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βnbc-BASES FOR COHOMOLOGY OF LOCAL SYSTEMS ON HYPERPLANE COMPLEMENTS

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In memory of Michitake Kita

ABSTRACT. We study cohomology with coefficients in a rank one local system on the complement of an arrangement of hyperplanes $\mathcal{A}$. The cohomology plays an important role for the theory of generalized hypergeometric functions. We combine several known results to construct explicit bases of logarithmic forms for the only non-vanishing cohomology group, under some nonresonance conditions on the local system, for any arrangement $\mathcal{A}$. The bases are determined by a linear ordering of the hyperplanes, and are indexed by certain "no-broken-circuits" bases of $\mathcal{A}$. The basic forms depend on the local system, but any two bases constructed in this way are related by a matrix of integer constants which depend only on the linear orders and not on the local system. In certain special cases we show the existence of bases of monomial logarithmic forms.

1. INTRODUCTION

Let V be the ℓ-dimensional affine space over $\mathbb{C}$, and let $\mathcal{A}$ be an arrangement of hyperplanes in V. We fix a linear order on $\mathcal{A}$ and write $\mathcal{A} = \{H_1, \dots , H_n\}$. Let $M = M(\mathcal{A}) = V - \bigcup_{i=1}^{n} H_i$ denote the complement of $\mathcal{A}$. Let $L = L(\mathcal{A})$ be the intersection poset of $\mathcal{A}$. By definition L is the set of nonempty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. By convention L includes V as its unique minimal element. Then L is a ranked poset, with $r(X) = \operatorname{codim}(X)$, and all maximal elements have the same rank [16, Lemma 2.4]. The rank of $\mathcal{A}$ is then the rank of any maximal element of L, denoted by r.

Let α_i be a polynomial function of degree one on V which vanishes exactly on H_i, for $1 \leq i \leq n$. Given a complex n-vector $\lambda = (\lambda_1, \dots , \lambda_n)$, consider the multivalued holomorphic function U_λ defined on M by $U_\lambda = \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n}$. Let $\mathcal{L}_\lambda$ be the rank one local system on M whose local sections are constant multiples of branches of $1/U_\lambda$.

The cohomology $H^r(M, \mathcal{L}_\lambda)$ is important for the theory of integrals of U_λ, the Aomoto-Gelfand theory of generalized hypergeometric functions, which itself has many important applications in diverse areas of mathematics. Consult [11, 3, 20] for references. In this context the cohomology $H^q(M, \mathcal{L}_\lambda)$ was the main subject of [2, 15, 10], among others.

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It is known that $H^q(M, \mathcal{L}_\lambda)$ vanishes for $q \neq r$ under certain nonresonance conditions on λ. The main goal of this note is to find an explicit basis for $H^r(M, \mathcal{L}_\lambda)$ for arbitrary $\mathcal{A}$ and generic λ. This basis depends on the choice of linear order on $\mathcal{A}$ and is, in some sense, the analogue of the **nbc** ("no broken circuits") basis for the ordinary cohomology $H^r(M, \mathbb{C})$ [4, 16]. The basis is parametrized by the β**nbc**-bases of $\mathcal{A}$, as defined by Ziegler in [23]. Our main result is obtained by combining the results of [10], [22], and [23].

When $\mathcal{A}$ is a general position arrangement, with real defining forms, our basis coincides with the basis which was first found by Aomoto [1, p. 292]. If $\mathcal{A}$ is a normal arrangement or in general position to infinity (both defined in section 4), again with real defining forms, then our basic forms coincide with forms constructed by Varchenko [19, 6.2] – see Remark 4.7. The construction in this paper applies in complete generality, without the restriction to real defining forms. The resulting basis is determined by the underlying combinatorial structure (the affine matroid), whereas the bases constructed in [1, 19] depend on the geometry of the associated real arrangement. In independent work, A. Douai [9] has produced a basis for $H^r(\mathcal{A})$ for general $\mathcal{A}$ and generic (i.e. transcendental) λ. The methods and the resulting basis are different from ours. In addition, construction of a basis for $H^r(\mathcal{A})$ forms a part of the thesis of H. Kanarek [13].

For real arrangements Varchenko [19] conjectures a formula for the determinant of a matrix of hypergeometric integrals $\int_{\Delta_i} U_\lambda \omega_{\Delta_j}$ whose rows are indexed by the bounded chambers Δ of the real form of $\mathcal{A}$, and whose columns are indexed by certain hypergeometric forms $\omega_\Delta \in H^r(M, \mathcal{L}_\lambda)$. There is a natural one-to-one correspondence between bounded chambers and β**nbc**-bases of $\mathcal{A}$. So our result provides a variation of Varchenko's conjecture in which the forms ω_Δ are replaced by our basic forms Ξ_B. These forms are easier to describe and satisfy a simple recursion, perhaps facilitating a proof of Varchenko's conjecture. This alternate formulation is adopted in [9]. In the cases where the determinant formula has been proven, the forms ω_Δ coincide with the Ξ_B, but not in general.

In the definition of generalized hypergeometric functions, the arrangement $\mathcal{A}$ is considered to be an independent variable varying over a parameter space Γ of arrangements with constant underlying matroid. The cohomology groups $H^r(M, \mathcal{L}_\lambda)$ comprise a vector bundle over Γ. Because of the combinatorial nature of our construction, we actually get a complete system of independent global sections, i.e., a global frame, of this vector bundle. This may allow one to give more explicit formulas for the associated generalized hypergeometric functions.

In the case of affine supersolvable arrangements, defined in Remark 2.6, it is particularly easy to write down the β**nbc**-bases of $\mathcal{A}$. This case includes the discriminantal arrangements of [18], also called Selberg-type arrangements in [3]. These arrangements are parametrized by the configuration space consisting of sets of n pairwise distinct point in $\mathbb{C}^n$; thus the arrangements do not in general have real defining forms. Schechtman and Varchenko use this setup to show that solutions of the Knizhnik-Zamolodchikov equations of conformal field theory are given by generalized hypergeometric integrals [18]. The β**nbc**-bases of global sections may then be used to construct explicit solutions of the KZ equations. Along the same lines, such a basis may be used to study the monodromy action on the twisted cohomology, in particular for calculation of the monodromy of the KZ equation. This is one focus of the dissertation [13].

Unless otherwise noted, we adopt the notation of [16], which should also be consulted for definitions and background material on arrangements.

