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JET COHOMOLOGY OF ISOLATED HYPERSURFACE SINGULARITIES AND SPECTRAL SEQUENCES

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Abstract. We study jet cohomology of isolated hypersurface singularities defined by partial differential forms and prove formulas to compute jet cohomology groups by linear algebra.

We use jet sheaves of finite order (or equivalently infinitesimal neighbourhood of finite order) and related exterior differential forms to define cohomologies for singularities of mappings and varieties. (See [2], [3],[4], [5],[6],[7] and this paper.) They are new invariances. They were first defined and studied in [2] for smooth functions. In this paper we study jet cohomology $H^p(\Omega^{\cdot}_{V,k-\cdot,0})$ defined by partial differential forms with respect to y for isolated hypersurface singularities.They can be computed by linear algebra (See Theorem 1 and Theorem 2.) and can distinguish singularities whose classical cohomological invariances and Milnor numbers coincide.

For example, if X is a quasi-homogeneous hypersurface with isolated singularity 0. $\Omega^{\cdot}_{X,0}$ is the complex defined by ordinary exterior differential forms. $H^p(\Omega^{\cdot}_{X,0}) = 0$, $p \geq 1$, and $H^0(\Omega^{\cdot}_{X,0}) = C$. (See [1].) But the cohomology groups defined by jets can tell. For example: (1) $f = x_1^5 + x_2^4$. Milnor number $\mu = 12$. If the order of jets $k = 3$, $dim_C \widetilde{H}^0(\Omega^{\cdot}_{V,3-\cdot,0}) = 6$. (2) $f = x_1^3 + x_2^7$. $\mu = 12$. If $k = 3$, $dim_C \widetilde{H}^0(\Omega^{\cdot}_{V,3-\cdot,0}) = 7$.

It is interesting that the cohomological groups defined by jets are determined by higher derivatives of f. (See Theorem 1 and Theorem 2.)

All results and proofs in this paper are true not only for complex holomorphic cases but also for real analytic cases.

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The main results of this paper are as follows.

Let

$$F = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^{\alpha}} (y - x)^{\alpha}$$

and

$$W^{N-1}_{f,k-N+1} = \frac{\Omega^{N-1}_{f,k-N+1,0}}{D\Omega^{N-2}_{f,k-N+2,0}}$$

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which is finitely dimensional (see [2]). (For $\Omega^p_{f,k-p,0}$ and other symbols see the following context or [2].)

Let

$$U^{N-1}_{f,k-N+1} = \{\omega \in W^{N-1}_{f,k-N+1} : (\delta^{-1}_{k-N+2} + f - F)\omega = 0, f\omega = 0\}.$$

$$R^{N-1}_{f,k-N+1} = \{(\theta, \eta) \in W^{N-1}_{f,k-N+1} \oplus W^{N-1}_{f,k-N+1} : f\theta + F\eta = 0\},$$

$$S^{N-1}_{f,k-N+1} = \{(-[\xi], [\xi]) \in W^{N-1}_{f,k-N+1} \oplus W^{N-1}_{f,k-N+1} : \xi \in H\Omega^{N-1}_{U,k-N+1,0}\},$$

$$T^{N-1}_{f,k-N+1} = \{(-F\omega, f\omega) \in W^{N-1}_{f,k-N+1} \oplus W^{N-1}_{f,k-N+1} : \omega \in W^{N-1}_{f,k-N+1}\},$$

$$V^{N-1}_{f,k-N+1} = (fW^{N-1}_{f,k-N+1} + FW^{N-1}_{f,k-N+1}) \cap H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}),$$

$$\Gamma^N_{f,k-N} = ker(\Xi^N_{f,k-N} \xrightarrow{f-F} \Xi^N_{f,k-N+1}), \quad \Xi^N_{f,l} = \{\omega \in \Omega^N_{f,l,0} : F\omega = 0\}.$$

They are all finitely dimensional.

Theorem 1. *If $f \in O_{U,0}$, 0 is its isolated singularity, f and $\frac{\partial f}{\partial x_i}$ have no common factors, $i = 1, \cdots, N$, then*

(a) *If $N \geq 4$,*

$$H^0(\Omega^{\cdot}_{V,k-\cdot,0}) = \frac{O_{U,0}}{fO_{U,0}},$$

$$H^p(\Omega^{\cdot}_{V,k-\cdot,0}) = 0, \quad 0 \leq p \leq N-4.$$

(b) *If $N \geq 4$,*

$$H^{N-3}(\Omega^{\cdot}_{V,k-\cdot,0}) \cong U^{N-1}_{f,k-N+1}.$$

If $N = 3$, the sequence

$$0 \longrightarrow \frac{O_{U,0}}{fO_{U,0}} \longrightarrow H^0(\Omega^{\cdot}_{V,k-\cdot,0}) \longrightarrow U^2_{k-2} \longrightarrow 0$$

is split exact.

(c) *If $N \geq 3$,*

$$H^{N-2}(\Omega^{\cdot}_{V,k-\cdot,0}) \cong \frac{\frac{R^{N-1}_{f,k-N+1}}{S^{N-1}_{f,k-N+1}+T^{N-1}_{f,k-N+1}}}{\frac{\Gamma^N_{f,k-N}}{DU^{N-1}_{f,k-N+1}}};$$

hence

$$dim_C H^{N-2}(\Omega^{\cdot}_{V,k-\cdot,0}) = dim_C R^{N-1}_{f,k-N+1} + dim_C DU^{N-1}_{f,k-N+1}$$

$$-(dim_C(S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1}) + dim_C \Gamma^N_{f,k-N}).$$

