$$

Then by Theorem 3.2, we have

$$
\begin{aligned}
P\Big\{T(b_k,\tau_k) \neq T(b_k)\Big\} &\leq P\Big\{\exists\, t \text{ such that } \ |t| > \tau_k \text{ and } |X(t)| \leq b_k\Big\} \\
&\leq K\left(\frac{b_k}{\tau_k^\alpha}\right)^{d-\frac{1}{\alpha}} \qquad (4.8)\\
&= Kk^{-3}\ .
\end{aligned}
$$

Hence by Lemma 2.4 (i), with probability 1, there exists $k_1 = k_1(\omega)$ such that $k \geq k_1$ implies

$$T(b_k,\tau_k) = T(b_k)\ .$$

Let

$$E_k = \Big\{T(b_k,\tau_k) < \lambda_k \phi(b_k)\Big\}\ ,$$

where $\lambda \leq \lambda_k \leq 2\lambda$ will be chosen later. By (4.1) and Lemma 4.1, we have

$$
\begin{aligned}
P(E_k) &\geq P\Big\{T(b_k) < \lambda_k \phi(b_k)\Big\} \\
&= P\Big\{T(1) < \frac{\lambda_k}{(\log\log\frac{1}{b_k})^{\frac{1}{2\alpha}}}\Big\} \qquad (4.9)\\
&\geq c_{23}\ k^{-\frac{K}{\lambda^{2\alpha}}}\ .
\end{aligned}
$$

Thus, if we take $\lambda = \lambda_0$ such that $K/\lambda_0^{2\alpha} \leq 1$, then

$$\sum_{k=1}^{\infty} P(E_k) = \infty\ . \qquad (4.10)$$

In order to prove

$$P\Big(\limsup_{k\to\infty} E_k\Big) = 1\ ,$$

it suffices to show that for any $\epsilon > 0$, (2.5) is satisfied.

Fix a positive integer k. For $j > k$, we need to estimate

$$P(E_k \cap E_j) = P\Big\{T(b_k,\tau_k) < \lambda_k\phi(b_k)\ ,\ T(b_j,\tau_j) < \lambda_j\phi(b_j)\Big\}\ .$$

We denote this probability by Q. In order to create independence, we follow [Ta2] and make use of the stochastic integral representation (1.1). We set $v = \sqrt{\tau_k\tau_j}$ and consider the following two processes:

$$X^1(t) = (Y_1^1(t),\cdots,Y_d^1(t))$$

and

$$X^2(t) = (Y_1^2(t),\cdots,Y_d^2(t))\ ,$$

where $Y_1^1,\cdots,Y_d^1$ are independent copies of

$$Z^1(t) = \int_{|x|\leq v} G(t,x)dW(x)\ ,$$

and $Y_1^2, \cdots, Y_d^2$ are independent copies of

$$Z^2(t) = \int_{|x|\geq v} G(t,x)dW(x) \ .$$

Then X^1 and X^2 are independent and $X(t) = X^1(t) + X^2(t)$. By Lemma 5.2 of [Ta2], for any $|t| \leq \tau_k$,

$$\begin{aligned} \|Z^1(t)\|_2 &\leq K\sqrt[4]{\tau_k\tau_j}\cdot\tau_k^{\alpha-\frac{1}{2}} \\ &= K\sqrt[4]{\frac{\tau_j}{\tau_k}}\cdot\tau_k^{\alpha} \ , \end{aligned} \tag{4.11}$$

and for any $|t| \leq \tau_j$,

$$\|Z^2(t)\|_2 \leq K\sqrt{\frac{\tau_j}{\tau_k}}\ \big(\tau_k\tau_j\big)^{\frac{\alpha}{2}} \ , \tag{4.12}$$

where $\|Z\|_2 = \big(E(Z^2)\big)^{\frac{1}{2}}$.

