

LÉVY GROUP ACTION AND INVARIANT MEASURES ON $\beta\mathbb{N}$

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ABSTRACT. For $f \in \ell^\infty(\mathbb{N})$ let Tf be defined by $Tf(n) = \frac{1}{n} \sum_{i=1}^n f(i)$. We investigate permutations g of \mathbb{N} , which satisfy $Tf(n) - Tf_g(n) \rightarrow 0$ as $n \rightarrow \infty$ with $f_g(n) = f(gn)$ for $f \in \ell^\infty(\mathbb{N})$ (i.e. g is in the Lévy group \mathcal{G}), or for f in the subspace of Cesàro-summable sequences. Our main interest are \mathcal{G} -invariant means on $\ell^\infty(\mathbb{N})$ or equivalently \mathcal{G} -invariant probability measures on $\beta\mathbb{N}$. We show that the adjoint T^* of T maps measures supported in $\beta\mathbb{N} \setminus \mathbb{N}$ onto a weak*-dense subset of the space of \mathcal{G} -invariant measures. We investigate the dynamical system $(\mathcal{G}, \beta\mathbb{N})$ and show that the support set of invariant measures on $\beta\mathbb{N}$ is the closure of the set of almost periodic points and the set of non-topologically transitive points in $\beta\mathbb{N} \setminus \mathbb{N}$. Finally we consider measures which are invariant under T^* .

INTRODUCTION

For $A \subset \mathbb{N}$ the density function is given by $d_A(n) = \frac{1}{n} |\{i \in A : i \leq n\}|$, and its upper density \overline{d}_A as $\overline{d}_A = \overline{\lim}_{n \rightarrow \infty} d_A(n)$. A set A has density α if $\lim_{n \rightarrow \infty} d_A(n) = \alpha$. For a function $f \in \ell^\infty(\mathbb{N})$ (the space of bounded real functions on \mathbb{N}) we define the operator T by $Tf(n) = \frac{1}{n} \sum_{i=1}^n f(i)$.

Any element f of $\ell^\infty(\mathbb{N})$ has a unique continuous extension to the Stone-Čech compactification $\beta\mathbb{N}$ of \mathbb{N} . The continuous extension of the characteristic functions χ_A of a subset A of \mathbb{N} is the characteristic function $\chi_{\overline{A}}$ of the closure \overline{A} of A in $\beta\mathbb{N}$. For a general f in $\ell^\infty(\mathbb{N})$ we will denote the continuous extension of a function f to $\beta\mathbb{N}$ again by f . This isometry between $\ell^\infty(\mathbb{N})$ and $C(\beta\mathbb{N})$ (the space of real continuous functions on $\beta\mathbb{N}$) allows us to express continuous functions on $\beta\mathbb{N}$ by their values on the integers, i.e. as bounded sequences on \mathbb{N} , to interpret T as a continuous operator on $C(\beta\mathbb{N})$, and to identify the dual spaces of $\ell^\infty(\mathbb{N})$ (finitely additive measures on \mathbb{N}) and of $C(\beta\mathbb{N})$ (regular Borel measures on $\beta\mathbb{N}$). In particular means on $\ell^\infty(\mathbb{N})$ correspond to probability measures on $\beta\mathbb{N}$ (see proof of Lemma 1 below).

This correspondence between finitely additive measures on \mathbb{N} and measures on $\beta\mathbb{N}$ becomes more transparent when we realize $\beta\mathbb{N}$ as the set of ultrafilters on \mathbb{N} , with ultrafilters being finitely additive 0-1 valued measures (cf. [11], 20.37 (b)): Then for an ultrafilter q the Dirac measure δ_q is the 0-1 valued Borel measure on $\beta\mathbb{N}$ corresponding to the finitely additive 0-1 valued measure q on \mathbb{N} .

For our approach however it will be more convenient to work with the usual topological definition of a filter on a set M as a nonempty collection \mathfrak{F} of subsets of

Received by the editors September 29, 1995.

1991 *Mathematics Subject Classification*. Primary 54H20.

Part of this work was carried out at Macquarie University with financial support from the Australian Research Council.

M satisfying (i) $\emptyset \notin \mathfrak{F}$, (ii) $A \in \mathfrak{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathfrak{F}$, (iii) $A \in \mathfrak{F}, A \subset B \subset M \Rightarrow B \in \mathfrak{F}$. An ultrafilter is a filter with the additional property that it is not properly contained in any other filter (i.e. it is maximal), or equivalently, $A \subset M \Rightarrow A \in \mathfrak{F}$ or $\mathbb{C}A \in \mathfrak{F}$.

We identify the natural numbers with their neighborhood filters, so $\mathbb{N} \subset \beta\mathbb{N}$. A base for the topology on $\beta\mathbb{N}$ is now given by the clopen sets. A subset of $\beta\mathbb{N}$ is clopen iff it is the closure \overline{A} of a subset A of \mathbb{N} , i.e. iff it is the set of all ultrafilters containing A . With this abuse of notation we can see $A \subset \mathbb{N}$ both as a subset of $\beta\mathbb{N}$ and as an element of an ultrafilter p in \overline{A} , i.e. we have $p \in \overline{A}$ iff $A \in p$. See [20] for a detailed exposition.

A permutation g of \mathbb{N} acts naturally as an automorphism on $\beta\mathbb{N}$: For $p \in \beta\mathbb{N}$ let the ultrafilter gp be defined as $gp = \{gA : A \in p\}$. So $gA \in gp \Leftrightarrow A \in p$ and $g_1(g_2p) = (g_1g_2)p$, i.e. the group of permutations of \mathbb{N} acts as a transformation group on $\beta\mathbb{N}$. Then for any clopen set \overline{A} , $A \subset \mathbb{N}$, we have

$$(1) \quad g\overline{A} = \{gp : p \in \overline{A}\} = \{gp : A \in p\} = \{gp : gA \in gp\} = \{q : gA \in q\} = \overline{gA}.$$

Since the clopen sets are a base for the topology on $\beta\mathbb{N}$, it follows that g is an automorphism.

A permutation g of \mathbb{N} also induces an isometry of $\ell^\infty(\mathbb{N}) : f \rightarrow f_g, f_g(n) = f(gn)$ and an isometry of $C(\beta\mathbb{N}) : f \rightarrow f_g, f_g(p) = f(gp), p \in \beta\mathbb{N}$, so g acts naturally on the space \mathcal{M} of (signed real regular Borel) measures on $\beta\mathbb{N}$ by $\langle g\mu, f \rangle = \langle \mu, f_g \rangle$ or equivalently by $g\mu(U) = \mu(g^{-1}U)$ for Borel sets U in $\beta\mathbb{N}$. A measure μ in \mathcal{M} is G -invariant (G being some group of permutations) if $\mu = \mu_g \forall g \in G$, i.e. if μ is in the annihilator $\{f - f_g : f \in C(\beta\mathbb{N}), g \in G\}^\perp$ (with respect to the duality $\langle \mathcal{M}, C(\beta\mathbb{N}) \rangle$).

We shall be interested in measures on $\beta\mathbb{N}$, which are invariant under certain permutations defined via invariance of density functions at infinity. The Lévy group \mathcal{G} is the group of permutations g satisfying $Tf - Tf_g$ vanishes on $\mathbb{N}^* \forall f \in C(\beta\mathbb{N})$. We construct a weak*-dense subspace of the space of \mathcal{G} -invariant measures and show that the bigger group \mathcal{G}_δ of permutations satisfying this requirement only for Cesàro-summable bounded sequences f allows no \mathcal{G}_δ -invariant measures.

We investigate the topological dynamics of the permutation group \mathcal{G} acting on $\beta\mathbb{N}$; similar problems of invariance under right translations (or more generally under motions, i.e. 1-1 maps of \mathbb{N} without periodic points) were discussed in [4], [12], [16] and [17]. See [2], II.5 for an overview.

Although right translation τ of \mathbb{N} is not a permutation of \mathbb{N} , there are elements g in \mathcal{G} which satisfy $gn = \tau n$ except for a set of density 0 (cf. proof of Theorem 7). Since sets of density 0 will turn out to be negligible for the questions considered, it is safe to think of translations as being a subsemigroup of the Lévy group. In particular the action of τ on the support set S of measures invariant under the Lévy group coincides with the action of certain elements of the Lévy group on S .

While the characterization of the support set S_τ of τ -invariant measures looks like our characterization of S (Theorem 4) with the notion of upper density replaced by that of upper Banach density, the orbit structures of the pertinent dynamical systems differ drastically: While we will show that points with minimal \mathcal{G} -orbit closure are dense in S , it is known that there are open sets in S_τ without points with minimal τ -orbit closure. On the other hand every minimal τ -orbit closure is the support set of some τ -invariant measure, whereas there are minimal \mathcal{G} -orbit closures which do not support any \mathcal{G} -invariant measure.

Some of these differences are due to the fact that the Lévy group is not amenable, whereas the semigroup of translations is amenable. An essential difference in the investigations of the dynamical systems $(\mathcal{G}, \beta\mathbb{N})$ and $(\tau, \beta\mathbb{N})$ stems from the fact that translations of \mathbb{N} suggest the consideration of \mathbb{N} as a semigroup under addition, which then allows some characterizations via algebraic properties of the left topological (right topological in the notation of [2]) semigroup $\beta\mathbb{N}$. So the set S is a right ideal of the semigroup $(\beta\mathbb{N}, +)$ cf. [5], 7.4 and minimal closed right ideals are just minimal orbit closures under τ cf. [1], 2.1.

Finally we characterize the support set of measures which are invariant under the adjoint T^* of T . This invariance is a stronger requirement than \mathcal{G} -invariance. The support set S_I of these measures allows a characterization similar to that of S or of S_I , again with an appropriate notion of density.

PERMUTATION GROUPS AND INVARIANT MEANS ON \mathbb{N}

We first clarify why the existence of positive invariant functionals is equivalent to the existence of nonzero invariant functionals for $\ell^\infty(\mathbb{N})$ resp. $C(\beta\mathbb{N})$. The proof of these facts is included for the reader's convenience.

We recall that a mean on \mathbb{N} is a positive normalized finitely additive measure (set function) on \mathbb{N} or equivalently a positive normalized functional on $\ell^\infty(\mathbb{N})$.

Lemma 1. *For a group G of permutations of \mathbb{N} the following are equivalent:*

- (a) *There exists a G -invariant mean on \mathbb{N} ;*
- (b) *There exists a nonzero G -invariant finitely additive measure on $\beta\mathbb{N}$;*
- (c) *There exists a G -invariant probability measure on $\beta\mathbb{N}$;*
- (d) *There exists a nonzero G -invariant (finite Borel) measure on $\beta\mathbb{N}$.*

Proof. The restriction map from $\beta\mathbb{N}$ to \mathbb{N} induces an isometry Φ between the Banach spaces $C(\beta\mathbb{N})$ and $\ell^\infty(\mathbb{N})$. The adjoint Φ^* then is an isometry between their dual spaces: $\Phi^* : \ell^\infty(\mathbb{N})^* \rightarrow C(\beta\mathbb{N})^*$. The dual of $\ell^\infty(\mathbb{N})$ is the space of finitely additive measures on \mathbb{N} (cf. [6], IV 8.16) and the dual of $C(\beta\mathbb{N})$ is the space \mathcal{M} of (finite real regular Borel) measures on $\beta\mathbb{N}$ by the Riesz representation theorem (cf. [6], IV 6.2), both spaces with the total variation norm.

We further note that, since for $f \in C(\beta\mathbb{N})$ and a permutation g of \mathbb{N} the action of g on f is defined by $f_g(p) = f(gp) \forall p \in \beta\mathbb{N}$, we have

$$(2) \quad (\Phi f)_g(n) = (\Phi f)(gn) = f(gn) = f_g(n) = \Phi(f_g)(n),$$

i.e. the action of G commutes with the isometry Φ and we may use the notation f_g both for the continuous extension of $f_g \in \ell^\infty(\mathbb{N})$ to $\beta\mathbb{N}$ and for the action of g on the continuous extension of $f \in \ell^\infty(\mathbb{N})$ to $\beta\mathbb{N}$.

Since the space of G -invariant measures on $\beta\mathbb{N}$ is just the annihilator of the set $\{f - f_g : f \in C(\beta\mathbb{N}), g \in G\}$ and the space of G -invariant finitely additive measures on \mathbb{N} is just the annihilator of the set $\{f - f_g : f \in \ell^\infty(\mathbb{N}), g \in G\}$, it follows that G -invariant measures on $\beta\mathbb{N}$ correspond to G -invariant finitely additive measures on \mathbb{N} via the isometry Φ^* , since by (2) Φ is a bijection between the above two sets.

Finally Φ is order preserving since the restriction to \mathbb{N} of a positive function on $C(\beta\mathbb{N})$ is positive, so means correspond to probability measures and (a) \Leftrightarrow (c) and (b) \Leftrightarrow (d) follow.

Since (c) \Rightarrow (d) is trivial, it remains to show (d) \Rightarrow (c): Let μ be a nontrivial G -invariant measure on $\beta\mathbb{N}$ and let $\beta\mathbb{N} = X_1 \cup X_2$ be the Hahn decomposition (cf. [11]) of $\beta\mathbb{N}$, i.e. μ is nonnegative when restricted to X_1 and nonpositive when restricted

to X_2 , where X_1, X_2 are disjoint Borel subsets of $\beta\mathbb{N}$. Assume μ is nonzero on X_1 (otherwise $-\mu$ is positive and invariant) and let μ_+ be the positive part of μ , i.e. $\mu_+(E) = \mu(E \cap X_1)$ for Borel sets E in $\beta\mathbb{N}$.

