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SPATIAL CHAOTIC STRUCTURE OF ATTRACTORS OF REACTION-DIFFUSION SYSTEMS

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Abstract. The dynamics described by a system of reaction-diffusion equations with a nonlinear potential exhibits complicated spatial patterns. These patterns emerge from preservation of homotopy classes of solutions with bounded energies. Chaotically arranged stable patterns exist because of realizability of all elements of a fundamental homotopy group of a fixed degree. This group corresponds to level sets of the potential. The estimates of homotopy complexity of attractors are obtained in terms of geometric characteristics of the potential and other data of the problem.

0. Introduction

We describe here a mechanism which causes complicated spatial behavior of solutions of PDE. This mechanism is robust, and conditions under which it works are very simple to verify. It may explain chaotic spatial patterns arising in many branches of mechanics, physics, material science, biology, etc. The main cause of such chaotic patterns due to our approach is the persistence of certain homotopy classes under the dynamics of the system. They are discrete-valued conserved quantities. These homotopy classes are determined in terms of nonlinearities involved in considered equations. The number of different conserved classes exponentially depends on parameters of the systems. Symbolic dynamics describing chaos is determined by homotopy groups. The essential feature of our approach is that it is based not on local properties of solutions of the equations (such as, for example, homoclinicity) but rather on studying global quantities, such as the energy of the system. Note that we describe in the considered situations regular middle-scale patterns which are irregularly (chaotically) distributed on a larger scale and not a high-frequency chaotic behavior which may happen in other problems.

We consider semilinear parabolic systems of the form

$$\partial_t u = a\partial_x^2 u - F'(u). \tag{0.1}$$

Here F is a function of class $C^1(\mathbb{R}^d)$, $d \geq 2$, F' is its gradient, $u = (u_1, \dots, u_d)$, $x \in [0, L] \subset \mathbb{R}$, $F(u) \geq 0 \ \forall\ u \in \mathbb{R}^d$. We impose Dirichlet boundary conditions

$$u|_{x=0} = u|_{x=L} = b, \qquad b \in \mathbb{R}^d, \tag{0.2a}$$

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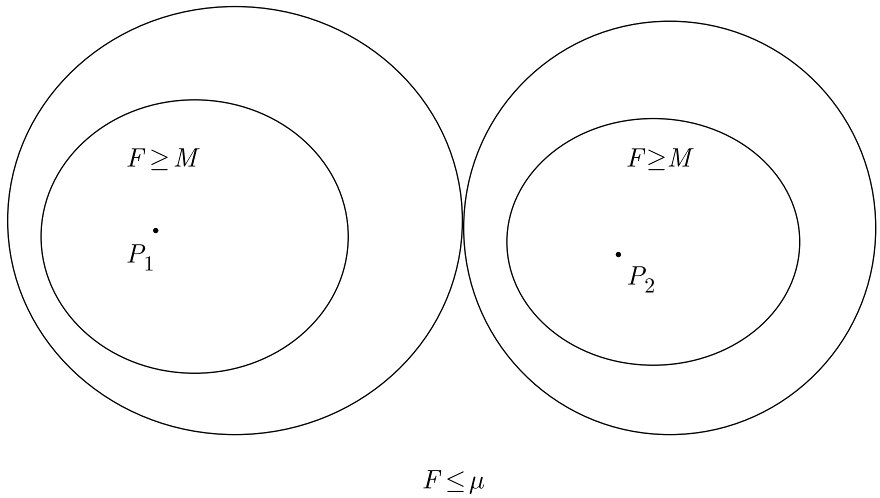

FIGURE 1

or periodic boundary conditions

$$u|_{x=0} = u|_{x=L}, \qquad \partial_x u|_{x=0} = \partial_x u|_{x=L}. \tag{0.2b}$$

Complicated spatial patterns persist under dynamics generated by this equation if variation of the potential F is large enough (for more precise conditions on F see below). The dynamics generated by (0.1) preserves homotopy classes determined by the potential F and initial data u_0 if the energy of initial data is not very large. Nontrivial homotopy classes correspond to complex spatial patterns of solutions, these patterns change with time but their homotopy type and, therefore, complexity is preserved. The homotopy type of these patterns is preserved also under continuous perturbations of initial data (with a bounded energy). The cause which generates persistent complicated spatial patterns is the structure of level sets and is a strong enough variation of the function F. We discuss it in the introduction in the case $d = 2$. To provide complicated solutions, the function F suffices to have at least two relatively sharp local maxima, the value of which should be large enough with respect to values of F in a low-energy domain $\mathcal{D}_\mu$, which surround them. To be more specific, assume that we have two points $P_1, P_2 \in \mathbb{R}^2$ ($P_i = (u_{i1}, u_{i2})$), $|P_1 - P_2| = 2R$, such that the potential F is large in r-neighborhood of these points ($r < R$),

$$F(u) \geq M \quad \text{when } |u - P_1| \leq r \text{ or } |u - P_2| \leq r. \tag{0.3}$$

The potential has to be small outside larger neighborhoods of these points,

$$F(u) \leq \mu \quad \text{when } |u - P_1| \geq R, |u - P_2| \geq R \tag{0.4}$$

(see Figure 1).

We denote the union of discs of radius r surrounding P_1 and P_2 by $\Omega(P, r)$, and its complement in $\mathbb{R}^2$ by $\Omega(P, r)$, $\Omega'(P, r) = \mathbb{R}^2 \backslash \Omega(P, r) = \{u : |u - P_i| > r, i = 1, 2\}$. The topology of Ω' is described by its fundamental group $\pi_1(\Omega'(P, r))$. This group corresponds to classes of continuous mappings of a circle S^1 into Ω', two homotopy equivalent mappings $u : S^1 \to \Omega'$ correspond to the same class $g(u)$.

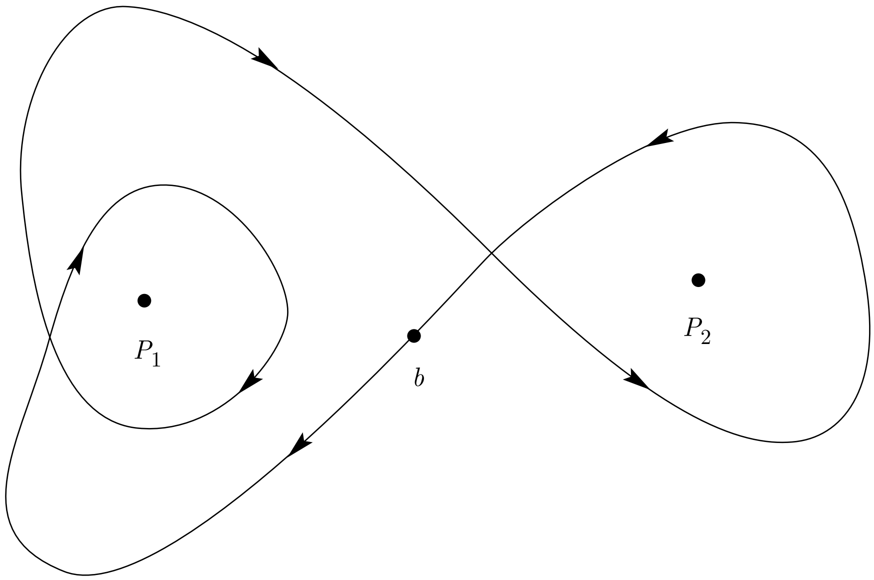

FIGURE 2

We shall consider a circle S^1 as a segment $[0, L]$ with endpoints glued together, so $u(0) = u(L)$. An example of the class is represented in Figure 2.

This class represents two clockwise turns around P_1 and one counterclockwise turn around P_2; it can be written as an element $g \in \pi_1(\Omega'(P, r), *)$ in the form

$$g = g_1^{-2} g_2, \tag{0.5}$$

where generators g_1 and g_2 correspond to counterclockwise turns around P_1 and P_2 respectively. For brevity, we shall consider mappings with a base point $* \in S^1$, $* = 0 = L$, and another base point $b \in \Omega'(P, r)$. (This case corresponds to Dirichlet boundary conditions. The case of periodic boundary conditions is similar but a little more complicated.) In this case the homotopy group is as usual denoted by $\pi_1(\Omega', *)$. Obviously, $\pi_1(\Omega'(P, r), *)$ does not depend on r, $0 < r < R$. An arbitrary finite sequence of non-zero integers $k_1, \dots, k_n$ determines an element

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \dots g_{i_n}^{k_n} \tag{0.6}$$

$i_{j+1} \neq i_j$. The sum $|k_1| + \dots + |k_n| = \deg g$ is called the degree of a monomial g. Obviously, g also can be represented as a product of $\deg g$ elements $g_1, g_2, g_1^{-1}, g_2^{-1}$, so the elements of the group $G = \pi_1(\Omega'(P, r), *)$ can be represented as finite words which consist of these four symbols and are subjected to a restriction: it should not include combinations $g_1 g_1^{-1}$, $g_1^{-1} g_1$, $g_2 g_2^{-1}$, $g_2^{-1} g_2$. If $\deg g$ is fixed, there is no less than $3^{\deg g}$ of such words of length $\deg g$. (More precisely, $4 \cdot 3^{\deg g - 1}$ in the case of Dirichlet boundary conditions.)

The structure of the word g contains information on the spatial structure corresponding to the function $u \in g$ (recall, that g is an equivalence class, therefore we write inclusion). For example, if points P_1 and P_2 have coordinates $(-R_1, 0)$ and $(R_1, 0)$ respectively and $b = (0, 0)$, then the function $u_1(x)$ which corresponds to the curve $u(x)$ described in Figure 2, has the graph of the form described in Figure 3.

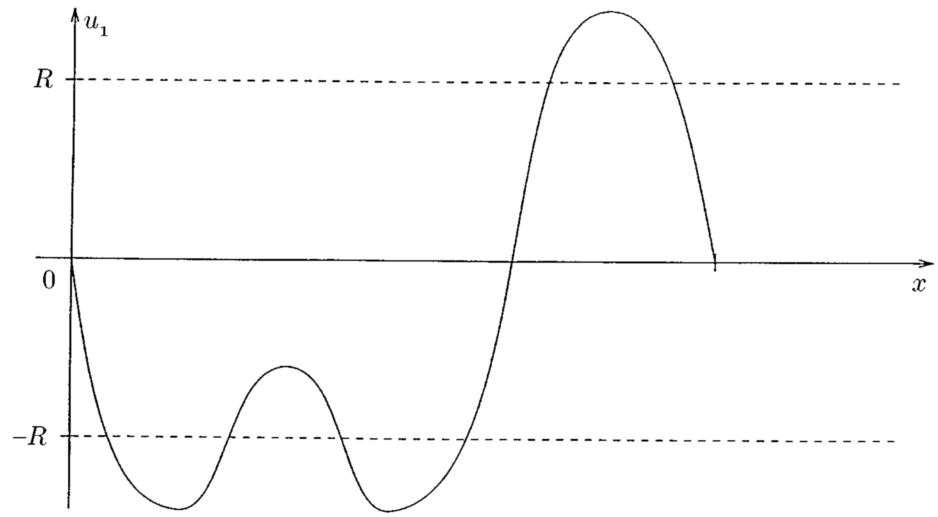

FIGURE 3

Sharp negative minima correspond to g_1 or g_1^{-1}, and sharp positive maxima to g_2 or g_2^{-1}. Analogously a graph of the form shown in Figure 4 corresponds to an element of form (0.6) $g = g_2^{k_1} g_1^{k_2} g_2^{k_3} g_1^{k_4}$ with $i_1 = 2$, $|k_1| = 3$, $i_2 = 1$, $|k_2| = 2$, $i_3 = 2$, $|k_3| = 1$, $i_4 = 1$, $|k_4| = 4$. (To see the difference between positive and negative k_j, we have to consider $u_2(x)$ as well.) One can see that neighboring terms $g_1^{\pm 1} g_2^{\pm 1}$ or $g_2^{\pm 1} g_1^{\pm 1}$ create transitions of amplitude of order R, and g_1 or g_2 itself are of order r (this can be easily justified if $M \gg \mu$, $R \gg r$). Hence, we have 4 transitions of order R and 10 oscillations of order r in this picture. So, if we have an element

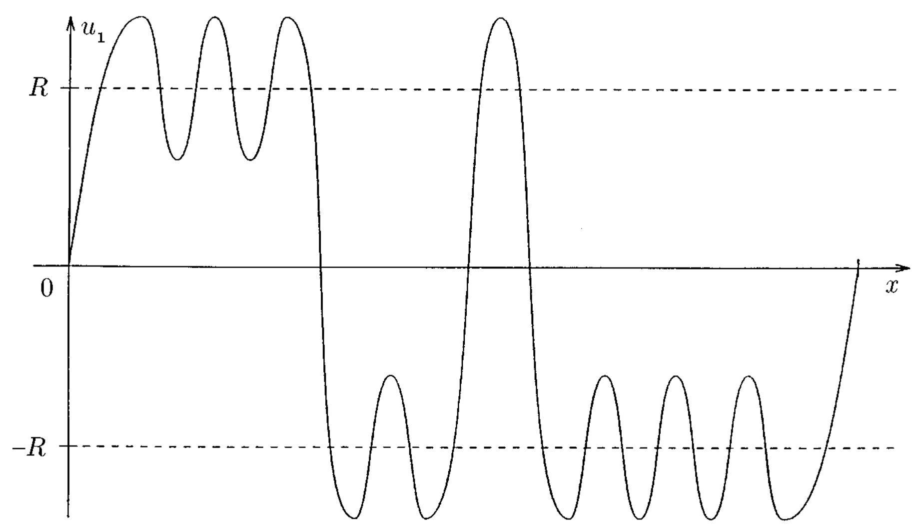

FIGURE 4

$g \in G$, its representative $u \in g$ has a homotopy stable spatial structure described by $u(x)$, which cannot be simpler than the structure of the element g itself.

Therefore, if we have at a fixed time t solution $u(x,t)$, such that $u(\cdot,t) \in g$ where $g \in G$ has a complex structure, then $u(x,t)$ has a complex structure as a function of x. If an attractor $\mathcal{A}$ includes N elements (which belong to different homotopy types), we call $\ln N$ homotopy complexity of the attractor.

To describe behavior of the system (0.1), it is very useful to consider the functional

$$\mathcal{P}(u) = \int_0^L \left(\frac{a}{2}\,|\partial_x u|^2 + F(u)\right) dx. \tag{0.7}$$

(It can be considered as the energy of the function u.) If $u(x,t)$ is a solution of (0.1), (0.2), then $\mathcal{P}(u(t))$ is a monotone non-increasing function of t.