If $N = 2$, the following sequences are exact:

$$H^0(\Omega^{\cdot}_{V,k-\cdot,0}) = \frac{Z^{0,0}_2}{B^{0,0}_1 + Z^{1,-1}_1},$$

$$0 \longrightarrow \frac{O_{U,0}}{fO_{U,0}} \longrightarrow \frac{Z^{0,0}_2}{B^{0,0}_1} \xrightarrow{\pi^{1,-1}} R^1_{f,k-1} \longrightarrow 0,$$

where $\overline{B}_1^{0,0} = B_1^{0,0} \cap ker(\pi^{1,-1})$. *It is split exact.*

$$0 \longrightarrow \frac{B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)}{\overline{B}_1^{0,0}} \longrightarrow \frac{B_1^{0,0} + Z_1^{1,-1}}{\overline{B}_1^{0,0}} \longrightarrow \Gamma^2_{f,k-2} \longrightarrow 0,$$

$$\begin{aligned} 0 \longrightarrow & U^1_{f,k-1} \cap H^1(\Omega^{\cdot}_{V,k-\cdot,0}) \longrightarrow \frac{B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)}{\overline{B}_1^{0,0}} \\ & \xrightarrow{\pi^{1,-1}} S^1_{f,k-1} + T^1_{f,k-1} \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} dim_C \frac{H^0(\Omega^{\cdot}_{V,k-\cdot,0})}{\frac{O_{U,0}}{fO_{U,0}}} = & dim_C R^1_{f,k-1} \\ & - (dim_C \Gamma^2_{f,k-2} + dim_C(S^1_{f,k-1} + T^1_{f,k-1}) + dim_C U^1_{f,k-1} \cap H^1(\Omega^{\cdot}_{V,k-\cdot,0})). \end{aligned}$$

(d)

$$H^{N-1}(\Omega^{\cdot}_{V,k-\cdot,0}) \cong \frac{W^{N-1}_{f,k-N}}{\delta_{k-N+1} V^{N-1}_{f,k-N+1}}.$$

By [2] $dim_C W^{N-1}_{f,l}, dim_C \Omega^N_{f,l,0} < \infty$. In [7] a canonical basis was given for $W^{N-1}_{f,l}$. The matrices δ^{-1}, Π were given with respect to the canonical basis. The linear subspace $\{[\xi] \in W^{N-1}_{f,l} : \xi \in H\Omega^{N-1}_{U,l,0}\}$ can be easily computed with respect to the canonical basis (see [7]). The process to compute the matrix $F\times : W^{N-1}_{f,l} \longrightarrow W^{N-1}_{f,l}$ was given in [7]. The matrix $f\times : W^{N-1}_{f,l} \longrightarrow W^{N-1}_{f,l}$ can be computed by the fact $\delta f = f\delta$ and a process similar to the process of computing $F\times$. $\Gamma^N_{f,l}$ can be computed by the matrices $F\times : \Omega^N_{f,l,0} \longrightarrow \Omega^N_{f,l,0}$ and $f\times : \Omega^N_{f,l,0} \longrightarrow \Omega^N_{f,l,0}$. Hence $H^p(\Omega^{\cdot}_{V,k-\cdot,0}), p = N-3, N-2, N-1$ can be computed by linear algebra.

Let $\widetilde{H}^0(\Omega^{\cdot}_{V,k-\cdot,0}) = \frac{H^0(\Omega^{\cdot}_{V,k-\cdot,0})}{\frac{O_{U,0}}{fO_{U,0}}}$ and $\widetilde{H}^p(\Omega^{\cdot}_{V,k-\cdot,0}) = H^p(\Omega^{\cdot}_{V,k-\cdot,0}), p \neq 0$. Define

$$\chi_{f,k} = \sum_{p=0}^{N} (-1)^p dim_C \widetilde{H}^p(\Omega^{\cdot}_{V,k-\cdot,0}).$$

Theorem 2. *Under the hypothesis of Theorem 1,*

$$\begin{aligned} \chi_{f,k} = & (-1)^{N-3} dim_C \Gamma^N_{f,k-N} + (-1)^{N-2} dim_C ker(\Lambda^N_{f,k-N,0} \xrightarrow{f} \Lambda^N_{f,k-N,0}) \\ = & (-1)^{N-2} \binom{k-2}{N-1} \nu \\ & + (-1)^{N-3} dim_C ker(\Xi^N_{f,k-N-1} \xrightarrow{\widetilde{f}(x)} \Xi^N_{f,k-N+1}) \\ & + (-1)^{N-2} dim_C ker(\Lambda^N_{f,k-N-1} \xrightarrow{\widetilde{f}(x)} \Lambda^N_{f,k-N+1}) \end{aligned}$$

where $\widetilde{f}(x) = \sum_{2 \leq |\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(y)}{\partial y^\alpha} (x-y)^\alpha$ *and*

$$\nu = dim_C \frac{O_{U,0}}{(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N}) O_{U,0}}.$$

$\chi_{f,k}$ can be computed by linear algebra too.

U is an open set in C^N. $0 \in U$. $(x_1, \cdots, x_N)$ are coordinates of U. V is a hypersurface defined by the holomorphic function f. 0 is an isolated singularity of V. O_U (or $O_U(x)$) is the sheaf of germs of holomorphic functions on U. $O_V = \frac{O_U}{fO_U}$.

$$Q_f(x) = \frac{O_{U,0}(x)}{(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N})O_{U,0}(x)}.$$

$(U^{(k)}, O_{U,k})$ and $(V^{(k)}, O_{V,k})$ are the k-th infinitesimal neighbourhoods of U and V with structure sheaves $O_{U,k}$ and $O_{V,k}$ respectively. Here k is a nonnegative integer.

$$O_{U,k} = \sum_{|\alpha| \le k} O_U (y-x)^\alpha,$$

where $y = (y_1, \cdots .y_N)$ and $\alpha = (\alpha_1, \cdots, \alpha_N)$. $O_{V,k} = \frac{O_{U,k}}{(fO_{U,k} + FO_{U,k})}$. Let $\Omega^1_{U,l} = \sum_{i=1}^N O_{U,l} dy_i$ be the differential form module with respect to $dy_1, \cdots, dy_N$. $\Omega^p_{U,l} = \bigwedge^p \Omega^1_{U,l}$. Let D be the partial differential with respect to $y_1, \cdots, y_N$.

