A DEFORMATION OF FLAT CONFORMAL STRUCTURES

HIROYASU IZEKI

ABSTRACT. We consider deformations of flat conformal structures from a viewpoint of connected sum decomposition of conformally flat manifolds.

1. INTRODUCTION

Let M be a smooth n-manifold and C a conformal class on M. If, for any point p of M, there exists a metric g in C such that g is flat on some neighborhood of p, then (M, C) is called *conformally flat*. A conformal class on M is called a *flat conformal structure* if (M, C) is conformally flat. A Riemannian manifold (M, g) is *conformally flat* if the conformal class containing g is a flat conformal structure. And M is said to be *conformally flat* if M admits a flat conformal structure. If $n = 2$ a flat conformal structure on an oriented surface S is nothing but a conformal structure on S. A flat conformal structure on a conformally flat manifold is a natural generalization of a conformal structure on a surface. The space of conformal structures on an oriented surface is understood as, for instance, the Teichmüller space of Riemann surfaces. The Teichmüller space of a compact oriented surface S with genus greater than one is identified with the quotient of the space of hyperbolic metrics on S by the identity component of the diffeomorphism group of S. To understand this space, the following description is useful.

Let (S, g) be a compact oriented surface of genus $h \geq 2$ with hyperbolic metric g. Then we can take $3h - 3$ disjoint simple closed geodesics with respect to g. Cutting S along these geodesics, we obtain $2h - 2$ hyperbolic surfaces with geodesic boundaries, and these are homeomorphic to S^2 with three disks removed. These hyperbolic surfaces with geodesic boundaries are called pairs of (*hyperbolic*) *pants*. Hyperbolic structures on a pair of pants are determined by the lengths of the three boundaries. On the other hand, S is obtained by gluing the boundaries of these pairs of pants by isometries. It is well-known that the lengths of the three boundaries and twisting parameters of identification of boundaries define coordinates on the Teichmüller space of S. In fact, it is easy to see that these coordinates consist of $6h - 6$ parameters (note that in order to identify boundaries they must have the same length), and it is well-known that the dimension of the Teichmüller space of a surface with genus h is just $6h - 6$. The coordinates of Teichmüller space defined as above are called the *Fenchel-Nielsen coordinates*. We refer the reader to [2] for a detailed exposition.

The subject of this paper is an attempt to describe the space of flat conformal structures on a compact conformally flat n-manifold ($n \geq 3$) in a way similar to the

description of the Teichmüller space in terms of the Fenchel-Nielsen coordinates. In order to realize such descriptions, we have to consider the following two problems:

1. Decompose a compact conformally flat manifold into small pieces with flat conformal structures.
2. Describe deformations of flat conformal structures by deformations of patching of these small pieces and deformations of flat conformal structures on these pieces.

In section 2, we consider Problem 1, using the C-prime decomposition (see section 2 or [8]) of compact conformally flat manifolds, which is a decomposition with respect to connected sums (recall that a connected sum of conformally flat manifolds is also conformally flat by a theorem of Kulkarni [12]). Then, in our case, a small piece corresponding to a pair of pants in Fenchel-Nielsen coordinates is a C-prime manifold (see Definition 2.1), which is a compact conformally flat manifold (without boundary). In order to obtain small pieces with flat conformal structures, since we decompose a manifold M with respect to connected sums, such as $M = M_1 \sharp M_2$, we have to extend a flat conformal structure on $M_i \setminus n$-disk, induced by restriction, to a flat conformal structure on M_i ($i = 1, 2$). It is not known (to the author) whether this extension is possible or not in the general case. But for a compact manifold with Kleinian structure, this extention is possible and we obtain a reasonable decomposition with respect to connected sums.

Theorem 2.8. *Let M be a compact connected n-manifold with $n \geq 3$, and C a Kleinian structure on M. Then there exist C-prime manifolds $P_1,...,P_k$ and a Kleinian structure C_i on P_i ($i = 1,...,k$) such that (M, C) is conformal to $(P_1, C_1)\sharp...\sharp(P_k, C_k)$.*

Here, (M, C) is called *Kleinian* if (M, C) is conformal to Ω/Γ for some open subset Ω of S^n and some discrete subgroup Γ of the conformal transformation group of standard S^n, which leaves Ω invariant and acts freely and properly discontinuously on Ω. Note that Ω/Γ has a natural flat conformal structure. A conformal class C on M is called a *Kleinian structure* (or a *uniformized conformal structure*), if (M, C) is Kleinian. See section 2 for the other notations. This theorem asserts that it is possible, in principle, to realize the description of the space of Kleinian structures in a way similar to the one using the Fenchel-Nielsen coordinates by means of the C-prime decomposition. In fact, we obtain a complete descriptiotion of the space of Kleinian structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ in such a way (see Appendix). Since, in this particular case, any flat conformal structure is Kleinian, this gives us a complete description of the space of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ following our program.

It should be mentioned that section 2 is based on a part of the author's previous paper [8], and can be considered as a supplement to [8].

In section 3, we construct a deformation of flat conformal structures on a connected sum of conformally flat manifolds from the viewpoint of Problem 2. Let (M_1, g_1) and (M_2, g_2) be compact conformally flat Riemannian manifolds and p_i a point of M_i ($i = 1, 2$). Then $(M_i \setminus \{p_i\}, g_i|_{M_i \setminus \{p_i\}})$ is conformal to a complete Riemannian manifold whose end is isometric to a half-infinite cylinder $[0, \infty) \times S^{n-1}(1)$ with radius 1. Cutting each end along the totally geodesic $(n-1)$-sphere of a half-infinite cylinder and gluing the resulting manifolds along these $(n-1)$-spheres of boundaries by an isometry, we obtain $M_1 \sharp M_2$ with a conformally flat metric. This Riemannian manifold contains a subset isometric to a finite cylinder $(0, l) \times S^{n-1}(1)$

for some positive real number l. Denote this Riemannian manifold by $(M_1 \sharp M_2, \hat{g}_l)$. By cutting each end along another totally geodesic sphere in this process, we can construct a Riemannian manifold containing a finite cylinder of different length, say l'. We denote this Riemannian manifold by $(M_1 \sharp M_2, \hat{g}_{l'})$.

Theorem 3.1. *If l' is greater than some constant depending only on l, n, M_1 and M_2, then $(M_1 \sharp M_2, \hat{g}_l)$ and $(M_1 \sharp M_2, \hat{g}_{l'})$ are not conformally equivalent, provided each M_i is not diffeomorphic to S^n.*

That is, by changing the length of a cylinder, we can deform a flat conformal structure on $M_1 \sharp M_2$, if each M_i is not diffeomorphic to S^n (see section 3 for a more precise statement). On the other hand, if either (M_1, g_1) or (M_2, g_2) is conformal to the standard S^n, then two Riemannian manifolds constructed as above are always conformally equivalent (see Remark 2 following Theorem 3.1). As a consequence of Theorem 3.1, we see that the moduli space of flat conformal structures on a compact, non-trivial, conformally flat and non-C-prime manifold is noncompact (see Corollary 3.2). Here, the topology of the moduli space of M is given by the quotient topology which comes from the compact C^∞-topology of the space of all conformally flat Riemannian metrics on M.

The deformation above is related to the bending deformation of Thurston (see [20] and [5]). By Johnson and Millson, the bending deformations for compact manifolds are studied in terms of the cohomology of group representations in [9]. They proved a result corresponding to Theorem 3.1 for infinitesimal or small deformations, which says that the bending deformation gives rise to a non-trivial deformation of flat conformal structures in the Teichmüller space of $M_1 \sharp M_2$ (section 5 of [9]). Here, the Teichmüller space of a manifold M is the quotient of the space of all flat conformal structures on M by the identity component of the group of diffeomorphisms of M. Theorem 3.1 tells us that the large bending deformation described above also gives us a non-trivial deformation in the moduli space, and we prove this by a purely geometric method. Here, we put special emphasis on the fact that our method also works for noncompact manifolds, if we consider the Teichmüller space instead of the moduli space, in contrast to the method of Johnson and Millson. See Theorem 3.4.

Though there are many other ways to deform connected sums defined in section 2, they do not always give us deformations of flat conformal structures (see Appendix). The result of Johnson and Millson mentioned above also suggests that, roughly speaking, a deformation of connected sums gives us a deformation of flat conformal structures on $M_1 \sharp M_2$, if there is no conformal transformation on each (M_i, C_i) which absorbs the deformation of connected sums. From this point of view, changing the length of a cylinder corresponds to essential conformal transformations, or more precisely, similarity transformations. And, as is expected from this, a theorem of Obata ([18]) and Lelong-Ferrand ([15], [16]) concerning the conformal transformation groups of compact Riemannian manifolds plays a key role in the proof of this theorem. Note that we do not restrict our attention to deformations of Kleinian structures here. See, for example, [1] for an exposition on deformations arising from those of discrete groups.

In the rest of this paper, we assume that a manifold M is connected with $n = \dim M \geq 3$. Moreover, if M is orientable, we also assume that M is oriented.

2. The C-prime decomposition of conformally flat manifolds

We start with a review of the C-prime decomposition of compact conformally flat manifolds defined in [8]. In this section, a manifold M is always assumed to be compact.

Definition 2.1. An n-manifold M is said to be *non-trivial* if M is not diffeomorphic to S^n. And a non-trivial conformally flat manifold M is called C-*prime* if M is not diffeomorphic to a connected sum of non-trivial conformally flat manifolds.

Proposition 2.2 ([8, Proposition 2.1])**.** *Every non-trivial conformally flat manifold is diffeomorphic to a connected sum $P_1\sharp...\sharp P_k$ of a finite number of C-prime manifolds.*

A decomposition $M = P_1\sharp...\sharp P_k$, where each P_i is C-prime, is called a C-*prime decomposition* of M. If M is an oriented 3-manifold admitting a Kleinian structure, then this decomposition is unique (see [8, Corollary 3.7]). On the other hand, $N\sharp N$ is diffeomorphic to $N\sharp(S^1 \times S^2)$, where N is the non-orientable S^2-bundle over S^1 (see for example [7, Lemma 3.17]). Since the non-orientable S^2-bundle over S^1 and $S^1 \times S^2$ are conformally flat and C-prime by Proposition 2.3 below, this is an example of a conformally flat manifold whose C-prime decomposition is not unique. It is not known whether the uniqueness holds for higher dimensional oriented manifolds.

Some sufficient conditions for a manifold to be C-prime are given as follows:

Proposition 2.3 ([8, Proposition 2.2])**.** *If the fundamental group of a non-trivial conformally flat n-manifold M is generated by one element, then M is C-prime.*

Proposition 2.4 ([8, Proposition 2.3])**.** *If the universal covering space of a conformally flat manifold M is diffeomorphic to $\mathbf{R}^n$, then M is C-prime.*

A manifold M is said to be *Kleinian* if M admits a Kleinian structure. For Kleinian manifolds, we have the following.

Proposition 2.5 ([8, Proposition 3.1])**.** *Let M be a non-trivial Kleinian manifold. If the homotopy group $\pi_{n-1}(M)$ is trivial, then M is C-prime.*

Take a subset A of S^n so that A is diffeomorphic to $(0,1) \times S^{n-1}$ and satisfies the following conditions:

(1) Each connected component of $S^n \setminus A$ is diffeomorphic to a closed n-disk. Denote them by D_1 and D_2.

(2) $E_i = D_i \cup A$ $(i = 1, 2)$ is diffeomorphic to the unit open n-disk and, under this identification, D_i corresponds to the closed disk of radius $1/2$.

Fix an orientation of S^n. Let C_i $(i = 1, 2)$ be a flat conformal structure on M_i and $\psi_i : E_i \longrightarrow (M_i, C_i)$ a conformal (orientation preserving) embedding. Note that E_i has a natural flat conformal structure induced by restricting the flat conformal structure of S^n.

Definition 2.6. A connected sum $(M_1, C_1)\sharp(M_2, C_2)$ of (M_1, C_1) and (M_2, C_2) is defined by

$$(M_1 \setminus \psi_1(D_1), C_1|_{M_1 \setminus \psi_1(D_1)}) \cup_{\psi_2 \circ \psi_1^{-1}|_{\psi_1(A)}} (M_2 \setminus \psi_2(D_2), C_2|_{M_2 \setminus \psi_2(D_2)}).$$

Remark. It is clear that this defines a flat conformal structure on $M_1\sharp M_2$ (moreover, if each C_i is a Kleinian structure on M_i, then the flat conformal structure

defined as above is also Kleinian by the proof of [13, Theorem 5.6], which states that a connected sum of Kleinian manifolds also admits a Kleinian structure). But this flat conformal structure depends on the choice of A, ψ_1 and ψ_2. Thus, more precisely speaking, we should denote the connected sum defined above by $(M_1, C_1) \natural_{(A, \psi_1, \psi_2)} (M_2, C_2)$ instead of $(M_1, C_1) \natural (M_2, C_2)$. In fact, varying these choices, we can deform a flat conformal structure, as we will see in the next section. But, since it is not convenient to use A, ψ_1 and ψ_2 as parameters of flat conformal structures, we omit them in what follows.

Theorem 2.7. *Let (M, C) be Kleinian, and suppose M is diffeomorphic to a connected sum $M_1 \natural M_2$, where M_1 and M_2 are not necessarily conformally flat. Then there exists a Kleinian manifold M_i', which is homeomorphic to M_i, and a Kleinian structure C_i on M_i' $(i = 1, 2)$ such that (M, C) is conformal to some $(M_1', C_1) \natural (M_2', C_2)$. If $n \leq 6$, M_i' is diffeomorphic to M_i.*

Proof. The first assertion follows from the proof of [8, Theorem 3.2]. Let S be a subset of M, which is an embedded S^{n-1}, such that $M \setminus S$ has two connected components, say L_i, where L_i is diffeomorphic to $M_i \setminus n$-disk. Note that L_i has a natural Kleinian structure induced by the restriction of a Kleinian structure C of M. In the proof of [8, Theorem 3.2], this Kleinian structure is extended to a Kleinian structure on $M_i' = L_i \cup n$-disk, and M_i' is clearly homeomorphic to M_i. Since $L_i \cup S$ is naturally embedded in M_i, if this embedding is extended to the n-disk, M_i' is diffeomorphic to M_i. If $n \leq 6$, it is possible to extend a diffeomorphism of the unit $(n-1)$-sphere of $\mathbf{R}^n$ to a diffeomorphism of the unit n-disk of $\mathbf{R}^n$ (see for example [3] and [17]). Applying this fact to the above embedding restricted to S, we see that the second assertion of the theorem follows. $\square$

Remark. We can not expect M_i' to be diffeomorphic to M_i if $n \geq 7$. Let M_1 be an exotic n-sphere Σ^n and M_2 be Σ^n with the opposite orientation $(n \geq 7)$. Then $M = \Sigma^n \natural (-\Sigma^n)$ is diffeomorphic to S^n and clearly Kleinian, where $-\Sigma^n$ denotes Σ^n with the opposite orientation. But Σ^n does not admit a flat conformal structure. In this case, M_i' must be S^n.