In section 2 we review the relevant combinatorial constructions as developed in [23]. These notions have been reformulated in more geometric language, and all references to the associated central arrangement $c\mathcal{A}$ have been eliminated. We define the broken circuit complex BC of $\mathcal{A}$, whose simplices are **nbc** sets of $\mathcal{A}$. The lexicographic order on the facets of BC is a shelling order, and leads immediately to the notion of β**nbc**-base. We observe that the β**nbc**-bases yield a natural basis for the cohomology $\tilde{H}^{r-1}(BC)$. The β**nbc**-bases of $\mathcal{A}$ can also be constructed inductively using deletion-restriction [23, Thm. 1.5], and we describe this process. Example 2.5 illustrates these ideas.

Let F be the Folkman complex of $\mathcal{A}$, defined as the simplicial complex of chains in $L - \{V\}$. There is a natural homotopy equivalence of BC with the Folkman complex F, which we use to construct a basis for $\tilde{H}^{r-1}(F)$.

In section 3 we review and study the results of Yuzvinsky [22]. Define logarithmic differential forms $\omega_i = \frac{d\alpha_i}{\alpha_i}$ on V. The Orlik-Solomon algebra $A^{\cdot}$ of L is the finite dimensional graded $\mathbb{C}$-algebra generated by the ω_i under exterior product. By convention $A^0 = \mathbb{C}$. Define a logarithmic 1-form ω_λ by

$$\omega_\lambda = d(\log(U_\lambda)) = \sum_{i=1}^{n} \lambda_i \omega_i.$$

Also define a map $\omega_\lambda\wedge : A^p \to A^{p+1}$ by $w_\lambda\wedge(\eta) := \omega_\lambda\wedge\eta$. Then $(\omega_\lambda\wedge)^2 = 0$, so $(A^{\cdot}, \omega_\lambda\wedge)$ is a complex. In [22] Yuzvinsky showed that there exists a natural isomorphism

$$\phi : H^q(A^{\cdot}, \omega_\lambda\wedge) \to \tilde{H}^{q-1}(F),$$

using sheaf theory on posets, under certain genericity conditions on λ. This shows in particular that $H^q(A^{\cdot}, \omega_\lambda\wedge) = 0$ for $q < r$.

We explicitly describe the inverse map ϕ^{-1}, for $q = r$ (Proposition 3.6). The map ϕ^{-1} turns out to coincide up to sign with the one induced by the map S^r introduced and studied by Schechtman and Varchenko in [18, 3.2]. The image of the β**nbc**-basis for $\tilde{H}^{r-1}(F)$ under ϕ^{-1} yields a basis for $H^r(A^{\cdot}, \omega_\lambda\wedge)$. On the other hand, Esnault, Schechtman, and Viehweg proved in [10] that the de Rham map

$$H^q(A^{\cdot}, \omega_\lambda\wedge) \to H^q(M, \mathcal{L}_\lambda)$$

is an isomorphism under appropriate nonresonance conditions on λ. A more general version of this result, with fewer restrictions on λ, appears in [17]. Thus we obtain a basis of logarithmic forms for $H^r(M, \mathcal{L}_\lambda)$. The explicit description of this basis is in Theorem 3.9.

More important than the result itself is the method of proof. Yuzvinsky's isomorphism allows questions about the local system cohomology to be carried over to the broken circuit complex, where powerful combinatorial tools apply. In particular, the parameter λ does not appear in BC. We can conclude that two bases constructed as above from different linear orderings of $\mathcal{A}$ are connected by a transition matrix which is independent of λ, and in fact has integral entries.

One might ask if the monomials $\omega_B = \prod_{k=1}^{r} \omega_{i_k}$, for $B = (H_{i_1}, \ldots, H_{i_r})$, $i_1 < \cdots < i_r$, themselves form a basis, as B ranges over the β**nbc**-bases of $\mathcal{A}$. In section 4 we show that this is the case, under the same nonresonance conditions on λ, when the linear ordering on $\mathcal{A}$ is *unmixed* (Def. 4.1). If $\mathcal{A}$ is in general position, or, more generally, if the hyperplane at infinity is generic relative to $\mathcal{A}$, then every linear order is unmixed. This is also the case when $\bigcup \mathcal{A}$ is a normal crossing divisor. There are some other special linear orders on arrangements of rank two for which these monomials form a basis. But without imposing some additional unnatural genericity requirements on λ, the monomials above will not form a basis for arbitrary orders, even in rank two.

2. Broken circuits and β-systems

In this section we establish some notation and develop the combinatorial tools needed for the proof of the main theorem in section 3. All this material is adapted from the references [6] and [23].

Let V be the ℓ-dimensional affine space over $\mathbb{C}$. Let $\mathcal{A}$ be an arrangement of hyperplanes in V. Fix a linear order on $\mathcal{A}$ by labelling the hyperplanes $H_1, \dots, H_n$. We will sometimes denote this linear order by $<$ when no confusion will result. The *intersection poset* L of $\mathcal{A}$ consists of the nonempty affine subspaces $X = \bigcap_{i \in I} H_i$ for $I \subseteq \{1, \dots n\}$, partially ordered by reverse inclusion. We will occasionally write H_I for $\bigcap_{i \in I} H_i$. The ambient space V, corresponding to $I = \emptyset$, is the unique minimal element of L. The maximal elements of L all have the same codimension [16, Lemma 2.4], which we denote by r. For $X \in L$ we set $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$.

A subset $\{H_i \mid i \in I\}$ of $\mathcal{A}$ is *dependent* if $\bigcap_{i \in I} H_i \neq \emptyset$ and $\operatorname{codim}(\bigcap_{i \in I} H_i) < |I|$. A subset of $\mathcal{A}$ which has nonempty intersection and is not dependent is called *independent*. Maximal independent sets are called *bases*; every base has cardinality r.

An inclusion-minimal dependent set is called a *circuit*. A *broken circuit* is a subset S of $\mathcal{A}$ for which there exists $H < \min(S)$ such that $\{H\} \cup S$ is a circuit. The collection of subsets of $\mathcal{A}$ having nonempty intersection and containing no broken circuits is a simplicial complex we call the *broken circuit complex* of $\mathcal{A}$, denoted by BC. This is a pure $(r-1)$-dimensional complex consisting of independent sets. Simplices of BC are called **nbc** sets, and facets (maximal simplices) of BC are bases of $\mathcal{A}$ called **nbc**-bases. We say an ordered base $(H_{i_1}, \dots, H_{i_r})$ is *standard* if $i_1 < \dots < i_r$. We denote by **nbc** the set of standard ordered **nbc**-bases of $\mathcal{A}$.

The *Folkman complex* of L is the simplicial complex of linearly ordered subsets $(X_0 > \dots > X_p)$ of $L - \{V\}$. It is also a pure $(r-1)$-dimensional simplicial complex, which we denote by F.