In [2]–[7] we defined three kinds of complexes. Let $\Omega^0_{f,k} = O_{U,k}$;

$$\Omega^p_{f,k-p} = \frac{\Omega^p_{U,k-p}}{DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1.$$

$\Lambda^0_{f,k} = \frac{O_{U,k}}{FO_{U,k}}$;

$$\Lambda^p_{f,k-p} = \frac{\Omega^p_{U,k-p}}{F\Omega^p_{U,k-p} + DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1.$$

$\Omega^0_{V,k} = O_{V,k}$;

$$\Omega^p_{V,k-p} = \frac{\Omega^p_{U,k-p}}{f\Omega^p_{U,k-p} + F\Omega^p_{U,k-p} + DF \wedge \Omega^{p-1}_{U,k-p}}, \quad p \ge 1.$$

We have three complexes

$$\Omega^{\cdot}_{f,k-\cdot} = \{\Omega^p_{f,k-p}, D\}_{p \ge 0},$$

$$\Lambda^{\cdot}_{f,k-\cdot} = \{\Lambda^p_{f,k-p}, D\}_{p \ge 0},$$

and

$$\Omega^{\cdot}_{V,k-\cdot} = \{\Omega^p_{V,k-p}, D\}_{p \ge 0}.$$

Remark (1). We use finite order jet sheaves not infinite order jet sheaves, because $O_{U,k}$, $k < \infty$, is a coherent O_U module. The D in the above complexes are induced by the partial differential with respect to y. Hence their cohomological groups are coherent O_U modules. For isolated singularities their stalks at the singular point are finitely dimensional vector spaces.

Remark (2). In the definitions of the above complexes we reduced the order of jets, e.g. $D : \Omega^p_{V,k-p} \longrightarrow \Omega^{p+1}_{V,k-p-1}$. It is due to the following facts. The natural inclusion $(U, O_U) \longrightarrow (U^{(k)}, O_{U,k})$ induces a relative differential module

$$\Omega^1_{U^{(k)}/U} = \sum_{i=1}^N O_{U,k} dy_i,$$

and differential

$$D : O_{U,k} \longrightarrow \Omega^1_{U^{(k)}/U}.$$

But $(y_i - x_i)^{k+1} = 0, \quad i = 1, \cdots, N$, in $O_{U,k}$. $(k+1)(y_i - x_i)^k dy_i = 0$ in $\Omega^1_{U^{(k)}/U}$. Hence $\Omega^1_{U^{(k)}/U}$ is not a free $O_{U,k}$ module. So we reduce the order and take

$$\Omega^1_{U,k-1} = \Omega^1_{U^{(k)}/U} \bigotimes_{O_{U,k}} O_{U,k-1} = \sum_{i=1}^{N} O_{U,k-1} dy_i.$$

In [2] (p. 256) we proved that $\Omega^1_{U,k-1} = \Omega^1_{U^{(k)}/U} \bigotimes_{O_{U,k}} O_{U,k-1}$ is a free $O_{U,k-1}$ module with free basis $dy_1, \cdots, dy_N$.

Let $\Pi_l : O_{U,l} \longrightarrow O_{U,l-1}$ be the natural projection. It induces natrural projections $\Pi_l : \Omega^p_{U,l} \longrightarrow \Omega^p_{U,l-1}$, $\Pi_l : \Omega^p_{f,l} \longrightarrow \Omega^p_{f,l-1}$, $\Pi_l : \Lambda^p_{f,l} \longrightarrow \Lambda^p_{f,l-1}$ and $\Pi_l : \Omega^p_{V,l} \longrightarrow \Omega^p_{V,l-1}$. They are all surjective. Let $H\Omega^p_{U,l} = \{\omega \in \Omega^p_{U,l} : \Pi_l \omega = 0\}$. We consider stalks at 0 of the above sheaves, e.g. $\Omega^p_{U,k-p,0}$ is the stalk of $\Omega^p_{U,k-p}$ at 0. Let

$$W^{N-1}_{f,l} = \frac{\Omega^{N-1}_{f,l,0}}{D\Omega^{N-2}_{f,l+1,0}}, \quad l \geq 0.$$

The following three basic lemmas were proved in [2] and [5].

Lemma A. *If* $\omega \in \Omega^p_{U,l,0}, \quad 0 \leq p \leq N-1, \quad DF \wedge \omega = 0$, *then there exists* $\eta \in \Omega^{p-1}_{U,l,0}$ *such that* $\omega = DF \wedge \eta$.

Lemma B. *The sequence* $0 \to O_{U,0} \to O_{U,k,0} \xrightarrow{D} \Omega^1_{U,k-1,0} \to \cdots \to \Omega^p_{U,k-p,0} \xrightarrow{D} \Omega^{p+1}_{U,k-p-1,0} \to \cdots \to \Omega^{N-1}_{U,k-N+1,0} \xrightarrow{D} \Omega^N_{U,k-N,0} \to 0$ *is exact.*

Lemma C. *If* f *and* $\frac{\partial f}{\partial x_i}$ *have no common factors in* $O_{U,0}$, $i = 1, \cdots, N$; $\theta, \eta \in O_{U,k,0}$, $k \geq 0$. *Then* $f\theta + F\eta = 0$ *if and only if there exist* $\zeta, \xi \in O_{U,k,0}$ *such that* $\Pi_k \xi = 0$ *and* $\theta = F\zeta + \xi, \eta = -f\zeta - \xi$.

Remark. The hypothesis of Lemma C is equivalent to f having no multi-factors i.e. if $f = f_1 \cdots f_n$ is an irreducible decomposition of f, $f_i \neq f_j$, $i \neq j$.

In [2] we proved

Theorem D.

$$H^0(\Omega^{\cdot}_{f,k-\cdot,0}) = \bigoplus_{i=0}^{k} O_{U,0} F^i,$$

,

$$H^p(\Omega^{\cdot}_{f,k-\cdot,0}) = 0, \quad p \neq 0, N-1,$$

$$H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \overset{\delta_{k-N+1}}{\cong} W^{N-1}_{f,k-N}.$$

If $[\omega] \in H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}), D\omega = DF \wedge \theta$, *then* $\delta_{k-N+1}[\omega] = [\theta]$.

$$dim_C H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) = \binom{k}{N+1} dim_C Q_f(x).$$

If $k \leq N, H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) = 0$.

Lemma 1. *Suppose* $\omega \in \Omega^{N-1}_{U,l,0}$, $\theta \in \Omega^{N-1}_{U,l-1,0}$. *Their cosets* $[\omega] \in W^{N-1}_{f,l}$, $[\theta] \in W^{N-1}_{l-1}$. *Then* $\delta_l[\omega] = [\theta]$ *if and only if* $D\omega = DF \wedge \theta + DF \wedge D\eta$, *where* $\eta \in \Omega^{N-2}_{U,l,0}$.