Using Theorem 2.7, we can prove the following theorem mentioned in the introduction in the same manner as the proof of Proposition 2.2 (see [8, Proposition 2.1]).

Theorem 2.8. *Let (M, C) be Kleinain. Then there exist C-prime manifolds $P_1, ..., P_k$ and a Kleinian structure C_i on P_i $(i = 1, ..., k)$ such that (M, C) is conformal to some $(P_1, C_1) \natural ... \natural (P_k, C_k)$.*

Remark. Let M be a Kleinian manifold, and fix a C-prime decomposition $M = P_1 \natural ... \natural P_k$ of M. If $n \leq 6$, we can decompose a Kleinian structure on M with respect to this decomposition. That is, we can decompose (M, C) as $(P_1, C_1) \natural ... \natural (P_k, C_k)$ for some Kleinian structure C_i on P_i. We can not expect this for the case $n \geq 7$ at the present stage, since a differentiable structure of P_i may change as we decompose a Kleinian structure C on M.

In the rest of this section, we consider a certain sufficient condition for a manifold to be C-prime, and show that most of familiar examples of conformally flat manifolds are C-prime. Note that, for a Kleinian structure C on M, there exists an open subset Ω of S^n such that (M, C) is conformal to a quotient of Ω by a discrete subgroup Γ of the conformal transformation group of standard S^n by definition.

Proposition 2.9. *Let M be a Kleinian manifold and C a Kleinian structure on M. If the number of connected components of $S^n \setminus \Omega$ is finite, then M is either C-prime or diffeomorphic to $\mathbf{R}P^n \sharp \mathbf{R}P^n$.*

Proof. Assume M is not C-prime. Then, by Theorem 2.7, there exist non-trivial Kleinian manifolds $(M_i, C_i) \cong \Omega_i / \Gamma_i$, $i = 1, 2$, such that (M, C) is conformal to a connected sum of (M_1, C_1) and (M_2, C_2). We show that unless the order of each Γ_i is less than three, the number of connected components of $S^n \setminus \Omega$ is infinite. If the order of each Γ_i is less than three, then each Ω_i must be compact and hence $\Omega_i = S^n$. Since M_i is non-trivial, each Γ_i is of order two. Thus each M_i is diffeomorphic to the real projective space $\mathbf{R}P^n$.

Now we may assume that Γ_1 is of order at least three and Γ_2 is of order at least two. Thus $M_1 \setminus n$-disk is covered by $\hat{\Omega}_1 = \Omega_1 \setminus$(at least three disks) and $M_2 \setminus n$-disk is covered by $\hat{\Omega}_2 = \Omega_2 \setminus$(at least two disks). By the construction of Ω_i in the proof of [8, Theorem 3.2], we see that Ω is reconstructed by gluing copies of $\hat{\Omega}_i$ along the boundaries made by removing disks (this construction is the same as what Kulkarni and Pinkall used in the proof of [13, Theorem 5.6]). Denote these disks removed from Ω_1 by D_1, D_2, D_3,..., and disks removed from Ω_2 by E_1, E_2,... (note that there are at least three D_i's and at least two E_i's). If $\Omega \cap D_i = D_i$, then either M_1 or M_2 must be diffeomorphic to S^n as in the proof of Proposition 2.5 (see [8, Proposition 3.1]). Therefore $\Omega \cap D_i \neq D_i$ and hence $S^n \setminus \Omega$ has at least three connected components. Put $\hat{\Omega}_2$ into D_1 and glue their boundaries suitably. Then at least one removed disk E_i remains, and in order to obtain Ω, we have to put a copy of $\hat{\Omega}_1$ into it and glue them along their boundaries. Then we see that $(S^n \setminus \Omega) \cap D_1$ has at least two connected components. In order to construct Ω, we have to repeat this process of putting copies of $\hat{\Omega}_i$ into D_1 infinitely. And when we put k copies of $\hat{\Omega}_1$ into D_1, the number of the connected components of $(S^n \setminus \Omega) \cap D_1$ is at least 2^k. Therefore $S^n \setminus \Omega$ has infinitely many connected components. This completes the proof. $\square$

Proposition 2.10. *Let M be a non-trivial manifold and not diffeomorphic to $\mathbf{R}P^n \sharp \mathbf{R}P^n$. If there exists a flat conformal structure C such that (M, C) is conformally covered by $S^n(1)$, $\mathbf{R}^n$, $H^n(-1)$, $\mathbf{R} \times S^{n-1}(1)$, $\mathbf{R} \times H^{n-1}(-1)$, or $S^p(1) \times H^q(-1)$ $(p, q \geq 2)$, then M is C-prime. Here $S^m(1)$ and $H^m(-1)$ denote the unit Euclidean m-sphere and hyperbolic m-space with constant curvature -1 respectively, and products are Riemannian products.*

Proof. If (M, C) is conformally covered by $\mathbf{R}^n$, $H^n(-1)$ or $\mathbf{R} \times H^{n-1}(-1)$ then the conclusion follows from Proposition 2.4. Since $\mathbf{R} \times S^{n-1}(1)$ and $S^p(1) \times H^q(-1)$ are conformally equivalent to $S^n \setminus$(two points) and $S^n \setminus S^{q-1}$ respectively, we can apply Proposition 2.9 in this case. $\square$

Corollary 2.11. *A non-trivial manifold admitting a locally homogeneous conformally flat metric is C-prime, except when it is $\mathbf{R}P^n \sharp \mathbf{R}P^n$.*

Proof. If (M, g) is locally homogeneous, then the Riemannian universal covering space $(\tilde{M}, \tilde{g})$ is homogeneous by a theorem of Singer([19]). Since the conformal transformation group $\mathrm{Conf}(\tilde{M}, \tilde{g})$ has positive dimension, either $(\tilde{M}, \tilde{g})$ is conformal to $S^n(1)$, $\mathbf{R}^n$, $H^n(-1)$, $\mathbf{R} \times S^{n-1}(1)$, $\mathbf{R} \times H^{n-1}(-1)$, or $(\tilde{M}, \tilde{g})$ is conformally mapped onto $S^n(1) \setminus \Lambda$ by the developing map of $(\tilde{M}, \tilde{g})$, where Λ is contained in a round k-sphere with $k \leq n - 2$, by a theorem of Goldman and Kamishima ([6]). In

the last case, M is conformally equivalent to $(S^n(1) \setminus \Lambda)/\Gamma$ and Λ is the limit set of a discrete subgroup Γ of the conformal transformation group $\mathrm{Conf}(S^n)$ of the standard sphere. Let S^k be a minimal round k-sphere containing Λ. Since if $k = 0$, $S^n(1) \setminus \Lambda$ is conformal to $\mathbf{R} \times S^{n-1}(1)$ or $\mathbf{R}^n$, we may assume that $1 \leq k \leq n-2$. Let G be the image of $\mathrm{Conf}(\tilde{M}, \tilde{g})$ by the map $\mathrm{Conf}(\tilde{M}, \tilde{g}) \longrightarrow \mathrm{Conf}(S^n)$ induced from the developing map of $(\tilde{M}, \tilde{g})$. Note that G acts transitively on the developing image of $(\tilde{M}, \tilde{g})$ even if the developing map of $(\tilde{M}, \tilde{g})$ is not injective. It is clear that G leaves the minimal round sphere S^k invariant. Thus if $S^k \setminus \Lambda$ is not empty, G can not act transitively on $S^n(1) \setminus \Lambda$, and hence the developing image of $\tilde{M}$ is $S^n(1) \setminus S^k$. Therefore (M, g) is conformally covered by $S^p(1) \times H^q(-1)$ $(p, q \geq 2)$ or $\mathbf{R} \times H^{n-1}(-1)$. This completes the proof. $\qquad \square$

3. Constructing a deformation

In this section we prove Theorem 3.1 mentioned in the introduction. Before we prove Theorem 3.1, we give a more precise statement of the theorem.

Let (M_i, g_i), $i = 1, 2$, be a compact conformally flat Riemannian manifold, where each M_i is not diffeomorphic to S^n. Let $d(M)$ be the smallest number of the generators of the fundamental group $\pi_1(M)$. We may assume $d(M_1) \geq d(M_2)$. Denote the integer part of $d(M_1)/d(M_2)$ by k_0. This number is well-defined since $d(M_i)$ is positive in our case by a theorem of Kuiper ([10]). Let p_i be a point of M_i. Then $(M_i \setminus \{p_i\}, g_i|_{M_i \setminus \{p_i\}})$ is conformal to a complete Riemannian manifold whose end is isometric to a half-infinite cylinder $[0, \infty) \times S^{n-1}(1)$. Denote them by $(\bar{M}_i, \bar{g}_i)$ and identify each end with $[0, \infty) \times S^{n-1}(1)$ by an isometry. Fix a positive real number l and define $X = (\bar{M}_1, \bar{g}_1) \setminus [l, \infty) \times S^{n-1}(1)$ and $Y = (\bar{M}_2, \bar{g}_2) \setminus [l, \infty) \times S^{n-1}(1)$ by using above identification of each end. Identifying $(0, l) \times S^{n-1} \subset X$ and $(0, l) \times S^{n-1} \subset Y$ by a suitable isometry, we obtain a compact conformally flat Riemannian manifold diffeomorphic to $M_1 \sharp M_2$. Denote this by $(M, \hat{g}_l)$. Similarly, define

$$\tilde{X} = \begin{cases} (\bar{M}_1, \bar{g}_1) \setminus [2l + (k_0 + 1)l', \infty) \times S^{n-1}(1) & \text{if } d(M_1) > d(M_2), \\ (\bar{M}_1, \bar{g}_1) \setminus [2l + 3l', \infty) \times S^{n-1}(1) & \text{if } d(M_1) = d(M_2), \end{cases}$$

and

$$\tilde{Y} = \begin{cases} (\bar{M}_2, \bar{g}_2) \setminus [2l + (k_0 + 1)l', \infty) \times S^{n-1}(1) & \text{if } d(M_1) > d(M_2), \\ (\bar{M}_2, \bar{g}_2) \setminus [2l + 3l', \infty) \times S^{n-1}(1) & \text{if } d(M_1) = d(M_2). \end{cases}$$

If $d(M_1) > d(M_2)$ (resp. $d(M_1) = d(M_2)$), by idendifying $(0, 2l + (k_0 + 1)l') \times S^{n-1}(1) \subset \tilde{X}$ (resp. $(0, 2l + 3l') \times S^{n-1}(1) \subset \tilde{X}$) with $(0, 2l + (k_0 + 1)l') \times S^{n-1}(1) \subset \tilde{Y}$ (resp. $(0, 2l + 3l') \times S^{n-1}(1) \subset \tilde{Y}$) suitably, we obtain a compact conformally flat Riemannian manifold diffeomorphic to M. Denote this by $(M, \hat{g}(l'))$ (see Figure 1). Now we can state Theorem 3.1.

Theorem 3.1. *If l' is greater than a certain constant $C(n, l)$ depending only on n and l, then $(M, \hat{g}_l)$ and $(M, \hat{g}(l'))$ are not conformally equivalent.*

Remark 1. Let C_i be a flat conformal structure on M_i containing g_i, and $\hat{C}_l$ and $\hat{C}(l')$ flat conformal structures on M containing $\hat{g}_l$ and $\hat{g}(l')$ respectively. Then it is clear that both $(M, \hat{C}_l)$ and $(M, \hat{C}(l'))$ are represented as $(M_1, C_1) \sharp (M_2, C_2)$, though these are not conformally equivalent. This is what we mentioned in the remark following Definition 2.6.
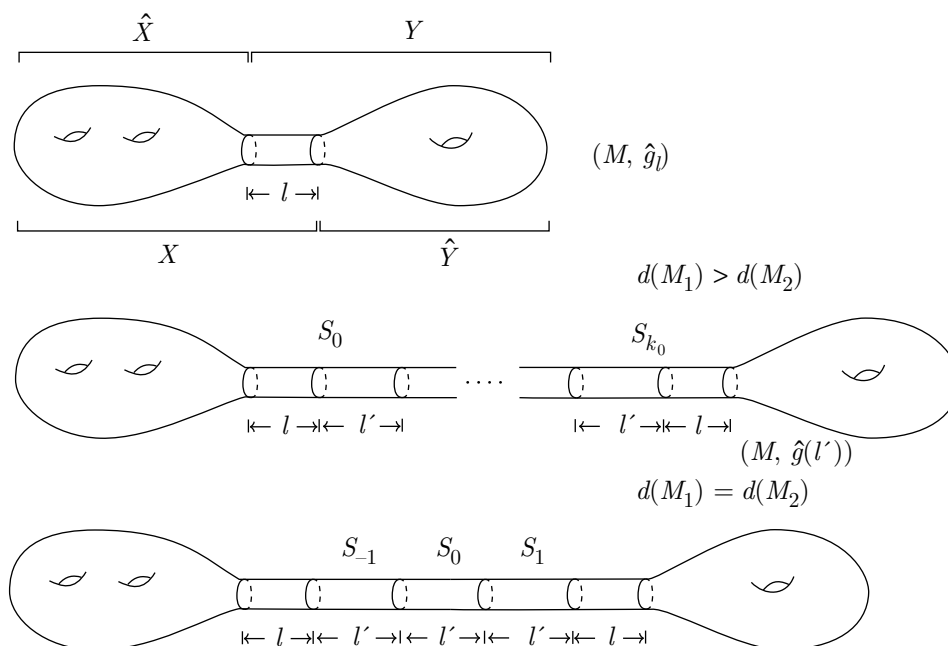
FIGURE 1

Remark 2. Assume M_2 is diffeomorphic to S^n. Then, by a theorem of Kuiper ([10]), (M_2, g_2) is conformal to the standard n-sphere $S^n(1)$. In this case, k_0, and thus, $(M, \hat{g}(l'))$ can not be defined as above. But we can define $(M, \hat{g}_{l''})$, which contains a subset isometric to $(0, l'') \times S^{n-1}(1)$ as follows: Choose l'' so that $l'' > l$, and let $Y_{l''} = (\bar{M}_2, \bar{g}_2) \setminus [l'', \infty) \times S^{n-1}(1) \subset Y$. Identifying $(l'' - l, l'') \times S^{n-1}(1) \subset Y_{l''}$ with $X \setminus \hat{X}$ by a suitable isometry, we obtain a conformally flat Riemannian manifold diffeomorphic to M. Denote this by $(M, g_{l''})$. Note that there exists a conformal transformation $\psi : Y \longrightarrow Y_{l''}$ with $\psi((0, l) \times S^{n-1}(1)) = (l'' - l, l'') \times S^{n-1}(1)$ by Liouville's theorem. Then it is easy to see that, by choosing such a conformal transformation suitably, we can construct a conformal diffeomorphism from $(M, \hat{g}_l)$ to $(M, \hat{g}_{l''})$.