The purpose of this section is to expose a natural basis for $\tilde{H}^{r-1}(F)$, given the linear ordering on $\mathcal{A}$. First we describe the structure of BC, following [23]. A *shelling order* for a simplicial complex K is a linear order $\prec$ on the set $\mathcal{F}$ of facets of K such that $\sigma \cap (\bigcup_{\tau \prec \sigma} \tau)$ is a nonempty union of facets of σ, for each $\sigma \in \mathcal{F} - \min(\mathcal{F})$. Suppose $\prec$ is a shelling order for K. A facet σ is called a *homology facet* if $\sigma \cap (\bigcup_{\tau \prec \sigma} \tau) = \partial\sigma$. It is easy to see that the union of the non-homology facets of K is a contractible subcomplex, and thus K has the homotopy type of a bouquet of spheres, one for each homology facet.

Theorem 2.1 ([5, Thm. 7.4.3]). *The lexicographic ordering of* **nbc** *is a shelling order for* BC.

If $B \in$ **nbc** and $H \in B$, then the facet $B - \{H\}$ of B lies in $B \cap (\bigcup_{B' \prec B} B')$ if and only if $B' := (B - \{H\}) \cup \{H'\}$ is an **nbc**-base for some $H' < H$. By [23, Lemma 1.2] it is enough that B' is a base for some $H' < H$. This determines the homology facets in the shelling of Theorem 2.1.

Definition 2.2. A base B is called a β**nbc**-*base* if B is an **nbc**-base and for every $H \in B$ there exists $H' < H$ in $\mathcal{A}$ such that $(B - \{H\}) \cup \{H'\}$ is a base.

Theorem 2.3 (Ziegler)**.** *The homology facets for the lexicographic shelling of BC are precisely the* β**nbc**-*bases of* $\mathcal{A}$.

The set of β**nbc**-bases of $\mathcal{A}$ is called a *beta-system* for $\mathcal{A}$. The set of standard ordered β**nbc**-bases of $\mathcal{A}$ will be denoted here by β**nbc**. The notation and terminology comes from the fact that the cardinality of β**nbc** is equal to Crapo's beta invariant of the matroid of $c\mathcal{A}$. In case $r = \ell$ and $\mathcal{A}$ is complexified real this is precisely the number of bounded chambers of the real form of $\mathcal{A}$.

There is also an inductive ("deletion-restriction") definition of β**nbc** provided by [23, Thm. 1.5]. We say $H \in \mathcal{A}$ is a *separator* if the rank of $\mathcal{A} - \{H\}$ is less than r. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ denote a triple of arrangements relative to the hyperplane H_n, as defined in [16]. For $H'' \in \mathcal{A}''$ let $\nu(H'')$ denote the smallest hyperplane of $\mathcal{A}$ containing H''. (Note that ν is injective.) We may apply ν to $(r-1)$-tuples of hyperplanes in the obvious way. Order $\mathcal{A}'$ as a subset of $\mathcal{A}$. Order $\mathcal{A}''$ by $H'' < K''$ if and only if $\nu(H'') < \nu(K'')$.

Theorem 2.4 (Ziegler)**.** *If* H_n *is a separator then* $\beta\mathbf{nbc}(\mathcal{A}) = \emptyset$. *Otherwise*

$$\beta\mathbf{nbc}(\mathcal{A}) = \beta\mathbf{nbc}(\mathcal{A}') \cup \{(\nu(B''), H_n) \mid B'' \in \beta\mathbf{nbc}(\mathcal{A}'')\}.$$

Example 2.5. If $\mathcal{A}$ is an arrangement of rank one, then $\mathbf{nbc} = \mathcal{A}$ and $\beta\mathbf{nbc} = \mathcal{A} - \min(\mathcal{A})$. Let $\mathcal{A}$ be the arrangement of rank two with defining equation

$$Q = (x+1)(x-1)(y+1)(y-1)(x-y),$$

with the linear order defined by the order of the factors in Q. Then

$$\mathbf{nbc} = \{(H_1, H_3), (H_1, H_4), (H_1, H_5), (H_2, H_3), (H_2, H_4), (H_2, H_5)\}$$

and

$$\beta\mathbf{nbc} = \{(H_2, H_4), (H_2, H_5)\}.$$

Here $\mathcal{A}' = \{H_1, H_2, H_3, H_4\}$ with $\beta\mathbf{nbc}(\mathcal{A}') = \{(H_2, H_4)\}$, and $\mathcal{A}'' = \{H_1 \cap H_5, H_2 \cap H_5\}$ so that $\beta\mathbf{nbc}(\mathcal{A}'') = \{H_{245}\}$ and $\nu(H_2 \cap H_5) = \min\{H_2, H_4, H_5\} = H_2$.

Remark 2.6. The previous example is a special case of a more general phenomenon. Let us say $\mathcal{A}$ is *supersolvable* if there is a sequence $\mathcal{A}_1 \subset \ldots \subset \mathcal{A}_r = \mathcal{A}$ such that the rank of $\mathcal{A}_p$ is equal to p and, for distinct $H, H' \in \mathcal{A}_p$ with $H \cap H' \neq \emptyset$, there exists $H'' \in \mathcal{A}_i$ with $i < p$ and $H'' \supset H \cap H'$. Examples of supersolvable affine arrangements include the discriminantal arrangements of [18].

Suppose $\mathcal{A}$ is supersolvable. Set $\mathcal{B}_p = \mathcal{A}_p - \mathcal{A}_{p-1}$. Order $\mathcal{A}$ so that the elements of $\mathcal{B}_{p-1}$ precede those of $\mathcal{B}_p$ for $1 < p \leq r$. It is proved in [6] that $B = (H_{i_1}, \ldots, H_{i_r}) \in$ **nbc** if and only if $H_{i_p} \in \mathcal{B}_p$ for all p. One immediately concludes that $B \in \beta$**nbc** if and only if $H_{i_p} \in \mathcal{B}_p - \min \mathcal{B}_p$ for all p.

For $B \in \mathbf{nbc}$ let $B^* \in C^{r-1}(BC)$ denote the cochain dual to B. Thus B^* is determined by the formula

$$\langle B^*, B' \rangle = \begin{cases} 1 & \text{if } B' = B, \\ 0 & \text{otherwise,} \end{cases}$$

for $B' \in \mathbf{nbc}$.

The next proposition follows immediately from Theorem 2.3, using the properties of shellable complexes outlined earlier.