For $g \in G$ we obtain using the G -invariance of μ :

$$0 \leq \mu_+(g^{-1}X_2) = \mu(g^{-1}X_2 \cap X_1) = \mu(X_2 \cap gX_1) \leq 0,$$

hence $\mu_+(g^{-1}X_2) = 0$. The same argument applies to g^{-1} , so $\mu_+(gX_2) = 0$ and it follows that $\mu_+(gX_1) = \mu_+(g^{-1}X_1) = \mu_+(\beta\mathbb{N})$. Thus for any Borel subset E of $\beta\mathbb{N}$ we get

$$(2) \quad \begin{aligned} \mu_+(E) &= \mu_+(E \cap g^{-1}X_1) = \mu(E \cap g^{-1}X_1 \cap X_1) \\ &= \mu(gE \cap X_1 \cap gX_1) = \mu_+(gE \cap gX_1) = \mu_+(gE). \end{aligned}$$

So μ_+ is G -invariant. Since μ_+ is positive and nonzero $\frac{1}{\|\mu_+\|}\mu_+$ is a G -invariant probability measure. \square

There are two natural permutation groups characterized by invariance of densities of natural numbers cf. [13], [14], [15], [3]:

Let \mathcal{G}_δ be the group of permutations g of \mathbb{N} satisfying

$$\lim_{n \rightarrow \infty} d_A(n) = \lim_{n \rightarrow \infty} d_{gA}(n) = \lim_{n \rightarrow \infty} d_{g^{-1}A}(n)$$

for all subsets A of \mathbb{N} which have a density, i.e. for which $\lim_{n \rightarrow \infty} d_A(n)$ exists. By [3], Theorem 1 an equivalent definition of \mathcal{G}_δ is given as the group of permutations g for which

$$\lim_{n \rightarrow \infty} Tf(n) = \lim_{n \rightarrow \infty} Tfg(n) = \lim_{n \rightarrow \infty} Tfg^{-1}(n)$$

for all f in the subspace of Cesàro-summable sequences in $\ell^\infty(\mathbb{N})$.

This group contains the proper subgroup of permutations g satisfying

$$\lim_{n \rightarrow \infty} d_A(n) - d_{gA}(n) = 0 \quad \forall A \subset \mathbb{N}.$$

The Lévy group \mathcal{G} (first considered in [13], 3.36) is defined as the group of permutations g of \mathbb{N} , for which

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{n : n \leq N, gn > N\}| = 0.$$

The following lemma identifies the Lévy group with the above subgroup of \mathcal{G}_δ :

Lemma 2. *The following are equivalent:*

- (a) $g \in \mathcal{G}$;
- (b) $\forall f \in \ell^\infty(\mathbb{N}) \quad Tf(n) - Tfg(n) \xrightarrow{n \rightarrow \infty} 0$;
- (c) $\forall A \subset \mathbb{N} \quad d_A(n) - d_{gA}(n) \xrightarrow{n \rightarrow \infty} 0$.

Proof. The definition of \mathcal{G} shows that (a) implies (b). Clearly (b) implies (c) since $d_A(n) = T\chi_A(n)$.

If $g \notin \mathcal{G}$, then there exists a sequence (n_i) with $n_{i-1}/n_i \rightarrow 0$ and $0 < \alpha \in \mathbb{R}$, such that the sets $E_i := \{n : n \leq n_i, gn > n_i\}$ satisfy $|E_i|/n_i > \alpha$. For $A := \bigcup_i E_i$ we obtain $d_A(n_i) > d_{E_i}(n_i) > \alpha$. Since $gE_j \subset]n_j, \infty[$ we obtain

$$d_{gA}(n_i) \leq d_{\bigcup_{j < i} E_j}(n_i) \leq \frac{n_{i-1}}{n_i} \xrightarrow{i \rightarrow \infty} 0$$

and $\overline{\lim}_{i \rightarrow \infty} d_A(n_i) - d_{gA}(n_i) \geq \alpha > 0$, so (c) implies (a). \square

We will also need the following generalization of [15], Lemma 2.4:

Lemma 3. *Let $A, B \subset \mathbb{N}$ such that $A, B, \mathbb{C}A, \mathbb{C}B$ are infinite sets. Then there is a $g \in \mathcal{G}$ with $B = gA$ if and only if $\lim_{n \rightarrow \infty} d_A(n) - d_B(n) = 0$.*

Proof. Let $A = \{a_1 < a_2 < \dots\}$, $\mathbb{C}A = \{a'_1 < a'_2 < \dots\}$ and $B = \{b_1 < b_2 < \dots\}$, $\mathbb{C}B = \{b'_1 < b'_2 < \dots\}$. We define the permutation g by $ga_i = b_i$, $ga'_i = b'_i$. We have

$$\begin{aligned} & \frac{1}{N} |\{n : n \leq N, gn > N\}| \\ &= \frac{1}{N} |\{n \in A : n \leq N, gn > N\}| + \frac{1}{N} |\{n \in \mathbb{C}A : n \leq N, gn > N\}| \\ &= \max(0, d_A(N) - d_B(N)) + \max(0, d_{\mathbb{C}A}(N) - d_{\mathbb{C}B}(N)). \end{aligned}$$

If $\lim_{n \rightarrow \infty} d_A(n) - d_B(n) = 0$, it follows that $\lim_{n \rightarrow \infty} d_{\mathbb{C}A}(n) - d_{\mathbb{C}B}(n) = 0$, and we see that $\lim_{N \rightarrow \infty} \frac{1}{N} |\{n : n \leq N, gn > N\}| = 0$, i.e. $g \in \mathcal{G}$.

The “only if” statement follows from Lemma 2 above. \square

We now show that the group \mathcal{G}_δ does not admit an invariant mean on $\ell^\infty(\mathbb{N})$. Since we are mainly interested in invariant measures this explains why we have to restrict our considerations concerning invariant measures to the Lévy group \mathcal{G} .

Theorem 1. *There exists no nontrivial \mathcal{G}_δ -invariant functional on $\ell^\infty(\mathbb{N})$.*

Proof. For $a + l \leq b$, $a, b, l \in \mathbb{N}$ let $g(a, a + l, b, b + l)$ be the permutation

$$x \mapsto \begin{cases} x + b - a & \text{for } x \in [a, a + l[, \\ x - (b - a) & \text{for } x \in [b, b + l[, \\ x & \text{else,} \end{cases}$$

that is $g(a, a + l, b, b + l)$ transposes the intervals of integers $[a, a + l[$ and $[b, b + l[$. We define permutations g_1, \dots, g_4 as the product of pairwise disjoint transpositions:

$$\begin{aligned} g_1 &= \prod_{i \geq 0} g(2^{2i}, 2^{2i+1}, 2^{2i+1}, 2^{2i+1} + 2^{2i}), \\ g_2 &= \prod_{i \geq 0} g(2^{2i+1}, 2^{2i+2}, 2^{2i+2}, 2^{2i+2} + 2^{2i+1}), \\ g_3 &= \prod_{i \geq 0} g(2^{2i}, 2^{2i+1}, 2^{2i+1} + 2^{2i}, 2^{2i+2}), \\ g_4 &= \prod_{i \geq 0} g(2^{2i+1}, 2^{2i+2}, 2^{2i+2} + 2^{2i+1}, 2^{2i+3}). \end{aligned}$$

An easy computation, or combining [3], Theorem 1 and [14], proof of Proposition 4.1, p. 106, shows that the permutation $\prod_i g(a_i, a_i + l_i, a_i + l_i, a_i + 2l_i)$ is in \mathcal{G}_δ for $a_i + 2l_i \leq a_{i+1}$. Therefore $g_1, g_2 \in \mathcal{G}_\delta$. Since

$$\begin{aligned} g_3 &= h_1 h_2 h_1 = g(2^{2i}, 2^{2i+1}, 2^{2i+1} + 2^{2i}, 2^{2i+2}) \quad \text{with} \\ h_1 &= g(2^{2i}, 2^{2i+1}, 2^{2i+1}, 2^{2i+1} + 2^{2i}) \\ h_2 &= g(2^{2i+1}, 2^{2i+1} + 2^{2i}, 2^{2i+1} + 2^{2i}, 2^{2i+2}) \end{aligned}$$

it follows that $g_3 \in \mathcal{G}_\delta$. $g_4 \in \mathcal{G}_\delta$ can be seen similarly. Note that \mathcal{G}_δ as defined in [14] is a semigroup, however since $g_i = g_i^{-1}$, we have $g_i \in \mathcal{G}_\delta$.

We define the functions f_i , $i = 1, 2, 3, 4$ by

$$f_i(l) = \begin{cases} 1, & g_i l > l, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that

$$(f_i - (f_i)_{g_i})(l) = \begin{cases} 1, & g_i l > l, \\ -1, & g_i l < l, \\ 0, & g_i l = l. \end{cases}$$

Hence for $y := \sum_{i=1}^4 f_i - (f_i)_{g_i}$ we have $y(l) = 1$ for $l > 1$ and $y(1) = 2$.

The space of \mathcal{G}_δ -invariant measures on $\beta\mathbb{N}$ can be considered as the annihilator $\{f - f_g : g \in \mathcal{G}_\delta, f \in C(\beta\mathbb{N})\}^\perp$ with respect to the duality $\langle \mathcal{M}, C(\beta\mathbb{N}) \rangle$. Therefore \mathcal{G}_δ -invariant measures vanish on y . Obviously any \mathcal{G}_δ -invariant functional has to vanish on $c_0(\mathbb{N})$ since the dual of $c_0(\mathbb{N})$ can be identified with $\ell^1(\mathbb{N})$ and 0 is the only \mathcal{G}_δ -invariant function in $\ell^1(\mathbb{N})$. In particular invariant measures vanish on the function $e_1 : e_1(1) = 1, e_1(n) = 0$ for $n > 1$. Since $y - e_1$ is just the constant function 1, we see that any \mathcal{G}_δ -invariant measure vanishes on constant functions. Since a positive normalized functional cannot vanish on constant functions we deduce that no mean on $\ell^\infty(\mathbb{N})$ is \mathcal{G}_δ -invariant. By Lemma 1 this implies that no nontrivial functional on $\ell^\infty(\mathbb{N})$ (nontrivial finitely additive measure on \mathbb{N}) is \mathcal{G}_δ -invariant. \square

We now give an explicit description of a weak*-dense subset of \mathcal{G} -invariant measures on $\beta\mathbb{N}$ corresponding to finitely additive \mathcal{G} -invariant set functions on \mathbb{N} . We remark that a similar characterization of invariant means for the translation semigroup was obtained in [12].

We denote the space of measures supported in \mathbb{N}^* by \mathcal{M}_∞ and the range of the restriction of the adjoint T^* of T to \mathcal{M}_∞ by \mathcal{R} , i.e. $\mathcal{R} := \{T^*\mu : \mu \in \mathcal{M}_\infty\}$. $\mathcal{M}_\mathcal{G}$ is the space of \mathcal{G} -invariant probability measures on $\beta\mathbb{N}$.

Since $\lim_{n \rightarrow \infty} Tf(n) - Tf_g(n) = 0$ for $g \in \mathcal{G}$ by Lemma 2, it follows that $Tf(x) = Tf_g(x) \forall x \in \mathbb{N}^*, g \in \mathcal{G}$. Therefore the adjoint T^* maps measures which are supported in \mathbb{N}^* to the space of \mathcal{G} -invariant measures on $\beta\mathbb{N}$, i.e.

$$(4) \quad \mathcal{R} \subseteq \mathcal{M}_\mathcal{G}$$

and T^* maps probability measures supported on \mathbb{N}^* to \mathcal{G} -invariant probability measures.

The following counterexample shows that this map is not surjective, i.e. there are \mathcal{G} -invariant probability measures which are not the image under T^* of a probability measure. Since this image is a weak*-compact set it follows that this image is not even dense.

For an ultrafilter q the ultrafilter $2q$ is defined by the filter base $\{2A : A \in q\}$. Let $q \in \mathbb{N}^*$ be an ultrafilter containing the set $\{2^{2^i} : i \in \mathbb{N}\}$. For any positive function f on \mathbb{N} with $\|f\| = 1$ we have

$$\frac{1}{2}Tf(n) \leq Tf(2n) \leq \frac{1}{2}(1 + Tf(n)).$$

Therefore

$$\frac{1}{2}\langle \delta_q, Tf \rangle \leq \langle \delta_{2q}, Tf \rangle \leq \frac{1}{2}(1 + \langle \delta_q, Tf \rangle)$$

and

$$1 \geq \langle T^*(2\delta_{2q} - \delta_q), f \rangle = \langle 2\delta_{2q} - \delta_q, Tf \rangle \geq 0,$$

so $T^*(2\delta_{2q} - \delta_q)$ is a probability measure as $\langle T^*(2\delta_{2q} - \delta_q), 1 \rangle = 1$. For the set $A = \bigcup_i [2^{2^i}, 2 \cdot 2^{2^i}[$ we obtain $\overline{\lim}_{n \rightarrow \infty} T\chi_A(n) = \lim_{i \rightarrow \infty} T\chi_A(2 \cdot 2^{2^i}) = 1/2$, and $\lim_{i \rightarrow \infty} T\chi_A(2^{2^i}) = 0$. Therefore $\langle \mu, T\chi_{\overline{A}} \rangle = \langle T^*\mu, \chi_{\overline{A}} \rangle \leq 1/2$ for any probability measure μ on \mathbb{N}^* and $\langle 2\delta_{2q} - \delta_q, T\chi_{\overline{A}} \rangle = 1$, so $\|T^*\mu - T^*(2\delta_{2q} - \delta_q)\| \geq 1/2$ for all probability measures μ supported in \mathbb{N}^* . Hence the probability measure $T^*(2\delta_{2q} - \delta_q)$ is not in the image of probability measures supported in \mathbb{N}^* under the map T^* .