Proof. If $\delta_l[\omega] = [\theta]$, it means $[\omega] \in H^{N-1}(\Omega^{\cdot}_{f,l+N-1-\cdot,0}) \subset W^{N-1}_{f,l}$. $D\omega = DF \wedge \zeta$. $[\zeta] = [\theta]$. Hence $\zeta = \theta + D\eta + DF \wedge \xi$. $D\omega = DF \wedge \theta + DF \wedge D\eta$. The inverse part is clear. $\square$

Take indeterminate elements A, B and DB. Define their degrees: $\deg A = 1$, $\deg B = 1$, $\deg DB = 2$. Define $\deg a = 0, \forall a \in O_{U,l,0}$ and $\deg dy_i = 1$, $i = 1, \cdots, N$.

Convention: $A^0 = B^0 = (DB)^0 = 1$, $A^n = B^n = (DB)^n = 0$, if $n \leq -1$.

$\Omega_k = \bigoplus_{p=0}^{N} \Omega^p_{U,k-p,0}$ is an exterior $O_{U,k,0}$-algebra. Let

$$\Omega^{p,q}_{k-p} = \sum_{\substack{l+r=p\\ s+t+r=q}} \Omega^l_{U,k-p,0} A^s B^t (DB)^r.$$

$\Omega^{p,q}_{k-p} = 0$, if $p \leq -1$ or $q \leq -1$. Let $\Omega_k(A,B) = \sum_{p,q} \Omega^{p,q}_{k-p}$. It is a commutative $O_{U,k,0}$-algebra. Here "commutative" means: if a, b are homogeneous elements of $\Omega_k(A,B)$, $ab = (-1)^{\deg a \cdot \deg b} ba$, e.g. $AB = -BA$, $AA = 0$, $BB = 0$. Hence

$$\Omega^{p,q}_{k-p} = \Omega^{p-q+2}_{U,k-p,0} AB(DB)^{q-2} + \Omega^{p-q+1}_{U,k-p,0} A(DB)^{q-1}$$

$$+ \Omega^{p-q+1}_{U,k-p,0} B(DB)^{q-1} + \Omega^{p-q}_{U,k-p,0} (DB)^q.$$

The partial exterior differential $D : \Omega_k \longrightarrow \Omega_k$ induces the differential $D : \Omega_k(A,B) \longrightarrow \Omega_k(A,B)$ as follows: Let $D(A) = 0$, $D(B) = DB$, $D(DB) = 0$. If $\omega \in \Omega^p_{U,l,0}$, $D(\omega B) = D\omega B + (-1)^p \Pi_l \omega DB$. $\deg D = 1$ and $DD = 0$ in $\Omega_k(A,B)$. $D : \Omega^{p,q}_{k-p} \longrightarrow \Omega^{p+1,q}_{k-p-1}$. For fixed q, $\{\Omega^{p,q}_{k-p}, D\}_{p \geq 0}$ is a complex.

$$\begin{aligned}
&D(\omega(p-q+2)AB(DB)^{q-2} + \theta(p-q+1)A(DB)^{q-1}\\
&\quad + \eta(p-q+1)B(DB)^{q-1} + \zeta(p-q)(DB)^q)\\
&= D\omega(p-q+2)AB(DB)^{q-2}\\
&\quad + ((-1)^{p-q+3}\Pi_{k-p}\omega(p-q+2) + D\theta(p-q+1))A(DB)^{q-1}\\
&\quad + D\eta(p-q+1)B(DB)^{q-1}\\
&\quad + ((-1)^{p-q+1}\Pi_{k-p}\eta(p-q+1) + D\zeta(p-q))(DB)^q.
\end{aligned}$$

Proposition. *For the differential algebra* $\Omega_k(A,B)$,

$$H^0(\Omega_k(A,B), D) = O_{U,0} + O_{U,0}A,$$

$$H^p(\Omega_k(A,B), D) = 0, \quad p \geq 1.$$

Proof. Take filtration

$$F_n\Omega_k(A,B) = \sum_{\substack{l,s,t\\ r \geq n}} \Omega^l_{U,k-l-r,0} A^s B^t (DB)^r.$$

It is clear that $F_n\Omega_k(A,B) \supset F_{n+1}\Omega_k(A,B)$ and $D(F_n\Omega_k(A,B)) \subset F_n\Omega_k(A,B)$.

$$E^n_0 = \frac{F_n}{F_{n+1}} = \sum_{l,s,t} \Omega^l_{U,k-l-n,0} A^s B^t (DB)^n.$$

$d_0^n : E_0^n \longrightarrow E_0^n$. By Lemma B,

$$E_1^n = \sum_{s,t} O_{U,0} A^s B^t (DB)^n.$$

$d_1^n : E_1^n \longrightarrow E_1^{n+1}$.

$$E_1 = \sum_n E_1^n = \sum_{t=0,1} O_{U,0}[DB]B^t$$

$$+(\sum_{t=0,1} O_{U,0}[DB]B^t)A,$$

where $O_{U,0}[DB]$ is the polynomial algbra of one variable $[DB]$ on $O_{U,0}$.

$$\sum_{t=0,1} O_{U,0}[DB]B^t$$

is the Koszul complex of one variable B on $O_{U,0}[DB]$ with differential $D(B) = DB$. Hence

$$E_2 = \sum_n E_2^n = O_{U,0} + O_{U,0}A. \quad \square$$

Corollary.

$$H^0(\Omega_{k-\cdot}^{\cdot,q}, D) = \begin{cases} O_{U,0}, & q = 0, \\ O_{U,0}A, & q = 1, \\ 0, & q \geq 2. \end{cases}$$

$$H^p(\Omega_{k-\cdot}^{\cdot,q}, D) = 0, \qquad p \geq 1.$$

Define the second differential $\partial : \Omega_k(A,B) \longrightarrow \Omega_k(A,B)$. $\deg\partial = -1$, $\partial\Omega_{U,l,0}^p = 0$, $p = 0, \cdots, N$, $\partial A = f$, $\partial B = F$, $\partial(DB) = -DF$. For homogeneous elements $a, b \in \Omega_k(A,B)$, $\partial(ab) = (\partial a) \cdot b + (-1)^{\deg a} a \cdot \partial b$. It is clear that $\partial D + D\partial = 0$ and $\partial\partial = 0$. ∂ induces $\partial^{p,q} : \Omega_{k-p}^{p,q} \longrightarrow \Omega_{k-p}^{p,q-1}$.