Proposition 2.7. *The set* $\{[B^*] \mid B \in \beta\mathbf{nbc}\}$ *forms a basis for* $\tilde{H}^{r-1}(BC)$.

The next step is to carry the basis of Proposition 2.7 over to F. The connection is provided by the following result.

Theorem 2.8 (Björner, Ziegler)**.** *The vertex map* $X \mapsto \min(\mathcal{A}_X)$ *induces a simplicial map* $\pi : F \longrightarrow BC$*, and* π *is a homotopy equivalence.*

Proof. This is a special case of Theorem 3.12 of [6]. □

It follows that F has the homotopy type of a bouquet of $(r-1)$-spheres [21].

Combining Theorem 2.8 and Proposition 2.7 we have that $\{\pi^*([B^*]) \mid B \in \beta\mathbf{nbc}\}$ forms a basis for $\tilde{H}^{r-1}(F)$. We need a slightly more precise result. For an $(r-1)$-simplex $\xi \in F^{(r-1)}$ let $\xi^* \in C^{r-1}(F)$ be the cochain dual to ξ. If $B = (H_{i_1}, \dots, H_{i_r})$ is an ordered base, set

$$\xi(B) = (X_1 > \ldots > X_r)$$

where $X_p = \bigcap_{k=p}^r H_{i_k}$, for $1 \le p \le r$.

Theorem 2.9. *The set* $\{[\xi(B)^*] \mid B \in \beta\mathbf{nbc}\}$ *forms a basis for* $\tilde{H}^{r-1}(F)$.

The proof will follow from Corollary 2.12 below.

Lemma 2.10. *Let* $B = (H_{i_1}, \dots, H_{i_r})$ *be a standard ordered base, with* $\xi(B) = (X_1 > \ldots > X_r)$*. Then* $B \in$ ***nbc*** *if and only if* $H_{i_k} = \min \mathcal{A}_{X_k}$ *for all* $1 \le k \le r$.

Proof. The base B contains a broken circuit with smallest element H_{i_k} if and only if $H_{i_k} \ne \min \mathcal{A}_{X_k}$. □

Lemma 2.11. *Let* $B = (H_{i_1}, \dots, H_{i_r}) \in$ ***nbc***.

(i) *Let* $\xi \in F^{(r-1)}$*. Then* $\pi(\xi) = B$ *if and only if* $\xi = \xi(B)$.

(ii) *Let* $B' = (H'_{i_1}, \dots, H'_{i_r})$ *be any ordered* ***nbc****-base. Then* $\xi(B) = \xi(B')$ *if and only if* $B = B'$.

Proof. (i) Sufficiency follows immediately from Lemma 2.10. Assume $\pi(\xi) = B$ and write $\xi = (X_1 > \ldots > X_r)$. Then $\mathcal{A}_{X_p} \supset \mathcal{A}_{X_{p+1}}$ for $1 \le p < r$. Then $H_{i_p} \cap \ldots \cap H_{i_r} \supseteq X_p$ for all $1 \le p \le r$, and it follows from a dimension count that $\xi = \xi(B)$.

(ii) Assume $\xi(B) = \xi(B')$. Suppose $B \ne B'$ and let p be maximal with $H_{i_p} \ne H'_{i_p}$. Since $H_{i_p} \cap \ldots \cap H_{i_r} = X_p = H'_{i_p} \cap \ldots \cap H'_{i_r}$ by assumption, this implies that $\{H_{i_p}, H'_{i_p}, \dots, H'_{i_r}\}$ is dependent. It then follows from Lemma 2.10 that $\{H'_{i_p}, \dots, H'_{i_r}\}$ contains a broken circuit, and this is a contradiction. □

Let $\pi^\sharp : C^{r-1}(BC) \to C^{r-1}(F)$ be the map on the cochain level induced by π.

Corollary 2.12. *For any* $B \in$ ***nbc****,* $\pi^\sharp(B^*) = \xi(B)^*$.

Proof. By definition $\langle \pi^\sharp(B^*), \xi\rangle = \langle B^*, \pi(\xi)\rangle$. By Lemma 2.11 (i), $\langle B^*, \pi(\xi)\rangle = \langle \xi(B)^*, \xi\rangle$, proving the equality of the lemma. □

Now Theorem 2.9 follows immediately from Corollary 2.12 together with Proposition 2.7 and Theorem 2.8.

3. β**nbc**-BASES FOR $H^r(M, \mathcal{L}_\lambda)$

We first introduce a topology on the intersection poset $L = L(\mathcal{A})$ as follows: $O \subseteq L$ is *open* if and only if O is a lower ideal, that is, if $X \in O$ and $Y \leq X$ imply $Y \in O$. For $X \in L$ let $L_X = \{Y \in L | Y \leq X\}$. Then L_X is an open set.

Let $0 \leq p \leq r$. Define a sheaf $\mathcal{F}^p$ on L whose stalk $\mathcal{F}^p(X)$ at $X \in L$ is defined to be equal to $A^p(\mathcal{A}_X)$, the degree p part of the Orlik–Solomon algebra $A^\cdot(\mathcal{A}_X)$ of the arrangement $\mathcal{A}_X$. If $Z \in L$ and $\mathrm{codim} Z = p$, put $A_Z = A^p(\mathcal{A}_Z)$. Then $A^p(\mathcal{A}_X) = \bigoplus A_Z$, summing over all $Z \in L$ of codimension p such that $Z \leq X$ [16, 3.73]. Thus if $Y \leq X$, then $\mathcal{F}^p(Y)$ is a direct summand of $\mathcal{F}^p(X)$ and there is a projection

$$\rho_{X,Y} : \mathcal{F}^p(X) \longrightarrow \mathcal{F}^p(Y).$$

These projections define a sheaf $\mathcal{F}^p$ on L, with $\Gamma(L_X, \mathcal{F}^p) = A^p(\mathcal{A}_X)$ [22, 3.2].

The results of this section require that λ satisfy certain genericity conditions. These are given below in Remark 3.2. At this point we must assume $\lambda_i \neq 0$ for all i. For each $X \in L$, let

$$\omega_\lambda(X) = \sum_{H_i \in \mathcal{A}_X} \lambda_i \omega_i.$$

Define a sheaf homomorphism

$$d_\lambda : \mathcal{F}^p \longrightarrow \mathcal{F}^{p+1}$$

by defining

$$d_\lambda(X) : \mathcal{F}^p(X) \longrightarrow \mathcal{F}^{p+1}(X)$$

as the left multiplication by $\omega_\lambda(X)$, for each $X \in L$. The sheaf $\mathcal{F}^0$ is the constant sheaf with $\mathcal{F}^0(X) = \mathbb{C}$ for any $X \in L$. Then $\ker(d_\lambda) \subset \mathcal{F}^0$ is the skyscraper sheaf $\mathbb{C}_V$ whose only nonzero stalk is at V and it equals $\mathbb{C}$. Thus we have a complex of sheaves

$$0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{F}^0 \xrightarrow{d_\lambda} \mathcal{F}^1 \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \mathcal{F}^r \longrightarrow 0.$$

Yuzvinsky [22] showed that this complex gives a flabby resolution of $\mathbb{C}_V$ under certain genericity conditions on λ. In order to formulate these non-resonance conditions we need the following definition.