Theorem 2. \mathcal{R} is a proper weak*-dense subset of the weak*-closed space $\mathcal{M}_{\mathcal{G}}$. The annihilator of the subspace $\mathcal{M}_{\mathcal{G}}$ with respect to the duality $\langle \mathcal{M}, C(\beta\mathbb{N}) \rangle$ is the space $\{f \in C(\beta\mathbb{N}) : Tf(p) = 0 \ \forall p \in \mathbb{N}^*\}$.

Proof. Clearly the annihilator \mathcal{R}^\perp of \mathcal{R} is $\{f \in C(\beta\mathbb{N}) : Tf(p) = 0 \ \forall p \in \mathbb{N}^*\} =: \mathcal{L}_0$. By [3], Theorem 2 any \mathcal{G} -invariant functional on $\ell^\infty(\mathbb{N})$ is a multiple of Cesàro-mean on the subspace of Cesàro-summable functions, so $\mathcal{M}_{\mathcal{G}} \subseteq \mathcal{L}_0^\perp$. Since

$$\mathcal{M}_{\mathcal{G}} = \{f - f_g : f \in C(\beta\mathbb{N}), g \in \mathcal{G}\}^\perp$$

it follows that $\mathcal{M}_{\mathcal{G}}$ is weak*-closed. With (4) and the bipolar theorem cf. [18], III.7.4 it follows that $\mathcal{R}^{\perp\perp} \subseteq \mathcal{M}_{\mathcal{G}}$. Therefore

$$\mathcal{M}_{\mathcal{G}} \supseteq \mathcal{R}^{\perp\perp} = \mathcal{L}_0^\perp \supseteq \mathcal{M}_{\mathcal{G}} \quad \text{and} \quad \mathcal{R}^{\perp\perp} = \mathcal{M}_{\mathcal{G}}.$$

The bipolar theorem now also gives the weak*-density of \mathcal{R} in $\mathcal{M}_{\mathcal{G}}$.

We next show that \mathcal{R} is a proper subset of $\mathcal{M}_{\mathcal{G}}$. Let $C_0(\beta\mathbb{N})$ be the closed subspace of functions in $C(\beta\mathbb{N})$ which vanish on \mathbb{N}^* . We first note that

$$(C(\beta\mathbb{N})/C_0(\beta\mathbb{N}))^* \cong C_0(\beta\mathbb{N})^\perp = \mathcal{M}_\infty$$

(cf. [6], II.4.18) and that T maps $C_0(\beta\mathbb{N})$ to $C_0(\beta\mathbb{N})$, so T induces a continuous map $\tilde{T} : C(\beta\mathbb{N})/C_0(\beta\mathbb{N}) \mapsto C(\beta\mathbb{N})/C_0(\beta\mathbb{N})$. We can therefore identify the restriction of T^* to \mathcal{M}_∞ with the adjoint of \tilde{T} .

Since $\mathcal{M}_{\mathcal{G}}$ being weak*-closed is clearly norm-closed, it follows by the closed range theorem cf. [11], VI 6.4, that the adjoint of \tilde{T} has no norm-closed range if \tilde{T} has no norm-closed range. To see that $\mathcal{R} \neq \mathcal{M}_{\mathcal{G}}$, it therefore remains to show that the range of \tilde{T} is not norm-closed.

For $\alpha > 1$ let

$$A_\alpha = \bigcup_{n \in \mathbb{N}} \left[2^{2^n} + 1, \left[2^{2^n} (1 + 1/\alpha) \right] \right],$$

where $[m, n]$ is the interval $\{m, m+1, \dots, n\}$ of integers and $[x]$ denotes the integer part of x , i.e. $[x] = \max\{n : n \in \mathbb{N}, n \leq x\}$ for $x \geq 1$. For $\psi_\alpha = \alpha\chi_{A_\alpha}$ we have $\lim_{i \rightarrow \infty} T\psi_\alpha(2^{2^i}) = 0$ and

$$\lim_{i \rightarrow \infty} T\psi_\alpha \left(\left[2^{2^i} (1 + 1/\alpha) \right] \right) = \overline{\lim}_{n \rightarrow \infty} T\psi_\alpha(n) = \frac{\alpha}{1 + \alpha}.$$

Therefore any function $\hat{\psi}_\alpha$ on \mathbb{N} with $\lim_{n \rightarrow \infty} (T(\psi_\alpha - \hat{\psi}_\alpha))(n) = 0$ satisfies

$$\begin{aligned} \overline{\lim}_{i \rightarrow \infty} \sup \left\{ \hat{\psi}_\alpha(n) : n \in \left[2^{2^i} + 1, \left[2^{2^i} (1 + 1/\alpha) \right] \right] \right\} &\geq \overline{\lim}_{i \rightarrow \infty} \frac{1}{[2^{2^i}/\alpha]} \sum_{n=2^{2^i}+1}^{[2^{2^i}(1+1/\alpha)]} \hat{\psi}_\alpha(n) \\ &= \overline{\lim}_{i \rightarrow \infty} \frac{1+1/\alpha}{1/\alpha} T\hat{\psi}_\alpha \left(\left[2^{2^i} (1 + 1/\alpha) \right] \right) - \frac{1}{1/\alpha} T\hat{\psi}_\alpha \left(2^{2^i} \right) = (1 + \alpha) \frac{\alpha}{1 + \alpha} = \alpha. \end{aligned}$$

Hence any element $\tilde{\psi}_\alpha$ in the quotient space $C(\beta\mathbb{N})/C_0(\beta\mathbb{N})$ with $\tilde{T}\tilde{\psi}_\alpha = T\psi_\alpha + C_0(\beta\mathbb{N})$ satisfies $\|\tilde{\psi}_\alpha\|_{C(\beta\mathbb{N})/C_0(\beta\mathbb{N})} \geq \alpha$. It follows that \tilde{T} is not an open mapping since α is arbitrary but $\|\tilde{T}\tilde{\psi}_\alpha\|_{C(\beta\mathbb{N})/C_0(\beta\mathbb{N})} = 1$. So by the open mapping theorem the range of \tilde{T} cannot be closed in $C(\beta\mathbb{N})/C_0(\beta\mathbb{N})$. \square

TOPOLOGICAL DYNAMICS OF $(\mathcal{G}, \beta\mathbb{N})$

We start with a definition that will be useful in considering \mathcal{G} -invariant subsets of \mathbb{N}^* :

A subset A of \mathbb{N} is *almost invariant* if the symmetric difference $A \Delta gA$ is a set of density 0 for all $g \in \mathcal{G}$.

A function $f \in C(\beta\mathbb{N})$ (resp. in $\ell^\infty(\mathbb{N})$) is *almost invariant* if for all $\epsilon > 0$ and $g \in \mathcal{G}$ the set $\{n \in \mathbb{N} : |f(n) - f(gn)| > \epsilon\}$ is of density 0. Clearly $A \subset \mathbb{N}$ is almost invariant iff the characteristic function χ_A is almost invariant.

The easiest nontrivial example of an almost invariant set (which we will need for the proof of Theorems 6 and 7) is the following:

Lemma 4. *Let $(m_i)_{i=0}^\infty, (n_i)_{i=0}^\infty$ be sequences in \mathbb{N} such that $m_i < n_i < m_{i+1}$ and $m_{i+1} > qm_i$ for $i = 0, 1, 2, \dots$ and some real constant $q > 1$. Then the set $A := \bigcup_{i=0}^\infty [m_i, n_i]$ is almost invariant.*

Proof. For fixed $g \in \mathcal{G}$ let ϕ be a positive function on \mathbb{R}_+ , which is decreasing to 0 and satisfies $n\phi(n) \geq |\{i \leq n : gi > n\}|$ and $(n-1)\phi(n-1) \leq n\phi(n)$, $n = 1, 2, \dots$

We can estimate the density function of the set $A \setminus gA$ at n by the number of points crossing the boundary of the set $A \cap [1, n]$ under the action of g :

Since

$$\begin{aligned} |[m, n] \setminus g([m, n])| &\leq |\{i \leq m-1 : gi > m-1\}| + |\{i \leq n : gi > n\}| \\ &\leq (m-1)\phi(m-1) + n\phi(n), \end{aligned}$$

we obtain

$$\begin{aligned} nd_{A \setminus gA}(n) &\leq \sum_{m_i \leq n} (m_i - 1)\phi(m_i - 1) + \sum_{n_i \leq n} n_i\phi(n_i) + (n)\phi(n) \\ &\leq \sum_{m_i \leq n} m_i\phi(m_i) + \sum_{n_i \leq n} n_i\phi(n_i) + n\phi(n) \leq 2 \sum_{n_i \leq n} n_i\phi(n_i) + 2n\phi(n). \end{aligned}$$

For $n \geq m_0$, $0 \leq i < I(n)$ with $I(n) := \sup\{i : m_i \leq n\}$, we have

$$q^i \leq m_i < n_i < m_{i+1} = (m_{i+1}q^{I(n)-i-1})q^{i+1-I(n)} \leq m_{I(n)}q^{i+1-I(n)} < nq^{i+1-I(n)}$$

and $n_i\phi(n_i) < nq^{i+1-I(n)}\phi(q^i)$ for $i < I(n)$. We thus obtain

$$d_{A \setminus gA}(n) \leq \frac{2}{n} \sum_{i=0}^{I(n)} n_i \phi(n_i) + 2\phi(n) < 2 \sum_{i=0}^{I(n)-1} q^{i+1-I(n)} \phi(q^i) + 4\phi(n).$$

To see that $\overline{d_{A \setminus gA}} = 0$, it remains to show that $S(k) := \sum_{i=0}^k q^{i-k} \phi(q^i)$ converges to 0 as $k \rightarrow \infty$: We have $S(k+1) = S(k)/q + \phi(q^{k+1})$, therefore

$$(5) \quad \limsup_{k \rightarrow \infty} S(k) = \limsup_{k \rightarrow \infty} S(k+1) = \limsup_{k \rightarrow \infty} S(k)/q + \phi(q^{k+1}) = \limsup_{k \rightarrow \infty} S(k)/q.$$

Since $S(k)$ is bounded by $\phi(1) \sum_i q^{-i}$, we obtain $\lim_{k \rightarrow \infty} S(k) = 0$.

So the set $A \setminus gA$ has density 0 $\forall g \in \mathcal{G}$. As $gA \setminus A = g(A \setminus g^{-1}A)$ it follows by Lemma 2 that $0 = \overline{d_{A \setminus g^{-1}A}} = \overline{d_{gA \setminus A}}$, hence $\overline{d_{A \Delta gA}} = 0$ and the set A is almost invariant. \square

Sets of positive upper density have been studied in combinatorial number theory, since they always contain subsets which exhibit a distinguished behaviour under certain translations. See [10] for an overview. The following Theorem 3, which will be used to describe the closure of the set of almost periodic points in $\beta\mathbb{N}$, shows that sets with positive upper density also contain subsets which have special properties under the action of the Lévy group.

We first prove

Lemma 5. *Let (c_i) be a sequence in \mathbb{N} with $c_{i+1} > 2c_i$.*

(a) *For a subset D of \mathbb{N} there exists a subsequence $(n_{D,i})$ of (n_i) and an absolutely continuous function ζ_D on the unit interval $I = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$, such that*

$$(6) \quad 0 \leq ((1+x)\zeta_D(x))' \leq 1 \quad \text{a.e. in } I$$

and $\lim_{i \rightarrow \infty} d_D([(1+x)c_{D,i}]) = \zeta_D(x)$ uniformly for $x \in I$.

(b) *If the sequence (c_i) satisfies furthermore $\frac{c_i}{c_{i+1}} \xrightarrow{i \rightarrow \infty} 0$ and if ζ is an absolutely continuous real-valued function on I with $\zeta(0) = 0$ and*

$$(7) \quad 0 \leq ((1+x)\zeta(x))' \leq \frac{1}{m} \quad \text{a.e. in } I$$

for some $m \in \mathbb{N}$, then there is a subset D of $\bigcup_i [c_i, 2c_i]$ with $|D \cap [n, n+m-1]| \leq 1$ $\forall n \in \mathbb{N}$, such that $\lim_{i \rightarrow \infty} d_D([(1+x)c_i]) = \zeta(x)$ uniformly for $x \in I$.