$$\begin{aligned} &\partial(\omega(p-q+2)AB(DB)^{q-2} + \theta(p-q+1)A(DB)^{q-1} \\ &\quad + \eta(p-q+1)B(DB)^{q-1} + \zeta(p-q)(DB)^q) \\ &= (-1)^{p-q+3}(q-2)\omega(p-q+2) \wedge DFAB(DB)^{q-3} \\ &\quad + ((-1)^{p-q+3}\omega(p-q+2)F + (-1)^{p-q+2}(q-1)\theta(p-q+1) \wedge DF)A(DB)^{q-2} \\ &\quad + ((-1)^{p-q+2}\omega(p-q+2)f + (-1)^{p-q+2}(q-1)\eta(p-q+1) \wedge DF)B(DB)^{q-2} \\ &\quad + (-1)^{p-q+1}(\theta(p-q+1)f + \eta(p-q+1)F + q\zeta(p-q) \wedge DF)(DB)^{q-1}. \end{aligned}$$

Let

$$K^{p,q} = \Omega_{k-p}^{p,-q}, \qquad 'd = D,\ ''d = \partial.$$

It is a double complex. $K^n = \sum_{p+q=n} K^{p,q}$. It is clear $K^{p,q} = 0$, if $p + q \leq -3$ or $p + q \geq N + 1$ or $p \geq k + 1$ or $p \leq -1$ or $q \leq -1$. $K^{p,0} = \Omega_{U,k-p,0}^p$.

Remark. This double complex is essentially the same double complex used in [5], but modified a little so as to make it clearer and easier to handle.

$$''E_1^{q,p} =' H^p(K^{\cdot,q},' d).$$

By the corollary,

$$''E_1^{q,0} = \begin{cases} O_{U,0}, & q=0, \\ O_{U,0}A, & q=-1, \\ 0, & q \neq 0,-1. \end{cases}$$

$$''E_1^{q,p} = 0, \quad p \neq 0.$$

$$''E_2^{q,p} = \begin{cases} \frac{O_{U,0}}{fO_{U,0}}, & p=q=0, \\ 0, & \text{otherwise.} \end{cases}$$

$$H^n(K^{\cdot}) = \begin{cases} \frac{O_{U,0}}{fO_{U,0}}, & n=0, \\ 0, & n \neq 0. \end{cases}$$

${}'E_0^{p,q} = K^{p,q} = \Omega^{p,-q}_{k-p}$. ${}'E_0^{p,0} = K^{p,0} = \Omega^{p,0}_{k-p} = \Omega^p_{U,k-p,0}$, ${}'E_1^{p,q} = {}''H^q(K^{p,\cdot}, {}''d) = H^{-q}(\Omega^{P,\cdot}_{k-p}, \partial)$. ${}'E_1^{p,0} = \Omega^p_{V,k-p,0}$. Hence ${}'E_2^{p,0} = H^p(\Omega_{V,k-\cdot,0})$.

Now we prove the theorem by computing ${}'E_2^{p,0}$. Let $d = D + \partial$ be the total differential of the double complex. ${}'E_2^{p,0} = \frac{{}'Z_2^{p,0}}{{}'Z_1^{p+1,-1} + {}'B_1^{p,0}}$. ${}'Z_2^{p,0} \supset {}'F_{p+2}K^p$, ${}'Z_1^{p+1,-1} \supset {}'F_{p+2}K^p$, . Hence

$${}'E_2^{p,0} = \frac{\frac{{}'Z_2^{p,0}}{{}'F_{p+2}K^p}}{\frac{{}'Z_1^{p+1,-1} + {}'B_1^{p,0}}{{}'F_{p+2}K^p}}.$$

Denote

$$Z_2^{p,0} = \{\, a^{p,0} + a^{p+1,-1} \in K^{p,0} + K^{p+1,-1} : Da^{p,0} + \partial a^{p+1,-1} = 0 \,\},$$

$$Z_1^{p+1,-1} = \{\, a^{p+1,-1} \in K^{p+1,-1} : \partial a^{p+1,-1} = 0 \,\},$$

$$\begin{aligned} B_1^{p,0} = \{ & (Db^{p-1,0} + \partial b^{p,-1}) + (Db^{p,-1} + \partial b^{p+1,-2}) : \\ & b^{p-1,0} \in K^{p-1,0}, b^{p,-1} \in K^{p,-1}, b^{p+1,-2} \in K^{p+1,-2} \}. \end{aligned}$$

Then

$${}'E_2^{p,0} = \frac{Z_2^{p,0}}{Z_1^{p+1,-1} + B_1^{p,0}}.$$

(a) If $1 \leq p \leq N-4$,

$$K^{p,0} = \Omega^p_{U,k-p,0},$$

$$K^{p+1,-1} = \Omega^{p+1}_{U,k-p-1,0}A + \Omega^{p+1}_{U,k-p-1,0}B + \Omega^p_{U,k-p-1,0}DB.$$

Suppose $a^{p,0} = \widetilde{\zeta}(p) \in K^{p,0}$ and $a^{p+1,-1} = \theta(p+1)A + \eta(p+1)B + \zeta(p)DB \in K^{p+1,-1}$.

$$
\begin{aligned}
0 =& Da^{p,0} + \partial a^{p+1,-1} \\
=& D\widetilde{\zeta}(p) + (-1)^{p+1}(\theta(p+1)f + \eta(p+1)F + \zeta(p) \wedge DF). \quad (1)\\
\Longleftrightarrow & fD\theta(p+1) + FD\eta(p+1) + (-1)^{p+1}\Pi_{k-p-1}\eta(p+1) \wedge DF \\
& + D\zeta(p) \wedge DF \quad (2)\\
\Longrightarrow & fD\theta(p+1) \wedge DF + FD\eta(p+1) \wedge DF = 0, \quad (3)\\
\Longleftrightarrow & D\theta(p+1) \wedge DF = \omega(p+3)F - \xi(p+3), \quad \Pi_{k-p-2}\xi(p+3) = 0, \\
& D\eta(p+1) \wedge DF = -\omega(p+3)f + \xi(p+3). \\
\Longrightarrow & \Pi_{k-p-2}\omega(p+3) \wedge DF = 0. \\
\Longleftrightarrow & \Pi_{k-p-2}\omega(p+3) = \Pi_{k-p-2}\mu(p+2) \wedge DF. \\
\Longrightarrow & D\Pi_{k-p-1}\theta(p+1) \wedge DF = F\Pi_{k-p-2}\mu(p+2) \wedge DF, \\
& D\Pi_{k-p-1}\eta(p+1) \wedge DF = -f\Pi_{k-p-2}\mu(p+2) \wedge DF, \quad (4)\\
\Longrightarrow & D\Pi_{k-p-1}\theta(p+1) = F\Pi_{k-p-2}\mu(p+2) + \Pi_{k-p-2}\lambda(p+1) \wedge DF.
\end{aligned}
$$