Definition 3.1. An element $X \in L - \{V\}$ is *dense* if $\mathcal{A}_X$ is not decomposable [17], that is, if $\mathcal{A}_X$ is not a product of two nonempty arrangements.

In [22] the condition $\bar{\chi}(A(\mathcal{A}_X)) \neq 0$ is considered. This condition is actually equivalent to the denseness of X by [8]. (Also see [17].)

Remark 3.2. For $X \in L$ set $\lambda(X) = \sum_{H_i \in \mathcal{A}_X} \lambda_i$. For the remainder of this paper we assume $\lambda(X) \neq 0$ for all dense $X \in L - \{V\}$. Note that this implies $\lambda_i \neq 0$ for all i.

Theorem 3.3 (Yuzvinsky)**.** *The complex of sheaves*

$$0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{F}^0 \xrightarrow{d_\lambda} \mathcal{F}^1 \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \mathcal{F}^r \longrightarrow 0$$

is a flabby resolution of the skyscraper sheaf $\mathbb{C}_V$*.*

Note that $\Gamma(L, \mathcal{F}^p) = A^p(\mathcal{A})$. By the standard sheaf cohomology theory [12], we have an isomorphism

$$\phi : H^p(A^{\cdot}, \omega_\lambda \wedge) \xrightarrow{\sim} H^p(L, \mathbb{C}_V).$$

Let $\mathcal{F}$ be any sheaf on L. Applying the canonical simplicial resolution from [12, 6.4], we can view $H^p(L, \mathcal{F})$ as the pth cohomology of the cochain complex $C^{\cdot}(L, \mathcal{F}) := \bigoplus_{p=0}^{r} C^p(L, \mathcal{F})$, where $C^p(L, \mathcal{F})$ is the set of all functions on the $(p+1)$-tuples $(X_0 > X_1 > \cdots > X_p)$ from L with values in $\mathcal{F}(X_0)$. The differential of the complex is given by

$$\begin{aligned}(df)(X_0 > X_1 > \cdots > X_p) \quad &= \quad (-1)^p \rho_{X_{p-1}, X_p} f(X_0 > \cdots > X_{p-1}) \\ &\quad + \sum_{j=0}^{p-1} (-1)^j f(X_0 > \cdots > \widehat{X_j} > \cdots > X_p).\end{aligned}$$

We often simply write $C^p(\mathcal{F})$ instead of $C^p(L, \mathcal{F})$.

Let $F = F(\mathcal{A})$ be the Folkman complex of $\mathcal{A}$, defined in section 2. Let $C^{p-1}(F, \mathbb{C})$ be the group of simplicial $(p-1)$–cochains on F for $1 \le p \le r$. Define $C^{-1}(F, \mathbb{C}) = \mathbb{C}$ and consider the augmented cochain complex $C^{\cdot}(F, \mathbb{C})$ with the usual coboundary maps. Consider the cochain maps

$$\gamma^p : C^{p-1}(F, \mathbb{C}) \longrightarrow C^p(L, \mathbb{C}_V) \ (0 \le p \le r)$$

given by

$$(\gamma^p(f))(X_0 > X_1 > \cdots > X_p) = \begin{cases} f(X_0 > \cdots > X_{p-1}) & \text{if } X_p = V, \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in C^{p-1}(F, \mathbb{C})$. Then $\gamma = \{\gamma^p\}$ defines an isomorphism between the two cochain complexes. Using γ, we identify the sheaf cohomology $H^p(L, \mathbb{C}_V)$ with the reduced simplicial cohomology $\tilde{H}^{p-1}(F, \mathbb{C})$. Then we have the following corollary [22, 4.1].

Corollary 3.4 (Yuzvinsky)**.** (i) $H^p(A^{\cdot}, \omega_\lambda \wedge) \simeq \tilde{H}^{p-1}(F, \mathbb{C}) = 0$ *unless* $p = r$.
(ii) $H^r(A^{\cdot}, \omega_\lambda \wedge) \simeq \tilde{H}^{r-1}(F, \mathbb{C})$.

We denote the isomorphism 3.4 (ii) by

$$\phi : H^r(A^{\cdot}, \omega_\lambda \wedge) \xrightarrow{\sim} \tilde{H}^{r-1}(F, \mathbb{C}).$$

We will describe the inverse isomorphism ϕ^{-1} explicitly.

Let $\xi = (X_1 > \cdots > X_r) \in F^{(r-1)}$. Let $\xi^* \in C^{r-1}(F, \mathbb{C})$ be the cochain dual to ξ. Define a linear map

$$\upsilon : C^{r-1}(F, \mathbb{C}) \longrightarrow A^r$$

by

$$\upsilon(\xi^*) := \omega_\lambda(X_1) \cdots \omega_\lambda(X_r).$$

Remark 3.5. The form $\omega_\lambda(X_1) \cdots \omega_\lambda(X_r)$ is called the flag form associated to the flag ξ, which was introduced and studied by Varchenko in [19, 6.1]. (See also [18, 3.2.3] and [20, 10.2.11].)

Proposition 3.6. *Suppose $\lambda(X) \neq 0$ for all dense $X \in L - \{V\}$. Then*

$$\phi^{-1} : \tilde{H}^{r-1}(F, \mathbb{C}) \longrightarrow H^r(A^{\cdot}, \omega_\lambda \wedge)$$

is induced by $(-1)^{\frac{r(r+1)}{2}} \upsilon$.