Proof. We first note that the definition of the density function gives for any $D \subset \mathbb{N}$

$$(8) \quad d_D(n+l) = \frac{n}{n+l} d_D(n) + \frac{1}{n+l} |D \cap [n+1, n+l]|,$$

hence

$$(9) \quad -\frac{l}{n+l} d_D(n) \leq d_D(n+l) - d_D(n) \leq -\frac{l}{n+l} d_D(n) + \frac{l}{n+l} \quad \text{for } n, l \in \mathbb{N}.$$

We define a sequence of functions (Ψ_i) on I by

$$\Psi_i(x) := ([xc_i] + 1 - c_i x) d_D([(1+x)c_i]) + (c_i x - [xc_i]) d_D([(1+x)c_i] + 1) \quad \forall x \in I,$$

i.e. Ψ_i is a piecewise linear function with values $\Psi_i\left(\frac{l}{c_i}\right) = d_D(c_i + l)$ at the nodes $\frac{l}{c_i}$, $l = 0, 1, \dots, c_i$. We now have by (9)

$$\left| \Psi_i\left(\frac{l}{c_i}\right) - \Psi_i\left(\frac{l+1}{c_i}\right) \right| = |d_D(c_i + l) - d_D(c_i + l + 1)| \leq \frac{1}{c_i + l + 1} < \frac{1}{c_i},$$

so Ψ_i is Lipschitz-continuous with Lipschitz-constant 1 and (Ψ_i) is an equicontinuous family of functions. It now follows from Arzelà-Ascoli's theorem (cf. [6], IV.6.7, I.6.13), that we may pass to a subsequence $(\Psi_{D,i})$ of (Ψ_i) , such that the functions $\Psi_{D,i}$ converge uniformly on I to some continuous function ζ_D . We denote the corresponding subsequence of (c_i) by $(c_{D,i})$. Since all functions Ψ_i are Lipschitz-continuous with Lipschitz constant 1, so is their uniform limit ζ_D . Therefore ζ_D is absolutely continuous and the derivative ζ'_D exists a.e. in I .

Since $d_D([(1+x)c_{D,i}]) = \Psi_{D,i}([xc_{D,i}]/c_{D,i})$ and $[xc_{D,i}]/c_{D,i} \xrightarrow{i \rightarrow \infty} x$ uniformly in I , it follows from the uniform convergence of the sequence $(\Psi_{D,i})$ to the continuous function ζ_D , that $\lim_{i \rightarrow \infty} d_D([(1+x)c_{D,i}]) = \zeta_D(x)$ uniformly for $x \in I$.

Assume that the sequences (n_i) and (l_i) satisfy $\frac{n_i}{c_i} \rightarrow 1+x$ and $\frac{l_i}{c_i} \rightarrow \epsilon$ for some $x \in I$ and $0 < \epsilon \leq 1-x$, so that $\lim_{i \rightarrow \infty} d_D(n_i) = \zeta_D(x)$ and $\lim_{i \rightarrow \infty} d_D(n_i + l_i) = \zeta_D(x + \epsilon)$. Then

$$\frac{l_i}{n_i + l_i} = \frac{l_i/c_{D,i}}{n_i/c_{D,i} + l_i/c_{D,i}} \xrightarrow{i \rightarrow \infty} \frac{\epsilon}{1+x+\epsilon}$$

and we obtain with (9)

$$-\frac{\epsilon}{1+x+\epsilon} \zeta_D(x) \leq \zeta_D(x+\epsilon) - \zeta_D(x) \leq -\frac{\epsilon}{1+x+\epsilon} \zeta_D(x) + \frac{\epsilon}{1+x+\epsilon}.$$

Hence

$$(1+x)\zeta_D(x) \leq (1+x+\epsilon)\zeta_D(x+\epsilon) \leq (1+x)\zeta_D(x) + \epsilon$$

and

$$0 \leq \frac{(1+x+\epsilon)\zeta_D(x+\epsilon) - (1+x)\zeta_D(x)}{\epsilon} \leq 1.$$

Therefore

$$0 \leq ((1+x)\zeta_D(x))' \leq 1$$

if this derivative exists in x . As we observed above ζ_D is differentiable a.e. in I , so (6) follows.

To prove part (b) of the lemma let ϱ for $n \in [c_i, 2c_i]$ be given by

$$(10) \quad \varrho(n) = \zeta\left(\frac{n-c_i}{c_i}\right).$$

We now define a subset D of $\bigcup_i [c_i, 2c_i]$ in the obvious way inductively as follows: $c_i \notin D$ and for $c_i < n \leq 2c_i$

$$n \in D \Leftrightarrow (d_D(n-1) < \varrho(n-1), [n-m+1, n-1] \cap D = \emptyset).$$

We claim that $\lim_{n \rightarrow \infty} d_D(n) - \varrho(n) = 0$ as $n \rightarrow \infty$ in $\bigcup_i [c_i, 2c_i]$. We first note that since ζ is absolutely continuous, so is the function $t \rightarrow (1+t)\zeta(t)$ on I , and we have (cf. [11], 18.16)

$$(1+x+y)\zeta(x+y) - (1+x)\zeta(x) = \int_x^{x+y} ((1+t)\zeta(t))' dt$$

for $x, y, x + y \in I$, so (7) yields

$$\frac{1+x}{1+x+y}\zeta(x) \leq \zeta(x+y) \leq \frac{1+x}{1+x+y}\zeta(x) + \frac{1}{m} \frac{y}{1+x+y}.$$

With (10) we obtain by setting $y = k/c_i$ and $x = (n - c_i)/c_i$

$$(11) \quad \frac{n}{n+k}\varrho(n) \leq \varrho(n+k) \leq \frac{n}{n+k}\varrho(n) + \frac{1}{m} \frac{k}{n+k}$$

for $c_i \leq n \leq 2c_i$, $0 \leq k \leq 2c_i - n$.

For $d_D(n) \geq \varrho(n)$ our definition of the set D implies $n+1 \notin D$, so $d_D(n+1) = \frac{n}{n+1}d_D(n)$ and we obtain with (11)

$$(12) \quad \begin{aligned} d_D(n+1) - \varrho(n+1) &\leq \frac{n}{n+1}d_D(n) - \frac{n}{n+1}\varrho(n) \\ &= \frac{n}{n+1}(d_D(n) - \varrho(n)) \leq d_D(n) - \varrho(n). \end{aligned}$$

Thus if we have $d_D(n_0) \leq \varrho(n_0)$ for some n_0 , $c_i \leq n_0 < n \leq 2c_i$ and $d_D(l) \geq \varrho(l)$ for $l = n_0 + 1, \dots, n$ we obtain from (9), (11) and (12):

$$(13) \quad \begin{aligned} d_D(n) - \varrho(n) &\leq d_D(n_0 + 1) - \varrho(n_0 + 1) \\ &\leq \frac{n_0}{n_0 + 1}d_D(n_0) + \frac{1}{n_0 + 1} - \frac{n_0}{n_0 + 1}\varrho(n_0) \leq \frac{1}{n_0 + 1} \leq \frac{1}{c_i}. \end{aligned}$$

Otherwise, if no such n_0 exists, i.e. if $d_D \geq \varrho$ in $[c_i, n]$, then $D \cap [c_i, n]$ is empty and d_D is decreasing in $[c_i, n]$. We then have $d_D(n) - \varrho(n) \leq d_D(c_i)$.

Together with (13) we obtain $\forall n \in [c_i, 2c_i]$

$$d_D(n) - \varrho(n) \leq \max(d_D(c_i), 1/c_i).$$

As $c_i/c_{i+1} \rightarrow 0$ and D is supported in $\bigcup_i [c_i, 2c_i]$ we have $d_D(c_i) \rightarrow 0$ and obtain

$$(14) \quad \overline{\lim}_{n \rightarrow \infty} d_D(n) - \varrho(n) \leq 0$$

as n tends to ∞ in $\bigcup_i [c_i, 2c_i]$.

To obtain a lower bound for $d_D - \varrho$ assume that $d_D(n) - \varrho(n) < 0$ for some $n \in [c_i, 2c_i]$. As $\varrho(c_i) = \zeta(0) = 0$ and $d_D(c_i) \geq 0$ there is an n_0 , $c_i \leq n_0 < 2c_i$ with $d_D(n_0) - \varrho(n_0) \geq 0$, such that $d_D(l) < \varrho(l)$ holds for $n_0 < l \leq n$. Our definition of the set D guarantees that for $n_0 < l \leq n - m$ there is exactly one element of $[l+1, l+m]$ in D , so $d_D(l+m) = \frac{l}{l+m}d_D(l) + \frac{1}{l+m}$ by (8), and we obtain with (11)

$$(15) \quad \begin{aligned} d_D(l+m) - \varrho(l+m) &\geq \frac{l}{l+m}d_D(l) + \frac{1}{l+m} - \frac{l}{l+m}\varrho(l) - \frac{1}{m} \frac{m}{l+m} \\ &= \frac{l}{l+m}(d_D(l) - \varrho(l)) \geq d_D(l) - \varrho(l) \end{aligned}$$

for $l \in [n_0 + 1, n - m]$.

For $n_0 < l \leq n_0 + m$ we have by (8) and (11)

$$d_D(l) - \varrho(l) \geq \frac{n_0}{l}d_D(n_0) - \frac{n_0}{l}\varrho(n_0) - \frac{1}{m} \frac{l - n_0}{l} \geq -\frac{1}{m} \frac{l - n_0}{l} \geq -\frac{1}{l} \geq -\frac{1}{c_i}.$$

Applying the estimate (15) repeatedly for $l = n - m, n - 2m, \dots, n - jm$, where $n_0 < n - jm \leq n_0 + m$ this yields $d_D(n) - \varrho(n) \geq d_D(n - jm) - \varrho(n - jm) \geq -\frac{1}{c_i}$ for $n \in [c_i, 2c_i]$ and we obtain

$$(16) \quad \overline{\lim}_{n \rightarrow \infty} d_D(n) - \varrho(n) \geq 0$$

as n tends to ∞ in $\bigcup_i [c_i, 2c_i]$.

It follows from (14) and (16) that $d_D(n) - \varrho(n)$ converges to 0 in $\bigcup_i [c_i, 2c_i]$.

As $\varrho([(1+x)c_i]) = \zeta\left(\frac{[xc_i]}{c_i}\right)$ and $\frac{[xc_i]}{c_i} \xrightarrow{i \rightarrow \infty} x$ uniformly on I the uniform convergence of $d_D([(1+x)c_i])$ to $\zeta(x)$ as $i \rightarrow \infty$ follows from the uniform continuity of ζ . \square

Corollary 1. *Let A_1, A_2 be disjoint subsets of \mathbb{N} and let (c_i) be a sequence in \mathbb{N} with $c_{i+1} > 2c_i$, such that*

$$d_{A_1}([(1+x)c_i]) \xrightarrow{i \rightarrow \infty} \zeta_{A_1}(x) \quad \text{and} \quad d_{A_2}([(1+x)c_i]) \xrightarrow{i \rightarrow \infty} \zeta_{A_2}(x) \quad \text{a.e. in } I.$$

Then

$$d_{A_1 \cup A_2}([(1+x)c_i]) \xrightarrow{i \rightarrow \infty} \zeta_{A_1 \cup A_2}(x) \quad \text{with } \zeta_{A_1 \cup A_2} = \zeta_{A_1} + \zeta_{A_2} \quad \text{a.e.}$$

Furthermore the functions ζ_{A_1}, ζ_{A_2} and $\zeta_{A_1 \cup A_2}$ are absolutely continuous on I and $((1+x)\zeta_{A_1}(x))' = 0$ a.e. in some Borel subset E of I if $((1+x)\zeta_{A_1 \cup A_2}(x))' = 0$, or if $((1+x)\zeta_{A_2}(x))' = 1$ a.e. in E .

Proof. Since for disjoint subsets A_1, A_2 of \mathbb{N} we have $d_{A_1 \cup A_2} = d_{A_1} + d_{A_2}$, the convergence to the function $\zeta_{A_1} + \zeta_{A_2}$ is obvious.