It is well known (see [BV]), that the system (0.1), (0.2) generates dynamics in Sobolev's space $H_1([0,L])$ and has a global attractor $\mathcal{A}$. For any initial function $u(0) \in H_1([0,L])$ the solution $u(t)$ after some time (which depends only on the norm of $u(0)$ in Sobolev space $H_1([0,L])$) gets arbitrarily close to the global attractor $\mathcal{A}$. Therefore, the solutions of (0.1) have complicated spatial structure for large t if the attractor consists of complex functions $v(x)$.

The sets $E_\beta = \{u \in H_1[0,L] : \mathcal{P}(u) \le \beta\}$ are invariant with respect to these dynamics, these sets have attractors $\mathcal{A}(\beta)$. The main observation which we use in this paper is that dynamics generated by (0.1) on E_β if β is not too large, has a number of conserved quantities, namely these quantities are homotopy classes $g(u(t)) = g \in \pi_1(\mathbb{R}^2 \backslash (P_1 \cup P_2))$ of solutions $u(t)$ of (0.1). Sufficient condition on β under which $g(u)$ is well-defined is

$$\beta < r\sqrt{8Ma} \tag{0.8}$$

(see Theorems 2.1, 2.6, 4.1). The sets $\{u \in E_\beta, g(u) = g\} = Y(g,\beta)$ are non-empty if the following sufficient condition (we assume here that $|P_1 - P_2| = 2R$, $b = (P_1 + P_2)/2$) is satisfied:

$$\deg\, g \le Q^{1/2}, \qquad Q = \frac{r\sqrt{8M}L}{\sqrt{a}(2\pi R)^2} - \frac{L^2\mu}{a(2\pi R)^2}. \tag{0.9}$$

(There exist at least $3^{\deg g}$ different elements $g \in G$ with given $\deg g$.) In particular, under these conditions for every element $g \in G$, $\deg\, g \le d_0$, there exists a stable component $\mathcal{A}(\beta, g) \subset \mathcal{A}_\beta$ of the attractor $\mathcal{A}_\beta$, such that all elements of $\mathcal{A}(\beta,g)$ consist of $v \in g$. Therefore, all possible functions of the form shown in Figure 4 are represented as limit (as $t \to +\infty$) spatial patterns of solutions of (0.1), all sequences of $(k_1, \ldots, k_n)$, describing the number of sharp maxima are realizable, provided $|k_1| + \cdots + |k_n| \le d_0$.

For example, consider the most simple situation of a singular potential. Let

$$F(u) \ge q(|u - P_1|^{-3} + |u - P_2|^{-3}) \tag{0.10}$$

when $u \in \Omega(P,r)$, with $r > 0$, $q > 0$ and let $F(u) \le \mu$ when $u \in \Omega''(P,r) = \{|u| \le 3R\} \cap \{\Omega'(P,r)\}$. Then we obtain $\sqrt{M}r \to \infty$ as $r \to 0$. We can take energy level β arbitrarily large and there exist functions $u \in g \cap E_\beta$ with arbitrarily large $\deg\, g$. Therefore, we obtain arbitrarily complex spatial patterns described by arbitrary g.

Hence, in the situation of a strongly singular potential satisfying (0.10), arbitrary configurations of the type described in Figure 4 are realizable as spatial forms of solutions $u(x,t)$ on the fixed interval $[0,L]$.

Note that the case of nonsingular potential is more complicated (and may be more interesting). In this case for solutions with large initial energy $\mathcal{P}(u(0))$ the homotopy class may not be defined. But $\mathcal{P}(u(t))$ is a strictly decreasing function of time, and after some time the condition $\mathcal{P}(u(t)) < \beta$ may be fulfilled. Such a behavior can be interpreted as the emerging of a (complicated) spatial middle-scale pattern of originally chaotic (on small scales) patterns without distinguishable middle-scale elements corresponding to homotopy groups. This recalls a process of crystallization under decreasing temperature.

We will discuss briefly now the formulae which give restrictions on deg g of homotopy classes to which solutions belong. These inequalities give estimates of complexity of solutions in very simple terms.

The right-hand side of (0.9) is maximal if we take $L = r\sqrt{2aM}/\mu$ and we obtain

$$Q = \frac{r^2 M}{\pi^2 R^2 \mu}.$$

The important feature of the situation when complex patterns persist is that there are at least two different levels M (high) and μ (low) of potential energy F and two different characteristic lengths r and R in the (u_1,u_2) plane. The length r is proportional to the minimal width of high-energy level core. The length R is proportional to the length of a cycle turning around a location of high potential energy but remaining in the low energy zone. The characteristic quantities are $\sqrt[4]{Mr^2} = \sigma_{\text{high}}$ and $\sqrt[4]{\mu R^2} = \sigma_{\text{low}}$; they are of dimensionality of the u. The chaotic pattern persists if $\sigma_{\text{high}} \gg \sigma_{\text{low}}$ and the complexity describing the chaos is proportional to the ratio $\frac{\sigma_{\text{high}}}{\sigma_{\text{low}}}$, deg $g \leq C\,\frac{\sigma_{\text{high}}}{\sigma_{\text{low}}}$ is the limitation on the degree of the homotopy element g (roughly speaking, on the number of crests in the graphs of u_1 and u_2).

In the situation when L in (0.2a) or (0.2b) is not prescribed, the quantities σ_{high} and σ_{low} which characterize the potential F play determining roles. If L is fixed, a new scaling parameter appears and one more quantity $\sigma_L = R\sqrt[4]{a}/\sqrt{L}$ of dimensionality of u is of importance. One can easily see that the quantity Q described by (0.9) is proportional to

$$Q_0 = \left(\frac{\sigma_{\text{high}}}{\sigma_L}\right)^2 - \left(\frac{\sigma_{\text{low}}}{\sigma_L}\right)^4$$

and complexity is proportional to

$$\sqrt{\left(\frac{\sigma_{\text{high}}}{\sigma_L}\right)^2 - \left(\frac{\sigma_{\text{low}}}{\sigma_L}\right)^4}.$$

Now we make remarks on equilibriums of (0.1). The steady states of (0.1) are solutions of system of ODE

$$a\partial_x^2 u - F'(u) = 0. \tag{0.11}$$

To use notations closer to standard for ODE, we put $x = \tau$, $\partial_x u = \dot{u} = P$, $-F(u) = U(u)$ and $H(u,p) = U(u) + \frac{a}{2}p^2$. Then (0.11) can be written in the form of Hamiltonian systems $\dot{u} = H_p$, $\dot{p} = -H_p$. Our results imply existence of a set of solutions of this system with positive entropy. Existence of many solutions of problems of type (0.11) was proved in many papers, see [G], [B], [R], [ACZ]. Our approach is different in the respect that we connect the number of solutions with values of $\mathcal{P}$ and not H, obtaining exponentially many different stable solutions, and we use the theory of dynamical systems in infinite dimensional spaces.

Another method was used in [N] where existence of many (but not exponentially many) solutions was proved for reaction-diffusion systems with large ratio of diffusion rates for different components.

Complex structure of the spatial domain can also cause existence of many stable solutions (see [JM], [H] where harmonic mappings of different topological types are constructed). In this approach, contrary to ours, topology of solutions is prescribed from the very beginning. In a way, our approach is closer to [BBH] where topology arises in the dynamics when singularities generate non-trivial topology of the spatial domain. Our approach is applicable to more general situations and gives, unlike in papers cited, exponentially many different types of stable solutions. We took the idea to connect complicated spatial behavior of solutions with symbolic dynamics from [AC]. In that work solutions on lattices were considered. Since we have solutions on a connected set in the present work, we use homotopy-class valued symbolic dynamics here. One has to note that the structure of nonlinearities and boundary conditions play very important role in our approach. Without this kind of assumption one cannot hope to obtain non-homogeneous stable solutions (see [KW]).

Generalizations of results of the paper are possible in many directions; we mention some of them. The first concerns the case of x-dependent potential $F(x,u)$, $x \in S^1$. Under appropriate conditions the homotopy classes of sections of fiber bundles over S^1 are preserved.

Another direction is the case of many spatial variables, $x \in \Omega \subset \mathbb{R}^n$, $n > 1$. In this case dynamics described by quasilinear systems preserves homotopy classes of mappings $u : \Omega \to \mathbb{R}^d$ (with appropriate boundary conditions and conditions on F and main part).

These and other generalizations will be published elsewhere.

1. A Semigroup Generated by a Parabolic Equation

We consider a semilinear parabolic system of equations

$$\partial_t u = a\partial_x^2 u - F'(u,x) \tag{1.1}$$

where $a > 0$, $0 < x < L$, $u = (u_1, u_2, \dots, u_d)$, $d \geq 2$, $F : \mathbb{R}^d \times [0,L] \to \mathbb{R}$ is a C^1-function (a potential), $F' = (\partial F/\partial u_1, \dots, \partial F/\partial u_d) = \nabla F$ is a Lipschitz function of u, $\partial_x u = \partial u/\partial x$, $\partial_t u = \partial u/\partial t$. We impose one of two boundary conditions: either Dirichlet,

$$u|_{x=0} = b_1, \qquad u|_{x=L} = b_2 \tag{1.2a}$$

(for simplicity we shall take $b_1 = b_2 = b$—the case $b_1 \neq b_2$ makes no difference in this section but may cause more lengthy discussion later); or the periodic (in this case F is assumed to be periodic in x as well),

$$u|_{x=0} = u|_{x=L} = 0, \qquad \partial_x u|_{x=0} = \partial_x u|_{x=L}. \tag{1.2b}$$

(We shall refer to these conditions as (1.2) if both are treated in the same way.) We assume that the potential F satisfies the condition

$$F(u,x) \geq 0 \qquad \forall\, u, x; \tag{1.3}$$

$$|F'(u,x) - F'(v,x)| \leq C(|u|,|v|)|u-v| \tag{1.4}$$

where $C(|u|,|v|)$ is bounded for $|u|,|v|$ bounded. We assume that $\inf_u F(x,u) = 0$ $\forall\, x \in [0,L]$. Since we can add any function of x to F not changing ∇F, this is not a restriction.

We denote by H the Hilbert space $L_2(\Omega) \times L_2(\Omega)$, $\Omega = (0,L)$; H_1 is the Sobolev space of functions $u = (u_1, u_2)$ which satisfy (1.2) with the bounded norm

$$\|u\|_1 = (\|\partial_x u\|^2 + \|u\|^2)^{1/2},$$

where $\|u\|$ is the norm in H,

$$\|u\|^2 = \int_\Omega |u_1(x)|^2 + \cdots + |u_d(x)|^2 dx.$$

The space H_2 consists of functions which satisfy (1.2) and $\partial_x^2 u \in H$.

Theorem 1.1. *Let $u_0 \in H_1$. Then there exists unique solution $u(t)$ of* (1.1), (1.2) *which belongs to $L_\infty([0,T],H_1) \cap L_2([0,T],H_2)$ for any $T > 0$.*

Proof. Local (in t) existence on a segment $[0,T_0]$ where T_0 depends on $\|u_0\|_1$ and uniqueness of solutions can be easily proved and is well-known (see for example, [H, BV, L]. To prove existence on $[0,T]$ for arbitrary T, we consider the energy functional

$$\mathcal{P}(u) = \int_\Omega {}^a\!/\!{}_2 |\partial_x u|^2 + F(u,x) dx. \tag{1.5}$$

Multiplying the equation (1.1) by $\partial_t u$ and integrating in α and t, we obtain the equation

$$\mathcal{P}(u(t_0)) + \int_0^{t_0} \|\partial_t u\|^2 dt = \mathcal{P}(u(0)). \tag{1.6}$$

Justification of this formula can be done as in [BV]. We do not need to impose growth conditions on F' here thanks to the Sobolev embedding $C(\bar\Omega) \supset H_1(\Omega)$ which holds in the one-dimensional case ($\Omega = (0,L)$).

From (1.6) we deduce

$$\mathcal{P}(u(t_0)) \leq \mathcal{P}(u(0)) \tag{1.7}$$

when $0 \leq t_0 \leq T_0$. Since

$$\|u\|_1^2 \leq \frac{2}{a}\,\mathcal{P}(u) + \|u\|^2, \tag{1.8}$$

the inequality (1.7) and the local existence imply in a usual way by the Gronwall inequality that the solution $u(t)$ exists for $0 \leq t < +\infty$ and satisfies (1.7) for all $t_0 > 0$.

Theorem 1.2. *The problem* (1.1), (1.2) *generates a semigroup of operators S_t : $H_1 \to H_1$. These operators are continuous for $t \geq 0$ and are compact on bounded sets for $t > 0$.*

Proof. Continuity of S_t easily follows from the Lipschitz condition (1.4) and the boundedness of $u(t)$, $t \in [0,T]$, in $C(\Omega)$ which follows from (1.7), (1.8) and the embedding $H_1(\Omega) \subset C(\Omega)$. Note that the mapping $u_0 \to u(t)$ is continuous from H_1 into the space $W(0,T)$ with the norm defined by

$$\|u\|_W^2 = \int_0^T \|u\|_2^2 + \|\partial_t u\|^2 dt.$$

The mapping $u(t) \to F'(u(t),x)$ is compact from $W(0,T)$ into $L_2([0,T],H)$. We consider (1.1) as a linear equation with F' a given function and an initial data u_0. Its solution $u(t) \in H_1$ with $t > 0$ fixed depends completely continuously on $u(0) \in H_1$, and continuously on $F'(u(t),\cdot) \in L_2([0,T],H)$. Therefore, it depends completely continuously on $u(0)$, and the theorem is proved.

Let us fix $\beta > 0$ and consider the set $E_\beta \subset H_1$,

$$E_\beta = \{v \in H_1 : \mathcal{P}(v) \le \beta\}. \tag{1.9}$$

According to (1.7), the set E_β is invariant with respect to S_t. Let $\{S_t\}$ be the semigroup generated by (1.1), (1.2a).

Theorem 1.3. *For every $\beta > 0$ there exists a global attractor $\mathcal{A}_\beta \subset E_\beta$ of the semigroup $\{S_t\}$ restricted to E_β.*

Proof. The existence of $\mathcal{A}_\beta$ follows from boundedness of E_β in H_1, compactness of S_t and continuity of S_t (see [BV, Ha]).

Note, that $\mathcal{A}_\beta$ may be a non-connected set. In fact, we shall show that under appropriate conditions on F there exists many disjoint components of $\mathcal{A}_\beta$.