Proof. The isomorphism ϕ is constructed out of a sequence of connecting homomorphisms. We have to trace back through this sequence. Let $\xi = (X_1 > \cdots > X_r) \in F^{(r-1)}$. Recall that $\xi^* \in C^{r-1}(F, \mathbb{C})$. Let β_0 be the corresponding element in $C^r(\mathbb{C}_V) \subset C^r(\mathcal{F}^0)$. Then

$$\beta_0(Y_1 > \cdots > Y_r > V) = \begin{cases} 1 & \text{if } (Y_1 > \cdots > Y_r) = (X_1 > \cdots > X_r), \\ 0 & \text{otherwise.} \end{cases}$$

Define $\beta_1 \in C^{r-1}(\mathcal{F}^0)$ by

$$\beta_1(Y_1 > \cdots > Y_r) = \begin{cases} (-1)^r & \text{if } (Y_1 > \cdots > Y_r) = (X_1 > \cdots > X_r), \\ 0 & \text{otherwise.} \end{cases}$$

Then obviously $d\beta_1 = \beta_0$. Let $\beta_2 = d_\lambda \beta_1 \in C^{r-1}(\mathcal{F}^1)$. Then

$$\beta_2(Y_1 > \cdots > Y_r) = \begin{cases} (-1)^r \omega_\lambda(X_r) & \text{if } (Y_1 > \cdots > Y_r) = (X_1 > \cdots > X_r), \\ 0 & \text{otherwise.} \end{cases}$$

Repeating this construction, we obtain $\beta_3 \in C^{r-2}(\mathcal{F}^1)$, $\beta_4 \in C^{r-2}(\mathcal{F}^2)$, and so on, until finally we reach $\beta_{2r} \in C^0(\mathcal{F}^r)$ such that

$$\beta_{2r}(Y_1) = \begin{cases} (-1)^{\frac{r(r+1)}{2}} \omega_\lambda(X_1)\omega_\lambda(X_2) \cdots \omega_\lambda(X_r) & \text{if } (Y_1) = (X_1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\beta_{2r} = (-1)^{\frac{r(r+1)}{2}} \upsilon(\xi^*) \in A^r$. By construction, $(-1)^{\frac{r(r+1)}{2}} \upsilon$ induces

$$\phi^{-1} : \tilde{H}^{r-1}(F, \mathbb{C}) \longrightarrow H^r(A^{\cdot}, \omega_\lambda \wedge)$$

as long as ϕ is an isomorphism. □

For $B \in \mathbf{nbc}(\mathcal{A})$, define

$$\Xi(B) := \upsilon(\xi(B)^*).$$

From Theorem 2.9 and Proposition 3.6, we have

Theorem 3.7. *The set*

$$\{[\Xi(B)] \in H^r(A^{\cdot}, \omega_\lambda \wedge) | B \in \beta\mathbf{nbc}(\mathcal{A})\}$$

is a basis for $H^r(A^{\cdot}, \omega_\lambda \wedge)$.

Define $\omega_B := \omega_{i_1} \cdots \omega_{i_r}$ for any ordered base $B = (H_{i_1}, \ldots, H_{i_r})$.

Remark 3.8. In [18, sections 2, 3] Schechtman and Varchenko introduced and studied the flag complex $\{\mathcal{F}^p\}$ (not to be confused with the sheaf $\mathcal{F}^p$) and homomorphisms $S^p : \mathcal{F}^p \to A^p$. The homomorphisms S^p yield the quasiclassical contravariant forms. The abelian group $\mathcal{F}^r$ is naturally identified as a certain factor group of the group $C_{r-1}(F, \mathbb{C})$ of $(r-1)$-chains of the Folkman complex F. Define an epimorphism $\pi : C^{r-1}(F, \mathbb{C}) \to \mathcal{F}^r$ by $\pi(\xi^*) = [\xi]$ for any $\xi \in F^{(r-1)}$. Then S^r and υ are related by $\upsilon = S^r \circ \pi$. It is shown in [7] that $\{[\xi(B)] \in \mathcal{F}^r | B \in \mathbf{nbc}\}$ is a basis for $\mathcal{F}^r$. Since the homomorphism S^r is an isomorphism by Theorem 3.7 of [18] (recall that we are assuming that $\lambda(X) \neq 0$ for all dense $X \in L$), the set

$$\begin{aligned}\{S^r([\xi(B)])|B \in \mathbf{nbc}\} &= \{S^r \circ \pi(\xi(B)^*)|B \in \mathbf{nbc}\}\\ &= \{\upsilon(\xi(B)^*)|B \in \mathbf{nbc}\}\\ &= \{\Xi(B)|B \in \mathbf{nbc}\}\end{aligned}$$

is a basis for A^r.

This setup, together with [18, Lemma 3.2.5] and Theorem 2.9, also yields a different proof of Theorem 3.7 as follows. It is easy to observe that the flag complex cohomology $H^r(\mathcal{F}^{\cdot})$ is naturally isomorphic to the $(r-1)$–dimensional reduced cohomology $\tilde{H}^{r-1}(F)$ of the Folkman complex F. So, the $\beta\mathbf{nbc}$-bases in Theorem 2.9 for $\tilde{H}^{r-1}(F)$ provide a basis for $H^r(\mathcal{F}^{\cdot})$. Since the quasiclassical bilinear forms $S^{\cdot}$ give a cochain isomorphism by Lemma 3.2.5 in [18], we obtain a basis for $H^r(A^{\cdot}, \omega_\lambda\wedge)$, the same basis as in Theorem 3.7.

The next theorem requires further restrictions on λ. Let $\mathbf{P}^\ell$ be the complex projective space, which is a compactification of $V = \mathbb{C}^\ell$. Consider the arrangement $\mathcal{A}_\infty$ of projective hyperplanes defined by

$$\mathcal{A}_\infty := \{\overline{H_1}, \overline{H_2}, \dots, \overline{H_\infty}\},$$

where $\overline{H_i}$ is the projective closure of H_i $(1 \leq i \leq n)$ and $\overline{H_\infty} := \mathbf{P}^\ell - \mathbb{C}^\ell$. Let $L(\mathcal{A}_\infty)$ be the collection of nonempty intersections of projective hyperplanes in $\mathcal{A}_\infty$. Cover $\mathbf{P}^\ell$ by the standard affine opens $U_0, U_1, \dots, U_\ell$, each of which is isomorphic to $\mathbb{C}^\ell$. Let $\mathcal{A}_i$ $(0 \leq i \leq \ell)$ be the arrangement in $U_i \simeq \mathbb{C}^\ell$ obtained by restricting each projective hyperplane in $\mathcal{A}_\infty$ to U_i. Let $X \in L(\mathcal{A}_\infty) - \{\mathbf{P}^\ell\}$. We say that X is *dense* if $X \cap U_i$ is dense in $\mathcal{A}_i$ for $0 \leq i \leq \ell$ with $X \cap U_i \neq \emptyset$. (See [17].) Define

$$\lambda_\infty := -\sum_{i=1}^{n} \lambda_i.$$

For $X \in L(\mathcal{A}_\infty) - \{\mathbf{P}^\ell\}$, let $\lambda(X)$ be the sum of λ_i, $i \in \{1, \dots, n, \infty\}$, with $X \subseteq \overline{H_i}$.