Assume that a compact conformally flat manifold M is non-trivial and not C-prime. Then, by definition, there are conformally flat manifolds M_1 and M_2 such that M is diffeomorphic to $M_1 \sharp M_2$. Take a positive real number l_1. Define l_j, $j = 1, 2, \ldots$, inductively as follows. Choose l'_{j-1} to be greater than $C(n, l_{j-1})$. Let $l_j = 2l_{j-1} + (k_0 + 1)l'_{j-1}$ (resp. $l_j = 2l_{j-1} + 3l'_{j-1}$), if $d(M_1) > d(M_2)$ (resp. $d(M_1) = d(M_2)$). Fix conformally flat Riemannian metrics g_1 and g_2 on M_1 and M_2 respectively, and a Riemannian metric $\hat{g}_{l_j}$ on M as above. Then, by Theorem 3.1, $(M, \hat{g}_{l_i})$ and $(M, \hat{g}_{l_j})$ are not conformally equivalent, if $i \neq j$. Therefore, $\{\hat{g}_{l_j}\}_{j=1,2,\ldots}$ defines an infinite sequence in the moduli space of flat conformal structures on M. It is easy to see that this sequence in the moduli space is a divergent sequence. Thus we have the following corollary to Theorem 3.1.

Corollary 3.2. *If M is compact, non-trival, conformally flat and not C-prime, then M carries infinitely many different flat conformal structures. Moreover, the moduli space of flat conformal structures on M is noncompact.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.3. *Suppose that there exists a conformal embedding ψ of $(-\epsilon, l' + \epsilon) \times S^{n-1}(1)$ into $(M, \hat{g}_l)$, $\epsilon > 0$, and that $(M, \hat{g}_l) \setminus \psi((0, l') \times S^{n-1}(1))$ has two connected components. Then there exists a positive constant $C(n, l)$ with the following property: If $l' > C(n, l)$, then there exists a closed interval $[\alpha, \beta] \subset (0, l')$ such that the closure of $\psi(A)$ is contained in either X or Y, where $A = (\alpha, \beta) \times S^{n-1}(1)$.*

The constant $C(n, l)$ in Lemma 3.3 is the same as $C(n, l)$ in Theorem 3.1. The proof of Lemma 3.3 is given in the next section. We note here that we do not have to assume M to be compact in Lemma 3.3.

Proof of Theorem 3.1. Take l' so that $l' > C(n, l)$. Assume that there exists a conformal diffeomorphism $\varphi : (M, \hat{g}(l')) \longrightarrow (M, \hat{g}_l)$, that is, $(M, \hat{g}_l)$ and $(M, \hat{g}(l'))$ are conformally equivalent. We prove the theorem by showing that this assumption leads to a contradiction.

Define subsets S_k of $(M, \hat{g}(l'))$ by

$$S_k = \begin{cases} [l + kl', l + (k+1)l'] \times S^{n-1}(1) \subset \tilde{X}, & 0 \leq k \leq k_0 \\ & \text{if } d(M_1) > d(M_2), \\ [l + (k+1)l', l + (k+2)l'] \times S^{n-1}(1) \subset \tilde{X}, & -1 \leq k \leq 1 \\ & \text{if } d(M_1) = d(M_2). \end{cases}$$

The natural inclusion $X \longrightarrow \tilde{X}$ (resp. $Y \longrightarrow \tilde{Y}$) defines an isometric embedding $i_X : X \longrightarrow (M, \hat{g}(l'))$ (resp. $i_Y : Y \longrightarrow (M, \hat{g}(l'))$). Define $\hat{X}$ and $\hat{Y}$ by $\hat{X} = (M, \hat{g}_l) \setminus Y \subset X$ and $\hat{Y} = (M, \hat{g}_l) \setminus X \subset Y$.

Clearly, a restriction of φ on some neighborhood of S_0 satisfies the assumption of Lemma 3.3. Thus we can take a subset A_0 of S_0 as A in the lemma. Then $(M, \hat{g}(l')) \setminus A_0$ has two connected components, and we denote them by $\tilde{X}_0$ and $\tilde{Y}_0$ so that $\tilde{X}_0 \supset i_X(X)$ and $\tilde{Y}_0 \supset i_Y(Y)$. By our choice of A_0, $\varphi(A_0)$ is contained in either X or Y. If $\varphi(A_0) \subset X$, then one out of two connected components of $(M, \hat{g}_l) \setminus \varphi(A_0)$ is contained in X, i.e., either $\varphi(\tilde{X}_0) \subset X$ or $\varphi(\tilde{Y}_0) \subset X$ holds. If $\varphi(A_0) \subset Y$, then either $\varphi(\tilde{X}_0) \subset Y$ or $\varphi(\tilde{Y}_0) \subset Y$ holds.

We divide the rest of the proof into several steps and label them as $(3.k.m)$. The first "3" stands for the number of this section, the second "k" stands for k of S_k and the last "m" denotes a number of possible cases which occur as we consider the embedding of S_k into $(M, \hat{g}_l)$ (these cases are listed in $(3.1.3)$).

$(3.0.1)$: $\varphi(A_0) \subset X$ *and* $\varphi(\tilde{X}_0) \subset X$. In this case, there is a conformal embedding

$$i_X \circ \varphi : \tilde{X}_0 \cup A_0 \longrightarrow \tilde{X}_0 \cup A_0$$

and $i_X \circ \varphi(\tilde{X}_0 \cup A_0) \subset i_X(X)$. Since $\tilde{X}_0 \cup A_0$ is conformal to (M_1, g_1) with a disk removed, we regard $\tilde{X}_0 \cup A_0$ as a subset of (M_1, g_1). Define D_0, D_1, D_2 and D_3 by

$$D_0 = (M_1, g_1) \setminus (\tilde{X}_0 \cup A_0),$$
$$D_1 = (M_1, g_1) \setminus \tilde{X}_0,$$
$$D_2 = (M_1, g_1) \setminus i_X \circ \varphi(\tilde{X}_0 \cup A_0),$$
$$D_3 = (M_1, g_1) \setminus i_X \circ \varphi(\tilde{X}_0).$$

Then we have $D_0 \subset D_1 \subset D_2 \subset D_3$. Since we take A_0 as in Lemma 3.3, A_0 is conformal to $(\alpha, \beta) \times S^{n-1}(1)$. Therefore $i_X \circ \varphi(A_0)$ is identified conformally with an open subset $S(\alpha_0)$ of $S^n(1)$ for some $\alpha_0 > 0$, where

$$S(\alpha_0) = \{x = (x_0, \ldots, x_n) \in S^n(1) \subset \mathbf{R}^{n+1}; \ |x_0| < \alpha_0\}.$$

We may assume that $\partial(i_X \circ \varphi(\tilde{X}_0 \cup A_0))$ is identified with $\{x \in S^n; \ x_0 = \alpha_0\}$, where $\partial(i_X \circ \varphi(\tilde{X}_0 \cup A_0))$ denotes the boundary of $i_X \circ \varphi(\tilde{X}_0 \cup A_0)$. Denote two connected components of $S^n \setminus S(\alpha_0)$ by $E(-\alpha_0)$ and $E(\alpha_0)$ so that $\partial E(\alpha_0) = \{x \in S^n; \ x_0 = \alpha_0\}$. By the above conformal identification of $i_X \circ \varphi(A_0)$ and $S(\alpha_0)$, we can glue $S(\alpha_0) \cup E(\alpha_0)$ to $(M_1, g_1) \setminus D_2 = i_X \circ \varphi(\tilde{X}_0 \cup A_0)$ and obtain a compact conformally flat manifold M_1'. Similarly, we can glue $S(\alpha_0) \cup E(-\alpha_0)$ to D_3 and obtain a compact conformally flat manifold M_2'. Clearly M_1 is diffeomorphic to $M_1' \sharp M_2'$ and $\pi_1(M_1) \cong \pi(M_1')$. Thus $\pi_1(M_2') = 1$ by van Kampen's theorem, and hence M_2' is conformal to S^n by a theorem of Kuiper ([10]). Therefore we have a conformal embedding $\tau : D_3 \longrightarrow S^n$, and $i_X \circ \varphi|_{A_0}$ defines a locally defined conformal diffeomorphism $\sigma : \tau(A_0) \longrightarrow \tau(i_X \circ \varphi(A_0))$. By Liouville's theorem σ is extended to a globally defined conformal transformation $\tilde{\sigma}$ on S^n. Note that $\tilde{\sigma}$ satisfies $\tilde{\sigma} \circ \tau(A_0) = \tilde{\sigma} \circ \tau(D_1 \setminus D_0) = \tau(D_3 \setminus D_2)$ and, in particular, $\tilde{\sigma} \circ \tau(D_0) = \tau(D_2)$ and $\tilde{\sigma} \circ \tau(D_1) = \tau(D_3)$. Define

$$\eta = \begin{cases} i_X \circ \varphi & \text{on } \tilde{X}_0 \cup A_0, \\ \tau^{-1} \circ \tilde{\sigma} \circ \tau & \text{on } D_1. \end{cases}$$

Then η is a well-defined conformal transformation on (M_1, g_1). In particular, $\eta(D_1) = D_3$. But since $D_3 \setminus i_X \circ \varphi(A_0) \supset D_1$, D_1 and D_3 never have the same volume with respect to any Riemannian metric on M_1. That is, η never be an isometry with respect to any Riemannian metric on M_1. Therefore η is an essential conformal transformation of (M_1, g_1). Thus, by a theorem of Obata ([18]) and Lelong-Ferrand ([15], [16]), (M_1, g_1) is conformal to $S^n(1)$. A contradiction.

(3.0.2): $\varphi(A_0) \subset Y$. In this case, either $\varphi(\tilde{X}_0) \subset Y$ or $\varphi(\tilde{Y}_0) \subset Y$ holds. If $\varphi(\tilde{Y}_0) \subset Y$, then this contradicts our assumption for the same reason as in (3.0.1). Assume $\varphi(\tilde{X}_0) \subset Y$. By regarding Y as a subset of (M_2, g_2), we see that $\varphi(\tilde{X}_0 \cup A_0) \subset (M_2, g_2)$. Then as we have seen in (3.0.1), we obtain compact conformally flat manifolds M' and M_1' such that $M_2 = M_1' \sharp M'$ and $\pi_1(M_1') \cong \pi_1(M_1)$. By van Kampen's theorem and the Gruško-Neumann theorem (see for example [14, p. 58]), we see that $d(M_2) = d(M_1) + d(M')$. If $d(M_1) > d(M_2)$, we have a contradiction. If $d(M_1) = d(M_2)$, replace a number "i" of M_i and rename suitably $X, Y, \tilde{X}, \tilde{Y}$, and so on. Then, $\varphi(A_0) \subset X$ and $\varphi(\tilde{Y}_0) \subset X$ hold. This is the case that we consider in the next step.
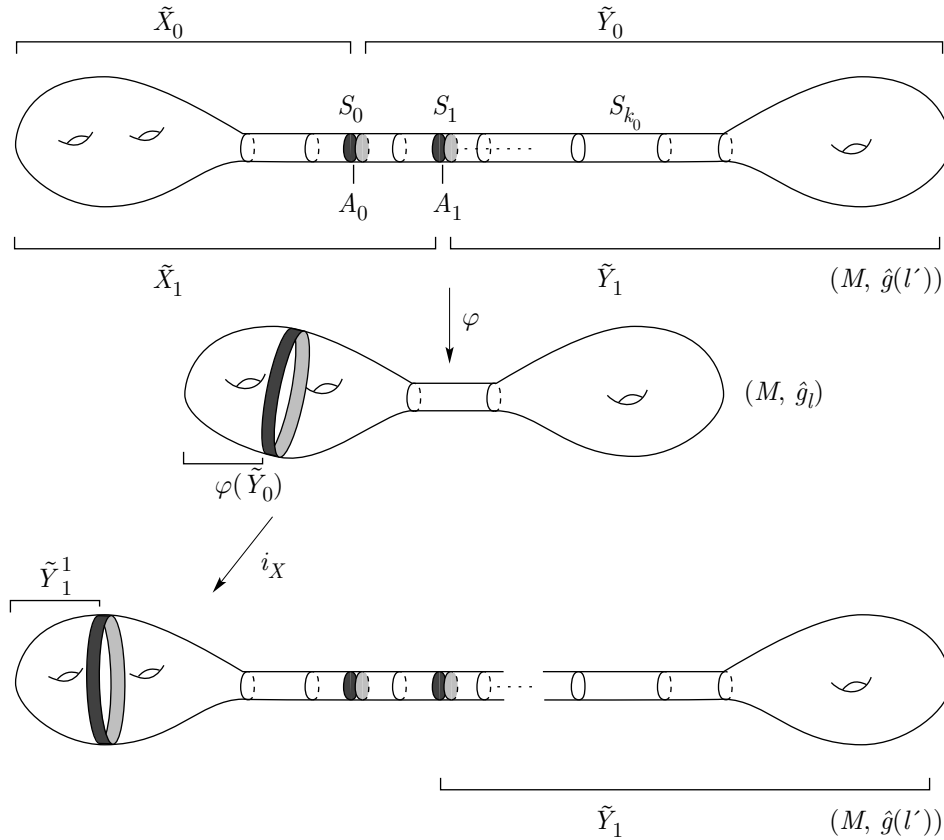
FIGURE 2

(3.0.3): $\varphi(A_0) \subset X$ *and* $\varphi(\tilde{Y}_0) \subset X$. Since $\tilde{Y}_0 \supset S_1$ and $\varphi(\tilde{Y}_0) \subset X$, we can define $i_X \circ \varphi$ on a neighborhood of S_1. Then it is easy to see that the conformal embedding $\varphi \circ i_X \circ \varphi : $ (a neighborhood of S_1) $\longrightarrow (M, \hat{g}_l)$ satisfies the assumption of Lemma 3.3. Take $A_1 \subset S_1$ as in Lemma 3.3. Denote two connected components of $(M, \hat{g}(l')) \setminus A_1$ by $\tilde{X}_1$ and $\tilde{Y}_1$ so that $\tilde{X}_1 \supset i_X(X)$ and $\tilde{Y}_1 \supset i_Y(Y)$. Note that $\tilde{X}_0 \subset \tilde{X}_1$, $\tilde{Y}_0 \supset \tilde{Y}_1$. Since $\varphi(\tilde{Y}_1 \cup A_1) \subset \varphi(\tilde{Y}_0) \subset X$, we can define $i_X \circ \varphi(A_1)$ and $i_X \circ \varphi(\tilde{Y}_1)$. Denote them by A_1^1 and $\tilde{Y}_1^1$, respectively. Note that $(M, \hat{g}(l')) \setminus A_1^1$ consists of two connected components $\tilde{Y}_1^1$ and $(M, \hat{g}(l')) \setminus (\tilde{Y}_1^1 \cup A_1^1)$. See Figure 2.