By [7] $H^{p+1}(\Lambda^{\cdot}_{f,k-\cdot,0}) = 0$, hence

$$
\begin{aligned}
&\Pi_{k-p-1}\theta(p+1) \\
&\quad = D\Pi_{k-p}\theta'(p) + (-1)^{p+2}\Pi_{k-p-1}\bar{\omega}(p+1)F + (-1)^{p+2}\Pi_{k-p-1}\bar{\theta}(p) \wedge DF \\
&\Longrightarrow \Pi_{k-p-2}\mu(p+2) \wedge DF = (-1)^{p+2}D\Pi_{k-p-1}\bar{\omega}(p+1) \wedge DF.
\end{aligned}
$$

Substitute into (4)

$$D(\Pi_{k-p-1}\eta(p+1) + (-1)^p f\Pi_{k-p-1}\bar{\omega}(p+1)) \wedge DF = 0.$$

By Theorem D, $H^{p+1}(\Omega^{\cdot}_{f,k-\cdot,0}) = 0$.

$$
\begin{aligned}
&\Pi_{k-p-1}\eta(p+1) + (-1)^p f\Pi_{k-p-1}\bar{\omega}(p+1) = D\Pi_{k-p}\eta'(p) \\
&\quad + (-1)^{p+2}\Pi_{k-p-1}\bar{\eta}(p) \wedge DF, \\
&\Longrightarrow \theta(p+1) = D\theta'(p) + (-1)^{p+2}F\bar{\omega}(p+1) \\
&\quad + (-1)^{p+1}\bar{\theta}(p) \wedge DF + \alpha(p+1), \quad \Pi_{k-p-1}\alpha(p+1) = 0, \\
&\quad \eta(p+1) = D\eta'(p) + (-1)^{p+1}f\bar{\omega}(p+1) \\
&\quad + (-1)^{p+1}\bar{\eta}(p) \wedge DF + \beta(p+1), \quad \Pi_{k-p-1}\beta(p+1) = 0.
\end{aligned}
$$

Substitute into (3)

$$
\begin{aligned}
& D(f\alpha(p+1) + F\beta(p+1)) \wedge DF = 0. \\
\Longleftrightarrow & D(\alpha(p+1) + \beta(p+1)) \wedge DF = 0. \\
\Longrightarrow & \alpha(p+1) + \beta(p+1) = D\phi'(p) + (-1)^{p+1}\bar{\phi}(p) \wedge DF. \\
\Longrightarrow & \theta(p+1) = D(\theta'(p) + \phi'(p)) + (-1)^{p+2}\bar{\omega}(p+1)F \\
& + (-1)^{p+1}(\bar{\theta}(p) + \bar{\phi}(p)) \wedge DF - \beta(p+1).
\end{aligned}
$$

Denote $\theta'(p)+\phi'(p)$ still by $\theta'(p)$ and $\bar{\theta}(p)+\bar{\phi}(p)$ by $\bar{\theta}(p)$.

$$\begin{aligned}\theta(p+1) &= D\theta'(p)+(-1)^{p+2}\bar{\omega}(p+1)F+(-1)^{p+1}\bar{\theta}(p)\wedge DF \\ &\quad -\beta(p+1), \quad \Pi_{k-p-1}\beta(p+1)=0, \\ \eta(p+1) &= D\eta'(p)+(-1)^{p+1}\bar{\omega}(p+1)f+(-1)^{p+1}\bar{\eta}(p)\wedge DF \\ &\quad +\beta(p+1).\end{aligned} \tag{5}$$

Substitute into (2)

$$D((-1)^{p+1}(f\bar{\theta}(p)+F\bar{\eta}(p))+(-1)^{p+1}\Pi_{k-p}\eta'(p)+\zeta(p))\wedge DF=0. \tag{6}$$

$$\begin{aligned}\Longrightarrow \zeta(p) &= (-1)^p\Pi_{k-p}\eta'(p)+(-1)^p(f\bar{\theta}(p)+F\bar{\eta}(p)) \\ &\quad + D\zeta'(p-1)+(-1)^p 2\bar{\zeta}(p-1)\wedge DF.\end{aligned}$$

Substitute into (1)

$$D(\widetilde{\zeta}(p)+(-1)^{p+1}(f\theta'(p)+F\eta'(p)+\zeta'(p-1)\wedge DF))=0.$$

$$\widetilde{\zeta}(p)=D\overset{*}{\zeta}(p-1)+(-1)^p(f\theta'(p)+F\eta'(p)+\zeta'(p-1)\wedge DF).$$

Let

$$\begin{aligned}b^{p-1,0} &= \overset{*}{\zeta}(p-1)\in K^{p-1,0}, \\ b^{p,-1} &= \theta'(p)A+\eta'(p)B+\zeta'(p-1)DB\in K^{p,-1}, \\ b^{p+1,-2} &= \bar{\omega}(p+1)AB+\bar{\theta}(p)ADB+\bar{\eta}(p)BDB \\ &\quad +\bar{\zeta}(p-1)(DB)^2\in K^{p+1,-2}. \\ \Longrightarrow a^{p,0} &= Db^{p-1,0}+\partial b^{p,-1}, \\ a^{p+1,-1} &= Db^{p,-1}+\partial b^{p+1,-2}+\beta(p+1)(B-A).\end{aligned}$$