For any $0 < \delta < 1$, let $\eta > 0$ satisfy

$$\frac{1}{(1-2\eta)^{\frac{1}{\alpha}}} = 1+\delta \ .$$

These numbers will be fixed for the moment. Clearly $\eta \geq \frac{\alpha}{4}\delta$. Since $|X^1(t)| \leq \eta b_k$ and $|X^2(t)| \leq (1-\eta)b_k$ imply $|X(t)| \leq b_k$, we have

$$\begin{aligned} \{T(b_k,\tau_k) < \lambda_k\phi(b_k)\} \subseteq \Big\{&\int_{|t|\leq\tau_k} 1_{B(0,(1-\eta)b_k)}(X^2(t))dt < \lambda_k\phi(b_k)\Big\} \\ &\cup \ \Big\{\max_{|t|\leq\tau_k}|X^1(t)| > \eta b_k\Big\} \ . \end{aligned} \tag{4.13}$$

Similarly,

$$\begin{aligned} \{T(b_j,\tau_j) < \lambda_j\phi(b_j)\} \subseteq \Big\{&\int_{|t|\leq\tau_j} 1_{B(0,(1-\eta)b_j)}(X^1(t))dt < \lambda_j\phi(b_j)\Big\} \\ &\cup \ \Big\{\max_{|t|\leq\tau_j}|X^2(t)| > \eta b_j\Big\} \ . \end{aligned} \tag{4.14}$$

By (4.13) and (4.14), we have that Q is less than

$$\begin{aligned} P\Big\{&\int_{|t|\leq\tau_k} 1_{B(0,(1-\eta)b_k)}(X^2(t))dt < \lambda_k\phi(b_k), \\ &\qquad \int_{|t|\leq\tau_j} 1_{B(0,(1-\eta)b_j)}(X^1(t))dt < \lambda_j\phi(b_j)\Big\} \\ &+ P\Big\{\max_{|t|\leq\tau_k}|X^1(t)| > \eta b_k\Big\} + P\Big\{\max_{|t|\leq\tau_j}|X^2(t)| > \eta b_j\Big\} \ . \end{aligned}$$

By the independence of X^1 and X^2, we have

$$\begin{aligned}
&P\Big\{\int_{|t|\le\tau_k} 1_{B(0,(1-\eta)b_k)}(X^2(t))dt < \lambda_k\phi(b_k),\\
&\qquad\qquad \int_{|t|\le\tau_j} 1_{B(0,(1-\eta)b_j)}(X^1(t))dt < \lambda_j\phi(b_j)\Big\}\\
&\quad = P\Big\{\int_{|t|\le\tau_k} 1_{B(0,(1-\eta)b_k)}(X^2(t))dt < \lambda_k\phi(b_k)\Big\}\\
&\qquad \cdot P\Big\{\int_{|t|\le\tau_j} 1_{B(0,(1-\eta)b_j)}(X^1(t))dt < \lambda_j\phi(b_j)\Big\}\\
&\quad \le P\Big\{\int_{|t|\le\tau_k} 1_{B(0,(1-2\eta)b_k)}(X(t))dt < \lambda_k\phi(b_k)\Big\}\\
&\qquad \cdot P\Big\{\int_{|t|\le\tau_j} 1_{B(0,(1-2\eta)b_j)}(X(t))dt < \lambda_j\phi(b_j)\Big\}\\
&\qquad + P\Big\{\max_{|t|\le\tau_k}|X^1(t)| > \eta b_k\Big\} + P\Big\{\max_{|t|\le\tau_j}|X^2(t)| > \eta b_j\Big\}\\
&\quad \hat{=} Q_1 + Q_2 + Q_3\ .
\end{aligned}$$

Hence,

$$Q \le Q_1 + 2Q_2 + 2Q_3\ . \tag{4.15}$$

Now we estimate $Q_1,\ Q_2$ and Q_3 respectively. First we consider

$$\begin{aligned}
&P\Big\{\int_{|t|\le\tau_k} 1_{B(0,(1-2\eta)b_k}(X(t))dt < \lambda_k\phi(b_k)\Big\}\\
&\le P\Big\{T((1-2\eta)b_k) < \lambda_k\phi(b_k)\Big\} + P\Big\{T((1-2\eta)b_k) \ne T((1-2\eta)b_k,\tau_k)\Big\}\\
&\le P\Big\{T(1) < \frac{1}{(1-2\eta)^{\frac{1}{\alpha}}}\cdot\frac{\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}}\Big\} + Kk^{-3}\ ,
\end{aligned} \tag{4.16}$$

where the last inequality follows from (4.1) and (4.8). By applying Lemma 4.2 to the distribution of $T(1)(\log\log\frac{1}{b_k})^{1/2\alpha}$, we can choose $\lambda_k \in [\lambda_0, 2\lambda_0]$ such that

$$P\Big\{\lambda_k \le T(1)(\log\log\frac{1}{b_k})^{\frac{1}{2\alpha}} < (1+\delta)\lambda_k\Big\} \le K\delta^{\frac{1}{2}} \quad \text{for any}\ \ 0<\delta<\frac{1}{8}\ . \tag{4.17}$$