(3.1.1): $\varphi(A_1^1) \subset X$ *and* $\varphi((M, \hat{g}(l')) \setminus (\tilde{Y}_1^1 \cup A_1^1)) \subset X$. In this case, $\varphi(\tilde{Y}_1^1) \supset ((M, \hat{g}_l) \setminus X) = \hat{Y}$. By the argument in $(3.k.1)$ below, we see that this case does not occur.

(3.1.2): $\varphi(A_1^1) \subset Y$. By the argument in $(3.k.2)$ below, we see that this contradicts our assumption.

(3.1.3): $\varphi(A_1^1) \subset X$ *and* $\varphi(\tilde{Y}_1^1) = \varphi \circ i_X \circ \varphi(\tilde{Y}_1) \subset X$. Since $S_2 \subset \tilde{Y}_1$, we can define $(i_X \circ \varphi)^2$ on a neighborhood of S_2. As in (3.0.3), the conformal embedding $\varphi \circ (i_X \circ \varphi)^2 : $ (a neighborhood of S_2) $\longrightarrow (M, \hat{g}_l)$ satisfies the assumption of Lemma

3.3, and hence we can take $A_2 \subset S_2$ as in the lemma. Denote $(i_X \circ \varphi)^2(A_2)$ by A_2^2, and two connected components of $(M, \hat{g}(l')) \setminus A_2$ by $\tilde{X}_2$ and $\tilde{Y}_2$ so that $\tilde{X}_2 \supset i_X(X)$ and $\tilde{Y}_2 \supset i_Y(Y)$. Since $\varphi \circ i_X \circ \varphi(\tilde{Y}_2) \subset \varphi \circ i_X \circ \varphi(\tilde{Y}_1) \subset X$, $(i_X \circ \varphi)^2$ is defined on $\tilde{Y}_2$. Let $\tilde{Y}_2^2 = (i_X \circ \varphi)^2(\tilde{Y}_2)$. Then $(M, \hat{g}(l')) \setminus A_2^2$ has two connected components $\tilde{Y}_2^2$ and $(M, \hat{g}(l')) \setminus (\tilde{Y}_2^2 \cup A_2^2)$. In general, if $\varphi \circ (i_X \circ \varphi)^j(\text{a neighborhood of } S_k) \subset X$ holds for j with $0 \le j \le k - 1$, then $(i_X \circ \varphi)^k$ is defined on a neighborhood of S_k and the conformal embedding

$$\varphi \circ (i_X \circ \varphi)^k : (\text{a neighborhood of } S_k) \longrightarrow (M, \hat{g}_l)$$

satisfies the assumption of Lemma 3.3. Thus we can take $A_k \subset S_k$ so that $\varphi \circ (i_X \circ \varphi)^k(A_k)$ is contained in either X or Y. Denote two connected components of $(M, \hat{g}(l')) \setminus A_k$ by $\tilde{X}_k$ and $\tilde{Y}_k$ so that $\tilde{X}_k \supset i_X(X)$ and $\tilde{Y}_k \supset i_Y(Y)$. Define A_k^j by

$$A_k^j = (i_X \circ \varphi)^j(A_k)$$

and let A_k^0 be A_k. If we can take A_j and A_k with $j \le k$, then $\tilde{Y}_j \supset \tilde{Y}_k$. Define $\tilde{Y}_k^j$ by

$$\tilde{Y}_k^j = (i_X \circ \varphi)^j(\tilde{Y}_k)$$

and let $\tilde{Y}_k^0$ be $\tilde{Y}_k$ when $(i_X \circ \varphi)^j$ is defined on $\tilde{Y}_k$, i.e., $\varphi \circ (i_X \circ \varphi)^i(\tilde{Y}_k) \subset X$ for i with $0 \le i \le j - 1$. If A_k^j and $\tilde{Y}_k^j$ are defined, then $(M, \hat{g}(l')) \setminus A_k^j$ has two connected components and they are $\tilde{Y}_k^j$ and $(M, \hat{g}(l')) \setminus (\tilde{Y}_k^j \cup A_k^j)$. Note that $(\tilde{Y}_k^j \cup A_k^j) \subset \tilde{X}_k$ in this case, since $\varphi(A_k^{j-1} \cup \tilde{Y}_k^{j-1}) \subset X$. In the case that A_k^k and $\tilde{Y}_k^k$ are defined, $\varphi(A_k^k)$ is contained in either X or Y by the definition of A_k, and there are three possibilities:

(1) $\varphi(A_k^k) \subset X$ and $\varphi(\tilde{Y}_k^k) \supset \hat{Y}$.

(2) $\varphi(A_k^k) \subset Y$.

(3) $\varphi(A_k^k) \subset X$ and $\varphi(\tilde{Y}_k^k) \subset X$.

In the case $k = 0$ and either (1) or (2) holds, we have seen, in (3.0.1) and (3.0.2), that this contradicts our assumption. If $k = 0$ and (3) holds, A_1^1 and $\tilde{Y}_1^1$ are defined and they satisfy (1), (2) or (3) with $k = 1$ as we have seen in (3.0.3). If either (1) or (2) holds, as we have mentioned in (3.1.1) and (3.1.2), then we can derive a contradiction by (3.k.1) or (3.k.2) below with $k = 1$. If (3) holds, A_2^2 and $\tilde{Y}_2^2$ are defined. In general, if (3) holds for k with $0 \le k \le j_0 - 1$, then, since $\tilde{Y}_k \supset \tilde{Y}_{j_0} \cup S_{j_0}$ and $\varphi(\tilde{Y}_k^k) \subset X$, $\tilde{Y}_{j_0}^j$ and $A_{j_0}^j$ are defined for $0 \le j \le j_0$ and one of the above three possibilities holds for $k = j_0$. Let k' be the smallest number for which (3) does not hold. Then (1) or (2) holds for $k = k'$. We have already seen the case $k' = 1$. In the case when (1) holds for $k' \ge 2$, since $\varphi(A_{k'}^j \cup \tilde{Y}_{k'}^j) \subset \varphi(\tilde{Y}_j^j) \subset X$, we can derive a contradiction by (3.k.1) below. In the case when (2) holds for $k' \ge 2$, we can derive a contradiction by (3.k.2) below.

Now assume that (3) holds for any k ($0 \le k \le k_0$). Since $\tilde{Y}_k \supset \tilde{Y}_{k_0} \cup A_{k_0}$ for k, $0 \le k \le k_0 - 1$, we have $\varphi \circ (i_X \circ \varphi)^k(\tilde{Y}_{k_0} \cup A_{k_0}) \subset X$ for any k. Thus

$$\tilde{X}_{k_0} \supset \bigcup_{k=0}^{k_0} (i_X \circ \varphi)^{k+1}(\tilde{Y}_{k_0} \cup A_{k_0}) = \bigcup_{k=1}^{k_0+1} (\tilde{Y}_{k_0}^k \cup A_{k_0}^k).$$

Since $\varphi(\tilde{Y}_{k_0} \cup A_{k_0}) \subset X$, $i_X \circ \varphi(\tilde{Y}_{k_0} \cup A_{k_0}) = (\tilde{Y}_{k_0}^1 \cup A_{k_0}^1)$ is contained in $\tilde{X}_{k_0}$. Thus $(\tilde{Y}_{k_0}^1 \cup A_{k_0}^1) \cap (\tilde{Y}_{k_0} \cup A_{k_0}) = \emptyset$ and hence $\varphi(\tilde{Y}_{k_0}^1 \cup A_{k_0}^1) \cap \varphi(\tilde{Y}_{k_0} \cup A_{k_0}) = \emptyset$. Since $\varphi(\tilde{Y}_{k_0}^1 \cup A_{k_0}^1) \subset X$, $(\tilde{Y}_{k_0}^2 \cup A_{k_0}^2) = i_X \circ \varphi(\tilde{Y}_{k_0}^1 \cup A_{k_0}^1)$ is contained in $\tilde{X}_{k_0}$, and hence
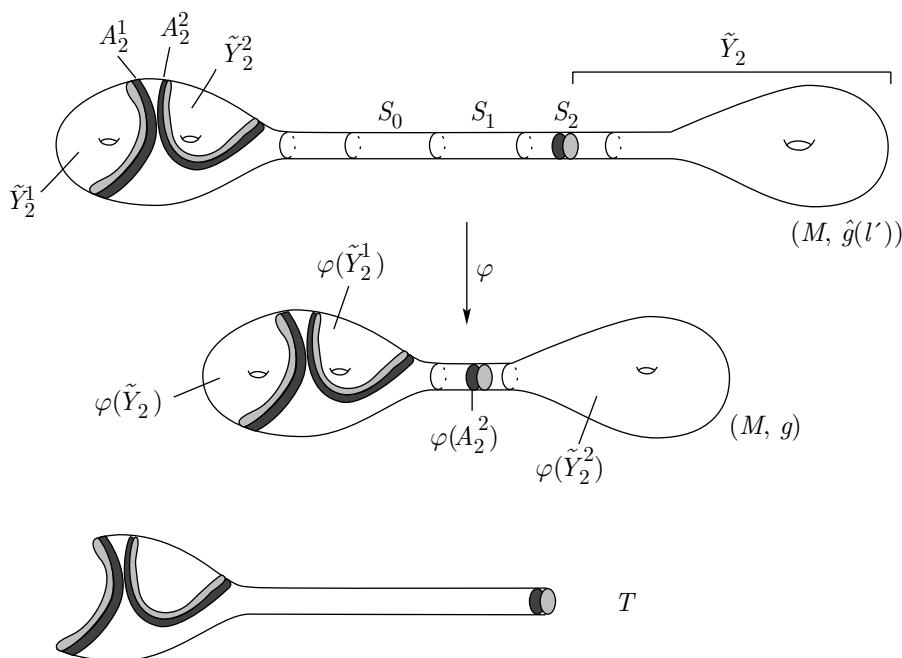
FIGURE 3

$\tilde{Y}_{k_0}^2 \cup A_{k_0}^2$, $\tilde{Y}_{k_0}^1 \cup A_{k_0}^1$ and $\tilde{Y}_{k_0} \cup A_{k_0}$ are disjoint. Proceeding this way, we see that $\tilde{Y}_{k_0}^k \cup A_{k_0}^k$ $(k = 0, \ldots, k_0 + 1)$ are mutually disjoint. Thus $\tilde{Y}_{k_0}^k \cup A_{k_0}^k$ is contained in $\tilde{X}_{k_0}$ for $k = 1, \ldots, k_0 + 1$, and we regard them as subsets of (M_1, g_1). Gluing disks to $\tilde{Y}_{k_0}^k \cup A_{k_0}^k$ $(k = 1, \ldots, k_0+1)$ and $(M_1, g_1) \setminus (\bigcup_{k=1}^{k_0+1} \tilde{Y}_{k_0}^k)$, we obtain compact conformally flat manifolds $M_2^{(k)}$ $(k = 1, \ldots, k_0 + 1)$ and M' such that

$$M_1 = M_2^{(1)} \sharp \ldots \sharp M_2^{(k_0+1)} \sharp M'$$

and $\pi_1(M_2^{(k)}) \cong \pi_1(M_2)$ as in (3.0.1). Then, by van Kampen's theorem and the Gruško-Neumann theorem, $d(M_1) = (k_0+1)d(M_2)+d(M')$ holds. This contradicts our definition of k_0. Thus there exists k' $(0 \le k' \le k_0)$ such that either (1) or (2) holds.