The second claim now follows from

$$((1+x)\zeta_{A_1 \cup A_2}(x))' = ((1+x)\zeta_{A_1}(x))' + ((1+x)\zeta_{A_2}(x))'$$

and Lemma 5 (a), since the functions $\zeta_{A_1}, \zeta_{A_2}, \zeta_{A_1 \cup A_2}$ all are absolutely continuous with their derivatives taking values between 0 and 1 only a.e. in I . \square

Theorem 3. *Let A be a subset of \mathbb{N} with positive upper density and set $m = \lceil 2/\overline{d}_A \rceil + 1$, then there are $g_1, \dots, g_m \in \mathcal{G}$ such that $\bigcup_{i=1}^m g_i A$ contains an almost invariant set B with $\overline{d}_B > 0$.*

Proof. Let (c_i) be a sequence in \mathbb{N} with $\lim_{i \rightarrow \infty} d_A(2c_i) = \overline{d}_A$, $c_{i+1} > 2c_i$ and $c_i/c_{i+1} \xrightarrow{i \rightarrow \infty} 0$. By Lemma 5 (a) there is a subsequence $(c_{A,i})$ of (c_i) and an absolutely continuous function ζ_A satisfying (6), such that

$$(17) \quad \lim_{i \rightarrow \infty} d_A([(1+x)c_{A,i}]) = \zeta_A(x) \quad \text{uniformly for } x \in I.$$

Then $\zeta_A(x) \leq \overline{d}_A \forall x \in I$ and $\zeta_A(1) = \overline{d}_A$. Since ζ_A is absolutely continuous, there exists a Borel subset E of I with positive Lebesgue measure, such that $\zeta'_A(x) \geq 0$ and $\zeta_A(x) \geq \overline{d}_A/2 > 1/m$ for $x \in E$. (In fact by the continuity of ζ_A there is an $\alpha < 1$ with $\zeta_A(x) \geq \overline{d}_A/2 = \zeta_A(1)/2 > 0$ for $\alpha \leq x \leq 1$. If ζ'_A were negative a.e. in $\{y : \alpha \leq y \leq 1\}$, then by [11], 18.16 we would obtain $\zeta_A(\alpha) > \zeta_A(1) = \overline{d}_A$ contradicting (17).) It follows with (6) that

$$(18) \quad ((1+x)\zeta_A(x))' > \frac{1}{m} \quad \text{for } x \in E, \quad 0 \leq ((1+x)\zeta_A(x))' \leq 1 \quad \text{a.e. in } I.$$

We define the absolutely continuous function

$$(19) \quad \zeta(x) := \frac{1}{m} \frac{1}{1+x} \int_0^x \chi_E(t) dt, \quad x \in I, \quad \text{so } ((1+x)\zeta(x))' = \frac{1}{m} \chi_E(x) \quad \text{a.e. in } I$$

by [11], 18.17 and inequality (7) is satisfied. Applying Lemma 5 (b) to the sequence $(c_{A,i})$ we find a subset B_0 of $\bigcup_i [c_{A,i}, 2c_{A,i}]$ with

$$(20) \quad \lim_{i \rightarrow \infty} d_{B_0}([(1+x)c_{A,i}]) = \zeta(x) \quad \text{uniformly in } I.$$

We now define the set B by

$$B = \{n \in \bigcup_i [c_{A,i}, 2c_{A,i}] : [n-m+1, n] \cap B_0 \neq \emptyset\}.$$

So B is essentially (up to the negligible boundaries of the intervals $[c_{A,i}, 2c_{A,i}]$) the union of m pairwise disjoint right translates of B_0 . We therefore have $d_B(n) - md_{B_0}(n) \xrightarrow{n \rightarrow \infty} 0$ and with (20)

$$(21) \quad \lim_{i \rightarrow \infty} d_B([(1+x)c_{A,i}]) =: \zeta_B(x) = m\zeta(x) \quad \text{uniformly in } I.$$

We have to prove that B is an almost invariant subset of \mathbb{N} . We first show that for $g \in \mathcal{G}$ and $C_g := gB \setminus B$ we have $\overline{d_{C_g}} = 0$:

Assume that

$$\lim_{i \rightarrow \infty} \sup_{n \in [\tilde{c}_i, 2\tilde{c}_i]} d_{C_g}(n) =: \Gamma > 0$$

for some subsequence (\tilde{c}_i) of $(c_{A,i})$. We then apply Lemma 5 (a) to the sequence (\tilde{c}_i) and the set C_g to obtain a subsequence $(c_{C_g,i})$ of (\tilde{c}_i) and an absolutely continuous function ζ_{C_g} for which

$$(22) \quad \lim_{i \rightarrow \infty} d_{C_g}([(1+x)c_{C_g,i}]) = \zeta_{C_g}(x) \quad \text{uniformly in } I, \quad \text{and} \quad \max_{x \in I} \zeta_{C_g}(x) = \Gamma > 0.$$

Since $B \cap C_g = \emptyset$ it follows from Corollary 1 that $((1+x)\zeta_{C_g}(x))' = 0$ a.e. on E , since $((1+x)\zeta_B(x))' = 1$ a.e. on E by (19), (21).

From Lemma 2 and (21) it follows that

$$(23) \quad \lim_{i \rightarrow \infty} d_{gB}([(1+x)c_{A,i}]) = \zeta_B(x) \quad \text{uniformly in } I.$$

Since gB is the disjoint union of the sets C_g and $gB \cap B$ it follows with (22) and (23) that

$$\lim_{i \rightarrow \infty} d_{gB \cap B}([(1+x)c_{A,i}]) = \zeta_B(x) - \zeta_{C_g}(x) \quad \text{uniformly in } I.$$

Applying Corollary 1 again to the disjoint sets C_g and $gB \cap B$ we see that $((1+x)\zeta_{C_g}(x))' = 0$ a.e. on $I \setminus E$, since $((1+x)\zeta_B(x))' = 0$ a.e. on $I \setminus E$ by (19), (21) and therefore $((1+x)(\zeta_B - \zeta_{C_g})(x))' = 0$ a.e. in $I \setminus E$.

Hence $((1+x)\zeta_{C_g})' = 0$ a.e. in I .

Since ζ_{C_g} is absolutely continuous by Lemma 5 and $\zeta_{C_g}(0) = 0$ it follows (cf. [11], 18.16) that $(1+x)\zeta_{C_g}(x) = \int_0^x ((1+t)\zeta_{C_g}(t))' dt = 0 \quad \forall x \in I$, which is a contradiction to $\Gamma > 0$ in (22).

Hence for all $g \in \mathcal{G}$ we have $\overline{d_{C_g}} = 0$. Since $B \setminus gB = g(g^{-1}B \setminus B) = gC_{g^{-1}}$ it follows with Lemma 2 that $\overline{d_{B \setminus gB}} = 0$. With $\overline{d_{C_g}} = 0$ this yields $\overline{d_{B \Delta gB}} = 0$, so B is almost invariant.

It remains to show that $B \subset \bigcup_{j=0}^{m-1} g_j A$ for some $g_0, g_1, \dots, g_{m-1} \in \mathcal{G}$.

We divide B into m mutually disjoint subsets B_0, B_1, \dots, B_{m-1} by translating B_0 : we set $B_j = \{n \in B : n-j \in B_0\} \cap (\bigcup_i [c_{A,i}, 2c_{A,i}])$.

It easily follows that $d_{B_j}(n) - d_{B_0}(n) \xrightarrow{n \rightarrow \infty} 0$, so by (20)

$$(24) \quad \lim_{i \rightarrow \infty} d_{B_j}([(1+x)c_{A,i}]) = \zeta(x) \quad \text{uniformly for } x \in I, \quad j = 0, \dots, m-1.$$

We now define inductively injective maps ρ_j from $B_j \setminus A$ to $A \setminus B_j$ as follows: For $k \in B_j \setminus A$ set

$$\rho_j(k) = \min\{l \in A \setminus B_j : l > k, l \notin \rho_j(\{n \in B_j \setminus A : n < k\})\}.$$

Since $((1+x)\zeta(x))' = \frac{1}{m}\chi_E$ a.e. by (19) and $\frac{1}{m}\chi_E \leq ((1+x)\zeta_A(x))'$ a.e. on I with the inequality being strict on E by (18), it follows from the absolute continuity of these functions and $\zeta_A(0) \geq \zeta(0) = 0$ that $\zeta_A(1) > \zeta(1)$ (cf. [11], 18.16). Hence $\lim_{i \rightarrow \infty} d_A(2c_{A,i}) > \lim_{i \rightarrow \infty} d_{B_j}(2c_{A,i})$, so $A \setminus B_j$ is infinite and this definition makes sense.

We define permutations g_j , $j = 0, \dots, m-1$ as the product of two-cycles:

$$g_j = \prod_{n \in B_j \setminus A} (n, \rho_j(n)), \quad \text{so } g_j(n) = \begin{cases} \rho_j(n), & n \in B_j \setminus A, \\ \rho_j^{-1}(n), & n \in \rho_j(B_j \setminus A), \\ n, & \text{otherwise,} \end{cases}$$

and claim that $g_j \in \mathcal{G}$. We have to show that

$$\phi_j(n) := \frac{1}{n} |\{l : l \leq n, g_j(l) > n\}| = \frac{1}{n} |\{l \in B_j \setminus A : l \leq n, \rho_j(l) > n\}|$$

tends to 0 for $j = 0, \dots, m-1$. We first note that by our definition of ρ_j we have

$$(25) \quad (n+1)\phi_j(n+1) - n\phi_j(n) = \begin{cases} 1, & n+1 \in B_j \setminus A, \\ -1, & n+1 \in A \setminus B_j, \quad \phi_j(n) > 0, \\ 0, & n+1 \in A \setminus B_j, \quad \phi_j(n) = 0, \\ 0, & n+1 \notin A \Delta B_j, \end{cases}$$

and it follows that if $\phi_j > 0$ in $[k+1, l-1]$ for some $l, k \in \mathbb{N}$, $k < l-1$, then $\phi_j(k) > 0$ or $k+1 \in B_j \setminus A$ and (25) yields

$$\begin{aligned} l\phi_j(l) - k\phi_j(k) &= |B_j \cap [k+1, l]| - |A \cap [k+1, l]| \\ &= l(d_{B_j}(l) - d_A(l)) - k(d_{B_j}(k) - d_A(k)). \end{aligned}$$

For $l \in [c_{A,i}, 2c_{A,i}]$ we set $k := \max(\{n \in \mathbb{N} : c_{A,i} \leq n \leq l, \phi_j(n) = 0\} \cup \{c_{A,i}\})$, so we have $k\phi(k) \leq c_{A,i}\phi(c_{A,i})$ and obtain

$$(26) \quad \begin{aligned} \phi_j(l) &\leq \frac{l}{c_{A,i}} \phi_j(l) = \frac{1}{c_{A,i}} (k\phi_j(k) + l(d_{B_j}(l) - d_A(l)) - k(d_{B_j}(k) - d_A(k))) \\ &\leq \phi_j(c_{A,i}) + \frac{l}{c_{A,i}} (d_{B_j}(l) - d_A(l)) - \frac{k}{c_{A,i}} (d_{B_j}(k) - d_A(k)). \end{aligned}$$

By (24) and (17) there is a sequence (ϵ_i) converging to 0, such that

$$\left| \zeta\left(\frac{n}{c_{A,i}} - 1\right) - d_{B_j}(n) \right| < \epsilon_i, \quad \left| \zeta_A\left(\frac{n}{c_{A,i}} - 1\right) - d_A(n) \right| < \epsilon_i$$

for all $n \in [c_{A,i}, 2c_{A,i}]$, $i \in \mathbb{N}$ and we obtain from (26)

$$(27) \quad \begin{aligned} \phi_j(l) &\leq \phi_j(c_{A,i}) + 8\epsilon_i + \frac{l}{c_{A,i}} \left(\zeta\left(\frac{l}{c_{A,i}} - 1\right) - \zeta_A\left(\frac{l}{c_{A,i}} - 1\right) \right) \\ &\quad - \frac{k}{c_{A,i}} \left(\zeta\left(\frac{k}{c_{A,i}} - 1\right) - \zeta_A\left(\frac{k}{c_{A,i}} - 1\right) \right). \end{aligned}$$

Now by the absolute continuity of ζ and ζ_A and (18), (19) we get (cf. [11], 18.17)

$$\begin{aligned} & \frac{l}{c_{A,i}} \left(\zeta \left(\frac{l}{c_{A,i}} - 1 \right) - \zeta_A \left(\frac{l}{c_{A,i}} - 1 \right) \right) - \frac{k}{c_{A,i}} \left(\zeta \left(\frac{k}{c_{A,i}} - 1 \right) - \zeta_A \left(\frac{k}{c_{A,i}} - 1 \right) \right) \\ &= \int_{\frac{k}{c_{A,i}} - 1}^{\frac{l}{c_{A,i}} - 1} ((1+s)\zeta(s))' - ((1+s)\zeta_A(s))' ds \\ &= \int_{\frac{k}{c_{A,i}} - 1}^{\frac{l}{c_{A,i}} - 1} \frac{1}{m} \chi_E(s) - ((1+s)\zeta_A(s))' ds < 0, \end{aligned}$$

so (27) yields

$$(28) \quad \phi_j(l) \leq \phi_j(c_{A,i}) + 8\epsilon_i \quad \forall l \in [c_{A,i}, 2c_{A,i}].$$

Since $B_j \subset \bigcup_i [c_{A,i}, 2c_{A,i}]$, it follows that ϕ_j is decreasing in $[2c_{A,i}, c_{A,i+1}]$. Since $c_{A,i}/c_{A,i+1} \rightarrow 0$ it follows that $\phi_j(c_{A,i}) \rightarrow 0$ as $i \rightarrow \infty$, and with (28) it follows that $\phi_j(n) \rightarrow 0$ as $n \rightarrow \infty$. So we have shown that $g_j \in \mathcal{G}$ for $j = 0, \dots, m-1$. \square

We define

$$S := \{p \in \beta\mathbb{N} : \text{the ultrafilter } p \text{ contains no set of density } 0\}.$$

This subset of $\beta\mathbb{N}$ has been used to investigate topological and algebraic properties of $\beta\mathbb{N}$. It can be shown that no point in S is the limit point of a countable number of P-points of \mathbb{N}^* (points p with the property that the countable intersection of neighborhoods of p is a neighborhood of p) cf. [20], 7.14. It is shown in [5], 7, that both S and $\mathbb{N}^* \setminus S$ are right ideals of the left topological semigroups $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$. It follows that the centers of both $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ are \mathbb{N} .

We need some properties of this set S , some of which are well known; cf. [5], 7.2, [20], 7.15.

Lemma 6. *S is a closed nonempty \mathcal{G} -invariant subset of \mathbb{N}^* with empty interior. If $A \subset \mathbb{N}$, then $\overline{A} \cap S \neq \emptyset$ iff $\overline{d}_A > 0$ and A is almost invariant iff the set $\overline{A} \cap S$ is \mathcal{G} -invariant.*

Proof. Since the finite intersection of subsets of \mathbb{N} with density 1 has density 1, the collection \mathfrak{F} of these sets is a filter. Any ultrafilter containing \mathfrak{F} is in S since the complement in \mathbb{N} of a set of density 0 has density 1, so no set of density 0 can be contained in an ultrafilter containing \mathfrak{F} .