Hence, by Lemma 4.1 and (4.17), the probability in (4.16) is less than

$$\begin{aligned}
&P\Big\{T(1) < \frac{\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}}\Big\}\cdot exp\Bigg(\frac{P\{\frac{\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}} \le T(1) < \frac{(1+\delta)\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}}\}}{P\{T(1) < \frac{\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}}\}}\Bigg)\\
&\le P\Big\{T(1) < \frac{\lambda_k\phi(b_k)}{b_k^{\frac{1}{\alpha}}}\Big\}\cdot exp\Big(Kk\ \delta^{\frac{1}{2}}\Big)\\
&\le \Big(P\{T(b_k,\tau_k) < \lambda_k\phi(b_k)\} + Kk^{-3}\Big)\cdot exp\Big(Kk\ \delta^{\frac{1}{2}}\Big)\ .
\end{aligned} \tag{4.18}$$

Combining (4.16) and (4.18), we have

$$\begin{aligned}
&P\Big\{\int_{|t|\le\tau_k} 1_{B(0(1-2\eta)b_k)}(X(t))dt<\lambda_k\phi(b_k)\Big\}\\
&\le\Big(P\{T(b_k,\tau_k)<\lambda_k\phi(b_k)\}+Kk^{-3}\Big)\cdot exp\Big(Kk\ \delta^{\frac12}\Big)+Kk^{-3}\\
&\le P\Big\{T(b_k,\tau_k)<\lambda_k\phi(b_k)\Big\}(1+\frac{K}{k^2})\cdot exp\Big(Kk\ \delta^{\frac12}\Big)\ .
\end{aligned}\tag{4.19}$$

Similarly, we have

$$\begin{aligned}
&P\Big\{\int_{|t|\le\tau_j} 1_{B(0(1-2\eta)b_j)}(X(t))dt<\lambda_j\phi(b_j)\Big\}\\
&\le P\Big\{T(b_j,\tau_j)<\lambda_j\phi(b_j)\Big\}(1+\frac{K}{j^2})\cdot exp\Big(Kj\ \delta^{\frac12}\Big)\ .
\end{aligned}\tag{4.20}$$

It follows from (4.19) and (4.20) that

$$\begin{aligned}
Q_1\le P\Big\{T(b_k,\tau_k)<\lambda_k\phi(b_k)\Big\}P\Big\{T(b_j,\tau_j)<\lambda_j\phi(b_j)\Big\}\\
\cdot(1+\frac{K}{k^2})(1+\frac{K}{j^2})exp\Big(Kj\ \delta^{\frac12}\Big)\ .
\end{aligned}\tag{4.21}$$

The same argument as in [Ta2] using (4.11) and (4.12) yields

$$P\Big\{\max_{|t|\le\tau_k}|X^1(t)|>\eta b_k\Big\}\le exp\Big(-K\eta^2\sqrt{\frac{\tau_k}{\tau_j}}\ \frac{1}{k^{2\theta\alpha}\ \log\frac{\tau_k}{\tau_j}}\Big)\tag{4.22}$$

and

$$P\Big\{\max_{|t|\le\tau_j}|X^2(t)|>\eta b_j\Big\}\le exp\Big(-K\eta^2(\frac{\tau_k}{\tau_j})^{1-\alpha}\ \frac{1}{j^{2\theta\alpha}\ \log\frac{\tau_k}{\tau_j}}\Big)\ .\tag{4.23}$$

Now we take

$$0<\beta<min\{\frac15,\ \frac13(1-\alpha)\}$$

and

$$\delta=\left(\frac{\tau_j}{\tau_k}\right)^\beta\ .$$

Then, by (4.15), (4.21), (4.22) and (4.23), we have

(4.24)

$$\begin{aligned}
Q\le P(E_k)P(E_j)(1+\frac{K}{k^2})(1+\frac{K}{j^2})\ exp\Big(Kj\ (\frac{\tau_j}{\tau_k})^{\frac{\beta}{2}}\Big)+c_{24}P(E_k)\\
\cdot\Big\{k\ exp\Big(-\frac{K}{k^{2\theta\alpha}\ \log\frac{\tau_k}{\tau_j}}(\frac{\tau_k}{\tau_j})^{\frac12-2\beta}\Big)+k\ exp\Big(-\frac{K}{j^{2\theta\alpha}\ \log\frac{\tau_k}{\tau_j}}(\frac{\tau_k}{\tau_j})^{1-\alpha-2\beta}\Big)\Big\}\ .
\end{aligned}$$