(3.k.1): $\varphi(A_k^j \cup \tilde{Y}_k^j) \subset X$ for $j = 0, \ldots, k-1$, $\varphi(A_k^k) \subset X$ and $\varphi(\tilde{Y}_k^k) \supset \hat{Y}$. Define T by

$$T = (M, \hat{g}(l')) \setminus \bigcup_{j=0}^{k} \tilde{Y}_k^j = \tilde{X}_k \cup A_k \setminus \bigcup_{j=1}^{k} \tilde{Y}_k^j.$$

Since $\varphi(\tilde{Y}_k^j \cup A_k^j) \subset X$ for $j = 0, \ldots, k-1$, $\tilde{Y}_k^j \cup A_k^j$ $(j = 0, \ldots, k)$ are mutually disjoint as in (3.1.3). Then, clearly, T is connected and T contains A_k^j for $j = 0, \ldots, k$ (see Figure 3). Since $\varphi(\tilde{Y}_k^j)$ is contained in X for $j = 0, \ldots, k-1$ and

$\varphi(\tilde{Y}_k^k)$ contains $\hat{Y}$, we obtain

$$\varphi(T) = (M, \hat{g}_l) \setminus \bigcup_{j=0}^{k} \varphi(\tilde{Y}_k^j) \subset X \setminus \bigcup_{j=0}^{k-1} \varphi(\tilde{Y}_k^j),$$

and hence $i_X \circ \varphi$ is defined on T. Moreover, since

$$i_X \circ \varphi(T) \subset i_X(X) \setminus (\bigcup_{j=0}^{k-1} i_X \circ \varphi(\tilde{Y}_k^j)) \subset \tilde{X}_k \setminus \bigcup_{j=1}^{k} \tilde{Y}_k^j,$$

$i_X \circ \varphi$ defines a conformal embedding $i_X \circ \varphi : T \longrightarrow T$. We denote this conformal embedding by ξ. Define $\hat{T}$ by

$$\hat{T} = T \setminus \bigcup_{j=0}^{k} A_k^j.$$

Then

$$\varphi(\hat{T}) = (M, \hat{g}_l) \setminus \bigcup_{j=0}^{k} \varphi(\tilde{Y}_k^j \cup A_k^j) \subset X \setminus \bigcup_{j=0}^{k-1} \varphi(\tilde{Y}_k^j \cup A_k^j).$$

Thus we have

$$\xi(\hat{T}) = i_X \circ \varphi(\hat{T}) \subset \tilde{X}_k \setminus \bigcup_{j=0}^{k-1} i_X \circ \varphi(\tilde{Y}_k^j \cup A_k^j) = \hat{T}.$$

Since A_k^k is contained in T, $\xi^j(A_k^k)$ is defined for $j \geq 0$ and $\xi^j(A_k^k) = A_k^{k+j}$ is contained in T. Clearly $T \setminus \xi(A_k^k)$ has at least two connected components. Since A_k^j $(j = 0, \ldots, k)$ are mutually disjoint, so are $A_k^{j+1} = \xi(A_k^j)$ $(j = 0, \ldots, k)$. In particular, A_k^{k+1} and A_k^j $(j = 1, \ldots, k)$ are mutually disjoint. Since $\varphi(A_k^k)$ is contained in X, $\xi(A_k^k) = i_X \circ \varphi(A_k^k)$ is contained in $i_X(X)$ and hence A_k^{k+1} and A_k are disjoint. This shows that

$$A_k^{k+1} \subset T \setminus \bigcup_{j=0}^{k} A_k^j = \hat{T}.$$

Thus $\xi^j(A_k^{k+1}) \subset \xi^j(\hat{T}) \subset \hat{T}$ holds for $j \geq 0$. Note that $\hat{T}$ is a closed set and hence $\hat{T}$ contains the closure of $\xi^j(A_k^{k+1})$ for $j \geq 0$. Then it is easy to see that $T \setminus \xi^j(A_k^{k+1})$ has exactly two connected components. The set $T \setminus A_k^{k+1}$ is described as follows: Since $(M, \hat{g}(l')) \setminus A_k^k$ has two connected components $\tilde{Y}_k^k$ and $(M, \hat{g}(l')) \setminus (\tilde{Y}_k^k \cup A_k^k)$, $(M, \hat{g}_l) \setminus \varphi(A_k^k)$ has two connected components $(M, \hat{g}_l) \setminus \varphi(\tilde{Y}_k^k \cup A_k^k)$ and $\varphi(\tilde{Y}_k^k)$. Note that $(M, \hat{g}_l) \setminus \varphi(\tilde{Y}_k^k \cup A_k^k)$ is contained in X, since $\varphi(\tilde{Y}_k^k) \supset \hat{Y}$ and $\varphi(A_k^k) \subset X$. Then two connected components of $(M, \hat{g}(l')) \setminus A_k^{k+1}$, say L and N, satisfy $L = i_X((M, \hat{g}_l) \setminus \varphi(\tilde{Y}_k^k \cup A_k^k)) \subset \tilde{X}_k$ and $N \supset A_k \cup \tilde{Y}_k$. Since $\tilde{Y}_k^j \cup A_k^j$ $(j = 0, \ldots, k)$ are mutually disjoint,

$$L \supset \bigcup_{j=0}^{k-1} i_X \circ \varphi(\tilde{Y}_k^j \cup A_k^j) = \bigcup_{j=1}^{k} (\tilde{Y}_k^j \cup A_k^j).$$

Denote two connected components of $T \setminus A_k^{k+1}$ by T_0 and T_0' so that $T_0 = T \cap L$ and $T_0' = T \cap N$. Note that T_0' contains A_k. Since $L \cup A_k^{k+1} = i_X((M, \hat{g}_l) \setminus \varphi(\tilde{Y}_k^k))$,

$$\xi(T) = i_X((M, \hat{g}_l) \setminus \bigcup_{j=0}^{k} \varphi(\tilde{Y}_k^j))$$

$$= (L \cup A_k^{k+1}) \setminus \bigcup_{j=1}^{k} \tilde{Y}_k^j$$

$$= (L \cup A_k^{k+1}) \cap ((M, \hat{g}_l) \setminus \bigcup_{j=0}^{k} \tilde{Y}_k^j).$$

Since $L \cap A_k^{k+1}$ is contained in $\tilde{X}_k$, we see that

$$\xi(T) = (L \cup A_k^{k+1}) \cap T$$

$$= T_0 \cup A_k^{k+1}.$$

In particular, $T_0 \supset \xi(\hat{T})$ and $T_0' = T \setminus \xi(T)$ hold. Denote two connected components of $T \setminus \xi^j(A_k^{k+1})$ by T_j and T_j' so that $T_j \supset \xi^j(T)$ and $T_j' \supset \xi^j(T_0')$. Then we see that

$$T_j \cup \xi^j(A_k^{k+1}) \supset \xi^{j+1}(T),$$

$$T_j \supset \xi^{j+1}(\hat{T}),$$

$$T_j' \supset A_k^j,$$

$$T_j' \supset \xi^j(T_0') = \xi^j(T) \setminus \xi^{j+1}(T)$$

hold. Take $\xi^i(A_k^{k+1}) = \xi^{k+1}(A_k^i)$ and $\xi^j(A_k^{k+1}) = \xi^{k+1}(A_k^j)$ for $0 \le i < j \le k$. Since A_k^j and A_k^i are disjoint, so are $\xi^{k+1}(A_k^j)$ and $\xi^{k+1}(A_k^i)$. Since $\xi^j(A_k^{k+1}) \subset \xi^j(\hat{T}) \subset \xi^{i+1}(\hat{T}) \subset T_i$, either $T_j \cup \xi^j(A_k^{k+1}) \subset T_i$ or $T_j' \cup \xi^j(A_k^{k+1}) \subset T_i$ holds. Assume $T_j \cup \xi^j(A_k^{k+1}) \subset T_i$. Then $T_j' \supset T_i' \cup \xi^i(A_k^{k+1})$. But since $\xi^i(A_k^{k+1}) = \xi^j(A_k^{k+i-j+1})$ and $1 \le k + i - j + 1 \le k$, $A_k^{k+i-j+1} = \xi(A_k^{k+i-j})$ is contained in $\xi(T)$. This shows $\xi^i(A_k^{k+1}) \subset \xi^j \circ \xi(T) = \xi^j(T_0 \cup A_k^{k+1}) \subset (T_j \cup \xi^j(A_k^{k+1}))$. A contradiction. Thus $T_j' \cup \xi^j(A_k^{k+1})$ is contained in T_i. In particular, we have

$$\begin{cases} (T_i' \cup \xi^i(A_k^{k+1})) \cap (T_j' \cup \xi^j(A_k^{k+1})) = \emptyset & \text{if } i \ne j, \\ T_i' \supset A_k^j & \text{if } i = j. \end{cases}$$

Since $\xi^{k+1}(T) \subset \xi^{j+1}(T) \subset T_j \cup \xi^j(A_k^{k+1})$ holds for $j = 0, \ldots, k$, $\bigcup_{j=0}^{k} T_j'$ is contained in $Y \setminus \xi^{k+1}(T)$. On the other hand, since T_j' contains $\xi^j(T) \setminus \xi^{j+1}(T)$ and $T \setminus \xi^{k+1}(T) = \bigcup_{j=0}^{k}(\xi^j(T) \setminus \xi^{j+1}(T))$, $T \setminus \xi^{k+1}(T)$ is contained in $\bigcup_{j=0}^{k} T_j'$. Thus, $T \setminus \xi^{k+1}(T) = \bigcup_{j=0}^{k} T_j'$. Identifying $S(\alpha_0)$ defined in (3.0.1) with each A_k^j, $j = 1, \ldots k$, we can glue $E(\alpha_0) \cup S(\alpha_0)$ to T and obtain a compact, connected and conformally flat manifold M' as in (3.0.1). Denote $E(\alpha_0) \cup S(\alpha_0)$ glued to A_k^j by D_j (see Figure 4). By the discussion above, we see that

$$\begin{cases} M' \setminus \xi^{k+1}(\hat{T}) = \bigcup_{j=0}^{k}(\xi^{k+1}(A_k^j) \cup T_j' \cup D_j), \\ (\xi^{k+1}(A_k^j) \cup T_j' \cup D_j) \cap (\xi^{k+1}(A_k^i) \cup T_i' \cup D_i) = \emptyset & \text{if } i \ne j. \end{cases}$$

Define $D_{j,m}$ for $j = 0, \ldots, k$ and $m = 0, \ldots, 3$ by $D_{j,0} = D_j \setminus A_k^j$, $D_{j,1} = D_j$, $D_{j,2} = T_j' \cup D_j$ and $D_{j,3} = \xi^{k+1}(T) \cup D_{j,2}$. Then we see that $D_{j,0} \subset D_{j,1} \subset D_{j,2} \subset D_{j,3}$ and

$$M' \setminus T = \bigcup_{j=0}^{k} D_{j,0},$$

$$M' \setminus \hat{T} = \bigcup_{j=0}^{k} D_{j,1},$$

$$M' \setminus \xi^{k+1}(T) = \bigcup_{j=0}^{k} D_{j,2},$$

$$M' \setminus \xi^{k+1}(\hat{T}) = \bigcup_{j=0}^{k} D_{j,3}.$$

Applying the same argument as in (3.0.1) to each j, we see that ξ^{k+1} is extended to a conformal transformation η on M'. Since $\eta(D_{j,0}) = D_{j,2}$ and $\eta(D_{j,1}) = D_{j,3}$ by our construction, η is an essential conformal transformation on M' as we have seen in (3.0.1). Therefore M' is conformal to the standard n-sphere $S^n(1)$. Note that $D_{j,m}$ is a closed round n-disk if $m = 0, 2$ and an open round n-disk if $m = 1, 3$. Thus, for some $\beta_0 > 0$, we may regard $M' \setminus (D_{0,1} \cup D_{1,1})$ as the closure of $S(\beta_0)$ defined in the same way as $S(\alpha_0)$ in (3.0.1). Since $\eta(M' \setminus (D_{0,1} \cup D_{1,1})) = M' \setminus (D_{0,3} \cup D_{1,3})$, and $D_{j,3} \setminus \xi^{k+1}(A_k^j) \supset D_{j,1}$, there exists $\beta_0' < \beta_0$ such that $D_{0,3} \supset E(-\beta_0') \supset$(the closure of $D_{0,1}$) $= E(-\beta_0)$ and $D_{1,3} \supset E(\beta_0') \supset$(the closure of $D_{1,1}$) $= E(\beta_0)$, that is, $\eta(M' \setminus (D_{0,1} \cup D_{1,1})) = \eta(S(\beta_0))$ is contained in $S(\beta_0')$. But by Lemma 4.13 in the next section, such an η does not exist. Thus case (1) of (3.1.3) does not occur if $k \geq 1$. See Figure 4.

(3.k.2): $\varphi(A_k^k) \subset Y$. Since $(M, \hat{g}(l')) \setminus A_k^k$ has two connected components $(M, \hat{g}(l')) \setminus (\tilde{Y}_k^k \cup A_k^k)$ and $\tilde{Y}_k^k$, either $\varphi(\tilde{Y}_k^k)$ or $\varphi((M, \hat{g}(l')) \setminus (A_k^k \cup \tilde{Y}_k^k))$ is contained in Y. If $\varphi(\tilde{Y}_k^k)$ is contained in Y, we have a conformal embedding $\varphi \circ (i_X \circ \varphi)^k : \tilde{Y}^k \longrightarrow Y$. If $\varphi((M, \hat{g}(l')) \setminus (\tilde{Y}_k^k \cup A_k^k))$ is contained in Y, then $\varphi(\tilde{Y}_k)$ is contained in Y, since $A_k^k \cup \tilde{Y}_k^k \subset \tilde{X}_k$. We can derive a contradiction in each case as in (3.0.1).

By the above discussion, we conclude that there is no conformal diffeomorphism between $(M, \hat{g}_l)$ and $(M, \hat{g}(l'))$. This completes the proof of Theorem 3.1.

Recall that Lemma 3.3 is also true for noncompact manifolds. Actually, a part of the proof of Theorem 3.1 also works for noncompact manifolds. Let (M_1, g_1) and (M_2, g_2) be conformally flat manifolds, possibly noncompact, which are not diffeomorphic to S^n nor $\mathbf{R}^n$. Let $M = M_1 \sharp M_2$ and $(M, \hat{g}_l)$ be as above. We do not assume that $d(M_1) \geq d(M_2)$ here, and we define $(M, \hat{g}(l'))$ by setting $k_0 = 0$ (so $(M, \hat{g}(l'))$ contains a subset isometric to $(0, 2l + l') \times S^{n-1}(1)$). Other notations are the same as above.

Then we have two conformally flat Riemannian manifolds, and there is a diffeomorphism φ_0 between them which maps $i_X(\hat{X})$ (resp. $i_Y(\hat{Y})) \subset (M, \hat{g}(l'))$ to $\hat{X}$ (resp. $\hat{Y}$).