But $\beta(p+1)(B-A)\in Z^{p+1,-1}$, because $\Pi_{k-p-1}\beta(p+1)=0$. Hence

$$H^p(\Omega^{\cdot}_{V,k-\cdot,0})=0, \quad 1\le p\le N-4.$$

If $p=0$, (5) and (6) are true. By Theorem D,

$$\zeta(0)=\Pi_k\eta'(0)+f\bar{\theta}(0)+F\bar{\eta}(0)+\sum_{i=0}^{k-1}a_iF^i, \quad a_i\in O_{U,0}.$$

Substitute into (1)

$$D\widetilde{\zeta}(0)=D(f\theta'(0)+F\eta'(0))+D(\sum_{i=0}^{k-1}\frac{1}{i+1}a_iF^{i+1}),$$

$$\begin{aligned}\widetilde{\zeta}(0) &= f\theta'(0)+F\eta'(0)+\sum_{i=0}^{k-1}\frac{1}{i+1}a_iF^{i+1}+a, \quad a\in O_{U,0}, \\ &= f\theta'(0)+F(\eta'(0)+\sum_{i=0}^{k-1}\frac{1}{i+1}a_iF^i)+a.\end{aligned}$$

We can rewrite

$$\zeta(0) = \Pi_k(\eta'(0) + \sum_{i=0}^{k-1} \frac{1}{i+1} a_i F^i) + f\bar{\theta}(0)$$
$$+ F(\bar{\eta}(0) + \sum_{i=0}^{k-2} \frac{i+1}{i+2} a_{i+1} F^i),$$
$$\eta(1) = D(\eta'(0) + \sum_{i=0}^{k-1} \frac{1}{i+1} a_i F^i) - \bar{\omega}(1) f - (\bar{\eta}(0) + \sum_{i=0}^{k-2} \frac{i+1}{i+2} a_{i+1} F^i) DF + \beta(1).$$

Denote $\eta'(0) + \sum_{i=0}^{k-1} \frac{1}{i+1} a_i F^i$ still by $\eta'(0)$ and $\bar{\eta}(0) + \sum_{i=0}^{k-2} \frac{i+1}{i+2} a_{i+1} F^i$ by $\bar{\eta}(0)$.

$$\widetilde{\zeta}(0) = f\theta'(0) + F\eta'(0) + a,$$
$$\theta(1) = D\theta'(0) + \bar{\omega}(1)F - \bar{\theta}(0)DF - \beta(1),$$
$$\eta(1) = D\eta'(0) - \bar{\omega}(1)f - \bar{\eta}(0)DF + \beta(1),$$
$$\zeta(0) = \Pi_k \eta'(0) + f\bar{\theta}(0) + F\bar{\eta}(0).$$

Let $b^{0,-1} = \theta'(0)A + \eta'(0)B$, and $b^{1,-2} = \bar{\omega}(1)AB + \bar{\theta}(0)ADB + \bar{\eta}(0)BDB$.

$$\Longrightarrow a^{0,0} = a + \partial b^{0,-1},$$
$$a^{1,-1} = Db^{0,-1} + \partial b^{1,-2} + \beta(1)(B - A).$$
$$\Longrightarrow H^0(\Omega^{\cdot}_{V,k-\cdot,0}) = \frac{O_{U,0}}{O_{U,0} \cap (B_1^{0,0} + Z_1^{1,-1})}.$$

If $b^{0,-1} \in K^{0,-1}$, $b^{1,-2} \in K^{1,-2}$ and $c^{1,-1} \in Z_1^{1,-1}$. $\partial b^{0,-1} + Db^{0,-1} + \partial b^{1,-2} + c^{1,-1} = a \in O_{U,o}$. Then

$$a = \partial b^{0,-1}.$$

If $b^{0,-1} = \theta'(0)A + \eta'(0)B$, we have $a = fb$, $b \in O_{U,0}$.

Conversely, for all fb, $b \in O_{U,0}$. Let $b^{0,-1} = bA$, $b^{1,-2} = 0$, $c^{1,-1} = 0$. We have $fb = \partial b^{0,-1}$. Hence

$$H^0(\Omega^{\cdot}_{V,k-\cdot,0}) = \frac{O_{U,0}}{fO_{U,0}}.$$

(b) $p = N - 3,\ N \geq 3$.

$$K^{N-3,0} = \Omega^{N-3}_{U,k-N+3,0},$$

$$K^{N-2,-1} = \Omega^{N-2}_{U,k-N+2,0}A + \Omega^{N-2}_{U,k-N+2,0}B + \Omega^{N-3}_{U,k-N+2,0}DB.$$

Suppose $a^{N-3,0} = \widetilde{\zeta}(N-3) \in K^{N-3,0}$ and $a^{N-2,-1} = \theta(N-2)A + \eta(N-2)B + \zeta(N-3)DB \in K^{N-2,-1}$.

$$\begin{aligned}
& 0 = Da^{N-3,0} + \partial a^{N-2,-1} \\
& \quad = D\widetilde{\zeta}(N-3) + (-1)^{N-2}(\theta(N-2)f + \eta(N-2)F + \zeta(N-3)\wedge DF). \\
(7) \quad & \Longleftrightarrow 0 = (-1)^{N-2}D(\theta(N-2)f + \eta(N-2)F + \zeta(N-3)\wedge DF). \\
& \quad = fD\theta(N-2) + FD\eta(N-2) \\
& \quad + ((-1)^{N-2}\Pi_{k-N+2}\eta(N-2) + D\zeta(N-3))\wedge DF. \\
& \Longrightarrow fD\theta(N-2)\wedge DF \\
(8) \quad & \quad + FD\eta(N-2)\wedge DF = 0.
\end{aligned}$$

$\Longleftrightarrow$ There exist

$$\omega(N), \xi(N) \in \Omega^N_{U,k-N+1,0},\ \Pi_{k-N+1}\xi(N) = 0,$$

such that

$$(9) \qquad \begin{cases} D\theta(N-2)\wedge DF = \omega(N)F - \xi(N), \\ D\eta(N)\wedge DF = -\omega(N)f + \xi(N). \end{cases}$$

By Lemma B, $\omega(N) = D\omega(N-1)$, $\xi(N) = D\xi(N-1)$, where $\omega(N-1)$, $\xi(N-1) \in \Omega^N_{U,k-N+2,0}$, $\Pi_{k-N+2}\xi(N-1) = 0$.