If $p \notin S$ there exists $B \subset \mathbb{N}$ with $\overline{d}_B = 0$ and $p \in \overline{B}$. Since \overline{B} is open and $\overline{B} \cap S = \emptyset$ it follows that S is closed. Clearly S contains no points of \mathbb{N} , i.e. $S \subset \mathbb{N}^*$. Since any subset of \mathbb{N} contains a subset of density 0 no clopen set \overline{A} , $A \subseteq \mathbb{N}$, is contained in S , so S has empty interior.

If $\overline{d}_A > 0$ and B has density 1 we have $\overline{d_{A \cap B}} = \overline{d}_A$ and $A \cap B \neq \emptyset$, hence $\mathfrak{F} \cap A$ is a filter base and $\overline{A} \cap S$ is nonempty since any filter contains an ultrafilter by Zorn's lemma. In particular S is nonempty.

If $\overline{A} \cap S \neq \emptyset$, then $\exists p \in \overline{A} \cap S$, hence $A \in p$, so $\overline{d}_A > 0$ since $p \in S$.

S is \mathcal{G} -invariant since by Lemma 2 $\overline{d}_A > 0$ iff $\overline{d_{gA}} > 0$, so $p \in S \Leftrightarrow gp \in S$ for $g \in \mathcal{G}$.

It follows that A is almost invariant iff $\overline{A \Delta gA} \cap S = \emptyset \quad \forall g \in \mathcal{G}$.

We next observe that for $A, B \subset \mathbb{N}$ we have

$$\begin{aligned}\overline{A \setminus B} &= \{p \in \beta\mathbb{N} : A \setminus B \in p\} \\ &= \{p \in \beta\mathbb{N} : A \in p \wedge B \notin p\} = \overline{A} \setminus \overline{B}.\end{aligned}$$

Since $g\overline{A} = \overline{gA}$ we have $\overline{A \Delta gA} = \overline{A \setminus gA} \cup \overline{gA \setminus A} = (\overline{A} \setminus \overline{gA}) \cup (\overline{gA} \setminus \overline{A})$, so

$$\begin{aligned}\overline{A \Delta gA} \cap S &= \emptyset \Leftrightarrow ((\overline{A} \setminus \overline{gA}) \cap S) \cup ((\overline{gA} \setminus \overline{A}) \cap S) = \emptyset \\ &\Leftrightarrow ((\overline{A} \cap S) \setminus (\overline{gA} \cap S)) \cup ((\overline{gA} \cap S) \setminus (\overline{A} \cap S)) = \emptyset \\ &\Leftrightarrow \overline{A} \cap S = \overline{gA} \cap S.\end{aligned}$$

Since S is \mathcal{G} -invariant $\overline{gA} \cap S = g(\overline{A} \cap S)$, so A is almost invariant iff $\overline{A} \cap S = g(\overline{A} \cap S)$ $\forall g \in \mathcal{G}$, i.e. iff $\overline{A} \cap S$ is \mathcal{G} -invariant. \square

Among the properties we will investigate is almost periodicity. We recall the pertinent definitions:

A subset A of a topological group G is *syndetic* if there is a compact subset K of G such that $G = KA$. A point p in the phase space X of the dynamical system (G, X) is *almost periodic* if for each neighborhood U of p there exists a syndetic subset A such that $gp \in U \forall g \in A$. Our group \mathcal{G} is discrete, so K will be finite. (For abelian group actions these points are called *uniformly recurrent* in [8].)

In the dynamical system (G, X) the point $p \in X$ has *minimal orbit closure* if \overline{Gp} does not contain nontrivial closed invariant subsets.

For our dynamical system $(\mathcal{G}, \beta\mathbb{N})$ these properties coincide since we have

Lemma 7. *For a dynamical system (G, X) , X compact, a point $p \in X$ is almost periodic iff p has minimal orbit closure.*

For the proof see [19], IV 1.2 or [9], 4.05, 4.10.

We are now ready to prove the main theorem of this section. We recall that the *support* $\text{supp}(\mu)$ of a (signed) measure μ is the smallest closed set with the property that any continuous function supported in the complement of this set is in the annihilator of μ . The *support set* of \mathcal{G} -invariant measures is the smallest closed set containing the support of any \mathcal{G} -invariant measure.

Theorem 4. *For $x \in \mathbb{N}^*$ the following are equivalent:*

- (i) $x \in S$;
- (ii) the \mathcal{G}_δ -orbit $\mathcal{G}_\delta x$ is not dense in \mathbb{N}^* ;
- (iii) the \mathcal{G} -orbit $\mathcal{G}x$ is not dense in \mathbb{N}^* ;
- (iv) x is in the support set of \mathcal{G} -invariant measures;
- (v) $Tf(x) = Tf(gx) \quad \forall f \in C(\beta\mathbb{N}), \forall g \in \mathcal{G}$;
- (vi) $f(x) = f(gx) \quad \text{for all almost invariant functions } f \in C(\beta\mathbb{N}) \text{ and } g \in \mathcal{G}$;
- (vii) x is in the closure of the set of points which are almost periodic with respect to \mathcal{G} .

Proof. We will first show (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). Using the equivalence of (i) and (iii) we then show (i) \Rightarrow (iv) \Rightarrow (iii), (i) \Rightarrow (v) \Rightarrow (iii), (i) \Rightarrow (vi) \Rightarrow (iii) and (i) \Rightarrow (vii) \Rightarrow (iii), where the implication (ii) \Rightarrow (iii) is obvious.

(i) \Rightarrow (ii): Assume y has dense orbit under the action of \mathcal{G}_δ . Taking any infinite set $B \subset \mathbb{N}$ with density 0, then $\exists g \in \mathcal{G}_\delta$ such that $gy \in \overline{B}$ since \overline{B} is open. Then

$y \in g^{-1}\overline{B} = \overline{g^{-1}B}$. This shows that the ultrafilter y contains a set of density 0 since \mathcal{G}_δ is density preserving hence $y \notin S$.

(iii) \Rightarrow (i): For $y \in \mathbb{N}^* \setminus S$ we can find an infinite subset A of \mathbb{N} of density 0 with $y \in \overline{A}$. Let B be an arbitrary infinite subset of \mathbb{N} and $B_0 \subset B$ an infinite subset of B of density 0. By Lemma 3 $\exists g \in \mathcal{G}$ such that $gA = B_0$. Therefore $gy \in \overline{B_0} \subset \overline{B}$. Since the collection of clopen sets \overline{B} , B an infinite subset of \mathbb{N} , is a base for the topology of \mathbb{N}^* , it follows that the \mathcal{G} -orbit of y is dense in \mathbb{N}^* .

(i) \Rightarrow (iv): Assume that $A \in x$ for some $A \subset \mathbb{N}$. Since $x \in S$ the function $T\chi_{\overline{A}}$ (which is just the continuous extension of d_A to $\beta\mathbb{N}$) does not vanish on \mathbb{N}^* , so there is a $q \in \mathbb{N}^*$, such that $T\chi_{\overline{A}}(q) = \langle T^*\delta_q, \chi_{\overline{A}} \rangle \neq 0$. By Theorem 2 the measure $T^*\delta_q$ is \mathcal{G} -invariant and has support intersecting \overline{A} . As the collection of sets \overline{A} , $x \in \overline{A}$, $A \subset \mathbb{N}$, is a neighborhood base for x , it follows that x is in the support set of \mathcal{G} -invariant measures.

(iv) \Rightarrow (iii): Since there exists an uncountable number of pairwise almost disjoint infinite subsets of \mathbb{N} (sets with pairwise finite intersection) cf. [20], 3.21, there exists an uncountable number of pairwise disjoint open subsets of \mathbb{N}^* . Therefore no measure on \mathbb{N}^* can have \mathbb{N}^* as its support. But if μ is \mathcal{G} -invariant, then the support of μ is \mathcal{G} -invariant. Therefore $\mathcal{G}y$ cannot be dense in \mathbb{N}^* for any y in the support of μ . So the support of any \mathcal{G} -invariant measure is contained in S by the equivalence of (i) and (iii). Since S is closed by Lemma 6, it follows that the support set of \mathcal{G} -invariant measures is contained in S .

(i) \Rightarrow (v): Let $g \in \mathcal{G}$, $f \in C(\beta\mathbb{N})$ with $Tf(y) = 1$, $Tf(gy) = 0$ for some $y \in \mathbb{N}^*$. Let $B_1 \subset \mathbb{N}$ be in the ultrafilter y , such that $Tf(n) > \frac{2}{3}$ for $n \in B_1$, and $B_0 \subset \mathbb{N}$ in the ultrafilter gy , such that $Tf(n) < \frac{1}{3}$ for $n \in B_0$. Now $y \in \overline{B_1}$ and $gy \in \overline{B_0}$ implies $y \in \overline{A}$ with $A := B_1 \cap g^{-1}B_0$. So $A \in y$ and $gA \in gy$. We want to show that A has density 0, i.e. $y \notin S$.

Since the sets A and gA are disjoint, there are numbers

$$M_1 \leq N_1 \leq \dots < M_i \leq N_i < M_{i+1} \leq N_{i+1} < \dots$$

defined by the properties

$$\begin{aligned} A &\subseteq \bigcup_i [M_i, N_i], & \{M_i, N_i\} &\in A, \\ gA &\subseteq \bigcup_i N_i, M_{i+1} \cup [1, M_1[, & gA \cap N_i, M_{i+1} & \neq \emptyset \quad \forall i. \end{aligned}$$

Since $N_i \in A$, $gA \cap N_i, M_{i+1} \neq \emptyset$ and $gA \subset B_0$, it follows that $Tf(N_i) > \frac{2}{3}$ and $Tf(N_i + k) < \frac{1}{3}$ for some $k < M_{i+1} - N_i$. From the estimate

$$|Tf(n) - Tf(n+i)| = \left| \left(\frac{1}{n} - \frac{1}{n+i} \right) \sum_{j=1}^n f(j) - \frac{1}{n+i} \sum_{j=n+1}^{n+i} f(j) \right| \leq \frac{2i\|f\|}{n+i}$$

it follows that

$$\frac{1}{3} < |Tf(N_i) - Tf(N_i + k)| \leq \frac{2k\|f\|}{N_i + k} < \frac{2k\|f\|}{N_i}.$$

Therefore $k > N_i/6\|f\|$ and we see that our sequences are lacunary, i.e.

$$N_{i+1} \geq M_{i+1} > N_i q \geq M_i q, \quad 1 < q = 1 + \frac{1}{6\|f\|}.$$

Since $gA \subset \bigcup_i N_i, M_{i+1}[$ we obtain the decomposition $A = A_1 \cup A_2$ with

$$A_1 := \{n \in A : gn < n\} = \bigcup_i \{n \in A \cap [M_i, N_i] : gn < M_i\},$$

$$A_2 := \{n \in A : gn > n\} = \bigcup_i \{n \in A \cap [M_i, N_i] : gn > N_i\}.$$

It now suffices to show that both A_1 and A_2 are sets of density 0.

For $M_K \leq N \leq M_{K+1}$ we have

$$(29) \quad \frac{1}{N} |\{n \in A_1 : n \leq N\}| \leq \frac{1}{M_K} \sum_{i=1}^K \phi_i M_i,$$

where $\phi_i = |\{n \in A \cap [M_i, N_i] : gn < M_i\}|/M_i \rightarrow 0$ as $i \rightarrow \infty$ since $g \in \mathcal{G}$. With the above lacunarity property $M_{i+1} > qM_i$ and $q > 1$ we obtain by an argument similar to (5) in the proof of Lemma 4

$$\lim_{K \rightarrow \infty} \frac{1}{M_K} \sum_{i=1}^K \phi_i M_i \leq \lim_{K \rightarrow \infty} \sum_{i=1}^K \phi_i q^{i-K} = 0,$$

which shows with (29) that A_1 has density 0.

For A_2 and $N_{K-1} \leq M < N_K$ we have

$$(30) \quad \frac{1}{N} |\{n \in A_2 : n \leq N\}| \leq \frac{1}{N_{K-1}} \sum_{i=1}^{K-1} \phi_i N_i + r_N,$$

where $\phi_i = |\{n \in A \cap [M_i, N_i] : gn > N_i\}|/N_i \rightarrow 0$ and

$$r_N = \frac{1}{N} |\{n \in A \cap [M_K, N] : gn > N_K\}| \leq \frac{1}{N} |\{n \in A \cap [M_K, N] : gn > N\}|.$$

Again since $g \in \mathcal{G}$ both ϕ_i and r_N converge to 0 as i respectively N go to ∞ . As above it follows that

$$\lim_{K \rightarrow \infty} \frac{1}{N_{K-1}} \sum_{i=1}^{K-1} \phi_i N_i \leq \lim_{K \rightarrow \infty} \sum_{i=1}^{K-1} \phi_i q^{i-K+1} = 0,$$

and with (30) we see that A_2 has density 0.

(v) \Rightarrow (iii): If the \mathcal{G} -orbit of x is dense in \mathbb{N}^* , then condition (v) implies that Tf is constant on \mathbb{N}^* , which is equivalent to f being Cesàro-summable. Therefore (v) can only hold for all f in $C(\beta\mathbb{N})$ if the \mathcal{G} -orbit of x is not dense.