In the case of periodic boundary conditions we have to impose additional conditions to provide boundedness of F_β:

$$F(u) \to +\infty \qquad \text{as } |u| \to \infty. \tag{1.10}$$

Theorem 1.4. *Let* (1.10) *hold in addition to conditions* (1.3), (1.4)*. Then for any $\beta > 0$ the semigroup $\{S_t\}$ generated by* (1.1), (1.2b) *has a global attractor $\mathcal{A}_\beta \subset E_\beta$.*

Proof. The proof is the same as the one of Theorem 1.3 if we have boundedness of E_β in H_1. To prove this boundedness, it suffices to show that the condition $\mathcal{P}(u) \le \beta$ implies

$$\left| \int_0^L u(x)dx \right| \le \mathcal{C} \tag{1.11}$$

for a constant $\mathcal{C} = \mathcal{C}(\beta)$. Assume that (1.11) were not true. Then we have a sequence u_j,

$$\int_0^L u_j(x)dx = \mathcal{C}_j, \qquad |\mathcal{C}_j| \to +\infty, \qquad \mathcal{P}(u_j) \le \beta.$$

Condition (1.10) implies that

$$F(u) \ge F_0(|u|), \qquad F_0(|u|) \to \infty \quad \text{as } |u| \to \infty.$$

Let

$$v_j = u_j - \frac{1}{L} \int_0^L u_j dx.$$

We have

$$\int_0^L |\partial_x v_j|^2 dx \le \beta, \qquad \int_0^L v_j dx = 0.$$

This implies compactness of v_j in $\mathcal{C}([0,L])$, and after relabeling we may assume that $v_j \to v$ in $\mathcal{C}([0,L])$. Obviously,

$$\int_0^L F(u_j)dx = \int_0^L F(v_j - \mathcal{C}_j/L)dx.$$

Since v_j are bounded in $\mathcal{C}([0,L])$,

$$|v_j - \mathcal{C}_j/L| \ge |\mathcal{C}_j|/L - |v_j| \to \infty$$

as $j \to \infty$ uniformly in x. This implies by (1.10)

$$\int_0^L F(u_j)dx \to \infty \quad \text{as } j \to \infty,$$

and this contradicts the condition $\mathcal{P}(u_j) \le \beta$. Hence, (1.11) is true. Boundedness of $\int_0^L |\partial_x u|^2 dx$, $\left|\int_0^L u dx\right|^2$ implies boundedness of u in $H_1([0,L])$, therefore E_β is bounded. This implies existence of the attractor as in Theorem 1.3.

Remark 1.1. If the condition $F(u)u \ge \alpha|u|^2 - C$, $\alpha > 0$, is imposed, then (1.1) and (1.2) have a global attractor $\mathcal{A}$ (see [BV]).

2. Preservation of Homotopy Classes by the Semigroup

In this section we give in case $d = 2$ sufficient conditions on the potential F under which the dynamics of S_t on E_β preserves classes of functions defined in homotopic terms. We consider in this and the following sections x-independent F (x-dependent F will be considered elsewhere). More general situations and case $d > 2$ are considered in Section 4.

For a given finite set P of points P_i, $i = 1, \dots, m$, let $d(P) = \inf_{i \ne j} |P_i - P_j|$. Let $\Omega(P,r) = \{y \in \mathbb{R}^2 : \exists\, P_i \in P, |y - P_i| \le r\}$, $\Omega' = \mathbb{R}^2 \backslash \Omega(P,r) = \Omega'(P,r)$.

Consider the set $\mathcal{D}_\mu \subset \mathbb{R}^2$

$$\mathcal{D}_\mu = \{y \in \mathbb{R}^2 : F(y) < \mu\}. \tag{2.1}$$

We assume that the potential F and the value μ satisfy the following condition.

Condition 2.1. There exist points $P_1, \dots, P_m \in \mathbb{R}^2 \backslash b$ and numbers $r > 0$, $M > 0$ such that $r < d(P)/2$ and

$$F(y) \ge M \text{ when } \operatorname{dist}(y,P) \le r. \tag{2.2}$$

Here and below

$$\operatorname{dist}(y,P) = \inf_i \{|y - P_i|\}.$$

We shall say that $C(P,r,M)$ holds if Condition 2.1 is fulfilled.

For a given set $\mathcal{D} \subset \mathbb{R}^d$ we denote by $\pi_1(\mathcal{D},*)$ the fundamental homotopy group of classes of mappings from the circle S^1 into $\mathcal{D}$ (see [FFG, M, W]). Let $G = \pi_1(\Omega'(P,r),*)$ be the fundamental homotopy group of $\Omega'(P,r)$ (with a base point $* = b \in \Omega'$ in the case of Dirichlet boundary conditions). Obviously, G is independent on $r \ge 0$ if $r < d(P,M)/2$ since $\Omega'(P,r_1)$ is a deformation retract of $\Omega'(P,r_2)$ when $0 < r_2 \le r_1$. The set $\mathcal{D}'_M = \mathbb{R}^d \backslash \mathcal{D}_M$ with large M may be considered a (soft) barrier for curves $u(t)$. We can take r_0 as maximal r such that $\Omega'(P,r)$ is a deformation retract of $\mathcal{D}'_\mu$. Obviously, $\Omega'(P,r)$ is homotopy equivalent to a bunch of circles $\bigvee_{j=1}^m S_j^1$ and elements of G can be uniquely represented in the form

$$g = \prod_{j=1}^{n} g_{i_j}^{k_j} \tag{2.3}$$

where g_i is a generator of j-th group $\pi_1(S_j^1, *)$, $k_j \in \mathbb{Z}$, $k_j \neq 0$, $i_j \neq i_{j+1}$. (In periodic case when we do not fix $*$, the representation is unique modulo cyclic permutations if $i_1 \neq i_n$.) We denote $\deg g = |k_1| + \cdots + |k_m|$.

Every continuous function $v(x)$, $x \in [0,L]$, which satisfies (1.2a) and takes values in $\Omega'(P,r)$ for all $x \in [0,L]$ belongs to a unique class of equivalence which is represented by $g = g(v) \in G$.

For $g \in G$, let $X(\Omega'(P,r), g)$ be the set of $u \in C([0,\ell])$ which take values in $\Omega'(P,r)$ and belong to g. Let

$$Y(P,r,g,\beta) = X(\Omega'(P,r), g) \cap E_\beta. \tag{2.4}$$

(Note that by Sobolev embedding theorems $E_\beta \subset H_1[0,L] \subset \mathcal{C}([0,L])$.) We shall prove below that under appropriate conditions on F and β the set $Y(P,g,\beta)$ is invariant with respect to S_t. Moreover, there exists $\delta > 0$ such that

$$S_t(Y(P,0,g,\beta)) \subset Y(P,\delta,g,\beta) \qquad \forall\, t \geq 0, \quad \forall\, g \in G. \tag{2.5}$$

(Note that the empty set is always invariant, $S_t\emptyset = \emptyset$.) We shall for brevity denote $Y(g,\beta) = Y(P,0,g,\beta)$.

Theorem 2.1. *Let Dirichlet boundary condition* (1.2a) *hold,* $R = \min_j |P_j - b|$. *Let Condition* (1.2a) *hold. Let* $r > \delta > 0$, $r \leq R$ *and*

$$\beta < \sqrt{8}(r-\delta)\sqrt{a}\sqrt{M} + \frac{2(R-r)^2 a}{L}. \tag{2.6}$$

Then for any $u \in E_\beta$

$$dist(u(x), P) > \delta, \qquad \forall\, x \in [0,L]. \tag{2.7}$$

Here and below $dist(v,P) = \inf_{z \in P} |v - z|$.

Proof. To prove (2.7), assume the contrary, that is

$$\exists\, P_1 \in P, \qquad |u(x_0) - P_1| = \delta, \qquad x_0 \in (0,L). \tag{2.8}$$

Let $r_1 = r - \delta$. Changing variable $v = u - P_1$, we obtain from (2.8) that

$$v(x_0) = P_0, \qquad |P_0| = \delta, \qquad v(0) = v(L) = b - P_1. \tag{2.9}$$

We choose such $x_0 = x^*$ that is minimal over all possible x_0 which satisfy (2.8), x_* being the maximal of such x_0. Let

$$U(\rho) = \inf_{|v| \leq \rho} F(P_1 + v). \tag{2.10}$$

Consider $\mathcal{P}(u)$,

$$\mathcal{P}(u) = \int_0^L {}^a\!/_2 |\partial_x u|^2 + F(u)dx = \int_0^L {}^a\!/_2 |\partial_x v|^2 + F(P_1 + v)dx.$$

Obviously $v(x) \neq 0$ for $x \in (0, x^*) \cup (x_*, L)$. Taking (2.10) into account, we see that

$$\mathcal{P}(u) \geq \int_0^{x^*} {}^a\!/\!_2 |\partial_x v|^2 + U(|v|)dx + \int_{x_*}^L {}^a\!/\!_2 |\partial_x v| + U(|v|)dx. \tag{2.11}$$

Using polar coordinates (ρ, ϕ) in $\mathbb{R}^2$, $\rho = |u|$, we see that

$$|\partial_x v|^2 = |\partial_x \rho|^2 + \rho^2 |\partial_x \phi|^2.$$

Therefore,

$$\mathcal{P}(u) \geq \int_0^{x_0^*} {}^a\!/\!_2 |\partial_x |v||^2 + U(|v|)dx + \int_{x_*}^L {}^a\!/\!_2 |\partial_x |v||^2 + U(|v|)dx.$$

Let

$$\begin{aligned} U^*(\rho) &= M \text{ when } 0 \leq \rho \leq r, \\ U^*(\rho) &= 0 \text{ when } \rho > r. \end{aligned} \tag{2.12}$$

Since Condition 2.1 holds,

$$U(|v|) \geq U^*(|v|),$$

and we have

$$\mathcal{P}(u) \geq \int_0^{x^*} {}^a\!/\!_2 |\partial_x |v||^2 + U^*(|v|)dx + \int_{x_*}^L {}^a\!/\!_2 |\partial_x |v||^2 + U^*(|v|)dx.$$

We apply Lemma 2.1 (see below) where $\ell = x^*$, and $r - \delta = r_1$, $R_1 = R - \delta$. We obtain:

$$\int_0^{x^*} {}^a\!/\!_2 |\partial_x |v||^2 + U^*(|v|)dx \geq \sqrt{2} r_1 \sqrt{a}\sqrt{M} + \frac{(R_1 - r_1)^2 a}{2x^*}.$$

Analogously,

$$\int_{x_*}^L {}^a\!/\!_2 |\partial_x |v||^2 + U^*(|v|)dx \geq \sqrt{2} r_1 \sqrt{aM} + \frac{(R_1 - r_1)^2 a}{2(L - x_*)}.$$

These inequalities and (2.6) imply that $\mathcal{P}(u) > \beta$, which contradicts the assumption $u \in E_\beta$. Therefore (2.7) holds, and the theorem is proved.

Lemma 2.1. *Let $U^*(\rho)$ be defined by* (2.12). *Let $w \in H_1([0, \ell])$ be a scalar function, $w(0) = 0$, $w(\ell) = R_1 \geq r_1 > 0$. Then*

$$\mathcal{P}(w, U^*) = \int_0^\ell {}^a\!/\!_2 |\partial_x w|^2 + U^*(|w|)dx \geq \sqrt{2} r_1 \sqrt{a}\sqrt{M} + \frac{(R_1 - r_1)^2 a}{2\ell}. \tag{2.13}$$

Proof. We take $R_1 > r_1$. Obviously, we can find a function $U_\epsilon \in C^2(\mathbb{R})$, $U_\epsilon(\rho) \leq U^*(\rho)\ \forall\ \rho$, $U_\epsilon(\rho) = M$ when $\rho \leq r_1 - \epsilon$, $U_\epsilon(\rho) = 0$ when $\rho \geq r_1$, $U'_\epsilon(\rho) \leq 0$, $U_\epsilon(\rho) \to U^*(\rho)$ when $\epsilon \to 0$. Let $w_\epsilon = w$ be a global minimizer of $\mathcal{P}^*(v, U_\epsilon)$. Such a minimizer exists. This easily follows from boundedness of the set $P(u) \leq \epsilon$ in $M_1([0, \ell])$, compactness of the embedding $M_1([0, \ell]) \subset \mathcal{C}([0, \ell])$ and continuity of U_ϵ. One also has to use bilinearity and positiveness of the main part of $P(v)$. This function w is a smooth solution of the Euler equation

$$-a\partial_x^2 w + U'_\epsilon(w) = 0, \qquad w(0) = 0, \qquad w(\ell) = R_1. \tag{2.14}$$

This equation admits the integral

$$-{}^a\!/\!_2 (\partial_x w)^2 + U_\epsilon(w) = -E. \tag{2.15}$$

Since $U'_\epsilon < 0$, $R_1 > r_1$, one can easily see that w is a convex growing function, and v is linear for $0 < x < x_1$ when $w < r_1 - \epsilon$ and for $x \geq x_2$ when $w \geq r_1$. We obtain that

$$w = x\sqrt{2(E+M)}/\sqrt{a} \quad \text{when } w \leq r_1 - \epsilon,$$
$$w = R + \sqrt{2E}(x-\ell)\sqrt{a} \quad \text{when } \ell \geq w \geq r_1.$$

We have also

$$\frac{\sqrt{2E}}{\sqrt{a}}(\ell - x_2) = R_1 - r_1, \qquad \frac{x_1\sqrt{2(E+M)}}{\sqrt{a}} = r_1 - \epsilon.$$

We see that

$$\int_0^\ell {}^a\!/_2|\partial_x w|^2 + U_\epsilon(w)dx \geq x_1(E+M) + x_1 M + E(\ell - x_2)$$
$$= \frac{a(r_1-\epsilon)^2}{2x_1} + x_1 M + \frac{(R_1-r_1)^2 a}{2(\ell - x_2)}$$
$$\geq \sqrt{2}(r_1-\epsilon)\sqrt{a}\sqrt{M} + \frac{(R_1-r_1)^2 a}{2\ell}.$$

Hence, $\mathcal{P}^*(v, U^*) \geq \mathcal{P}^*(v, U_\epsilon) \geq \sqrt{2}(r_1-\epsilon)\sqrt{M}\sqrt{a} + (R_1-r_1)^2 a/(2\ell)$. Since ϵ is arbitrarily small, we obtain (2.13). If $R_1 = r_1$, we can pass in (2.13) to the limit as $R_1 \to r_1$.

Remark. In fact, from the proof of Lemma 2.1, we see that we have the inequality

$$\mathcal{P}(w, U^*) \geq \min_{0 \leq x_1 \leq \ell} \frac{ar_1^2}{2x_1} + x_1 M + \frac{(R_1-r_1)^2 a}{2(\ell - x_1)}.$$

The inequality (2.13) gives a convenient lower estimate of the right-hand side. If $R_1 \leq r_1$, we have $\mathcal{P}(w, U^*) \geq r_1\sqrt{2Ma}$.

Corollary 2.1. *Under conditions of Theorem* 2.1, (2.5) *holds.*

Proof. By (1.7) $S_t E_\beta \subset E_\beta$. Since $u(x,t)$ is continuous in x, t, two curves $u(x,t_1)$, $u(x,t_2)$ are homotopy equivalent. Using (2.7), we obtain (2.5).