By the choice of τ_k, we have

$$\frac{\tau_k}{\tau_j}=\left(\frac{k}{j}\right)^\theta exp\Big(\frac{1}{\alpha}(j^2-k^2)\Big)\ .$$

Hence, by (4.24), for any $\epsilon > 0$, there exist a constant $K > 0$ and a positive integer k_0 such that for any $k \geq k_0$ and any $J > k$, we have

$$\sum_{j=k+1}^{J} P(E_k \cap E_j) \leq P(E_k)\left(K + (1+\epsilon) \sum_{j=k+1}^{J} P(E_j)\right) . \tag{4.25}$$

It follows from (4.10), (4.25) and Lemma 2.4 that

$$P\left(\limsup_{k\to\infty} E_k\right) \geq \frac{1}{1+2\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$P\left\{T(b_k) < \lambda_k \phi(b_k) \quad i.\ o.\right\} = 1 .$$

Therefore, with probability 1,

$$\liminf_{r\to 0} \frac{T(r)}{\phi(r)} \leq \gamma_2, \tag{4.26}$$

where $\gamma_2 = 2\lambda_0$. By (4.5), (4.26) and the zero-one law in [PT], we obtain (4.5). This completes the proof of Theorem 4.1. □

Since for any $t_0 \in \mathbf{R}$, $X(t+t_0) - X(t_0)$ $(t \in \mathbf{R})$ is also a fractional Brownian motion of index α, we have the following corollary.

Corollary 4.1. *Let $X(t)$ $(t \in \mathbf{R})$ be a fractional Brownian motion of index α in $\mathbf{R}^d$ and $1 < \alpha d$. Then for any $t_0 \in \mathbf{R}$, with probability 1,*

$$\liminf_{r\to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} = \gamma .$$

5. Packing Measure of Fractional Brownian Motion

We are ready to calculate the packing measure of a transient fractional Brownian motion in $\mathbf{R}^d$.

Proposition 5.1. *Let $X(t)$ $(t \in \mathbf{R})$ be a fractional Brownian motion in $\mathbf{R}^d$ of index α. If $1 < \alpha d$, then there exist positive constants C_1 and C_2 such that, with probability 1,*

$$C_1 \leq \phi\text{-}p\big(X([0,1])\big) \leq C_2 , \tag{5.1}$$

where $\phi(s) = s^{\frac{1}{\alpha}} / (\log\log \frac{1}{s})^{\frac{1}{2\alpha}}$.

Proof. We define a random Borel measure μ on $X([0,1])$ as follows. For any Borel set $B \subseteq \mathbf{R}^d$, let

$$\mu(B) = L_1\{t \in [0,1],\ X(t) \in B\} .$$

Then $\mu(\mathbf{R}^d) = \mu\big(X([0,1])\big) = 1$. By Corollary 4.1, for each fixed $t_0 \in (0,1)$, with probability 1

$$\begin{aligned} &\liminf_{r\to 0} \frac{\mu(B(X(t_0), r))}{\phi(r)} \\ &\qquad \leq \liminf_{r\to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} = \gamma . \end{aligned} \tag{5.2}$$

Let $E(\omega) = \{X(t_0) : \; t_0 \in (0,1) \text{ and (5.2) holds } \}$. Then $E(\omega) \subseteq X([0,1])$. A Fubini argument shows $\mu\big(E(\omega)\big) = 1$, a.s. Hence by Lemma 2.1, we have

$$\phi\text{-}p\big(E(\omega)\big) \geq K_1^{-3}\gamma^{-1} \ .$$

This proves the left hand inequality of (5.1) with $C_1 = K_1^{-3}\gamma^{-1}$.