(i) \Rightarrow (vi): Assume f is not constant on the \mathcal{G} -orbit of x , say $f(x) + 3\epsilon < f(gx)$ for some $\epsilon > 0$ and $g \in \mathcal{G}$. Then the set $C := \{n : f(n) < f(x) + \epsilon, f(gn) > f(gx) - \epsilon\}$ is contained in the ultrafilter x , so C has positive upper density \overline{d}_C since $x \in S$. But $\{n : |f(n) - f(gn)| > \epsilon\} \supset C$, so f is not almost invariant.

(vi) \Rightarrow (iii): If A is an infinite subset of \mathbb{N} with density 0, it follows by Lemma 2 that gA and therefore $A \triangle gA$ have density 0, so $\chi_{\overline{A}}$ and $\chi_{\mathbb{C}\overline{A}}$ are almost invariant functions. Now (vi) implies $\mathcal{G}x \subset \overline{A}$ if $x \in \overline{A}$ and $\mathcal{G}x \subset \mathbb{C}\overline{A}$ if $x \in \mathbb{C}\overline{A}$. In either case the \mathcal{G} -orbit of x is not dense in \mathbb{N}^* since both \overline{A} and $\mathbb{C}\overline{A} = \overline{\mathbb{C}A}$ are open.

(i) \Rightarrow (vii): Let \overline{A} , $A \subset \mathbb{N}$ be a neighborhood of x . Since $x \in S$ we have $\overline{d}_A > 0$. By Theorem 3 there are $g_1, \dots, g_m \in \mathcal{G}$ such that $B \subset \bigcup_{i=1}^m g_i A$ for an almost invariant set $B \subset \mathbb{N}$ with $\overline{d}_B > 0$. $\overline{B} \cap S$ is nonempty, compact and \mathcal{G} -invariant by Lemma 6. A standard application of Zorn's lemma shows that any nonempty compact invariant set contains a minimal orbit closure (cf. [19], II 3.12 or [9],

2.22), so it contains almost periodic points by Lemma 7. If $y \in \overline{B} \cap S$ is almost periodic, then it follows from Lemma 7 that gy is almost periodic for $g \in \mathcal{G}$. Now for some i the point $g_i^{-1}y$ is in \overline{A} since $\overline{B} \subset \bigcup_{i=1}^m g_i \overline{A}$ and $y \in \overline{B}$. Hence each clopen neighborhood \overline{A} of x contains an almost periodic point. Since these neighborhoods are a base for the neighborhood filter of x we are done.

(vii) \Rightarrow (iii): If a point $y \in \mathbb{N}^*$ has dense \mathcal{G} -orbit, then the orbit closure of y is not minimal since S is a closed invariant proper subset of \mathbb{N}^* , so y is not almost periodic by Lemma 7. By Lemma 6 and the equivalence of (i) and (iii) the set S of points without dense \mathcal{G} -orbit is closed in $\beta\mathbb{N}$, so the closure of the set of almost periodic points is contained in S . \square

Theorem 5. $f \in \ell^\infty(\mathbb{N})$ is almost invariant iff its continuous extension to $\beta\mathbb{N}$ is constant on the \mathcal{G} -orbits in S .

Proof. Since (i) implies (vi) in Theorem 4 it remains to show that functions which are not almost invariant are not constant on some \mathcal{G} -orbit in S . If for some $\epsilon > 0$ the set $\{n : f(n) < f(gn) - 3\epsilon\}$ has positive upper density, then for some $k \in \mathbb{Z}$ and $g \in \mathcal{G}$ the set $C := \{n : f(n) < k\epsilon, f(gn) > (k+1)\epsilon\}$ has positive upper density. So $\overline{C} \cap S \neq \emptyset$ by Lemma 6 and we have $f(x) \leq k\epsilon, f(gx) \geq (k+1)\epsilon$ for $x \in \overline{C} \cap S$, so f is not constant on the \mathcal{G} -orbit of $x \in \overline{C} \cap S$. \square

With Theorem 4 we therefore have:

Corollary 2. The function Tf is almost invariant $\forall f \in \ell^\infty(\mathbb{N})$.

For the dynamical system $(\tau, \beta\mathbb{N})$, τ being the shift operator, the support set of τ -invariant measures was investigated in [17] and [4]. Lemma 1.3 in [4] gives a characterization using a different notion of upper density that strongly resembles the equivalence of (i) and (iv) in Theorem 4 above. Concerning the equivalence of (iv) and (vii) in Theorem 4 we remark that for the shift operator (or more generally certain amenable semigroup actions on the Stone-Čech compactification of the semigroup) there always exist open subsets in the support set of invariant measures, which do not contain almost periodic points cf. [4], [7]. So (iv) \Rightarrow (vii) is false in the setting of the dynamical system $(\tau, \beta\mathbb{N})$.

NON-MINIMAL ORBIT CLOSURES IN S

In view of Theorem 4 it is natural to ask whether every point in S is almost periodic, or equivalently, if all orbit closures in S are minimal. We will show that there are non-minimal orbit closures in S and that there are minimal orbit closures in S which do not support any invariant measure. This property contrasts the case of amenable group actions on a compact space, where any closed nonempty invariant set supports an invariant measure cf. [19], IV 6.11.

We first define ultrafilters with non-minimal orbit closures: For $i, j, n = 0, 1, \dots$ let

$$(31) \quad \begin{aligned} M_j &:= \{2^j k : k \in \mathbb{N}\}, & J_{n,j} &= \left[2^{\frac{n(n+1)}{2}+j}, 2^{\frac{n(n+1)}{2}+j+1} \right], \\ I_{n,j} &:= J_{n,j} \cap M_j, & E_i &:= \bigcup_{n=i}^{\infty} \bigcup_{j=i}^n I_{n,j}, \end{aligned}$$

i.e. E_0 consists of successive intervals of doubling length with the “densities” in these intervals being $2^0, 2^0, 2^{-1}, 2^0, 2^{-1}, 2^{-2}, 2^0, 2^{-1}, 2^{-2}, 2^{-3}, \dots$ respectively. E_i then is just E_0 with all intervals with density greater than 2^{-i} being removed.

Clearly any set E_i has positive upper density and $E_i \cap E_j = E_j$ for $j \geq i$. Together with the collection of all sets of density one, the sets E_i define a filter base for a filter \mathfrak{F}_0 with all ultrafilters containing \mathfrak{F}_0 being in S .

Theorem 6. *No ultrafilter containing \mathfrak{F}_0 is almost periodic with respect to \mathcal{G} .*

Proof. Let p be an ultrafilter containing \mathfrak{F}_0 . To show that p is not an almost periodic point of $\beta\mathbb{N}$, we have to show that for some neighborhood U of p the set $\{g : gp \in U\}$ is not syndetic. Our neighborhood will be $\overline{E_0}$. To prove that the set $\mathcal{A} := \{g : gp \in \overline{E_0}\}$ is not syndetic, we have to find for any finite collection g_0, g_1, \dots, g_m of elements in \mathcal{G} a $g' \in \mathcal{G}$ with $g'p \notin \bigcup_{j=0}^m g_j \mathcal{A}p$. As $\mathcal{A}p \subset \overline{E_0}$ this will follow, if we show that $g'p \notin \bigcup_{j=0}^m g_j \overline{E_0}$.

We choose $i \in \mathbb{N}$ such that $2^i > m + 1$ and define sets $A_0 := g_0 E_i$, $A_1 := g_1 E_i \setminus g_0 E_i, \dots, A_l := g_l E_i \setminus \left(\bigcup_{j=0}^{l-1} g_j E_i \right)$, $1 \leq l \leq m$, so $\bigcup_{l=0}^m g_l E_i$ is the disjoint union of the sets A_l , $l = 0, 1, \dots, m$. It follows that

$$(32) \quad d_{\bigcup_{l=0}^m g_l E_i}(n) = \sum_{l=0}^m d_{A_l}(n) \quad \text{for } n \in \mathbb{N}.$$

We define for $0 \leq l \leq m$ sets $B_l := \tau^l g_l^{-1} A_l \subset \tau^l E_i$, τ being the right shift operator $\tau C = \{n \in \mathbb{N} : n - 1 \in C\}$ and $B := \bigcup_{l=0}^m B_l \subset \bigcup_{l=0}^m \tau^l E_i$. Now since $d_{\tau^l C}(n) = \frac{n-l}{n} d_C(n-l)$, for $C \subset \mathbb{N}$, it follows with Lemma 2, that $\lim_{n \rightarrow \infty} d_{B_l}(n) - d_{A_l}(n) = 0$. Since the sets $\tau^l E_i$, are pairwise disjoint for $l = 0, \dots, m$ by our choice of i , so are the sets B_l , hence $d_B(n) = \sum_{l=0}^m d_{B_l}(n)$ and with (32) we obtain

$$\lim_{n \rightarrow \infty} d_B(n) - d_{\bigcup_{j=0}^m g_j E_i}(n) = \sum_{l=0}^m \lim_{n \rightarrow \infty} d_{B_l}(n) - d_{A_l}(n) = 0.$$

By Lemma 3 there exists $g \in \mathcal{G}$ with $gB = \bigcup_{j=0}^m g_j E_i$. Since $B \subset \bigcup_{l=0}^m \tau^l E_i$ we have $(\tau^{m+1} E_i) \cap B = \emptyset$ and $(g\tau^{m+1} E_i) \cap \left(\bigcup_{j=0}^m g_j E_i \right) = \emptyset$. Applying Lemma 3 again, we see that $\tau^{m+1} E_i = \tilde{g} E_i$, $\tilde{g} \in \mathcal{G}$, so $g\tau^{m+1} E_i = g' E_i$ for some $g' \in \mathcal{G}$ and $\bigcup_{j=0}^m g_j E_i \cap g' E_i = \emptyset$, hence $\bigcup_{j=0}^m g_j \overline{E_i} \cap g' \overline{E_i} = \emptyset$. Since $p \in \overline{E_i}$ it follows that $g'p \notin \bigcup_{j=0}^m g_j \overline{E_i}$.

We define with (31) a set $F_i := \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{i-1} J_{k,l}$. By Lemma 4 F_i is almost invariant. Since $E_i \cap F_i = \emptyset$, it follows that $p \notin \overline{F_i}$, and since $p \in S$, we conclude with Lemma 6 that $g'p \notin \overline{F_i}$. But $E_0 \subset F_i \cup E_i$, so $\bigcup g_j \overline{E_0} \cap S \subset (\overline{F_i} \cup g_j \overline{E_i}) \cap S$. Since we have shown that $g'p \in S$ is neither in $\bigcup_{j=0}^m g_j \overline{E_i}$ nor in $\overline{F_i}$, it follows that $g'p \notin \bigcup_{j=0}^m g_j \overline{E_0}$. \square

Theorem 7. *No invariant measure is supported in the \mathcal{G} -orbit closure of any ultrafilter p containing \mathfrak{F}_0 . Therefore there exist minimal \mathcal{G} -orbit closures in S which do not support any \mathcal{G} -invariant measure.*

Proof. It is sufficient to consider probability measures only since the positive part μ_+ of an invariant measure μ is invariant (cf. (3) in the proof of Lemma 1). So let μ be a \mathcal{G} -invariant probability measure supported in $\overline{\mathcal{G}p}$. By Lemma 4 the set $D_i := \bigcup_{n=0}^{\infty} \bigcup_{j=i}^n J_{n,j}$ is almost invariant for $i \in \mathbb{N}$. Clearly $D_i \in \mathfrak{F}_0$, so $p \in \overline{D_i} \cap S$

and $\overline{\mathcal{G}p} \subseteq \overline{D_i}$ by Theorem 4 since $\chi_{\overline{D_i}}$ is an almost invariant function and $p \in S$. Then $\mu(\overline{E_0}) = \mu(\overline{E_i})$ since $\overline{E_0} \setminus \overline{E_i}$ is disjoint from $\overline{D_i}$, and hence disjoint from $\overline{\mathcal{G}p}$. As the permutation g_0 defined by $g_0 n = n + 1$ for $n \neq 2^l - 1$, $g_0(2^l - 1) = 2^{l-1}$, $l \geq 1$, is in \mathcal{G} and the sets $g_0^k \overline{E_i}$, $k = 0, \dots, 2^i - 1$ are pairwise disjoint, it follows that $\mu(\overline{E_0}) = \mu(\overline{E_i}) < 2^{-i}$. Since i was arbitrary, it follows that $\mu(\overline{E_0}) = 0$ and p is not in the support of μ as $p \in \overline{E_0}$.

The second statement follows since every orbit closure on a compact space contains a point with minimal orbit closure by Zorn's lemma. \square

Corollary 3. \mathcal{G} is not amenable.