Theorem 2.2. *Under conditions of Theorem* 2.1, *the semigroup* $\{S_t\}$ *restricted to* $Y(g,\beta)$ *has the attractor* $\mathcal{A}(g,\beta) \subset Y(g,\beta)$. *The attractor* $\mathcal{A}_\beta$ *can be represented in the form*

$$\mathcal{A} = \bigcup_{g \in G} \mathcal{A}(g,\beta)$$

where $\mathcal{A}(g,\beta)$ *are disjoint sets. If* $Y(g,\beta)$ *is not empty,* $\mathcal{A}(g,\beta)$ *is not empty.* $\mathcal{A}(g,\beta)$ *is stable and contains a stable set* $\mathcal{N}_0(\mathcal{A}(g,\beta))$ *of* t*-independent solutions of* (1.1), (1.2), *it includes a connected component* $X \subset \mathcal{N}_0(\mathcal{A}(g,\beta))$.

Proof. Since $E_\beta \cap g$ is invariant by Corollary 2.1, the existence of $\mathcal{A}(g,\beta)$ is proved exactly like the existence of $\mathcal{A}(\beta)$. By (2.7),

$$E_\beta = \bigcup_g Y(g,\beta). \tag{2.16}$$

By Theorem 1.1 there exists $\delta > 0$ such that

$$|u(x) - P_j| \geq \delta > 0 \qquad \forall\, x \in L, \quad P_j \in P, \quad u \in E_\beta. \tag{2.17}$$

From (2.17) it follows that every $Y(g,\beta)$ is open and closed in $H_1 \cap E_\beta$. Therefore $\mathcal{A}(g,\beta)$, $\mathcal{A}(g',\beta)$ lie at a finite distance if $g \neq g'$, $\mathcal{A}(g,\beta) \neq \emptyset$, $\mathcal{A}(g',\beta) \neq \emptyset$. $\mathcal{A}(g,\beta)$ is stable, see [BV]. Let

$$\underline{\mathcal{P}}(g,\beta) = \inf_{u \in \mathcal{A}(g,\beta)} \mathcal{P}(u).$$

Since $\mathcal{A}(g,\beta)$ is compact, for non-empty $\mathcal{A}(g,\beta)$ the set

$$\mathcal{N}(g,\beta) = \{u \in \mathcal{A}(g,\beta) : \mathcal{P}(u) = \underline{\mathcal{P}}(g,\beta)\} \tag{2.17a}$$

is non-empty and compact. Let $X = \mathcal{N}(g,\beta)$. First, X consists of equilibrium points of $\{S_t\}$. Let $\sigma(X)$ be a small neighborhood of $\mathcal{A}$. Indeed, if $u_0 \in \sigma(X) \subset Y(g,\beta)$, $u(t) = S_t u_0$ is attracted to $\mathcal{A}(g,\beta)$. We have

$$\mathcal{P}(u(t)) \leq \mathcal{P}(u_0).$$

Let $Y' = \mathrm{cl}[S_1(Y(a,\beta))]$ be the closure in $H_1([0,L])$, obviously (see Theorem 1.2) Y is a compact invariant subset of $Y(g,\beta)$. Let $Y'(\gamma) = Y' \cap \{u : \mathcal{P}(u) = \gamma\}$,

$$\mathcal{P}_1 = \sup_{u \in Y'(\gamma)} \mathcal{P}(u), \qquad \mathcal{P}_2 = \inf_{u \in Y'(\gamma)} \mathcal{P}(u).$$

Since $\mathcal{P}(S_t u)$ is a decreasing function of γ and $\mathcal{A}(\gamma,\beta)$ is the attractor, we have

$$\mathcal{P}_1 = \underline{\mathcal{P}}(g,\beta).$$

The set $Y'(\mathcal{P}_2)$ consists of equilibrium points of S_t, since $\mathcal{P}(S_t u)$ is a strictly decreasing function of t on non-equilibrium points.

The sets $Y'(\gamma)$ upper semicontinuously depend on γ (this follows from compactness of Y' and continuity of $\mathcal{P}$). Therefore, if σ is a given neighborhood of $Y'(\mathcal{P}_2)$, there exists $\delta > 0$ such that $Y'(\gamma) \subset \sigma$ when $|\gamma - \mathcal{P}_2| \leq \delta$. Since the set $\{u \in Y' : \mathcal{P}(u) \leq \mathcal{P}_2 + \delta\}$ is invariant, this implies stability of $Y'(\mathcal{P}_2)$. Using the fact that a trajectory $u(t)$ is a continuous function of t, we easily conclude that every (maximal connected) component X of $Y'(\mathcal{P}_2)$ is stable, and Theorem 2.2 is proved.

Definition 2.1. For a set $\mathcal{D} \subset \mathbb{R}^2$, $* \in \mathcal{D}$ let $\mathcal{L}(\mathcal{D},*)$ be the set of functions $u(*)$, $u : [0,L] \to \mathcal{D}$, $u(0) = u(L) = *$, such that $u \in H_1([0,L])$.

Remark 2.2. Since $H_1([0,L]) \subset \mathcal{C}([0,L])$ by Sobolev's embedding theorem, any $u \in \mathcal{L}(\mathcal{D},*)$ is continuous and belongs to a class $g \in \pi_1(\mathcal{D},*)$. (As usual, we denote by $\pi_1(\mathcal{D},*)$ the fundamental homotopy group of $\mathcal{D}$ with a base point $*$.) Moreover, since the arc length $\ell(u)$ of the curve is estimated by

$$\ell(u) = \int_0^L |\partial_x u| dx \leq \sqrt{L} \left(\int_0^L |\partial_x u|^2 dx \right)^{1/2},$$

any $u \in \mathcal{L}(\mathcal{D},*)$ has a finite length $\ell(u)$.

Definition 2.2. For a set $\mathcal{D} \subset \mathbb{R}^2$, $* \in \mathcal{D}$ any $g \in \pi_1(\mathcal{D},*)$ let

$$\ell(g,\mathcal{D}) = \inf_{u \in g \cap \mathcal{L}(\mathcal{D},*)} \ell(u).$$

Remark 2.3. Obviously, if $* \in \mathcal{D}' \subset \mathcal{D}$ and $\mathcal{D}'$ is a deformation retract of $\mathcal{D}$, then $\pi_1(\mathcal{D}',*) = \pi_1(\mathcal{D},*)$ and $\ell(g,\mathcal{D}') \geq \ell(g,\mathcal{D})$.

Lemma 2.2. *Let $g \in \pi_1(\mathcal{D}, *)$. If $\ell(g, \mathcal{D}) < \infty$, then for any $\epsilon > 0$ there exists $u \in \mathcal{L}(\mathcal{D}, *) \cap g$ such that*

$$\mathcal{P}(u) \leq L\mu(\mathcal{D}, F) + \frac{a}{2L}\left(\ell(g, \mu)\right)^2 + \epsilon \tag{2.17b}$$

where

$$\mu(\mathcal{D}, F) = \sup_{u \in \mathcal{D}} F(u).$$

Proof. According to the definition of $\ell(g, \mathcal{D})$ for any $\delta > 0$ there exists $u \in \mathcal{L}(\mathcal{D}, *)$ such that

$$\ell(u) \leq \ell(g, \mathcal{D}) + \delta.$$

Let $s(y)$ be the length of the arc of the curve $u(x)$, $0 \leq x \leq y$, $y \leq L$,

$$s(y) = \int_0^y |\partial_x u(\theta)| d\theta.$$

Let $y(s)$, $0 \leq s \leq \ell(u)$, be the inverse function. The function $\tilde{u}(x) = u\left(y\left(x\,\frac{\ell(u)}{L}\right)\right)$ belongs to $\mathcal{L}(\mathcal{D}, *)$. We have

$$\partial_x \tilde{u}(x) = u' \cdot y' \cdot \frac{\ell(u)}{L}.$$

Since

$$u'(y) \cdot y'(s) = u'(y)/s'(y) = 1,$$

we obtain

$$\partial_x \tilde{u}(x) = \ell(u)/L.$$

We have

$$\begin{aligned}\mathcal{P}(\tilde{u}) &= \int_0^L \left[\frac{a}{2}\,|\partial_x \tilde{u}|^2 + F(\tilde{u})\right] dx \\ &\leq L\mu(\mathcal{D}, F) + \frac{a}{2L}\,\ell(u)^2 \leq L\mu(\mathcal{D}, F) + \frac{a(\ell(g, \mathcal{D}) + \delta)^2}{2L}.\end{aligned}$$

This implies (2.17b).

Remark 2.4. If $\mathcal{D}$ is closed and its boundary is piecewise smooth, $\ell(g, \mathcal{D}) = \ell(u_0)$ where $u_0(x)$ is a geodesic curve which belongs to the class g. Obviously, $u_0(x)$ consists of line segments and arcs which belong to $\partial\mathcal{D}$.

Remark 2.5. If $* \in \mathcal{D}' \subset \mathcal{D}$, then the natural mapping $\mathcal{T} : \pi_1(\mathcal{D}', *) \to \pi_1(\mathcal{D}, *)$ is well-defined. If $\mathcal{D}'$ is a deformation retract of $\mathcal{D}$, this mapping is isomorphism, and we identify g and $\mathcal{T}g$.

Theorem 2.3. *Let μ be such that the domain $\mathcal{D}_\mu$ defined by* (2.1) *lies in $\Omega'(P, r)$ with $r > 0$ and $\mathcal{D}_\mu$ is a deformation retract of $\Omega'(P, r)$. Let $v_i : [0, L] \to \mathcal{D}_\mu$ be C^1-mappings which represent the generators g_i of the homotopy group $G = \pi(\mathcal{D}_\mu) = \pi(\Omega'(P, 0), *)$ and the curves v_i have finite lengths. Let the lengths of the curves $v_i(x)$ be equal to ℓ_i. Let $g \in G$ be defined by* 2.3). *Then there exists $u \in X(\Omega'(P, r))$ such that $u \in E_\beta$,*

$$\beta \leq L\mu + \frac{a}{2L}\left(\sum_{j=1}^{n} |k_j| \ell_{i_j}\right)^2 \tag{2.18}$$

where k_j are the same as in (2.3)

Proof. Let $s(y)$ be the length of the curve $v_i(x)$ between 0 and $v_i(y)$,

$$s_i(y) = \int_0^y |\partial_x v_i'(\theta)| d\theta.$$

Let $y_i(s)$ be the inverse of $s_i(y)$, let

$$\tilde{v}_i(s) = v_i(y_i(s)), \qquad 0 \le s \le \ell_i.$$

Let

$$\bar{\ell} = \sum_{j=1}^{n} |k_j| \ell_{i_j},$$

be the total length of the curve corresponding to g, defined by (2.3). The segment $[0, \bar{\ell}]$ is covered by $N = \deg g = |k_1| + \cdots + |k_n|$ segments of length ℓ_{i_j} in the order determined by $i_j, |k_j|$. We denote the end points of these segments by x_j, $j = 0$, N. x_p may be explicitly written as

$$x_p = \sum_{j=1}^{\sigma-1} |k_j| \ell_{i_j} + k\ell_{i\sigma}, \qquad p = \sum_{j=1}^{\sigma-1} |k_j| + k,$$

where $0 \le k \le |k_\sigma|$. Obviously, $g_{i\sigma}$ is at p-th position in the product (2.3), the function $i = i(p)$, $1 \le i \le m$, is well defined, $i(p)$ is the number of the generator g_i which is at p-th position in (2.3). We put

$$u(x_p + x) = \tilde{v}_{i(p)}(x), \qquad 0 \le x \le \ell_{i(p)}.$$

The function $u(x)$, $0 \le x \le \bar{\ell}$, is continuous for all x and continuously differentiable at every point $x \ne x_q$, $q = 0, \ldots, N$.

Let $\bar{u}(x)$ be $u(x\bar{\ell}/L)$, $0 \le x \le L$. This function represents the homotopy class $g \in G$ (see, for example [FFG, M, W]).

Obviously, $|\partial_x u| = \bar{\ell}/L$ at $x \ne x_p$. Computing $\mathcal{P}(u)$, we obtain

$$\mathcal{P}(u) = \frac{a}{2} L^{-1} \bar{\ell}^2 + \mu L = \frac{a}{2} L^{-1} \left(\sum_{j=1}^{n} |k_j| \ell_{i_j} \right)^2 + \mu L.$$

This implies (2.18). □

Theorem 2.4. *Let periodic boundary condition* (1.2b) *hold. Let g be defined by* (2.3) *with $n \ge 2$. Let* (2.6) *hold with $R = d(P)$. Then* (2.5) *holds.*

Proof. We repeat literally the proofs of Theorem 2.1 and Corollary 2.1. The only difference is that instead of (2.9) we have

$$v(x_0) = P_0, \quad |P_0| = \delta, \quad |v(x_1)| \ge R \tag{2.19}$$

where $x_1 \in [0, L]$, $x_0 \in [0, L]$. We have (2.19) because otherwise $v(x_1) \le d(P)$ $\forall$ $x_1 \in [0, L]$ and the curve $u(x)$ is homotopic to an element $g_1^{k_1}$.

Remark 2.6. In the case $g = g_1^{k_1}$ the following inclusion holds:

$$S_t(Y(P, r, g, \beta)) \subset Y(P, \delta, g, \beta) \qquad \forall\, t \ge 0$$

if $\beta < r\sqrt{Ma}/2$. (For the proof see Theorem 4.2.)

Remark 2.7. One can easily see that the function constructed in Theorem 2.3 satisfies periodic boundary conditions as well, since functions from H_1 admit discontinuous derivatives.