To prove the right hand inequality of (5.1), we let $\mathcal{J}_k$ be the family of dyadic intervals of length 2^{-k} in $[0,1]$. For each $x \in [0,1]$, let $I_k(x)$ be the dyadic interval in $\mathcal{J}_k$ which contains x and let

$$a_k(x) = \sup_{s,\ t \in I_k(x)} |X(t) - X(s)| \ .$$

For any $k \geq 1$ and any $I \in \mathcal{J}_k$, by Lemma 2.2, we have

$$P\bigg\{ \sup_{s,\ t \in I} |X(t) - X(s)| \geq u 2^{-k\alpha} \bigg\} \leq exp\bigg(-\frac{u^2}{K} \bigg) \ .$$

Take $u = \sqrt{K\lambda \log k}$, where $\lambda > 6 + \frac{1}{2\alpha}$. We have

$$P\bigg\{ \sup_{s,\ t \in I} |X(t) - X(s)| \geq \sqrt{K\lambda \log k}\ 2^{-k\alpha} \bigg\} \leq \frac{1}{k^\lambda} \ .$$

Denote

$$M_k = \#\bigg\{ I \in \mathcal{J}_k, \quad \sup_{s,\ t \in I} |X(t) - X(s)| \geq \sqrt{K\lambda \log k}\ 2^{-k\alpha} \bigg\};$$

then

$$P\bigg\{ M_k \geq 2^k \cdot \frac{1}{k^{\lambda - 5}} \bigg\} \leq \frac{1}{k^5} \ .$$

By Lemma 2.4 (i), with probability one, for k large enough,

$$M_k < \frac{2^k}{k^{\lambda - 5}} \ . \tag{5.3}$$

Let Ω_0 be the event that (5.3) holds eventually, and let Ω_1 be the event that (2.5) holds. Then $P(\Omega_0 \cap \Omega_1) = 1$. Fix an $\omega \in \Omega_0 \cap \Omega_1$, and let $k_0 = k_0(\omega)$ be a positive integer such that $k \geq k_0$ implies (5.3).

For any $0 < \epsilon < 2^{-k_0}$ and any ϵ-packing $\{\overline{B}(X(t_i), r_i)\}$ of $X([0,1])$, we will show that for some absolute constant $C_2 > 0$,

$$\sum_i \phi(2r_i) \leq C_2 \text{ a.s.} \tag{5.4}$$

For each i, let

$$k_i = \inf\{k : \ a_k(t_i) \leq r_i\} \ .$$

Thus $a_{k_i - 1}(t_i) > r_i$ and the $\{I_{k_i}(t_i)\}$ are disjoint, so

$$\sum_i 2^{-k_i} \leq 1 \ . \tag{5.5}$$

Let Γ_1 be the set of the positive integer i's with

$$a_{k_i - 1}(t_i) \leq \sqrt{K\lambda \log(k_i - 1)}\ 2^{-(k_i - 1)\alpha}.$$

By (5.5), we obtain

$$\sum_{i\in\Gamma_1}\phi(2a_{k_i-1}(t_i))\le c_{25}\sum_i 2^{-k_i} \le c_{25}\ . \tag{5.6}$$

Let Γ_2 be the set of the i's satisfying

$$a_{k_i-1}(t_i)\ge\sqrt{K\lambda\log(k_i-1)}\ 2^{-(k_i-1)\alpha}\ .$$

By (5.3) and the uniform modulus of continuity of X, we have

$$\begin{aligned}\sum_{i\in\Gamma_2}\phi\Big(2a_{k_i-1}(t_i)\Big) &\le \sum_{i\in\Gamma_2}\frac{2^{k_i-1}}{(k_i-1)^{\lambda-5}}\cdot\phi\Big(K\sqrt{k_i-1}\ 2^{-\alpha(k_i-1)}\Big)\\ &\le c_{26}\sum_{k=1}\frac{1}{k^{\lambda-5-\frac{1}{2\alpha}}}=c_{27}\ .\end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7), we have

$$\sum_i\phi\Big(2a_{k_i-1}(t_i)\Big)\le c_{25}+c_{27}\ .$$

This implies (5.4), and by (2.2) we have

$$\phi\text{-}P\Big(X([0,1])\Big)\le C_2\ .$$

The right hand inequality of (5.1) follows from (2.4). □

Since fractional Brownian motion $X(t)$ $(t\in\mathbf{R})$ is ergodic (cf. [T]), the same proof as that of Proposition 5.1 in [T] yields the following theorem.

Theorem 5.1. *Let $X(t)$ $(t\in\mathbf{R})$ be a fractional Brownian motion of index α in $\mathbf{R}^d$. If $1<\alpha d$, then there exists a constant K, $0<K<\infty$, such that, with probability 1,*

$$\phi\text{-}p(X([0,t])=Kt\quad\textit{for any}\ \ t>0\ ,$$

where $\phi(s)=s^{\frac{1}{\alpha}}/(\log\log\frac{1}{s})^{\frac{1}{2\alpha}}$.

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