Recall the definitions of M and the local system $\mathcal{L}_\lambda$ from section 1. Combining Theorem 3.7 with [17, Theorem 4.1] and [10], we obtain

Theorem 3.9. *Suppose that none of the $\lambda(X)$ is a nonnegative integer for dense $X \in L(\mathcal{A}_\infty) - \{\mathbf{P}^\ell\}$. Then the set*

$$\{[\Xi(B)] \in H^r(M, \mathcal{L}_\lambda)|B \in \beta\mathbf{nbc}(\mathcal{A})\}$$

is a basis for the local system cohomology $H^r(M, \mathcal{L}_\lambda)$.

Example 3.10. Let $\mathcal{A}$ be the arrangement in Example 2.5 with defining equation

$$Q = (x+1)(x-1)(y+1)(y-1)(x-y),$$

with the linear order defined by the order of the factors in Q. Then

$$\beta\mathbf{nbc}(\mathcal{A}) = \{B_1, B_2\},$$

where $B_1 = (H_2, H_4)$ and $B_2 = (H_2, H_5)$. We have $\xi(B_1) = (H_{245} > H_4)$ and $\xi(B_2) = (H_{245} > H_5)$. Thus

$$\Xi(B_1) = (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_4\omega_4 = \lambda_2\lambda_4\omega_{24} - \lambda_4\lambda_5\omega_{45},$$

$$\Xi(B_2) = (\lambda_2\omega_2 + \lambda_4\omega_4 + \lambda_5\omega_5)\lambda_5\omega_5 = \lambda_2\lambda_5\omega_{25} + \lambda_4\lambda_5\omega_{45}.$$

Let $\lambda_\infty = -\lambda_1 - \cdots - \lambda_5$. Suppose that none of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_2+\lambda_4+\lambda_5, \lambda_1+\lambda_3+\lambda_5, \lambda_1+\lambda_2+\lambda_\infty, \lambda_3+\lambda_4+\lambda_\infty$ is a nonnegative integer. Then $\Xi(B_1)$ and $\Xi(B_2)$ give a basis for $H^2(M, \mathcal{L}_\lambda)$.

A different linear order on $\mathcal{A}$ may give another basis for $H^r(M, \mathcal{L}_\lambda)$. However we have the following proposition which is used in [9].

Proposition 3.11. *Under the same hypotheses as Theorem 3.9, the transition matrix between two bases for $H^r(M, \mathcal{L}_\lambda)$ obtained from different linear orders on $\mathcal{A}$ is an integral unimodular matrix independent of λ.*

Proof. The transition matrix is also the transition matrix between the two bases for $\tilde{H}^{r-1}(F, \mathbb{Z})$. □

4. Special cases

In this section we assume that none of the $\lambda(X)$ is a nonnegative integer for dense $X \in L(\mathcal{A}_\infty) - \{\mathbf{P}^\ell\}$.

Recall

$$\omega_B = \omega_{i_1} \cdots \omega_{i_r}$$

for any ordered base $B = (H_{i_1}, \dots, H_{i_r})$. Since ω_B is a "monomial" while $\Xi(B)$ is a sum of several ω_B, it might be desirable that the set $\{[\omega_B] \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$ form a basis for $H^r(M, \mathcal{L}_\lambda)$. This will not hold in general, even for arrangements of rank two. In this section, we will prove that these monomial forms give a basis in two special situations:

- when the linear order on $\mathcal{A}$ is unmixed (Def. 4.1),
- when the linear order on $\mathcal{A}$ is admissible (Def. 4.8) and $r = 2$.

The first case contains arrangements in general position, arrangements in general position to infinity, and normal arrangements, among others.

Definition 4.1. For each maximal $X \in L$, let

$$\mathbf{nbc}_X := \{B \in \mathbf{nbc} \mid \bigcap B = X\}.$$

We say that X is *unmixed* if either $\mathbf{nbc}_X \subseteq \beta\mathbf{nbc}$ or $\mathbf{nbc}_X \cap \beta\mathbf{nbc} = \emptyset$. A linear order on $\mathcal{A}$ is *unmixed* if any maximal $X \in L$ is unmixed.

Theorem 4.2. *If the linear order on $\mathcal{A}$ is unmixed, then the set $\{[\omega_B] \mid B \in \beta\mathbf{nbc}(\mathcal{A})\}$ is a basis for $H^r(M, \mathcal{L}_\lambda)$.*

Proof. Let $X \in L$ be an arbitrary maximal element with $\mathbf{nbc}_X \subseteq \beta\mathbf{nbc}$. Then each $\Xi(B),\ B \in \mathbf{nbc}_X$, belongs to

$$\sum_{B' \in \mathbf{nbc}_X} \mathbb{C}\omega_{B'} \subseteq \sum_{B' \in \beta\mathbf{nbc}} \mathbb{C}\omega_{B'}.$$

The result follows from Theorem 3.9 and a dimension argument. □

Example 4.3. An arrangement $\mathcal{A}$ is said to be *in general position* if (i) $n = |\mathcal{A}| \geq \ell+1$, (ii) $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_k}) = k$ whenever $1 \leq k \leq \ell$, and (iii) $H_{i_1} \cap \cdots \cap H_{i_k} = \emptyset$ whenever $k > \ell$. Let $X \in L$ be a maximal element. Then $\mathbf{nbc}_X$ is a singleton. Thus any linear order on $\mathcal{A}$ is unmixed. We have

$$\beta\mathbf{nbc}(\mathcal{A}) = \{(H_{i_1}, \dots, H_{i_\ell}) \mid 1 < i_1 < \cdots < i_\ell \leq n\},$$

$$\Xi(B) = \lambda_{i_1} \cdots \lambda_{i_\ell} \omega_B,$$

where $B = (H_{i_1}, \dots, H_{i_\ell}) \in \beta\mathbf{nbc}(\mathcal{A})$. This basis coincides with the basis constructed in [1, p. 292].

Example 4.4. An arrangement $\mathcal{A}$ is said to be *normal* [19, 1.4] if $|\mathcal{A}_X| = \operatorname{codim}(X)$ for all $X \in L(\mathcal{A})$. (This is equivalent to saying that $\bigcup \mathcal{A}$ is a normal crossing divisor in V.) Arrangements in general position are normal. Let $X \in L$ be a maximal element. Then $\mathbf{nbc}_X$ is again a singleton. Thus any linear order on $\mathcal{A}$ is unmixed. We have

$$\Xi(B) = \lambda_{i_1} \cdots \lambda_{i_r} \omega_B,$$

where $B = (H_{i_1}, \dots, H_{i_r}) \in \beta\mathbf{nbc}(\mathcal{A})$.