Theorem 2.5. *Let $b \in \mathcal{D} \subset \mathcal{D}_M = \{u : F(u) \leq M\}$. Let $\Omega(P,r) \subset \mathbb{R}^2 \backslash \mathcal{D}_M$, $M > \mu_0$. Let $g \in \pi_1(\mathcal{D},*)$. Let*

$$L\mu(\mathcal{D},F) + \frac{a\ell(g,\mathcal{D})^2}{2L} < \sqrt{8}r\sqrt{Ma} + \frac{2(R-r)^2}{L}, \tag{2.20}$$

$\mu(\mathcal{D},F)$ is the same as in Lemma 2.2. *Let $g_* = \mathcal{T}_* g$, where $\mathcal{T} : \pi_1(\mathcal{D},*) \to \pi_1(\Omega'(P,r),*)$ is the homomorphism generated by the embedding $\mathcal{D} \subset \Omega'(P,r) = \mathbb{R}^2 \backslash \Omega(P,r)$. Then there exists such $\beta > 0$ that the component $\mathcal{A}(\mathcal{T}_* g,\beta)$ of the global attractor $\mathcal{A}$ is not empty, $\mathcal{A}(\mathcal{T}_* g,\beta) \neq \emptyset$.*

Proof. By (2.20) we can find such β and $\delta_1, \delta_2 > 0$ that

$$L\mu(\mathcal{D},F) + \frac{a\ell(g,\mathcal{D})^2}{2L} < \beta - \delta_1,$$

$$\beta < \sqrt{8}(r-\delta_2)\sqrt{Ma} + \frac{2(R-r)^2}{L}.$$

By Lemma 2.2 there exists $u \in g$ such that $\mathcal{P}(u) < \beta$, therefore $u \in E_\beta \cap X(\Omega'(P,r),\mathcal{T}_* g) = Y(\mathcal{T}_* g,\beta)$. By Theorem 2.1 and Corollary 2.1 we have (2.5) with g replaced by $\mathcal{T}_* g$. By Theorem 2.2 we obtain that $\mathcal{A}(\mathcal{T}_* g,\beta) \neq \emptyset$ and the theorem is proved.

Theorem 2.6. *Let $\delta > 0$, let β satisfy* (2.6)*. Then the mapping*

$$\mathcal{B} : u \to g(u),$$

where $g = g(u) \in \pi_1(\Omega'(P,0,g),*)$ is the homotopy class of u, is well-defined. This mapping is continuous on E_β.

Proof. The first statement directly follows from Theorem 2.1 with $t = 0$, $\delta = 0$. To obtain the second statement note that if $\|u - \tilde{u}\|_1 < \epsilon$, then by Sobolev embedding theorem

$$|u(x) - \tilde{u}(x)| \leq \epsilon \qquad \forall\, x \in [0,L].$$

Consider

$$u_\tau = (1-\tau)u + \tau\tilde{u}, \qquad \tau \in [0,1].$$

If ϵ is small enough,

$$u_\tau(x) \in \Omega'(P,\delta-\epsilon,g) \qquad \forall\, x \in [0,L].$$

At the same time, since the functional $\mathcal{P}$ is continuous on $H_1([0,L])$, (see [BV]), $\mathcal{P}(u_\delta)$ depends continuously on τ, and

$$\mathcal{P}(u_\tau) \leq \beta + C\epsilon \quad \text{when } \tau \in [0,1].$$

Therefore, (2.7) holds with δ replaced by $\delta/2$ if ϵ is small enough. Hence, by (2.5), $g(u_\tau)$ is well-defined (since $\Omega'(P,\delta)$ is a deformation retract of $\Omega'(P,\delta-\epsilon)$, g does not depend on δ), and since u_τ is a homotopy, $g(u_\tau) = g(u)$. So, g is continuous, and the theorem is proved.

Remark 2.8. The mapping $\mathcal{B} : E_\beta \to \pi_1(\Omega'(P,r),*)$ generates decomposition of E_β into disjoint basins K_g, $g \in G$, $K_g = \mathcal{B}^{-1}(g)$.

Theorem 2.7. *Let* (2.6) *hold. Then the mapping*

$$\mathcal{B} : E_\beta \to G$$

maps E_β *onto subset* $G_1 \in G$*, which consists of elements* g *defined by* (2.3) *and satisfying*

$$\left(\sum |k_j| \ell_{i_j}\right)^2 \le \frac{2L\beta}{a} - \frac{2L^2\mu}{a}. \tag{2.21}$$

Proof. By Theorem 2.3, if (2.21) holds, then there exists $u \in g \cap E_\beta$. That means $\mathcal{B}(u) = g$, and the theorem is proved.

3. Examples of Existence of Complicated Spatial Structure of Attractors

Now we give a simple example which illustrates Theorems 2.1–2.5 and their corollaries. We identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Example 3.1. Consider the equations (1.1), (1.2a) with $a = 1$ and the potential $F_\epsilon : \mathbb{C} \to \mathbb{R}$ which is defined by the formula

$$F_\epsilon(u) = M_1|u - z_1|^{-p} + M_1|u - z_2|^{-p} \tag{3.1}$$

for $|u - z_i| \ge \epsilon > 0$, $p > 2$. We assume that $F_\epsilon(u)$ belongs to $C^2(\mathbb{R}^2)$ and

$$F_\epsilon(u) \ge M_1\epsilon^{-p} \quad \text{when } |u - z_j| \le \epsilon. \tag{3.2}$$

(This potential can be considered as a singular potential modified in the ϵ-neighborhoods of singular points.) Here we take

$$z_1 = (-R, 0) = -R, \qquad z_2 = (R, 0) = R, \qquad R \ge 2.$$

We take $P_1 = z_1$, $P_2 = z_2$ in notations of Section 2, $P = \{P_1, P_2\}$, $m = 2$. The set $\Omega'(P, 1)$ is the plane with two deleted disks of radius 1. Now we define the generators of the fundamental homotopy group $\pi_1(\Omega'(P, 1))$. The first generator g_1 coincides with the homotopy class of mappings $u : S^1 \to \Omega'(P, 1)$ (we put $S^1 = [0, 2\pi]$ with points 0 and 2π considered as the same point, it is the designated point $* \in S^1$) which map $[0, 2\pi]$ into a curve turning one time around z_1 in counter-clockwise direction. The fixed representative of this class is

$$g_{10}(x) = z_1 + Re^{ix}. \tag{3.3}$$

Obviously, $g_{10}(*) = 0$, 0 coincides with the designated point $* \in \mathbb{R}^2 = \mathbb{C}$. The generator g_2 corresponds to curves around z_2 in counter-clockwise direction, and is represented by

$$g_{20}(x) = z_2 - Re^{ix}. \tag{3.4}$$

The group G is not commutative and consists of all monomials generated by g_1 and g_2. For example, the monomial $g = g_1^2 g_2 g_1^{-1}$ of degree 4 corresponds to two counter-clockwise turns around z_1 followed by one counter-clockwise turn around z_2 and one clockwise turn around z_1. This element is represented by the function $g_0(x)$,

$$g_0(x) = z_1 + Re^{4ix} \text{ on } [0, \pi];$$
$$g_0(x) = z_2 - Re^{4ix} \text{ on } [\pi, 3\pi/2];$$
$$g_0(x) = z_1 + Re^{-4ix} \text{ on } [3\pi/2, 2\pi].$$

Now we give a condition on r and M which guarantees that Condition 2.1 holds. By (3.1) and (3.2) if $r \geq \epsilon$ we have

$$F_\epsilon(u) \geq M_1 r^{-p} \tag{3.5}$$

when $\text{dist}(u, P) \leq r \leq 1$ where $p > 2$. Therefore Condition 2.1 holds with $M = r^{-p}$, for $1 \geq r \geq \epsilon$.

Now for given β condition (2.6) of Theorem 2.1 is fulfilled if

$$\beta \leq \sqrt{8}(r - \delta) r^{-p/2} \sqrt{M_1} \tag{3.6}$$

if $1 > r > \epsilon$. Since $\delta > 0$ is arbitrary, we can take $\delta = \epsilon/2$, $r = \epsilon$. Theorem 2.1 implies

$$S_t(Y(g, \beta)) \subset Y(g, \beta) \qquad \forall\, t \geq 0 \tag{3.7}$$

if

$$\beta \leq \epsilon^{1-p/2} \sqrt{2M_1}. \tag{3.8}$$

Therefore the sets $\mathcal{A}(g, \beta)$ and $\mathcal{N}(g, \beta)$ are well-defined for such β. Note that if $\mathcal{N}(g, \beta_0) \neq \emptyset$, then $\mathcal{N}(g, \beta) = \mathcal{N}(g, \beta_0)$ if $\beta \geq \beta_0$. Indeed, $\mathcal{A}(g, \beta) \supset \mathcal{A}(g, \beta_0)$ since $E(\beta_0) \subset E(\beta)$. Hence, $\mathcal{N}(g, \beta_0) \subset \mathcal{N}(g, \beta)$ for $\beta \geq \beta_0$. At the same time, by (2.17) $\mathcal{N}(g, \beta) \subset E(\underline{\beta})$, $\underline{\beta} = \underline{\mathcal{P}}(g, \beta)$ and since $\mathcal{N}(g, \beta_0) \subset \mathcal{A}(g, \beta)$, we have $\underline{\beta} \leq \beta_0$. This implies $\mathcal{N}(g, \beta) \subset \mathcal{N}(g, \beta_0)$.

Consider now the question: for which g are the sets $\mathcal{A}(g, \beta)$ and $\mathcal{N}(g, \beta)$ nonempty? A sufficient condition is given by Theorem 2.3. For a g given by (2.3) (where we take $m = 2$) and generators g_1 and g_2 defined above, we have a curve $u_g \in E_\beta$, $u_g \in g$ where

$$\beta \leq L\mu + \frac{1}{2L} \left(\sum_{j=1}^{n} |k_j| 2\pi R \right)^2 . \tag{3.9}$$

(Obviously, for generators corresponding to g_{10}, g_{20} defined by (3.3), (3.4) the lengths $\ell_i = 2\pi R$.) Obviously, g_{10} and g_{20} take their values in $\{F_\epsilon(u) \leq 2R^{-p}\}$, therefore we can take

$$\mu = 2 \cdot R^{-p}, \tag{3.9a}$$

Therefore, combining (3.8) and (3.9) we see that $\mathcal{N}(g, \beta) \neq \emptyset$ if

$$L\mu + 2\pi^2 R^2 L^{-1} \deg^2 g \leq \sqrt{2M_1} \epsilon^{1-p/2}. \tag{3.10}$$

This gives a restriction on a degree of a monomial g:

$$R^2 (\deg g)^2 \leq \frac{L}{2\pi^2}\, \epsilon^{1-p/2} \sqrt{2M_1} - \frac{L^2}{2\pi^2} \mu. \tag{3.11}$$

Note that the number N_n of different monomials of a degree n is estimated by

$$N_n \geq 4 \cdot 3^{n-1}.$$

Therefore, the homotopy complexity $\sigma = \ln \sum_0^n N_n \geq \ln(2 \cdot 3^n - 1) \geq n \ln 3$. Hence, we have obtained the estimate from below for the homotopy complexity of the attractor of the system (1.1), (1.2) with the potential F_ϵ satisfying (3.1), (3.2):

$$\sigma^2 \geq \frac{L \ln^2 3}{R^2 2\pi^2} \left[\sqrt{2M_1} \epsilon^{1-p/2} - \mu L \right]. \tag{3.12}$$

For a given small ϵ the maximum of the right-hand side is at $L = \frac{\sqrt{2M_1}}{2\mu}\,\epsilon^{1-p/2}$, we obtain

$$\sigma \geq \frac{\ln 3}{2\pi}\,\epsilon^{1-p/2}\,\frac{M_1^{1/2}}{\sqrt{\mu}R}. \tag{3.13}$$

Obviously, if $\epsilon \to 0$ the complexity σ tends to $+\infty$ since $p > 2$.

Note that we can replace (3.5) by a condition $F_\epsilon(u) \geq M$ when $\operatorname{dist}(u, P) \leq M^{-1/p}$ and (3.8) takes the form

$$\beta \leq \sqrt{2}M^{-(1-p/2)/p} = \sqrt{2}M^{1/2-1/p}.$$

Example 3.2. We can consider in exactly the same way the case when instead of explicit formulae (3.1), (3.2) we have more general potential F_ϵ which satisfies conditions

$$F_\epsilon(u) \geq M_2\epsilon^{-2} \qquad \text{when } |u - z_j| \leq \epsilon \tag{3.2a}$$

for a fixed ϵ, $0 < \epsilon \leq 2$. We also impose a condition

$$F_\epsilon(u) \leq \mu \qquad \text{when } |u - z_j| = R \tag{3.2b}$$

which replaces (3.9a). Repeating with obvious modifications computations of Example 3.1 (we can put $M_2 = M_1\epsilon^{-p+2}$), we can conclude analogously to (3.11) that there exists β such that $\mathcal{N}(g, \beta) \neq \emptyset$ if

$$\deg^2 g \leq \frac{L}{2\pi^2R^2}\sqrt{2M_2} - \frac{L^2}{2\pi^2R^2}\mu \tag{3.11a}$$

and the complexity σ is estimated as follows:

$$\sigma^2 \geq \frac{L\ln^2 3}{2\pi^2R^2}\,\left[\sqrt{2M_2} - \mu L\right].$$

If ϵ and M_2 are fixed and the dependence of μ on R is such that $\mu \leq \mu_0 R^{-p}$, then choosing $L = \sqrt{2M_2}/(2\mu)$ we obtain $\sigma^2 \geq \frac{\ln^2 3}{4\pi^2\mu_0}\,R^{p-2}M_2$, and complexity is large for large R.

Remark 3.1. In Example 3.1 we have solutions with large deg g thanks to large values of F as $r = |u - P_i| \to 0$. In Example 3.2 the same effect is thanks to small values of $F(u)$ when $|u|$ is large, in both cases the critical exponent is $|u|^{-2}$.

Example 3.3. Singular potential.

Now we shall discuss the case $\epsilon = 0$. Note, that $S_tE_\beta \subset E_\beta\ \forall\ t \geq 0$, therefore by (2.5)

$$|u(x) - P_j| \geq \delta \qquad \forall\ P_j \in P, \quad x \in [0, L].$$

Therefore, if $\epsilon < \delta$, we have $u(t)$ is at the same time a solution of (1.1), (1.2) with $F = F_\epsilon$ and $F = F_0$ where

$$F_0(u) = |u - P_1|^{-p} + |u - P_2|^{-p}. \tag{3.14}$$

Let

$$X = \{u_0 \in M_1 \text{ such that } u_0(x) \notin P\ \forall\ x \in [0, L]\}. \tag{3.15}$$

From the above discussion we conclude that for any $u_0 \in X$ there exists $\beta > 0$ such that $u_0 \in E_\beta$ and $u(t) \in E_\beta \cap X\ \forall\ t \geq 0$. From the definition of X we see, that the homotopy class $g(u_0) = g \in G$, $G = \pi_1(\mathbb{R}^2\backslash\{P_1, P_2\})$, is well-defined. Since $u(t, x)$ is a continuous function of $(t, x) \in \mathbb{R}_+ \times [0, L]$, $u(t) \in g(u_0)\ \forall\ t \geq 0$, the attractor $\mathcal{A}(g,\beta)$ is not empty and contains a stable equilibrium set $\mathcal{N}(g,\beta)$ which is not empty for large β (see Example 3.1), and is independent of β.

Remark 3.2. Existence of steady-state solutions of $\partial_x^2 u = F'(u)$ corresponding to arbitrary homotopy class $g \in \pi_1(\mathbb{R}^1 - \{P_1, P_2\}^*)$ and to $\pi_1(\mathbb{R}^d - P)$ for $d \geq 2$ for the case of strong forces (that is of singularities of kind of (3.14)) was proved in [G].