Proof. Amenability is equivalent to the existence of invariant means on any compact \mathcal{G} -space cf. [19], IV 6.11. \square

T^* -INVARIANT MEASURES

It is easily seen that the measure δ_1 is the only T^* -invariant probability measure supported in \mathbb{N} . Let

$$\mathcal{M}_I = \{\mu \in \mathcal{M}(\beta\mathbb{N}) : \langle \mu, f \rangle = \langle T^* \mu, f \rangle = \langle \mu, Tf \rangle \quad \forall f \in C(\beta\mathbb{N})\} \cap \mathcal{M}_\infty$$

be the space of T^* -invariant measures in \mathcal{M}_∞ (space of measures supported in \mathbb{N}^*). Clearly \mathcal{M}_I is a weak*-closed subspace of \mathcal{M} . Since weak and norm-closure of a linear subspace coincide (cf. [6], V.3.13), it follows from the bipolar theorem [18], III.7.4, that

$$(33) \quad \mathcal{M}_I^\perp = (\text{Span}(\text{Ran}(I - T) \cup \{e_i\}))^{\perp\perp} = \overline{\text{Span}(\text{Ran}(I - T) \cup \{e_i\})}$$

(norm-closure), where $e_1 \in C(\beta\mathbb{N})$ is given by $e_1(1) = 1$, $e_1(n) = 0$, $n > 1$. The support of any measure in \mathcal{M}_I is disjoint from \mathbb{N} , so it follows from Theorem 2 that $\mathcal{M}_I \subseteq \mathcal{M}_\mathcal{G}$.

We denote the set of probability measures in \mathcal{M}_∞ by \mathcal{P} . As T^* maps \mathcal{P} to \mathcal{P} , the compact convex set \mathcal{P} has a fixed point under the map T^* by the Markov-Kakutani fixed point theorem, and \mathcal{M}_I is not empty. We can moreover give an explicit description (similar to Theorem 2) of a subset of \mathcal{M}_I which has a weak*-dense linear hull in \mathcal{M}_I :

Let

$$\mathcal{L}_I = \left\{ f \in C(\beta\mathbb{N}) : \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f(p) = 0 \quad \forall p \in \mathbb{N}^* \right\}.$$

Clearly \mathcal{L}_I is a norm (and therefore weakly) closed subspace of $C(\beta\mathbb{N})$ containing \mathcal{L}_0 (as defined in the proof of Theorem 2). For arbitrary f in $C(\beta\mathbb{N})$ the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f(p)$$

need not exist for all p . For fixed $p \in \beta\mathbb{N}$ the sequence $(c_N(p))_{N=1}^\infty$ with $c_N(p) := \frac{1}{N} \sum_{n=1}^N T^n f(p)$ can be considered as an element of $\ell^\infty(\mathbb{N})$. The continuous extension to $\beta\mathbb{N}$ can be evaluated at a point q of $\beta\mathbb{N}$. In order to distinguish between our phase space $\beta\mathbb{N}$ and this functional on $\ell^\infty(\mathbb{N})$ we shall use Z. Frolík's notation $q\text{-}\lim_{N \in \mathbb{N}} c_N(p)$ for the evaluation of the continuous extension of $(c_N(p))$ to $\beta\mathbb{N}$ at the point q . If $p, q \in \mathbb{N}^*$ and $f \in C(\beta\mathbb{N})$ then $q\text{-}\lim_{N \in \mathbb{N}} \frac{1}{N} \sum_{n=1}^N T^n f(p)$ is a positive linear functional of norm 1 on $C(\beta\mathbb{N})$, which we denote by $\nu_{p,q}$.

It is easily seen that $\nu_{p,q}$ is a T^* -invariant probability measure. We now show that the linear hull \mathcal{H} of the set $\{\nu_{p,q} : p, q \in \mathbb{N}^*\}$ is weak*-dense in \mathcal{M}_I :

Let $f \in \mathcal{H}^\perp$, i.e.

$$q\text{-}\lim_{N \in \mathbb{N}} \frac{1}{N} \sum_{n=1}^N T^n f(p) = 0 \quad \forall p, q \in \mathbb{N}^*,$$

then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f(p) = 0 \quad \forall p \in \mathbb{N}^*,$$

so $f \in \mathcal{L}_I$. Therefore $\mathcal{H}^\perp \subseteq \mathcal{L}_I$.

On the other hand, for $\mu \in \mathcal{M}_I$ and $f \in \mathcal{L}_I$, we obtain

$$\langle \mu, f \rangle = \langle T^* \mu, f \rangle = \frac{1}{N} \left\langle \sum_{n=1}^N T^{*n} \mu, f \right\rangle = \left\langle \mu, \frac{1}{N} \sum_{n=1}^N T^n f \right\rangle,$$

which by Lebesgue's convergence theorem yields $\langle \mu, f \rangle = \langle \mu, 0 \rangle = 0$, so $\mathcal{M}_I^\perp \supseteq \mathcal{L}_I$. Since $\mathcal{M}_I \supseteq \mathcal{H}$ we have $\mathcal{M}_I^\perp \subseteq \mathcal{H}^\perp$, so we obtain

$$(34) \quad \mathcal{H}^\perp \subseteq \mathcal{L}_I \subseteq \mathcal{M}_I^\perp \subseteq \mathcal{H}^\perp,$$

and it follows that $\mathcal{H}^{\perp\perp} = \mathcal{M}_I^{\perp\perp}$. Again by the bipolar theorem we see that $\mathcal{H}^{\perp\perp}$ is the weak*-closure of \mathcal{H} and $\mathcal{M}_I^{\perp\perp} = \mathcal{M}_I$, since \mathcal{M}_I being the annihilator of $\text{Ran}(I - T)$ is weak*-closed. With (33) we thus have shown:

Theorem 8. \mathcal{M}_I is a weak*-closed subspace of \mathcal{M}_G with annihilator $\mathcal{M}_I^\perp = \mathcal{L}_I = \overline{\text{Span}(\text{Ran}(I - T) \cup \{e_i\})}$. \mathcal{H} is weak*-dense in \mathcal{M}_I .

We now determine the support set of measures in \mathcal{M}_I and get a similar characterization as the equivalence of (i) and (iv) in Theorem 4 :

Corollary 4. If for $A \subset \mathbb{N}$ we have $\langle \nu_{p,q}, \chi_{\overline{A}} \rangle = 0 \forall p, q \in \mathbb{N}^*$, then \overline{A} is disjoint from the support set of \mathcal{M}_I .

Proof. Since $\nu_{p,q}$ is a probability measure and $\chi_{\overline{A}}$ is positive, it follows that the support of $\nu_{p,q}$ does not intersect \overline{A} for $p, q \in \mathbb{N}^*$. Therefore any continuous function on $\beta\mathbb{N}$ vanishing outside of \overline{A} is in \mathcal{H}^\perp . But $\mathcal{H}^\perp = \mathcal{M}_I^\perp$ by Theorem 8 or (34), so the open set \overline{A} is disjoint from the support of any measure in \mathcal{M}_I , and the claim follows. \square

Let S_I be the set of points x in $\beta\mathbb{N}$ with the property that no clopen neighborhood \overline{A} , $A \subset \mathbb{N}$, of x satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n \chi_{\overline{A}}(p) = 0 \quad \forall p \in \mathbb{N}^*,$$

i.e. for no clopen neighborhood \overline{A} , $A \subset \mathbb{N}$, of x is the function $\chi_{\overline{A}}$ in \mathcal{L}_I .

Theorem 9. S_I is the support set of \mathcal{M}_I .

Proof. If $x \notin S_I$, then there exists a clopen neighborhood \overline{A} of x with

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n \chi_{\overline{A}}(p) = 0$$

for all $p \in \mathbb{N}^*$, therefore $\langle \nu_{p,q}, \chi_{\bar{A}} \rangle = 0 \quad \forall p, q \in \mathbb{N}^*$. It follows from Corollary 4 that x is not in the support set of \mathcal{M}_I .

If $x \in S_I$, then for any clopen neighborhood \bar{A} of x , there exists a measure $\nu_{p,q}$, such that $\langle \nu_{p,q}, \chi_{\bar{A}} \rangle \neq 0$, i.e. \bar{A} intersects the support of $\nu_{p,q}$ for some $p, q \in \mathbb{N}^*$. Since the clopen sets containing x are a neighborhood base of x it follows that x is in the support set of \mathcal{M}_I . \square

Theorem 10. S_I is a nonempty proper closed subset of S .

Proof. Clearly $S_I \subseteq S$ and S_I is closed and nonempty by Theorem 9, as support sets are by definition closed and nonempty if invariant measures exist.

To prove that $S_I \neq S$ we show that there exists a subset A of $\mathbb{N} \setminus \{1\}$ with $\overline{d_A} > 0$ such that $\chi_{\bar{A}} \in \overline{\text{Ran}(I - T)}$ (norm-closure): By Lemma 6 $\bar{A} \cap S \neq \emptyset$, but by Theorem 8 $\chi_{\bar{A}} \in \mathcal{M}_I^\perp$, so Corollary 4 with Theorem 9 show that $\bar{A} \cap S_I = \emptyset$, hence $S_I \neq S$.

For any $A \subset \mathbb{N} \setminus \{1\}$ we can define inductively a function F with $(I - T)F(n) = \chi_A(n)$. However F will not be bounded in general, since F must be constant on $\mathbb{C}A$ and increasing on A (see the computations below). We therefore define the function f on \mathbb{N} , such that f decreases on $\mathbb{C}A$ in order to guarantee boundedness, but still satisfies $f - Tf - \chi_A \in c_0$. Now $g \in c_0(\mathbb{N})$ is in $\overline{\text{Ran}(I - T)}$ by Theorem 8 if $g(1) = 0$, so this will be sufficient to show that $\chi_{\bar{A}} \in \overline{\text{Ran}(I - T)}$.

The set A will be $\bigcup_i [N_i, 2N_i]$ for a suitably chosen sequence (N_i) . Clearly A has upper density $\overline{d_A} \geq 1/2$, so \bar{A} intersects S by Lemma 6.

Let $N_1 = 3$ and $f(1) = f(2) = 0$. Assume N_1, \dots, N_n have been chosen and f has been defined on $[1, l - 1]$ for some l , $N_n \leq l \leq 2N_n$. We now define $f(l)$ by

$$(35) \quad f(l) := Tf(l - 1) + \frac{l}{l - 1},$$

and obtain

$$(36) \quad \begin{aligned} f(l) - Tf(l) &= \frac{l - 1}{l} f(l) - \frac{1}{l} \sum_{j=1}^{l-1} f(j) \\ &= \frac{l - 1}{l} (f(l) - Tf(l - 1)) = \frac{l - 1}{l} \frac{l}{l - 1} = 1. \end{aligned}$$

For $N_n < l \leq 2N_n$ this yields

$$\begin{aligned} f(l) &= Tf(l - 1) + \frac{l}{l - 1} \\ &= f(l - 1) - 1 + \frac{l}{l - 1} = f(l - 1) + \frac{1}{l - 1}. \end{aligned}$$

Therefore f is increasing in $[N_n, 2N_n]$ and we obtain

$$(37) \quad \begin{aligned} f(2N_n) &= f(N_n) + \sum_{l=N_n}^{2N_n-1} \frac{1}{l} \leq f(N_n) + \log(2N_n - 1) - \log(N_n - 1) \\ &\leq f(N_n) + 2. \end{aligned}$$

For $l > 2N_n$ and $f(l - 1) > \frac{1}{n}$ we define

$$(38) \quad f(l) := Tf(l - 1) - \frac{l}{l - 1} \frac{1}{n},$$

and obtain

$$(39) \quad f(l) - Tf(l) = \frac{l-1}{l} (f(l) - Tf(l-1)) = -\frac{l-1}{l} \frac{1}{n} \frac{l}{l-1} = -\frac{1}{n}.$$

For $l = 2N_n + 1$ we have with (38), (36)

$$(40) \quad f(2N_n + 1) = Tf(2N_n) - \frac{2N_n + 1}{2N_n} \frac{1}{n} = f(2N_n) - 1 - \frac{2N_n + 1}{2N_n} \frac{1}{n},$$

and for $l > 2N_n + 1$, $f(l-1) > \frac{1}{n}$ (38) and (39) yield

$$(41) \quad f(l) = Tf(l-1) - \frac{l}{l-1} \frac{1}{n} = f(l-1) + \frac{1}{n} - \frac{l}{l-1} \frac{1}{n} = f(l-1) - \frac{1}{n} \frac{1}{l-1} > 0.$$

Therefore there exists a smallest $l_n > N_n$ such that $f(l_n - 1) \leq \frac{1}{n}$ and we set $N_{n+1} := l_n$ and continue by defining f on $[N_{n+1}, 2N_{n+1}]$ as above.

The function f so constructed satisfies $(f - Tf)(l) = 1$ on $A = \bigcup_i [N_n, 2N_n]$ by (36) and $|(f - Tf)(l)| = \frac{1}{n}$ on the intervals $]2N_n, N_{n+1}[$ by (39). It follows that $f - Tf - \chi_A \in c_0$.

We have $f(N_{n+1} - 1) < \frac{1}{n}$ by our choice of N_{n+1} , hence

$$Tf(N_{n+1} - 1) - \frac{1}{n} = f(N_{n+1} - 1) < \frac{1}{n}$$

by (39), so

$$f(N_{n+1}) = Tf(N_{n+1} - 1) + \frac{N_{n+1}}{N_{n+1} - 1} < \frac{2}{n} + \frac{N_{n+1}}{N_{n+1} - 1} < 4$$

by (35) and $f(2N_{n+1}) < 6$ by (37). Since f is increasing on $[N_n, 2N_n]$ and decreasing on $]2N_n, N_{n+1}[$ by (40) and (37), it follows with (41) that $0 \leq f(n) < 6 \forall n \in \mathbb{N}$, i.e. $f \in \ell^\infty(\mathbb{N})$. \square

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