Theorem 3.4. *If $l' > C(n, l)$, where $C(n, l)$ is the constant in Lemma 3.3, then there is no conformal diffeomorphism between $(M, \hat{g}(l'))$ and $(M, \hat{g}_l)$ isotopic to φ_0.*
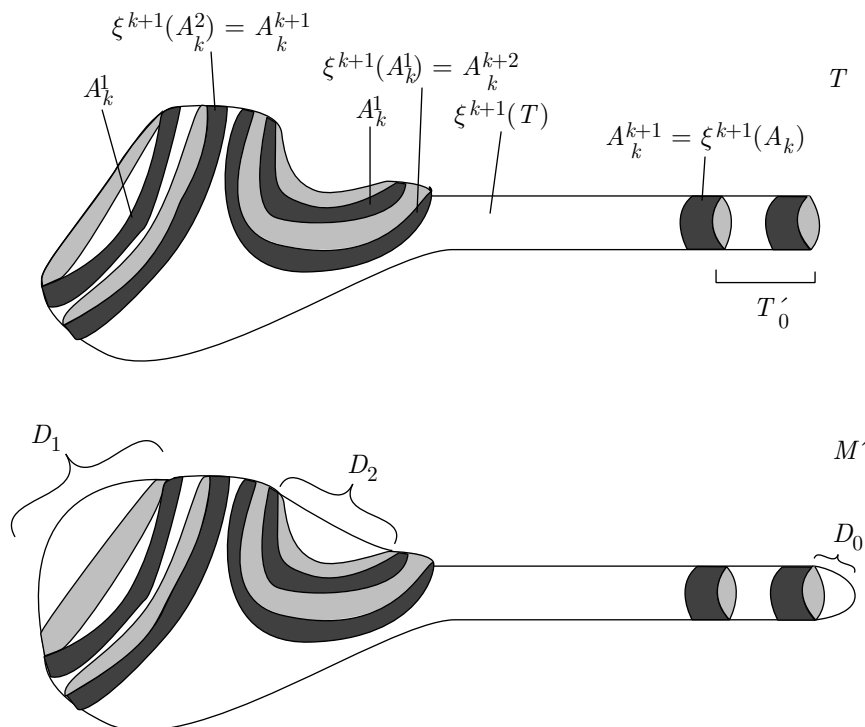
$$\xi^{k+1}(A_k^2) = A_k^{k+1}$$

$$\xi^{k+1}(A_k^1) = A_k^{k+2}$$

A_k^1

A_k^1

$\xi^{k+1}(T)$

T

$A_k^{k+1} = \xi^{k+1}(A_k)$

T_0'

D_1

D_2

M'

D_0

FIGURE 4

In other words, $\hat{g}_l$ and $\hat{g}(l')$ represent distinct elements in the Teichmüller space of flat conformal structures on M.

Proof. Assume that there is a conformal diffeomorphism $\varphi : (M, \hat{g}(l')) \longrightarrow (M, \hat{g}_l)$, which is isotopic to φ_0. By Lemma 3.3, there is a subset A_0 of S_0 such that $\varphi(A_0)$ is contained in either X or Y. We may assume $\varphi(A_0) \subset X$ by interchanging M_1 and M_2, if necessary. Since φ is isotopic to φ_0, $\varphi(\tilde{X}_0)$ is contained in X. Then $i_X \circ \varphi$ is defined on $\tilde{X}_0 \cup A_0$ and $i_X \circ \varphi(A_0 \cup \tilde{X}_0) \subset \tilde{X}$. Since $\varphi \circ \varphi_0^{-1}$ is isotopic to the identity map, it is easy to see that $X \setminus \varphi(\tilde{X}_0 \cup A_0)$ must be compact. Therefore, $\tilde{X}_0 \setminus i_X \circ \varphi(\tilde{X}_0 \cup A_0)$ is also compact. As in (3.0.1), we can regard $\tilde{X}_0 \cup A_0$ as a subset of (M_1, g_1). Through this, define D_0, D_1, D_2 and D_3 as in (3.0.1). Let M_1' and M_2' be as in (3.0.1). Then, since $\tilde{X}_0 \setminus i_X \circ \varphi(\tilde{X}_0 \cup A_0)$ is compact, M_2' is compact and simply connected by the same reasoning as in (3.0.1). Hence M_2' is conformally equivalent to S^n. Thus we can extend the conformal diffeomorphism $i_X \circ \varphi : \tilde{X}_0 \cup A_0 \longrightarrow i_X \circ \varphi(\tilde{X}_0 \cup A_0)$ to a global conformal transformation η on (M_1, g_1), and η is an essential conformal transformation as in (3.0.1). If M_1 is compact, then M_1 must be diffeomorphic to S^n, and this contradicts our assumption. Therefore, M_1 is noncompact. Let $(\tilde{M}_1, \tilde{g}_1)$ be the Riemannian universal covering of (M_1, g_1). Fix a lift $\tilde{D}_i$ of D_i, $i = 0$, 1, 2 and 3, so that $\tilde{D}_i \subset \tilde{D}_j$ for $i < j$. Then there is a conformal transformation $\tilde{\eta}$ on $(\tilde{M}_1, \tilde{g}_1)$, which is a lift of η, such that $\tilde{\eta}(\tilde{D}_1) = \tilde{D}_3$. Fix a developing map Φ of $(\tilde{M}_1, \tilde{g}_1)$. Since the $\tilde{D}_i$'s are conformally equivalent to a round ball of S^n, Φ is injective on $\tilde{D}_3$. Then, it is clear that the conformal

transformation ξ of S^n satisfying $\Psi \circ \tilde{\eta} = \xi \circ \Psi$ is a loxodromic transformation of S^n whose repulsive fixed point q_1 is in $\Psi(\tilde{D}_3)$. Since $\Psi \circ \tilde{\eta}^k = \xi^k \circ \Psi$, $(\tilde{M}_1, \tilde{g}_1)$ contains a subset conformally equivalent to $\mathbf{R}^n \cong S^n \setminus q_2$, where q_2 is the attractive fixed point of ξ. If this set U is a proper subset of $\tilde{M}_1$, then the boundary of U is mapped locally homeomorphically onto a point q_2 by Ψ. Therefore, $\tilde{M}_1$ is diffeomorphic to S^n. But here, since we assume M_1 to be noncompact, $(\tilde{M}_1, \tilde{g}_1)$ must be conformally equivalent to $\mathbf{R}^n$. On the other hand, the set U can be considered as a lift of the limit of $\eta^k(D_1)$ as $k \to \infty$, thus (M_1, g_1) itself must be conformal to $\mathbf{R}^n$. A contradiction. This completes the proof. $\qquad\square$

Remark. If M_1 or M_2 is diffeomorphic to either S^n or $\mathbf{R}^n$, then $(M, \hat{g}_l)$ and $(M, \hat{g}(l'))$ may be conformally equivalent. This can be seen as in Remark 2 following Theorem 3.1.

The same argument which shows Corollary 3.2 leads to the following corollary.

Corollary 3.5. *Let $M = M_1 \sharp M_2$, where M_1 and M_2 are conformally flat, possibly noncompact, manifolds and each of them is diffeomorphic to neither S^n nor $\mathbf{R}^n$. Then the Teichmüller space of flat conformal structures on M is noncompact.*

4. Proof of lemmas

To prove Lemma 3.3, we need some basic facts on a conformal invariant called the *modulus*, which is related to the theory of quasi-conformal mappings (see [21] for a detailed exposition on this subject). Let (M, g) be a Riemannian manifold and Δ an interval. A continuous map $\gamma : \Delta \longrightarrow M$ is called a *path*. A path γ is called a *closed path* if Δ is a closed interval. If Δ' is a subinterval of Δ, $\gamma|_{\Delta'}$ is called a *subpath* of γ.

Definition 4.1. Let $\Delta = [a, b]$ and $A = \{a_0 < a_1 < \cdots < a_{j+1}; \ a_0 = a, \ a_{j+1} = b\}$ a subdivision of Δ. Denote the distance between p and q with respect to a metric g by $\mathrm{dist}_g(p, q)$. The length $l(\gamma)$ of γ is defined by

$$l(\gamma) = \sup_A \sum_{i=0}^{j} \mathrm{dist}_g(\gamma(a_i), \gamma(a_{i+1})),$$

where the supremum is taken over all subdivisions of Δ.

Definition 4.2. A path γ is said to be *rectifiable* if the length of γ is finite. And a path γ is called *locally rectifiable* if every closed subpath of γ is rectifiable.

If γ is rectifiable, then γ is parametrized by arclength. In what follows, we assume that a rectifiable path is parametrized by arclength, and denote this parameter by s. Then, for a rectifiable path γ, the line integral along γ makes sense. These notions are also defined for a locally rectifiable path in a natural way. Let Γ be a path family.

Definition 4.3. A Borel function $\rho : M \longrightarrow [0, \infty]$ is called an *admissible function* of Γ, if, for any locally rectifiable path γ contained in Γ, ρ satisfies

$$\int_\gamma \rho \, ds \geq 1.$$

The set of all admissible functions of Γ is denoted by $\mathcal{F}(\Gamma)$.

Definition 4.4. For a path family Γ, the *modulus* $\mathcal{M}(\Gamma)$ is defined by

$$\mathcal{M}(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_M \rho^n dV_g,$$

where dV_g denotes the volume element of g.

Let E, F and G be Borel sets of (M, g). Define $\Delta(E, F; G)$ by

$$\Delta(E, F; G) = \{\text{closed non-constant paths } \gamma; \ \gamma(0) \in E,$$
$$\gamma(l(\gamma)) \in F, \gamma(t) \in G \text{ for } t \in (0, l(\gamma))\}.$$

The following are obvious consequences of the above definitions. For the proofs, see [21].

Proposition 4.5. *If Γ is a subfamily of Γ', then $\mathcal{M}(\Gamma) \leq \mathcal{M}(\Gamma')$.*

Proposition 4.6. *If every path γ contained in Γ has a subpath belonging to Γ', then $\mathcal{M}(\Gamma) \leq \mathcal{M}(\Gamma')$.*

Proposition 4.7.

$$\mathcal{M}(\Delta(E, F; G)) = \inf_{\rho \in \mathcal{F}(\Delta(E,F;G))} \int_G \rho^n dV_g.$$

Proposition 4.8. *Let G be an open subset of a Riemannian manifold (M, g) and $\Gamma = \Delta(E, F; G)$ for subsets E and F of G. Assume that there exists a conformal embedding φ of $(G, g|_G)$ into a Riemannian manifold (N, h). Then $\mathcal{M}(\Gamma) = \mathcal{M}(\varphi(\Gamma))$.*

Proposition 4.8 asserts the conformal invariance of $\mathcal{M}(\Gamma)$. In the proof of Lemma 3.3 below, the above properties of the conformal invariant $\mathcal{M}(\Gamma)$ are used frequently.

Proof of Lemma 3.3. We use the same notations as in the proof of Theorem 3.1. Denote $X \cap Y$ by S. Then S is isometric to $(0, l) \times S^{n-1}(1)$. If $\psi(\{0\} \times S^{n-1}(1))$ is contained in X, then there exists $\delta > 0$ such that $\psi((0, \delta) \times S^{n-1}(1))$ is contained in X. Thus, there exist α and β such that $[\alpha, \beta] \subset (0, \delta)$, and hence we get a subset A of S as in Lemma 3.3. Similarly, if one of $\psi(\{0\} \times S^{n-1}(1)) \subset Y$, $\psi(\{l'\} \times S^{n-1}(1)) \subset X$ or $\psi(\{l'\} \times S^{n-1}(1)) \subset Y$ holds, we see that A in Lemma 3.3 exists. Therefore, to prove Lemma 3.3, it suffices to show that if we choose $l' > C(n, l)$ and both $\psi(\{0\} \times S^{n-1}(1)) \cap \hat{X} \neq \emptyset$ and $\psi(\{0\} \times S^{n-1}(1)) \cap \hat{Y} \neq \emptyset$ hold, then either $\psi(\{l'\} \times S^{n-1}(1)) \subset X$ or $\psi(\{l'\} \times S^{n-1}(1)) \subset Y$ holds. Now assume that $\psi(\{0\} \times S^{n-1}(1)) \cap \hat{X} \neq \emptyset$, $\psi(\{0\} \times S^{n-1}(1)) \cap \hat{Y} \neq \emptyset$, $\psi(\{l'\} \times S^{n-1}(1)) \cap \hat{X} \neq \emptyset$ and $\psi(\{l'\} \times S^{n-1}(1)) \cap \hat{Y} \neq \emptyset$ hold. We prove Lemma 3.3 by deriving a contradiction in the case that l' is greater than a certain constant depending only on n and l.

The set S is conformally equivalent to a subset $S(a)$ of $S^n(1)$ defined as in the proof of Theorem 3.1 for some positive real number a. That is, there exists a conformal diffeomorphism $i : S \longrightarrow S(a)$. Note that $\{0\} \times S^{n-1}(1)$ and $\{l'\} \times S^{n-1}(1)$ are totally geodesic hypersurfaces of $(-\epsilon, l' + \epsilon) \times S^{n-1}(1)$. Since ψ and i are conformal embeddings, a connected component T_0 of $i(\psi(\{0\} \times S^{n-1}(1)) \cap S)$ and a connected component $T_{l'}$ of $i(\psi(\{l'\} \times S^{n-1}(1)) \cap S)$ are intersections of round $(n-1)$-spheres of $S^n(1)$ and $S(a)$. Thus, there exist hyperplanes V_0 and $V_{l'}$ of $\mathbf{R}^{n+1}$ such that $T_0 = S(a) \cap V_0$ and $T_{l'} = S(a) \cap V_{l'}$. By our assumption, we may assume
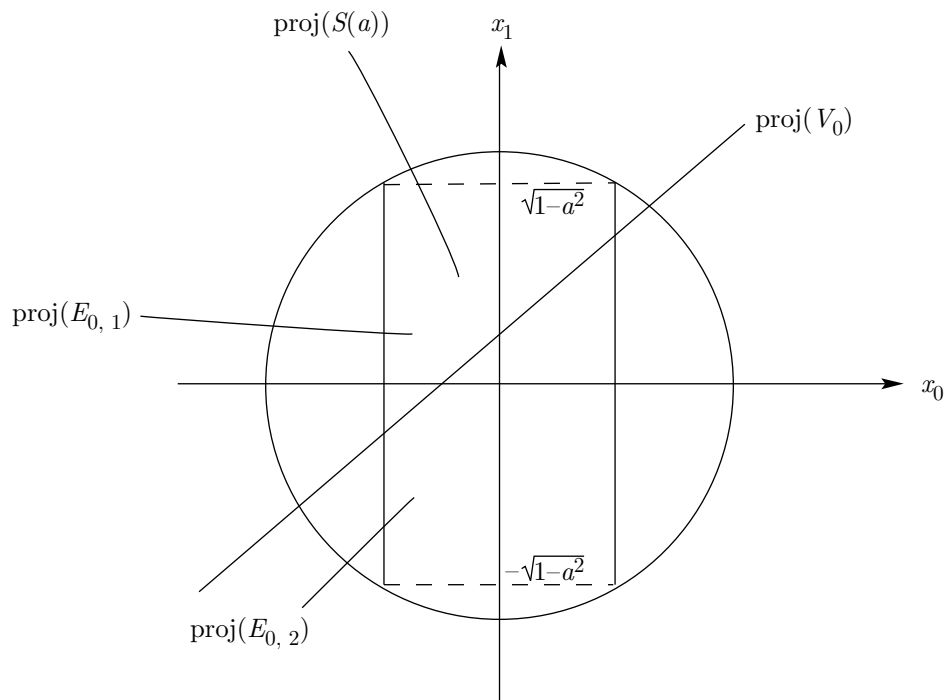
FIGURE 5

that these hyperplanes satisfy

$$(4.9) \quad \begin{cases} V_0 \cap \{x = (x_0, \ldots, x_n) \in S^n(1) \subset \mathbf{R}^{n+1};\ x_0 = -a\} \neq \emptyset \\ V_0 \cap \{x = (x_0, \ldots, x_n) \in S^n(1) \subset \mathbf{R}^{n+1};\ x_0 = a\} \neq \emptyset \\ V_{l'} \cap \{x = (x_0, \ldots, x_n) \in S^n(1) \subset \mathbf{R}^{n+1};\ x_0 = -a\} \neq \emptyset \\ V_{l'} \cap \{x = (x_0, \ldots, x_n) \in S^n(1) \subset \mathbf{R}^{n+1};\ x_0 = a\} \neq \emptyset. \end{cases}$$