Now we give a generalization of a preceding example. Consider a potential F_ϵ, $0 < \epsilon < r_0$ which is defined by the formula

$$F_\epsilon(u) \geq r^{-p_j} \text{ when } |u - z_j| \leq r, \quad 0 < \epsilon_0 \leq \epsilon \leq r_0, \tag{3.16}$$

$z_j \in P = \{z_1, \ldots, z_m\}$. It is assumed that $F \in C^2(\mathbb{R}^2)$.

Theorem 3.1. *There exists a constant* C_0 *such that* (3.7) *holds if*

$$\beta \leq C_0 \epsilon^{(1-p/2)}, \qquad p = \min(p_1, \ldots, p_m). \tag{3.17}$$

If

$$\deg g \leq C_2 L \epsilon^{1-p/2} - C_3 L^2,$$

then $Y(g,\beta) \neq \emptyset$, $\mathcal{A}(g,\beta)$, $\mathcal{N}(g,\beta) \neq \emptyset$. *Here* C_2, C_3 *do not depend on* ϵ.

The proof is analogous to considerations of Example 3.1.

The case of a singular potential (Example 3.3) can be generalized as follows. Let $F_0 \in C^2(\mathbb{R}^2 \backslash P)$, $P = \{P_1, \ldots, P_m\}$. Let

$$\lim_{u \to P_1} F(u)|u - P_i|^2 = +\infty, \qquad i = 1, \ldots, m.$$

Let

$$X = \{u_0 \in H_1 : u_0(x) \notin P \;\forall\, x \in [0, L]\}.$$

Theorem 3.2. *Let* $u_0 \in X$. *Then there exists* $\beta > 0$ *and a solution* $u(x,t)$ *of* (1.1), (1.2) *which belongs to* $E(\beta) \cap X \;\forall\; t > 0$. *For every* $g \in G = \pi_1(\mathbb{R}^2 \backslash P)$ *there exists* β *such that* $Y(g,\beta)$, $\mathcal{A}(g,\beta)$, $\mathcal{N}(g,\beta)$ *are not empty.*

The proof of Theorem 3.2 is quite similar to considerations of Example 3.3.

Remark 3.3. Note that the existence of steady-state solutions of (0.1) in the limit case of a singular potential when $|F(u)| \geq \mathcal{C}$, $F(u)/|u - P_i|^2 \to \infty$ as $u \to P_i$ was studied in a number of papers (see [G], [ACZ], [BR] and papers cited there). Our results (restricted to steady states) differ in the following respect. We prove for this special case not only existence of steady-state solutions representing every homotopy class $g \in \pi_1\{\mathbb{R}^d \backslash P\}$, but estimate the number of trajectories which belong to a class with bounded energy $\mathcal{P}$. These estimates, as one can see, include not only characteristics of F near P, but also far from P (at the small energy level μ).

In the next example r, R and M are fixed, and $\mu = 0$. This gives a possibility to obtain large deg g if L is large.

Example 3.4. Let

$$F(u) = q(R_1 - |u - P_1|_\epsilon)^2 (|u - P_2|_\epsilon - R_2)^2 \tag{3.18}$$

where $|\zeta|_\epsilon$ is a smoothened $|\zeta|$, $|\zeta|_\epsilon = |\zeta|$ if $|\zeta| \geq \epsilon$, $\epsilon \geq |\zeta|_\epsilon > \epsilon/2$ for $|\zeta| \leq \epsilon$, $|\zeta|_\epsilon \in \mathcal{C}^2(\mathbb{R})$. In this case $\mu = 0$,

$$\mathcal{D}_\mu = \{|u - P_1| = R_1\} \cup \{|u - P_2| = R_2\}. \tag{3.19}$$

We assume that $|P_1 - P_2| = R_1 + R_2$. Obviously, $\pi_1(\mathcal{D}_\mu)$ is the same free group with generators g_1 and g_2 as in Example 2.1.

Theorem 3.3. *Let $R_1 = R_2 = R$, $r = R/2$, let $F(u)$ be defined by (3.18) where ϵ is small enough. Let $g \in \pi_1(P')(P' - \mathbb{R}^2 \backslash P)$, deg $g \geq 1$. Let*

$$\text{deg } g < \sqrt{\frac{L}{\sqrt{a}}} \sqrt[4]{\frac{qR^2}{2}}. \tag{3.20}$$

Then $Y(\beta, g) \neq \emptyset$, $\mathcal{A}(\beta, g) \neq \emptyset$.

Proof. According to (3.10), $\mu = 0$. One can easily see that $\mathcal{D}_\mu = S^1 \vee S^1$ is a deformation retract of P'. Since

$$(|u - P_2|_\epsilon - R)^2 \geq (3R/2 - R)^2$$

when $|u - P_1| \leq r = R/2$, $|P_1 - P_2| = 2R$, we can take $M = qR^4/16$ in (2.6), $\beta = r\sqrt{8aM}$. Obviously $F(u) \leq qR^4/16$ when $\text{dist}(u, P) \leq r$. Therefore, (2.21) takes the form

$$2\pi \text{deg } g \leq \left(\frac{2L}{\sqrt{a}}\right)^{1/2} \sqrt[4]{qR^2/8}.$$

Let

$$\lambda = \sqrt{\frac{L}{\sqrt{a}}}, \qquad q' = \sqrt[4]{qR^2}.$$

We have the restriction on g

$$2\pi \text{deg } g < \frac{\lambda q'}{\sqrt[4]{2}}.$$

Remark 3.4. Obviously if q, R, r, g are fixed, we can find solution u of homotopy type g taking $L/\sqrt{a}$ sufficiently large. Therefore (1.1) with F defined by (3.10) has solutions which represent any element $g \in \pi_1(P', *)$. The same is true for

$$F(u) = q(R^2 - |u - P_1|^2)^2(R^2 - |u - P_2|^2).$$

Remark 3.5. One can see considering (2.6) and (2.18) like we did in the above examples that the high complexity of attractors (which corresponds to high values of deg g) may be obtained in following situations:

(a) we have large values of Mr^2 (situation close to a singular);
(b) we have small values of μR^2, this can be achieved if $|P_1 - P_2|$ is large, and the potential F rapidly decays far from P_1 and P_2 (more rapidly than $|u - P_i|^{-2}$);
(c) we have zero (or very close to zero) μ on the homotopy nontrivial set, surrounding P_1, P_2. In this case $L/\sqrt{a}$ should be large. (If L is fixed, the situation is close to a singular limit $\partial_t u = -F'(u)$.)

4. Higher Order Systems

We consider here the case when $d \geq 2$, that is $u = (u_1, \dots, u_d)$ takes its values in $\mathbb{R}^d$, $d \geq 2$. Results of Section 1 hold in this case. Formulations and results of Section 2 should be slightly modified.

We shall use the following notations:

$$\mathcal{D}_\mu = \{y \in \mathbb{R}^d : F(y) < \mu\}, \qquad \mathcal{D}'_\mu = \mathbb{R}^d \backslash \mathcal{D}_\mu.$$

We assume that there exists $M > \mu > 0$, a local compact set $P \subset \mathcal{D}'_M$ such that sets $P' = \mathbb{R}^d \backslash P$ and $\mathcal{D}_\mu$ are homotopy equivalent, $\mathcal{D}_\mu$ is a deformation retract of P',

$$\{u \in \mathbb{R}^m : \text{dist}(u, P) \leq r\} = \Omega(P, r) \subset \mathcal{D}'_M. \tag{4.0}$$

One can easily see that these definitions agree with definitions given in Section 2.

We assume that the set $\Omega'(P, r) = \mathbb{R}^d \backslash \Omega(P, r)$ and $\mathcal{D}_\mu$ is a deformation retract of P'. In this case for $b = * \in \mathcal{D}_\mu \subset \Omega'(P, r) \subset P'$ we have

$$\pi_1(\mathcal{D}_\mu, *) = \pi_1(\Omega'(P, r), *) = \pi_1(P', *) = G.$$

For $g \in G$ we put

$$X(\Omega'(P, r)) = \{u \in H_1([0, \ell]) : u(0) = u(L) = b, u(x) \in \Omega'(P, r) \ \forall \, x \in [0, L]\},$$
$$X(\Omega'(P, r), g) = X(\Omega'(P, r)) \cap g, \qquad Y(P, r, g, \beta) = E_\beta \cap X(\Omega'(P, r), g).$$

Here E_β is defined by (1.9). We assume that conditions imposed in Section 1 on F hold. For $\mathcal{D}'_\mu$ given, let $r_0(\mu)$ be the supremum of all r such that $\exists$ such P that $\mathcal{D}'_\mu$ is a deformation retract of $\Omega(P, r) \subset \mathcal{D}'_\mu$.

Theorem 4.1. *Let $R = dist(b, P)$. Let $u \in H_1([0, L])$, $u(0) = b$, and*

$$\mathcal{P}(u) \leq \beta < \sqrt{2}(r - \delta)\sqrt{M} + \frac{(R - r)^2}{2L}. \tag{4.1}$$

Then

$$\text{dist}(u(x), P) \geq \delta > 0 \qquad \forall \, x \in [0, L]. \tag{4.2}$$

Proof. To prove (4.2), assume the contrary. Then there exists $u_0 \in E_\beta$, $x^* > 0$ such that

$$\text{dist}(u(x^*), P) \leq \delta. \tag{4.3}$$

Since $u(x)$ is continuous in x we can take x^* as first moment when (4.3) holds, therefore

$$\text{dist}(u(x), P) > \delta \qquad \forall \, x \in [0, x^*[. \tag{4.4}$$

By (4.3) and local compactness of P there exists a point $P_0 \in P$ such that

$$|u(x^*) - P_0| = \delta \tag{4.5}$$

Let $v(x) = u(x - x^*) - P_0$. Consider the function $\partial_x |v(x)|$. Since $v(x) \neq 0$ when $x < x^*$, this function is smooth for $0 \leq x < x^*$ (we use smoothness of solutions of

(1.1) for $t > 0$). We have

$$\begin{aligned}\partial_x |v| &= \partial_x \left[\sum_{j=1}^{m} |v_i|^2\right]^{1/2} \\ &= |v|^{-1} \sum_{i=1}^{m} v_i \partial_x v_i \leq |v|^{-1} \left(\sum_{i=1}^{m} v_i^2\right)^{1/2} \left(\sum_{i=1}^{m} |\partial_x v_i|^2\right)^{1/2} \\ &= |\partial_x v|.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{P}(u) &= \int_0^L \left[\frac{a}{2}\, |\partial_x u|^2 + F(u)\right] dx \geq \int_0^{x^*} \left[\frac{a}{2}\, |\partial_x v|^2 + F(P_0 + v)\right] dx \\ &\geq \int_0^{x^*} \left[\frac{a}{2}\, |\partial_x |v||^2 + F(P_0 + v)\right] dx \geq \int_0^{x^*} \left[\frac{a}{2}\, |\partial_x |v||^2 + U^*(P)\right] dx,\end{aligned} \tag{4.6}$$

where $U^*(P)$ is defined by (2.12). (We are using that $|v| \leq r$ implies $P + v \in \Omega(P, r) \subset \mathcal{D}_M$.) Using Lemma 2.1 as in the proof of Theorem 2.1, we obtain that (4.6) and (4.2) imply $\mathcal{P}(u) > \beta$, which contradicts $u \in E_\beta$. This contradiction proves the theorem.

Let b_1, b_2 be two points in P'. Let S_0 be a fixed curve in P' which connects b_2 and b_1. Obviously, any continuous curve $u(x)$, $x \in [0, L]$ which satisfies $u(0) = b_1$, $u(L) = b_2$ determines together with S_0 a cycle in P'. (We take $0 = * \in [0, L]$ as a base point.) Therefore we have put in correspondence with $u(x)$ a unique element $g(u) \in \pi_1(P', *)$, $* = b_1 \subset P'$ (this element depends on the choice of S_0).

Corollary 4.1. *Let*

$$E_\beta = \{u \in M_1([0, L]) : u(0) = b_1, u(L) = b_2\}.$$

Let S_t be the semigroup in E_β defined in Theorem 1.2. Let β satisfy (4.1) where $R = \operatorname{dist}\{\{b_1 \cup b_2\}, P\}$. Then $g(S_t(u))$ is well-defined for all $u \in E_\beta$, $t \geq 0$ and is constant in t. Every set $Y(\beta, g)$, $g \in \pi_1(P', *)$ is invariant. If this set is non-empty, it contains the attractor $\mathcal{A}(\beta, g)$ and a stable component of equilibria $\mathcal{N}(\beta, g) \subset \mathcal{A}(\beta, g)$.

Corollary 4.2. *Let β satisfy conditions of Corollary 4.1. Then $S_t u_0$ is homotopy stable in $u_0 \in E_\beta$. Namely, there exists such $\epsilon > 0$ that if u_θ, $\theta \in [0, 1]$ is a continuous (in $H_1([0, L])$) curve connecting u_0 and u_1, $u_\theta \in E_{\beta+\epsilon}$ $\forall\ \theta \in [0, 1]$, then $g(S_t u_1) = g(S_t u_0)$ $\forall\ t \geq 0$.*

Proof. Condition (4.1) holds if β is replaced by $\beta + \epsilon$, δ by $\delta/2$. Since $S_t u$ continuously depends on u for any fixed t (see [BV]), we have a homotopy $S_t u_\theta$, $\theta \in [0, 1]$, between $S_t u_0$ and $S_t u_1$. By Theorem 4.1 $S_t u_\theta(x) \in P'$ $\forall\ x \in [0, L]$, which implies that homotopy classes of $S_t u_1$ and $S_t u_0$ coincide. □

Consider now the case of periodic boundary conditions (1.2b).

Theorem 4.2. *Let*

$$\beta < r\sqrt{M}\sqrt{a}/\sqrt{2}. \tag{4.2}$$

Then for every nontrivial $g \in \pi_1(\Omega'(P,r))$ (4.2) *holds true with some* $\delta > 0$. (*In the definition of* $X(\Omega'(P,r)')$ *we impose periodic conditions* (1.2b) *on* u *instead of* (1.2a).)

Proof. The proof is analogous to the case of Dirichlet boundary conditions (1.2a), therefore we show only the modifications needed.

We consider the case of periodic boundary conditions (1.2b). In this case we take $G = \pi_1(P')$ the fundamental group without base point. In the case when g includes powers of different generators, we can apply the same proof to obtain (4.2). The only modification is that we take $R = \frac{1}{2}\min_{i\neq j} \operatorname{dist}(\Omega_i, r_j)$ where Ω_i are connected components of $\Omega(P,r)$.