Example 4.5. An arrangement $\mathcal{A}$ is said to be *in general position to infinity* [22, 6.2] if $H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset$ whenever $1 \leq k \leq \ell$. This implies that there are no parallels among the elements of L, so that the hyperplane at infinity is generic relative to $\mathcal{A}$. In particular, general position arrangements are in general position to infinity.

Let $X \in L$ be a maximal element. Note that $\mathcal{A}_X$ is a central generic arrangement [16, 5.22]. We have $\mathbf{nbc}_X \cap \beta\mathbf{nbc} = \emptyset$ if and only if $X \subseteq H_1$. Otherwise $\mathbf{nbc}_X \subseteq \beta\mathbf{nbc}$. Thus any linear order on $\mathcal{A}$ is unmixed. We have

$$\beta\mathbf{nbc}(\mathcal{A}) = \{(H_{i_1}, \dots, H_{i_r}) \mid 1 < i_1 < \cdots < i_r \leq n, i_1 = \min \mathcal{A}_X, X = H_{i_1 \cdots i_r}\}.$$

Example 4.6. Let $H_1 \in \mathcal{A}$ be generic, that is, H_1 transversely intersects Y unless $Y \in L(\mathcal{A} - \{H_1\})$ is maximal in $L(\mathcal{A} - \{H_1\})$. Let $X \in L$ be a maximal element. Then $\mathbf{nbc}_X \cap \beta\mathbf{nbc} = \emptyset$ if and only if $X \subseteq H_1$. Otherwise $\mathbf{nbc}_X \subseteq \beta\mathbf{nbc}$. Thus any linear order on $\mathcal{A}$ in which H_1 is the first hyperplane is unmixed. In this case, we have

$$\beta\mathbf{nbc}(\mathcal{A}) = \{(H_{i_1}, \dots, H_{i_r}) \in \mathbf{nbc}(\mathcal{A}) \mid 1 < i_1 < \cdots < i_r \leq n\}.$$

Remark 4.7. Suppose that $\mathcal{A}$ is complexified real and $r = \ell$. Then the number of bounded chambers of the real form of $\mathcal{A}$ is equal to $|\beta\mathbf{nbc}|$. In this case Varchenko [19, 6.2] associated a differential ℓ–form $\eta_\Delta \in A^\ell(\mathcal{A})$ (which is not necessarily a monomial) to each bounded chamber Δ. Recall the definition of U_λ from section 1 and define $\omega_\Delta = U_\lambda \eta_\Delta$. The form ω_Δ is called the hypergeometric form associated to Δ. The hypergeometric integrals give a determinant $\det \left[\int_\Gamma \omega_\Delta\right]$. The main results in [19] are beautiful formulas for the determinant when $\mathcal{A}$ is in general position (Thm. 1.1), normal (Thm. 1.4), or in general position to infinity (Thm. 6.1). In particular, these determinants are nonzero, so that in these cases the set $\{\eta_\Delta\}$ gives a basis for $H^\ell(M, \mathcal{L}_\lambda)$. The relationship between $\{\eta_\Delta\}$ and $\{\Xi(B)\}$ is intriguing. They are not the same in general, but coincide when $\mathcal{A}$ is normal or in general position to infinity.

Finally we specialize to the case $r = 2$.

Definition 4.8. The linear order on $\mathcal{A}$ is called *admissible* if there exists an integer ν such that H_i and H_1 are parallel if and only if $1 \leq i < \nu$.

If the linear order on $\mathcal{A}$ is admissible, then it is not difficult to see that

$$\beta\mathbf{nbc}(\mathcal{A}) = \{(H_{i_1}, H_{i_2}) \in \mathbf{nbc} \mid 1 < i_1 < i_2 \neq \nu\}.$$

Proposition 4.9. *Suppose $\mathcal{A}$ is an arrangement of rank 2 with admissible linear order. Then the set $\{[\omega_B] \mid B \in \beta\mathbf{nbc}\}$ gives a basis for $H^2(M, \mathcal{L}_\lambda)$.*

Proof. Since $\dim H^2(M, \mathcal{L}_\lambda) = |\beta\mathbf{nbc}|$, it suffices to show that the set $\{[\omega_B] \mid B \in \beta\mathbf{nbc}\}$ spans $H^2(M, \mathcal{L}_\lambda)$. Define

$$N := \sum_{B \in \beta\mathbf{nbc}} \mathbb{C}\omega_B + d_\lambda(A^1).$$

We want to show $N = A^2$. By Theorem 3.9, it is enough to show that $\Xi(B) \in N$ for all $B \in \beta\mathbf{nbc}$. Let $B \in \beta\mathbf{nbc}$ and $X = \bigcap B$. If X is unmixed, then

$$\Xi(B) \in \sum_{B' \in \mathbf{nbc}_X} \mathbb{C}\omega_{B'} \subseteq \sum_{B' \in \beta\mathbf{nbc}} \mathbb{C}\omega_{B'} \subseteq N.$$

Suppose that X is mixed. Then $B = (H_i, H_j) \in \beta\mathbf{nbc}$ with $1 < i < \nu < j$ and $X = H_i \cap H_\nu = H_i \cap H_j$. Note that $(H_i, H_p) \in \beta\mathbf{nbc}$ for all $p > \nu$. Thus $\omega_p\omega_q = \omega_i\omega_q - \omega_i\omega_p \in N$ if $\nu \notin \{p, q\}$, and $H_p \cap H_q = X$. Therefore we have the following congruence relations modulo N:

$$\begin{aligned} \Xi(B) &= \omega_\lambda(X) \wedge \lambda_j\omega_j \equiv \lambda_\nu\omega_\nu \wedge \lambda_j\omega_j \\ &= \lambda_\nu\lambda_j(\omega_{ij} - \omega_{i\nu}) \equiv -\lambda_\nu\lambda_j\omega_{i\nu} \equiv \lambda_j d_\lambda(\omega_i) \equiv 0. \end{aligned}$$

This proves $\Xi(B) \in N$. □

Proposition 4.9 was independently proved by M. Kita [14]. He gives a direct proof which doesn't use Theorem 3.9.

If the linear order is not admissible, the set $\{[\omega_B] \mid B \in \beta\mathbf{nbc}\}$ does not give a basis for $H^2(M, \mathcal{L}_\lambda)$ in general unless we impose additional unnatural genericity conditions on λ.

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