Denote two connected components of $S(a) \setminus T_0$ by $E_{0,1}$ and $E_{0,2}$ so that $E_{0,2}$ contains $T_{l'}$. Similarly, denote two connected components of $S(a) \setminus T_{l'}$ by $E_{l',1}$ and $E_{l',2}$ so that $E_{l',2}$ contains T_0. Then $E_{0,2}$ contains $E_{l',1}$, and $E_{l',2}$ contains $E_{0,1}$. Let v be a vector of $\mathbf{R}^{n+1}$ perpendicular to V_0. Then, by (4.9), the x_0-axis and v are linearly independent. Denote the 2-dimensional subspace spanned by the x_0-axis and v by Q. We may assume the x_1-axis is contained in Q. Let $\mathrm{proj} : \mathbf{R}^{n+1} \longrightarrow Q$ be the orthogonal projection. Then $\mathrm{proj}(V_0)$ is a line in Q, and the x_1-coordinate of $\mathrm{proj}(V_0) \cap \mathrm{proj}(S(a))$ takes its value between $-(1-a^2)^{1/2}$ and $(1-a^2)^{1/2}$ (see Figure 5). Fix a positive real number ϵ_0 so that $\epsilon_0 < a$. We may assume $x = (0, 1, 0 \ldots, 0)$ is contained in $E_{0,1}$, and hence

$$B_1 = \{x = (x_0, \ldots, x_n) \in S(a);\ x_1 \geq (1 - (a - \epsilon_0)^2)^{1/2}\}$$

is a round n-disk contained in $E_{0,1}$. We get a round n-disk B_2 contained in $E_{l',1}$ in the same way. Define path families Γ_0 and Γ by

$$\Gamma_0 = \Delta(\psi(\{0\} \times S^{n-1}(1)),\ \psi(\{l'\} \times S^{n-1}(1));\ \psi((-\epsilon, l' + \epsilon) \times S^{n-1}(1))),$$
$$\Gamma = \Delta(T_0, T_{l'}; S(a)).$$

For any $\gamma \in \Gamma$, $i^{-1}(\gamma)$ is a path between $\psi(\{0\} \times S^{n-1}(1))$ and $\psi(\{l'\} \times S^{n-1}(1))$. Since $(M, g) \setminus \psi((0, l') \times S^{n-1}(1))$ has two connected components, there is a subpath of $i^{-1}(\gamma)$ contained in Γ_0. By Proposition 4.6 and Proposition 4.8, we see that

$$(4.10) \qquad \mathcal{M}(\Gamma_0) \geq \mathcal{M}(i^{-1}(\Gamma)) = \mathcal{M}(\Gamma).$$

Define a path family Γ' by

$$\Gamma' = \Delta(B_1, B_2; S(a)).$$

Then, by the definition of B_1 and B_2, every $\gamma' \in \Gamma'$ intersects T_0 and $T_{l'}$. Thus there is a subpath of γ' which belongs to Γ. By Proposition 4.6,

$$(4.11) \qquad \mathcal{M}(\Gamma) \geq \mathcal{M}(\Gamma').$$

Since the center of B_i, $i = 1, 2$, is contained in $S(0) = \{x = (x_0, \ldots x_n) \in S^n(1) \subset \mathbf{R}^{n+1};\ x_0 = 0\}$, we can take the shortest geodesic c between the centers of B_1 and B_2 so that c is contained in $S(0)$. Let p be the midpoint of c and q the antipodal point of p. Note that q is contained in $S(0) \subset S(a)$. Define Γ'' by

$$\Gamma'' = \Delta(B_1, B_2; S(a) \setminus \{q\}).$$

Since $\Gamma'' \subset \Gamma'$, $\mathcal{M}(\Gamma') \geq \mathcal{M}(\Gamma'')$ by Proposition 4.5. Let π_q be the stereographic projection from q. By Proposition 4.8,

$$(4.12) \qquad \mathcal{M}(\Gamma') \geq \mathcal{M}(\Gamma'') = \mathcal{M}(\Delta(\pi_q(B_1),\ \pi_q(B_2);\ \pi_q(S(a) \setminus \{q\}))).$$

We may assume $\pi_q(c)$ corresponds to the y_1-axis of $\mathbf{R}^n = \pi_q(S^n \setminus \{q\})$ and $\pi_q(S(0))$ corresponds to the $(y_1, \ldots, y_{n-1})$-subspace. Let the Euclidean distance between two connected components of the boundaries of $\pi_q(S(a - \epsilon_0))$ be $2r_0$. Then the intersections of the y_n-axis and the boundaries of $\pi_q(S(a - \epsilon_0))$ are $(0, \ldots, 0, \pm r_0)$, and $\pi_q(B_1)$ and $\pi_q(B_2)$ are disks centered at points of y_1-axis and each radius is greater than r_0. Set

$$F = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 = r_0^2\}.$$

Then, for some positive real numbers h_1 and h_2 with $h_1 < h_2$,

$$F \cap \partial(\pi_q(B_1)) = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 = r_0^2,\ y_1 = h_1, h_2\},$$

$$F \cap \partial(\pi_q(B_2)) = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 = r_0^2,\ y_1 = -h_1, -h_2\},$$

where ∂ denotes the boundary of a set. Define F_1, F_2 and G by

$$F_1 = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 \leq r_0^2,\ y_1 = h_1\},$$

$$F_2 = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 \leq r_0^2,\ y_1 = -h_1\},$$

$$G = \{y = (y_1, \ldots, y_n) \in \mathbf{R}^n;\ y_2^2 + \cdots + y_n^2 \leq r_0^2,\ |y_1| \leq |h_1|\}.$$

Then, clearly, $F_1 \subset \pi_q(B_1)$, $F_2 \subset \pi_q(B_2)$ and $G \subset \pi_q(S(a) \setminus \{q\})$ hold and hence $\Delta(F_1, F_2; G)$ is contained in $\Delta(\pi_q(B_1),\ \pi_q(B_2);\ \pi_q(S(a) \setminus \{q\}))$. By our choices of B_1, B_2 and p, each of $\pi_q(B_1)$ and $\pi_q(B_2)$ is tangent to $\partial(\pi_q(S(a - \epsilon_0)))$ at two points, and these two points are contained in the unit closed ball of $\mathbf{R}^n$, which is the image of the closure of the southern hemisphere by π_q (see Figure 6). Since the Euclidean distance between these two tangent points is greater than $2r_0$, we have $|h_1| < 1$. Moreover, since $\mathcal{M}(\Delta(F_1, F_2; G)) = \text{Vol}_{n-1}(F_1)(2h_1)^{1-n}$ (see for example [21, p. 52]), where $\text{Vol}_{n-1}(F_1)$ denotes the $(n-1)$-dimensional Euclidean volume of F_1, we have

$$\mathcal{M}(\Delta(\pi_q(B_1),\ \pi_q(B_2);\ \pi_q(S(a) \setminus \{q\}))) \geq \text{Vol}_{n-1}(F_1)(2h_1)^{1-n} > 2^{1-n} \text{Vol}_{n-1}(F_1)$$
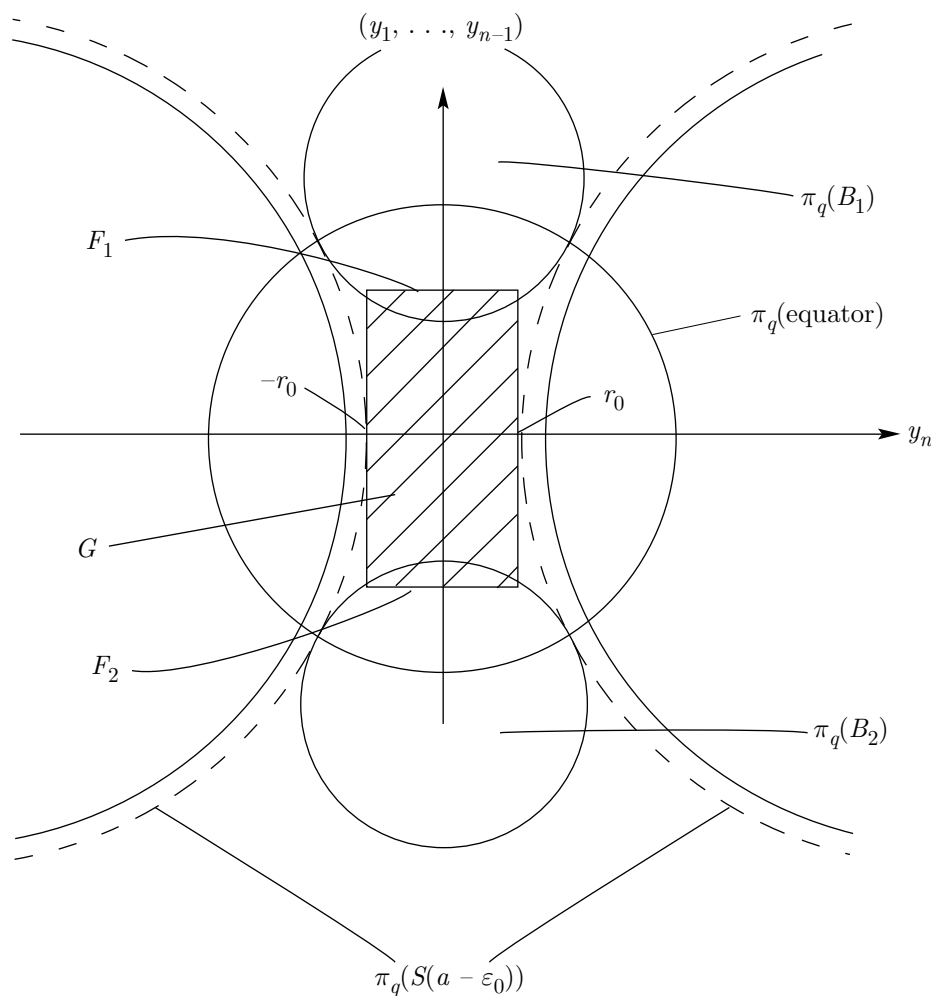
FIGURE 6

by Proposition 4.5. Combining with (4.10), (4.11) and (4.12), we have

$$\mathcal{M}(\Gamma_0) > 2^{1-n} \operatorname{Vol}_{n-1}(F_1).$$

Since the radius r_0 of F_1 is determined by a and ϵ_0 and we can choose ϵ_0 as, for instance, $a/10$, r_0 is determined by a. Thus we may consider that r_0 is determined by l. Set $K(n,l) = 2^{1-n} \operatorname{Vol}_{n-1}(F_1)$. For any $\gamma_0 \in \psi^{-1}(\Gamma_0) = \Delta(\{0\} \times S^{n-1}(1), \{l'\} \times S^{n-1}(1); (-\epsilon, l' + \epsilon) \times S^{n-1}(1))$, clearly $l(\gamma_0) \geq l'$ holds. Thus, for any $\gamma_0 \in \psi^{-1}(\Gamma_0)$, $f = l'^{-1}$ defined on $(-\epsilon, l' + \epsilon) \times S^{n-1}(1)$ satisfies

$$\int_{\gamma_0} f \, ds = l'^{-1} l(\gamma_0) \geq 1,$$

and hence $f \in \mathcal{F}(\psi^{-1}(\Gamma_0))$. Define $\hat{\Gamma}_0$ by

$$\hat{\Gamma}_0 = \Delta(\{0\} \times S^{n-1}(1), \{l'\} \times S^{n-1}(1); [0, l'] \times S^{n-1}(1)).$$

Then, by Proposition 4.6, we have

$$\mathcal{M}(\hat{\Gamma}_0) \geq \mathcal{M}(\psi^{-1}(\Gamma_0))$$

and clearly $f|_{[0,l'] \times S^{n-1}(1)}$ belongs to $\mathcal{F}(\hat{\Gamma}_0)$. Note that

$$\mathcal{M}(\hat{\Gamma}_0) \leq \int_{[0,l'] \times S^{n-1}(1)} f^n ds$$
$$= l'^{-n} \operatorname{Vol}(S^{n-1}(1)) l'$$
$$= l'^{1-n} \operatorname{Vol}(S^{n-1}(1)).$$

Define a positive constant $C(n,l)$ by

$$C(n,l) = \{\operatorname{Vol}(S^{n-1}(1))/K(n,l)\}^{1/n-1}$$

and choose l' so that l' is greater than $C(n,l)$. Then we have

$$l'^{1-n} \operatorname{Vol}(S^{n-1}(1)) < K(n,l)$$

and hence

$$\mathcal{M}(\psi^{-1}(\Gamma_0)) \leq \mathcal{M}(\hat{\Gamma}_0) < K(n,l) < \mathcal{M}(\Gamma_0).$$

This contradicts Proposition 4.8. This completes the proof of Lemma 3.3. $\square$

Using the modulus of path families, we can also prove the following lemma which is used in $(3.k.1)$ of the previous section.