The case when g includes only one generator g_1, $g = g_1^{k_1}$ is a little different. In this case we have (4.3)–(4.5) as well. Let $X = \bigcup_{x\in[0,L]} U(x)$. Since $X \subset \Omega'(P, r/2)$, the estimate diam $X < r/2$ implies u is contractible in $\Omega'(P)$. Hence, it suffices to prove the theorem for diam $X \geq r/2$. If (4.5) holds, we have $|u(x^*) - u(x_1)| \geq r/4$. Repeating the proof of Theorem 2.1 with $R = r$, replaced by $r/2$, $r_1 = r/4 - \delta$, we by Lemma 2.1 come to $\mathcal{P}(u) \geq \sqrt{8aM}(r/4 - \delta)$ which gives a contradiction with (4.7) as well if δ is small. Therefore we have proved the assertion of the theorem for any nontrivial g.

Remark 4.1. We have the inequality

$$\sup_{y,x\in[0,L]} |u(x) - u(y)| \leq L^{1/2}\left[\int_0^L |\partial_x u|^2 dx\right]^{1/2}. \tag{4.7}$$

Therefore, boundedness of $\mathcal{P}(u)$ implies in case of the Dirichlet boundary condition the restriction

$$|u(x) - b| \leq \frac{\sqrt{2}L^{1/2}}{a^{1/2}}\beta \qquad \forall\, u \in E_\beta. \tag{4.8}$$

Therefore, we can replace in the formulation of Theorem 4.1 groups $\pi_1(\mathbb{R}^d\backslash P, *)$ by $\pi_1(B_\beta\backslash P, *)$ where

$$B_\beta = \left\{z \in \mathbb{R}^d : |z - b| \leq \frac{\sqrt{2}L^{1/2}}{a^{1/2}}\beta\right\}.$$

When periodic boundary conditions (1.2b) are imposed, (4.7) also may imply restrictions, but their form depends on structure of P.

Note that Definitions 2.1 and 2.2 are good for $d > 2$ as well.

Theorem 4.3. *The assertion of Theorem* 2.2 *holds for* $d \geq 2$.

Note, that Remarks 2.4 and 2.5 are valid in case $d > 2$.

Theorem 4.4. *Assertions of Theorem* 2.5 *hold if we replace* $\mathbb{R}^2$ *by* $\mathbb{R}^d$*, and* $\Omega(P,r)$ *is defined by* (4.0).

5. Localization of Support of Homotopy Elements

For every nontrivial element g of a fundamental homotopy group $\pi_1(P', *)$, $P' \subset \mathbb{R}^2\backslash\{P_1 \cup P_2\}$ where P_1 and P_2 are points (see Section 3) we have a representation

$$g = \prod_{j=1}^{n} g_{i_j}^{k_j}$$

where $i_j = 1$ or $i_j = 2$, $k_j \in \mathbb{Z}\backslash 0$. At the same time, if $u \in g$, it is a function of $x \in [0, L]$. We will give here an algorithm which describes a location of every $N = |k_1| + \cdots + |k_n|$ of elements of decomposition of g for every $u \in g$. The support of g_i is a segment $[y_i^-, y_i^+]$, this segment is determined uniquely by u. Two neighboring segments $[y_i^-, y_i^+]$ may overlap, but the distance between $[y_i^-, y_i^+]$ and $[y_{i+2}^-, y_{i+2}^+]$ is positive.

We will consider separately cases when u satisfies Dirichlet and periodic boundary conditions. The case of periodic condition is more complicated, therefore we consider it in more detail. We consider L-periodic functions $u(x)$ which are elements of $\pi_1(P')$. Obviously, the translated function $u(x+y)$ is also periodic. At the same time, the decomposition (2.3) is not unique when we do not fix base points. We impose the condition that if $i_1 = i_n$, then k_1 and k_n have the same sign. It is possible to make cyclic permutations in this class not changing the homotopy class. We shall show connection between translations and permutations.

Now we proceed to define support of elements of $g(u)$. We use notations of Section 3. Let $Q_1 = \{|u_1| < 4R, |u_2| < 2R\}$ be a rectangle which contains circles $|u - P_i|$ inside. There exists a homotopy $T_1 : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$, $T_1(u,t)$ is continuous. $T_1(u, 0) = u$, T_1 is one-to-one for any fixed t and $T_1(\mathbb{R}^2, 1) = Q_1$. T_1 is identical on $\{|u_1| \le 2R, |u_2| \le R\} = Q_2$. Let $\phi(\theta, t)$, $\theta \in [0, 2]$, $[t \in 0, 1]$ be a function which is defined by

$$\phi(\theta, t) = \theta \qquad \text{when } 0 \le \theta \le 1, \quad 0 \le t \le 1,$$

$$\phi(\theta, t) = 1 + (\theta - 1)(1 - t) \qquad \text{when } 1 \le \theta \le 2, \quad 0 \le t \le 1.$$

Obviously, this function is continuous. For $u \in Q_1$, $t \in [0, 1]$, let a homotopy T_2 be defined by

$$T_2(u, t) = \frac{u}{|u|}\,\phi(|u|, t), \qquad u \ne 0,$$

$$T_2(0, t) = 0.$$

Obviously, T_2 is identical on Q_2, T_2 is a retraction of Q_1 on Q_2.

Note that circles $|u - P_i| = R$ lie in Q_2 ($P_1 = (-R, 0)$, $P_2 = (R, 0)$), their upper halves ($u_2 \ge 0$) can be explicitly written as graphs of a function $u_2 = \psi_1(u_1)$, $-2R \le u_1 \le 2R$, $0 \le \psi_1 \le R$. We define for $u \in Q_2$, $u_2 > 0$,

$$T_3(u_1, u_2, t) = (u_1, u_2) \qquad \text{if } u_2 \le \psi_1(u_1),$$

$$T_3(u_1, u_2, t) = (u_1, u_2 t + \psi_1(u_1)(1 - t))$$

if $\psi_1(u_1) \le u_2 \le R$. Similarly we define T_3 for $u_2 < 0$. This is a retraction of Q_2 on the set $Q_3 = \{u : \mathrm{dist}\{u, P_1 \cup P_2\} \le R\}$. Let $Q_5 = \{u : \mathrm{dist}\{u, P_1 \cup P_2\}\}$ be the union of two circles (with a common point 0). For $0 \le \rho < R$, let $Q_4 = \{u \in Q_3 : \mathrm{dist}\{u, P_1 \cup P_2\} > \rho\}$, $\mathcal{D}(\rho) = \{u \in \mathbb{R}^2 : \mathrm{dist}\{u, P_1 \cup P_2\} > \rho\}$. We define a homotopy T_4 on $Q_4(\rho)$, $\rho \ge 0$ separately for $u_1 \ge 0$, $u_1 \le 0$. For $u_1 \ge 0$ let

$$T_4(P_2 + v, t) = P_2 + [Rt + (1 - t)|v|]v/|v|;$$

for $u_1 < 0$ this homotopy is defined in a similar way replacing P_2 by P_1. Obviously, $T_4(Q_4, 1) = Q_5$.

Applying homotopies T_1, T_2, T_3, T_4 consecutively we obtain resulting retraction T_5 of $\mathcal{D}(0)$ onto Q_5.

Consider now a curve $u(x)$, $x \in S^1 = [0, L]$, $u \in C(S^1)$, $u \in \mathcal{D}(0) = P' = R^2 \backslash P$ $\forall\, x \in S^1$. Obviously, $u \in \pi_1(P')$. $v(x) = T_5 u(x)$ is a continuous curve as well; it belongs to the same class g and $v(x)$ takes values in Q_5.

Now we fix $u(x)$ and let $v(x) = T_5 u(x)$. Let Z be the set of zeros of v,

$$Z = \{x \in [0, L] : v(x) = 0\}. \tag{5.1}$$

It is a compact subset of $[0, L]$. For any two points $x_1, x_2 \in Z$ obviously $v(x_1) = v(x_2) = 0$, and restrictions of v on $[x_1, x_2]$ belongs to one of the classes of $\pi_1(Q_5, *)$. We denote this class (for v fixed) by $g[x_1, x_2]$. Consider pairs $x_1, x_2 \in Z$ such that $0 \le x_1 < x_2 \le L$. We say that a segment $[x_1, x_2]$ is contractible if v on this segment belongs to a trivial homotopy class (we denote it by 1), $g[x_1, x_2] = 1 \in \pi_1(Q_5, *)$ where $* = 0$ is the base point. Obviously, if $[x_1, x_2]$ and $[x_2, x_3]$ are two segments with a common point x_2, we have

$$g[x_1, x_2] g[x_2, x_3] = g[x_1, x_3]. \tag{5.2}$$

Hence, if both $g[x_1, x_2]$ and $g[x_2, x_3]$ are trivial, so does $g[x_1, x_3]$.

Since $v(x)$ is uniformly continuous, there exists such $\delta_0 > 0$ that if $|x_1 - x_2| < \delta_0$, $x_1, x_2 \in Z$, then $g[x_1, x_2] = 1$. Our goal is to write the element $g(v) = g \in \pi_1(P')$ in the form

$$\prod_{j=1}^{\deg g} g_{i_j}^{\delta_1}, \qquad \delta_j = \pm 1, \tag{5.3}$$

where all terms are uniquely determined by $v = T_5 u$ and $g_{ij} = g[y_j^-, y_j^+]$. We shall define also segments $[y_j^-, y_j^+]$ which correspond to elements in this decomposition of $g = g(u) = g(v) \in \pi_1(P')$.

Let $x_1^- = \inf\{x \in Z\}$, $x_1^+ = \sup\{x \in Z : x_1^- + L \ge x \ge x_1^-, g[x_1^-, x] = 1\}$. We define x_j^-, x_j^+, $j \ge 2$ by induction

$$x_j^- = \inf\{x \in Z, x_1^+ + L \ge x > x_{j-1}^+\},$$
$$x_j^+ = \sup\{x \in Z, x_{j-1}^- \le x \le L + x_1^-, g[x_1^-, x] = 1\}.$$

Obviously,

$$x_j^+ \ge x_j^-, \qquad x_{j-1}^+ + \delta_0 \le x_j^- \tag{5.4}$$

for all j. Therefore, there is only a finite number N of such segments $[x_j^-, x_j^+]$, $N \le L/\delta_0$, the segments $[x_j^-, x_j^+]$ are uniquely defined by the above algorithm. Obviously, $g[x_{j-1}^+, x_j^-] \ne 1$, hence deg $g[x_{j-1}^+, x_j^-] \ge 1$. At the same time, $(x_{j-1}^+, x_j^-) \cap Z = \emptyset$. This follows from the definition of x_j^-. This easily implies that

$$\deg g[x_{j-1}^+, x_j^-] = 1. \tag{5.5}$$

We denote $f_j = g[x_{j-1}^+, x_j^-]$, $j = 1, \dots, N$. We see that $\prod_k^\ell f_j \ne 1$ for any $\ell \le k \le N$ by the definition of x_j^+ and by (5.2). Therefore $g = \prod_{j=1}^N f_j$, $f_j = g_{k_j}^{\delta_j}$, $\delta_j = \pm 1$, $j = 1, \dots, N$ where g_i is the generator.

A cancelling of terms is possible thanks to cyclic permutations, that is, we may have

$$f_j = f_{N+1-j}^{-1}, \qquad j = 1, \dots, k.$$

Let $\bar{k}$ be the maximum of such k, $\bar{k} \ge 0$ ($\bar{k} = 0$ if such f_i do not exist). Obviously, $N - 2\bar{k} = \deg g$. We put $i_j = k_{(j+\bar{k})}$, and obtain the representation (5.3). It is uniquely defined by v. We have segments $[x^+_{j-1}, x^-_j]$, $j = \bar{k}+2, \dots, \deg g + \bar{k} + 1$, every segment corresponds to a generator powered by 1 or -1.

We consider segments

$$[x^-_{\bar{k}+j-1}, x^+_{\bar{k}+j}] = [y^-_j, y^+_j],$$

$j = 2, \dots, \deg g$,

$$[y^-_1, y^+_1] = [x^-_{N-\bar{k}}, x^+_{\bar{k}+1}].$$

We will call $[y^-_j, y^+_j]$ the support of the element $g^{\delta_j}_{k_j}$ in (2.3). Therefore, we have proved the following theorem.

Theorem 5.1. (*Support of homotopy elements*). *There u is defined the mapping*

$$\mathcal{L} : u \to \{[y^-_j, y^+_j], j = 1, \dots, \deg g(u)\}$$

such that if the homotopy class of u *is defined by* (5.3)*, the restriction of* u *on* $[y^-_j, y^+_j]$ *belongs to has the homotopy class* $g^{\delta_j}_{i_j}$, $[y^-_i, y^+_i] \cap [y^-_{i+1}, y^+_{j+1}] \neq \emptyset$, $j = 1, \dots, \deg g$; $[y^-_i, y^+_i] \cap [y^-_{j+2}, y^+_{j+2}] = \emptyset$, $j = 1, \dots, \deg g$; $y^-_{j+2} - y^+_j \geq \delta > 0$, δ *is bounded uniformly on any compact set of* $u \in C(S^1)$.

Remark 5.1. One can easily see that the segments we have defined lie on the circle $S^1 = \mathbb{R}/L\mathbb{Z}$.

Remark 5.2. Using the mapping $\mathcal{L}$, one can define the first element $g^{\delta_1}_{i_1}$ in the decomposition (5.3) in a following way. We take $j_0 = \min\{j : y^-_j \geq 0\}$, and let corresponding $g^{\delta_{i_0}}_{i_{j_0}}$ be the first term in (5.3). Therefore, using the described algorithm, we can for every continuous $u(x)$, which takes values in $\mathbb{R}^2 \backslash \{P_1 \cup P_2\}$, obtain a representation (5.3) which we shall denote $\bar{g}(u)$ (any cyclic transformation of $\bar{g}$ determines the same homotopy class).

Remark 5.3. Since $F(u)$ does not depend on x, for any equilibrium point of (1.1), (1.2b) translated function $u(x+z)$ is a solution as well. Therefore, if $u \in \mathcal{A}(g, \beta)$, then all translations of u belong to $\mathcal{A}(g, \beta)$ as well. Therefore, all words $\bar{g}$ which represent g are realizable as $\bar{g}(u)$, $u \in \mathcal{A}(g, \beta)$.

Now we consider the case of Dirichlet boundary conditions. In this case we have a base point $* = 0 = L$, and a base point $b = 0$. The construction of support of g_i is essentially the same with simplifications.

We denote x^-_j, x^+_j as follows: $x^-_1 = \inf\{x \in Z\}$;

$$x^-_j = \inf\{x \in Z, x^+_{j-1} < x \leq L\},$$
$$x^+_j = \inf\{x \in Z, x^-_{j-1} \leq x \leq L, g[x^-_1, x] = 1\}.$$

Also, we do not cancel f_j and f_{N-j} if they are inverse. So the mapping $\mathcal{L}$ is well-defined, and the assertion of Theorem 5.1 holds in this case as well.

The first element in (5.3) is determined homotopy invariantly (unlike in the case of periodic boundary conditions).

Remark 5.4. Localization of homotopy elements is possible in case $m > 2$ as well. We have to connect b by paths with circles $|u - P_i| = r$. The set which consists of these paths and circles (a path together with a circle is a cycle around P_i) is now Q_5. All other considerations are quite similar, and the mapping $\mathcal{L} : u \to \{[y^-_j, y^+_j], j = 1, \dots, \deg g(u)\}$ is well defined. (In the case of periodic boundary conditions we make cancellations as in the case $m = 2$.)

Remark 5.5. If the argument x varies increasing from 0 to L, it passes subsequently through segments $[y_j^-, y_j^+]$. Therefore, generators $g[y_j^-, y_j^+]^{\pm 1}$ appear subsequently. Estimates which we give depend only on $\deg g$, therefore after any given at j-th step $g_{i_j} = g[y_j^-, y_j^+]^{\pm 1}$ any new element $g_{i_{j+1}}$ may appear (excluding $g_{i_j}^{-1}$). Therefore, transitions from one pattern element g_{i_j} to the next $g_{i_{j+1}}$ are chaotic. Since we have no natural measure describing probability of such transitions, we shall describe it (as is usually done in such situations, see [A]) using topological symbolic dynamics. We will do it in the next section.

6. Calculation of the Homotopic Complexity for Periodic Boundary Conditions

We present here a piece of symbolic dynamics which allows us to describe complexity of the attractors. Start from the description of words in the fundamental group. Under assumptions of Section 2, the group $= \pi_1(P', *)$ is the fundamental group of a bunch of circles $\bigvee_{j=1}^m S_j^1$ with a base point and each element of this group (except of the trivial one) is a nonreducible word $g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_n}^{k_n}$ where $k_j \in \mathbb{Z}\backslash\{0\}$, $j = 1, \ldots, n$, and g_i is a generator of the j-th group $\pi_1(S_j^1, *)$. At that, $\deg\, g = |k_1| + \cdots + |k_n|$. To treat periodic boundary conditions, we consider homotopies without base point. Now we describe corresponding homotopy group G. Here we denote $G = \pi_1(P')$, elements of G are classes of equivalence. Two elements $P, g_0 \in \pi_1(P', *)$ are equivalent, if $P = g g_0 g^{-1}$, $g \in \pi_1(P', *)$. Obviously P and g_0 are equivalent if corresponding words are cyclic permutations one of another. Our first problem is to calculate the number of admissible words of fixed degree N taking into account the following restrictions: admissible words cannot contain combinations $g_i g_i^{-1}$ and $g_i^{-1} g_i$ for each $i = 1, \ldots, m$. The language of symbolic dynamics is an adequate tool for that. Indeed, let us introduce the following topological Markov chain. It has $2m$ states denoted by symbols $1, \ldots, 2m$. We identify symbols $1, \ldots, m$ with generators $g_1, \ldots, g_m$ and symbols $m+1, \ldots, 2m$ with elements $g_1^{-1}, \ldots, g_m^{-1}$ correspondingly. There is a transition $i \to j$, $i, j \in \{1, \ldots, 2m\}$, if the pair (i, j) does not correspond to pair $g_k g_k^{-1}$ or $g_k^{-1} g_k$, $k \in \{1, \ldots, m\}$. Then, as usual, consider the matrix of transition, say $A = (a_{ij})_{i,j=1}^{2m}$, so that $a_{ij} = 1$ if there exists a transition $i \to j$, and $a_{ij} = 0$ if not. For example for $m = 2$ this matrix has the following form

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Recall that we call homotopy complexity of the set of functions the log of number of homotopy classes to which the functions belong. Then denote by Ω_m the set of infinite sequences $\underline{\omega} = (\ldots \omega_{-1} \omega_0 \ldots \omega_k \ldots)$, $\omega_k \in \{1, \ldots, 2m\}$, with the following property $a_{\omega_k \omega_{k+1}} = 1$, $k \in \mathbb{Z}$, and endow it with the metric

$$\operatorname{dist}(\underline{\omega}', \underline{\omega}'') = \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|}} |\omega_k' - \omega_k''|.$$

After that Ω_m becomes a compact Cantor-like metric space and the shift map $\sigma : \Omega_m \to \Omega_m$ defined by $(\sigma\underline{\omega})_k = \omega_{k+1}$ if $(\underline{\omega})_k = \omega_k$, becomes a homeomorphism.

The dynamical system $(\sigma^n, \Omega_m)_{n\in\mathbb{Z}}$ is called the topological Markov chain (or subshift of finite type) determined by the matrix of transitions A (see, for instance [A]). It is very simple to express the number of admissible words of length N, say K_N, using the matrix A. Let us identify an admissible word $g = g_{i_1}^{k_1} g_{i_2}^{k_2} \dots g_{i_n}^{k_n}$, deg $g = N$ with the corresponding word $\omega_0'\omega_1' \cdots \omega_{N-1}'$, $\omega_i' \in \{1, \dots, 2m\}$, and, then, with the periodic sequence $\underline{\omega} = (\dots \omega_0 \dots \omega_{N-1} \dots \omega_k \dots)$ where $\omega_0' = \omega_0, \dots, \omega_{N-1}' = \omega_{N-1}$; $\omega_N = \omega_0', \dots, \omega_{-1} = \omega_{N-1}', \dots$. It is clear that $\sigma^N \underline{\omega} = \underline{\omega}$, so $\underline{\omega}$ is a N-periodic point of the map σ. And therefore the number K_n of admissible words of deg N coincides with the number of N-periodic points of the topological Markov chain (σ^n, Ω_{2m}). It is known, see for example [A], that $K_N = \lambda_1^N + \lambda_2^N + \cdots + \lambda_{2m}^N$ where $\lambda_1, \lambda_2, \dots, \lambda_{2m}$ are eigenvalues of the matrix of transitions A. Simple calculations show us that

$$(\lambda_1, \dots, \lambda_{2m}) = \Bigg(\underbrace{1, \dots, 1}_{m}, \underbrace{-1, -1, \dots, -1}_{m-1}, 2m-1\Bigg).$$

Therefore

$$\begin{aligned} K_N &= m + (m-1)\cdot(-1)^N + (2m-1)^N \\ &= \begin{cases} (2m-1)^N + 1 & \text{if } N \text{ is odd,} \\ (2m-1)^N + (2m-1) & \text{if } N \text{ is even.} \end{cases} \end{aligned} \tag{6.1}$$

For $m = 2$, $K_N = 3^N + 2 + (-1)^N$. The topological entropy of the shift map $\sigma|_{\Omega_{2m}}$ equals $\ln(2m-1)$. It follows from (6.1) that the number of all admissible words g with $1 \le \deg g \le N$ is

$$\begin{aligned} &\sum_{k=1}^{N} m + (m-1)(-1)^k + (2m-1)^k \\ &\quad = Nm + \frac{1}{2}\left[(m-1)(1+(-1)^N) + \frac{(2m-1)}{m-1}((2m-1)^N - 1)\right]. \end{aligned} \tag{6.2}$$

We call localized complexity the log of number of words $\mathcal{L}u$ for all u in a set of functions ($\mathcal{L}$ is defined in Theorem 5.1). Taking into account Remark 5.3, we see that the localized homotopic complexity of an attractor which contains solutions of all homotopic classes up to deg $g \le N$ and their translations can be estimated as follows:

$$\begin{aligned} \mathcal{K}(A) &= \ln\left[Nm + \frac{1}{2}\left[(m-1)(1+(-1)^N) + \frac{2m-1}{m-1}((2m-1)^N - 1)\right]\right] \\ &\cong N\ln(2m-1). \end{aligned} \tag{6.3}$$

For $m = 2$

$$\mathcal{K}(A) = \ln\left(2N + \frac{1}{2}(3^{N+1} + (-1)^N - 2)\right). \tag{6.4}$$

Let us calculate now the number of all admissible words g with $N_1 \le \deg g \le N_2$. This number, say $\mathcal{K}_{N_1N_2}$, is obtained directly from the formula (6.1):

(6.5)

$$\mathcal{K}_{N_1N_2} = \mathcal{K}_{N_2} - \mathcal{K}_{N_1} = (2m-1)^{N_2} - (2m-1)^{N_1} + (m-1)((-1)^{N_2} - (-1)^{N_1}).$$

For $m = 1$

$$\mathcal{K}_{N_1N_2} = 3^{N_2} - 3^{N_1} + 2((-1)^{N_2} - (-1)^{N_1}). \tag{6.6}$$

The number $\ln \mathcal{K}_{N_1N_2}$ is, in fact, the homotopic complexity in the fixed basis of a piece of the attractor $\mathcal{A}$ corresponding to solutions with energy between two levels: E_1 which is related to deg $g = N_1$ and E_2 – related to deg $g = N_2$.

Now, consider the problem of homotopic complexity of attractors (without localization $\mathcal{L}$ of homotopy elements of a basis). Instead of words determined by elements of the fundamental group $\pi_1(P', *)$ we need to calculate the number of homotopy classes of free homotopies. It means that a word g' corresponding to the symbolic word $(\omega'_0, \dots, \omega'_{N-1})$ and a word g'' corresponding to the word $(\omega''_0, \dots, \omega''_{N-1})$ represent the same class of free homotopies iff the first of them is a result of a cyclic permutation of the second one.

At first, let us assume that N is a prime number. Then each N-periodic trajectory of the shift map σ contains strictly N, N-periodic points, and every class of free homotopies of degree N corresponds to one and only one N-periodic trajectory. Therefore, the number of classes of free homotopies with degree N, say $\widetilde{\mathcal{K}}_N$, coincides with the number of N-periodic trajectories and

$$\widetilde{\mathcal{K}}_N = \frac{1}{N}\mathcal{K}_N = \frac{1}{N}(m + (m-1)(-1)^N + (2m-1)^N). \tag{6.7}$$

The homotopic complexity of a piece of an attractor $\mathcal{A}$ which contains solutions corresponding to words g with deg $g = N$ is the following number

$$\begin{aligned}\widetilde{\mathcal{K}}(\mathcal{A}) = \ln \widetilde{\mathcal{K}}_N &= \ln \frac{1}{N}(m + (m-1)(-1)^N + (2m-1)^N) \\ &\cong N \ln(2m-1) - \ln N.\end{aligned} \tag{6.8}$$

Assume now that N is not a prime number: $N = N_1 \cdot N_2$, $N_1 > 1$, $N_2 > 1$. Then an N-periodic point could be the N_1-periodic one. Therefore, the corresponding trajectory contains at most N_1 points. Since $N_1 < N$ then the number of periodic trajectories in this case is greater than $\frac{1}{N}\mathcal{K}_N$, and we have the following estimation for the homotopic complexity of the piece $\mathcal{A}_N$ of an attractor corresponding to the word with deg $g = N$.

$$\begin{aligned}&\ln \frac{1}{N}(m + (m-1)(-1)^N + (2m-1)^N) \\ &\qquad \le \widetilde{\mathcal{K}}(\mathcal{A}_N) \le \ln(m + (m-1)(-1)^N + (2m-1)^N).\end{aligned} \tag{6.9}$$

This estimation works for every $N \ge 1$, so after summation we obtain the following estimation of homotopic complexity of an attractor $\mathcal{A}$ which contains solutions of homotopic type g, deg $g \le N$:

$$\begin{aligned}&\ln \sum_{k=1}^{N} \frac{1}{k}(m + (m-1)(-1)^k + (2m-1)^k) \\ &\qquad \le \widetilde{\mathcal{K}}(\mathcal{A}) \le \ln \sum_{k=1}^{N}(m + (m-1)(-1)^k + (2m-1)^k).\end{aligned} \tag{6.10}$$

In fact for any number N we can find not only estimations like (6.9), (6.10) but rigorous formulas. These formulas look very cumbersome, so let us only explain how to do it. We consider 2 examples. Let $N = N_1^2$, first, and N_1 be a prime. We may separate all N-periodic points into two sets: the first one contains only N-periodic points each of those is N_1-periodic, the second one contains others. The numbers of trajectories (in other words the number of different classes of free homotopies) corresponding to the first class is $\frac{1}{N_1}\widetilde{\mathcal{K}}_{N_1}$ and corresponding to the second one is $\frac{1}{N}(\mathcal{K}_N - K_{N_1})$. Using formula (6.1) we have that the corresponding complexity can be represented in the following form (compare with (6.9))

$$\begin{aligned}\widetilde{\mathcal{K}}(\mathcal{A}_N) = \ln\Bigg\{&(2m-1)^N \cdot \frac{1}{N} + (2m-1)^{N_1}\left(\frac{1}{N_1} - \frac{1}{N}\right)\\ &+ (m-1)\left((-1)^N \cdot \frac{1}{N} + (1)^{N_1} \cdot \left(\frac{1}{N_1} - \frac{1}{N}\right)\right) + \frac{1}{N_1}\, m\Bigg\}.\end{aligned} \tag{6.11}$$

Consider the second example. Let $N = N_1 \cdot N_2$ where N_1, N_2 be different primes. In the same way as above we separate all N-periodic points into three sets: (1) the N_1-periodic, (2) the N_2-periodic, (3) the others. The number of trajectories corresponding to the first set is $\frac{1}{N_1}\, K_{N_1}$, the second one is $\frac{1}{N_2}\, K_{N_2}$ and the third one is $\frac{1}{N}(K_N - K_{N_1} - K_{N_2})$. Therefore, the corresponding homotopic complexity using formulas (6.1) and (6.7) may be written in the form

$$\begin{aligned}\widetilde{\mathcal{K}}(\mathcal{A}_N) = \ln\Bigg[&\sum_{i=1}^{2} \frac{1}{N_i}(m + (m-1)(-1)^{N_i} + (2m-1)^{N_i}\\ &+ \frac{1}{N}(m + (m-1)^N + (2m-1)^N\\ &- \sum_{i=1}^{2} m + (m-1)(-1)^{N_i} + (2m-1)^{N_i}\Bigg].\end{aligned}$$

In the same way we can derive the formula of complexity for the general case $N = N_1^{\alpha_1} \cdot N_2^{\alpha_2} \cdots N_s^{\alpha_s}$.

Remark 6.1. One can easily see by (6.1), that for any N

$$\widetilde{\mathcal{K}}(\mathcal{A}_N) \geq \ln[K_N/N] = \ln[m + (m-1)(-1)^N + (2m-1)^N] - \ln N.$$

For a prime N this estimate is precise (see (6.7)).

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