Lemma 4.13. *There is no conformal transformation of $S^n(1)$ which sends the closure of $S(\beta_0)$ into $S(\beta_0')$ for $\beta_0' < \beta_0$.*

Proof. Assume that there exists a conformal transformation η of $S^n(1)$ which satisfies $\eta(S(\beta_0)) \subset S(\beta_0')$ with $\beta_0' < \beta_0$. We denote the interior of two connected components of $S^n(1) \setminus S(\beta_0)$ by $D_{0,1}$ and $D_{1,1}$ as in the previous section. Define two path families Γ and Γ' by

$$\Gamma = \Delta(\eta(\partial D_{0,1}), \ \eta(\partial D_{1,1}); \ \eta(S^n \setminus (D_{0,1} \cup D_{1,1}))),$$
$$\Gamma' = \Delta(\partial E(-\beta_0'), \ \partial E(\beta_0'); \ \text{the closure of } S(\beta_0')).$$

By our assumption, every $\gamma' \in \Gamma'$ has a subpath belonging to Γ, and hence, by Proposition 4.6, $\mathcal{M}(\Gamma') \leq \mathcal{M}(\Gamma)$. Let π_p be the stereographic projection from $p = (1,0,\dots,0) \in S^n(1) \subset \mathbf{R}^{n+1}$. An $(n-1)$-round sphere defined by $\{x = (x_0,\dots x_n) \in S^n \subset \mathbf{R}^{n+1}; \ x_0 \text{ is constant}\}$ is projected to a sphere of $\mathbf{R}^n$ centered at the origin by π_p. Thus, we have

$$\pi_p(\partial D_{0,1}) = S^{n-1}(b_1) \subset \mathbf{R}^n = \pi_p(S^n(1) \setminus \{p\}),$$
$$\pi_p(\partial D_{1,1}) = S^{n-1}(b_2),$$
$$\pi_p(\partial E(-\beta_0')) = S^{n-1}(b_1'),$$
$$\pi_p(\partial E(\beta_0')) = S^{n-1}(b_2'),$$

with $b_1 < b_1' < b_2' < b_2$. Denote the closed disk with radius r of $\mathbf{R}^n$ by $D^n(r)$ and the interior of $D^n(r)$ by $\operatorname{int} D^n(r)$. Since

$$\mathcal{M}(\Gamma') = \mathcal{M}(\Delta(S^{n-1}(b_1'), \ S^{n-1}(b_2'); \ D^n(b_2') \setminus \operatorname{int} D^n(b_1')))$$
$$= \operatorname{Vol}(S^{n-1}(1))\{\log(b_2'/b_1')\}^{1-n}$$

and

$$\mathcal{M}(\Delta(\partial D_{0,1}, \ \partial D_{1,1}; \ S^n(1) \setminus (D_{0,1} \cup D_{1,1}))$$
$$= \mathcal{M}(S^{n-1}(b_1), \ S^{n-1}(b_2); \ D^n(b_2) \setminus \operatorname{int} D^n(b_1))$$
$$= \operatorname{Vol}(S^{n-1}(1))\{\log(b_2/b_1)\}^{1-n},$$

by Proposition 4.8 and [21, p. 53], we have

$$\mathcal{M}(\Gamma) \geq \mathcal{M}(\Gamma') > \mathcal{M}(\Delta(\partial D_{0,1}, \ \partial D_{1,1}; \ S^n(1) \setminus (D_{0,1} \cup D_{1,1}))) = \mathcal{M}(\eta^{-1}(\Gamma)).$$

Then it is clear that this contradicts Proposition 4.8. This completes the proof of Lemma 4.13. □

APPENDIX. THE SPACE OF FLAT CONFORMAL STRUCTURES ON $\mathbf{R}P^n \sharp \mathbf{R}P^n$

In this Appendix, we examine the space of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$, which is the simplest example of non-C-prime conformally flat manifolds, following the program mentioned in the introduction. Note that there exists a 2-fold covering $S^1 \times S^{n-1} \longrightarrow \mathbf{R}P^n \sharp \mathbf{R}P^n$. Since any flat conformal structure on $S^1 \times S^{n-1}$ is Kleinian (see [4] and [11]), for any flat conformal structure C, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, C)$ is also Kleinian and conformal to the quotient of $S^n \setminus \{$two points$\}$ by a discrete subgroup Γ of the conformal transformation group of S^n, where Γ is isomorphic to $\pi_1(\mathbf{R}P^n \sharp \mathbf{R}P^n) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. Then, by Theorem 2.7, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, C)$ is conformal to some $(M_1, C_1) \sharp (M_2, C_2)$, where each M_i is homeomorphic (diffeomorphic if $n \leq 6$) to $\mathbf{R}P^n$ and the C_i's are Kleinian structures on M_i. Since $\pi_1(M_i)$, $i = 1, 2$, are finite, each M_i must be diffeomorphic to $\mathbf{R}P^n$ by a theorem of Kuiper ([10]). Moreover each C_i contains a constant curvature metric, that is, each C_i is the standard (and hence unique) conformal structure on $\mathbf{R}P^n$. We denote this flat conformal structure by C_0. Then, for any C, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, C)$ is conformal to some $(\mathbf{R}P^n, C_0) \sharp (\mathbf{R}P^n, C_0)$. And we see that deformations of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ arise from varying A, ψ_1 and ψ_2 in Definition 2.6. Precisely speaking, for any $(\mathbf{R}P^n \sharp \mathbf{R}P^n, C)$, there exists $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_l)$ constructed from two copies of $(\mathbf{R}P^n, g_0)$ as in section 3 such that $\hat{g}_l$ is contained in C, where g_0 denotes the metric of constant curvature 1 on $\mathbf{R}P^n$ (see below). In this case, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_l)$ is conformal to $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_{l'})$ if and only if $l = l'$. In particular, the space of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ is parametrized by l and hence 1-dimensional.

Since $S^n \setminus \{$two points$\}$ is conformal to $\mathbf{R} \times S^{n-1}(1)$, for any flat conformal structure C on $\mathbf{R}P^n \sharp \mathbf{R}P^n$, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, C)$ is conformal to $\mathbf{R} \times S^{n-1}(1)/\Gamma$, where $\Gamma \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ is a discrete subgroup of the conformal transformation group G of $\mathbf{R} \times S^{n-1}(1)$. Let γ_1 and γ_2 be the generators of Γ, each of which is of order 2. Note that G coincides with the isometry group of $\mathbf{R} \times S^{n-1}(1)$ and, for any $\gamma \in G$, the action of γ is written as $\gamma(r, p) = (\pm r + d, Ap)$ for some real number d and some $A \in O(n)$, where (r, p) denotes a point of $\mathbf{R} \times S^{n-1}(1)$ with $r \in \mathbf{R}$ and $p \in S^{n-1}(1)$. Since γ_1 and γ_2 act freely on $\mathbf{R} \times S^{n-1}(1)$, we see easily that γ_1 and γ_2 are written as

$$\begin{cases} \gamma_1(r, p) = (-r + 2a, \bar{p}), \\ \gamma_2(r, p) = (-r + 2b, \bar{p}), \end{cases}$$

where $\bar{p}$ denotes the antipodal point of p. We may assume that a is greater than b. Since G coincides with the isometry group, $\mathbf{R} \times S^{n-1}(1)/\Gamma$ defines a Riemannian manifold, which is obtained as closing the boundaries of $[0, a-b] \times S^{n-1}(1)$ by the identification $p = \bar{p}$, where p is a point of $\{0\} \times S^{n-1}(1)$ or $\{a-b\} \times S^{n-1}(1)$. We may regard this Riemannian manifold as $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_l)$ constructed in section 3 with $l = a-b$. Let C and C' be flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$. Then there exists a discrete subgroup Γ (resp. Γ') of G such that the metric on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ defined by $\mathbf{R} \times S^{n-1}(1)/\Gamma$ (resp. $\mathbf{R} \times S^{n-1}(1)/\Gamma'$) is contained in C (resp. $\tilde{C}$). Denote generators of Γ and Γ' by γ_i and γ_i' $(i = 1, 2)$ respectively. As we have seen above, we may assume that γ_i and γ_i' are written as

$$\begin{cases} \gamma_1(r, p) = (-r + 2a, \bar{p}), \\ \gamma_2(r, p) = (-r + 2b, \bar{p}), \end{cases}$$

$$\begin{cases} \gamma_1'(r, p) = (-r + 2a', \bar{p}), \\ \gamma_2'(r, p) = (-r + 2b', \bar{p}), \end{cases}$$

for $a > b$ and $a' > b'$. Then $\mathbf{R} \times S^{n-1}(1)/\Gamma$ (resp. $\mathbf{R} \times S^{n-1}(1)/\Gamma'$) contains a subset isometric to $(0, a-b) \times S^{n-1}(1)$ (resp. $(0, a'-b') \times S^{n-1}(1)$). Suppose C and C' are conformally equivalent. Then, since there exists $\gamma \in G$ such that $\Gamma = \gamma^{-1}\Gamma'\gamma$ and G is just the isometry group, $\mathbf{R} \times S^{n-1}(1)/\Gamma$ is isometric to $\mathbf{R} \times S^{n-1}(1)/\Gamma'$ and hence $a - b = a' - b'$. Therefore the correspondence between a flat conformal structure C and $a - b$ defines a map F from the moduli space of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ (i.e., the quotient space of the space of all flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ by the group of diffeomorphisms) to positive real numbers, and this map F is clearly surjective. On the other hand, if $F(C) = F(C')$, then $\mathbf{R} \times S^{n-1}(1)/\Gamma = (\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_{F(C)})$ and $\mathbf{R} \times S^{n-1}(1)/\Gamma' = (\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_{F(C')})$ are clearly isometric, and hence C and C' are conformally equivalent. This shows that F is injective and hence bijective. Moreover we have a representative $\hat{g}_{F(C)}$ of C. Therefore, as we mentioned in the previous paragraph, $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_l)$ and $(\mathbf{R}P^n \sharp \mathbf{R}P^n, \hat{g}_{l'})$ constructed as above are conformally equivalent if and only if $l = l'$. Since $\hat{g}_l$ depends continuously on l, we see that the map F from the moduli space of flat conformal structures on $\mathbf{R}P^n \sharp \mathbf{R}P^n$ to positive real numbers is a homeomorphism. Thus l defines a global coordinate of the moduli space.

In our construction of conformally flat metrics on a connected sum of conformally flat manifolds (in section 3), it seems that there are more parameters for deformations of flat conformal structures other than l (for example, choices of points removed and isometries of a cylinder gluing X and Y). But, for $\mathbf{R}P^n \sharp \mathbf{R}P^n$, deformations of flat conformal structures do not arise from these choices. In this case, it seems that the conformal symmetries of $\mathbf{R}P^n$ absorb variations of these choices, just as an essential conformal transformation on S^n absorbs a deformation arising from changing the length of a cylinder as we have seen in Remark 2 following Theorem 3.1.

References

[1] B. N. Apanasov, *Discrete Groups in Space and Uniformization Probrems*, Math. and its Appl., Vol 40, Kluwer Academic Publ. Group, Dordrecht, Holland, 1991. MR **93h:**57026
[2] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag, Berlin Heidelberg, 1992. MR **94e:**57015

[3]   J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Math. 53, Springer-Ferlag, Berlin Heidelberg, 1968. MR **37:**4824

[4]   W. M. Goldman, *Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds*, Trans. Amer. Math. Soc. **278** (1983), 573–583. MR **84j:**53043

[5]   ———, *Projective structures with Fuchsian holonomy*, J. Diff. Geom. **25** (1987), 297–326. MR **88i:**57006

[6]   W. M. Goldman and Y. Kamishima, *Conformal automorphisms and conformally flat manifolds*, Trans. Amer. Math. Soc. **323** (1991), 797–810. MR **91e:**53034

[7]   J. Hempel, *3-manifolds*, Annals of Math. Studies 86, Princeton Univ. Press, Princeton, New Jersey, 1976. MR **54:**3702

[8]   H. Izeki, *On the decomposition of conformally flat manifolds*, J. Math. Soc. Japan **45** (1993), 105–119. MR **93h:**53036

[9]   D. Johnson and J. J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, in Discrete Groups in Geometry and Analysis, Papers in Honor of G. D. Mostow on His Sixtieth Birthday, Progress in Math. 67 (R. Howe, eds.), Birkhäuser, Boston, Basel, 1987, pp. 48–106. MR **88j:**22010

[10] N. H. Kuiper, *On conformally flat spaces in the large*, Ann. of Math. **50** (1949), 916–924. MR **11:**133b

[11] N. H. Kuiper, *On compact conformally Euclidean spaces of dimension > 2*, Ann. of Math. **52** (1950), 478–490. MR **12:**283c

[12] R. S. Kulkarni, *On the principle of uniformization*, J. Diff. Geom. **13** (1978), 109–138. MR **81k:**53009

[13] R. S. Kulkarni and U. Pinkall, *Uniformization of geometric structures with applications to conformal geometry*, in Lecture Notes in Math. 1209 (1986), Springer-Verlag, Berlin Heidelberg, 190–209. MR **88b:**53036

[14] A. G. Kurosh, *The Theory of Groups*, Vol. 2, Chelsea, New York, 1956. MR **18:**188f

[15] J. Lelong-Ferrand, *Transformations conformes et quasi-conformes des variété riemanniennes compactes* (*démonstration de la conjecture de A. Lichenerowicz*), Acad. Roy. Beg. Sci. Mem. Coll. in 8° (2) **39** (1971), no. 5. MR **48:**1100

[16] J. Lelong-Ferrand, *Geometrical interpretation of scalar curvature and regularity of conformal homeomorphism*, in Differential Geometry and Relativity (1976), Reidel Publ. Co., Dordrecht, Holland, 91–105. MR **56:**1222

[17] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton Univ. Press, Princeton, New Jersey, 1965. MR **33:**8352

[18] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Diff. Geom. **6** (1971), 247–258. MR **46:**2601

[19] I. M. Singer, *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math. **13** (1960), 685–697. MR **24:**A1100

[20] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University Lecture Notes, 1979.

[21] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319, Springer-Verlag, Berlin Heidelberg, 1988. MR **89k:**30021

Graduate School of Polymathematics, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan

E-mail address: `izeki@math.nagoya-u.ac.jp`