$$\begin{aligned}
& D\theta(N-2)\wedge DF = D(F\omega(N-1) - \xi(N-1)) - DF\wedge \Pi_{k-N+2}\omega(N-1), \\
& D\eta(N-2)\wedge DF = D(-f\omega(N-1) + \xi(N-1)). \\
\Longleftrightarrow & D(F\omega(N-1) - \xi(N-1)) = D\theta(N-2)\wedge DF + DF\wedge \Pi_{k-N+2}\omega(N-1). \\
& -f\omega(N-1) + \xi(N-1) = \eta(N-2)\wedge DF + D\lambda(N-2). \\
\Longleftrightarrow & \delta_{k-N+2}[F\omega(N-1) - \xi(N-1)] = [\Pi_{k-N+2}\omega(N-1)] \in W^{N-1}_{f,k-N+1}. \\
& f[\omega(N-1)] - [\xi(N-1)] = 0, \quad \text{in } W^{N-1}_{f,k-N+2}. \\
\Longleftrightarrow & (\delta^{-1}_{k-N+2} + f - F)[\Pi_{k-N+2}\omega(N-1)] = 0, \\
& f[\Pi_{k-N+2}\omega(N-1)] = 0.
\end{aligned}$$

Define a map

$$\pi^{N-2,-1} : Z_2^{N-3,0} \longrightarrow W^{N-1}_{f,k-N+1},$$

$\pi^{N-2,-1}(a^{N-3,0} + a^{N-2,-1}) = [\Pi_{k-N+2}\omega(N-1)]$. $\pi^{N-2,-1}(Z_2^{N-3,0}) \subset U^{N-1}_{f,k-N+1}$.

Conversely, if $\omega \in W^{N-1}_{f,k-N+1}$, $(\delta^{-1}_{k-N+2} + f - F)\omega = 0$, $f\omega = 0$. By the surjectivity of Π_{k-N+2}, there is $\omega(N-1) \in \Omega^N_{U,k-N+2,0}$, $[\Pi_{k-N+2}\omega(N-1)] = \omega$. By the above inference, there are $\theta(N-2)$, $\eta(N-2) \in \Omega^{N-2}_{U,k-N+2,0}$, such that

$$\begin{aligned}
& fD\theta(N-2)\wedge DF + FD\eta(N-2)\wedge DF = 0. \\
\Longleftrightarrow \quad & D(f\theta(N-2)\wedge DF + F\eta(N-2)\wedge DF) = 0. \\
\Longleftrightarrow \quad & (f\theta(N-2) + F\eta(N-2))\wedge DF = D\mu(N-2),
\end{aligned}$$

where $\mu(N-2) \in \Omega^{N-2}_{U,k-N+3,0}$. By $H^{N-2}(\Omega^{\cdot}_{f,k+1-\cdot,0}) = 0$,

$$\mu(N-2) = (-1)^{N-1}\widetilde{\zeta}(N-3) \wedge DF + D\nu(N-3).$$
$$\Longrightarrow (f\theta(N-2) + F\eta(N-2)) \wedge DF = (-1)^{N-1}D\widetilde{\zeta}(N-3) \wedge DF.$$
$$\Longrightarrow D\widetilde{\zeta}(N-3) = (-1)^{N-2}(f\theta(N-2) + F\eta(N-2) + \zeta(N-3) \wedge DF).$$

Hence there exist

$$a^{N-3,0} + a^{N-2,-1} \in Z_2^{N-3,0},$$

such that

$$\pi^{N-2,-1}(a^{N-3,0} + a^{N-2,-1}) = \omega,$$

i.e.

$$\pi^{N-2,-1} : Z_2^{N-3,0} \longrightarrow U^{N-1}_{f,k-N+1}$$

is surjective.

If $(Db^{N-4,0} + \partial b^{N-3,-1}) + (Db^{N-3,-1} + \partial B^{N-2,-2}) \in B_1^{N-3,0}$,

$$b^{N-3,-1} = \overset{*}{\theta}(N-3)A + \overset{*}{\eta}(N-3)B + \overset{*}{\zeta}(N-4)DB,$$

$$\begin{aligned} b^{N-2,-2} = {} & \widetilde{\omega}(N-2)AB + \widetilde{\theta}(N-3)ADB \\ & + \widetilde{\eta}(N-3)BDB + \widetilde{\zeta}(N-4)(DB)^2. \end{aligned}$$

$$\begin{aligned} & Db^{N-3,-1} + \partial b^{N-2,-2} \\ & \quad = (D\overset{*}{\theta}(N-3) + (-1)^{N-1}\widetilde{\omega}(N-2)F + (-1)^{N-2}\widetilde{\theta}(N-3) \wedge DF)A \\ & \quad + (D\overset{*}{\eta}(N-3) + (-1)^{N-2}\widetilde{\omega}(N-2)f + (-1)^{N-2}\widetilde{\eta}(N-3) \wedge DF)B \\ & \quad + ((-1)^{N-3}\Pi_{k-N+3}\overset{*}{\eta}(N-3) + D\overset{*}{\zeta}(N-4) \\ & \quad + (-1)^{N-3}(\widetilde{\theta}(N-3)f + \widetilde{\eta}(N-3)F + 2\widetilde{\zeta}(N-4) \wedge DF))DB \end{aligned}$$

$$D(D\overset{*}{\theta}(N-3) + (-1)^{N-1}\widetilde{\omega}(N-2)F$$

$$+(-1)^{N-2}\widetilde{\theta}(N-3) \wedge DF) \wedge DF = (-1)^{N-1}FD\widetilde{\omega}(N-2).$$

$$D(D\overset{*}{\eta}(N-3) + (-1)^{N-2}\widetilde{\omega}(N-2)f + (-1)^{N-2}\widetilde{\eta}(N-3) \wedge DF) \wedge DF$$

$$= (-1)^{N-2}fD\widetilde{\omega}(N-2).$$

Hence $\pi^{N-2,-1}(B^{N-3,0}) = 0$.

If $a^{N-2,-1} \in Z_1^{N-2,-1}$, $a^{N-2,-1} = \theta(N-2)A + \eta(N-2)B + \zeta(N-3)DB$.

(10) $\partial a^{N-2,-1} = \theta(N-2)f + \eta(N-2)F + \zeta(N-3) \wedge DF = 0$,

(11)

$$\Longrightarrow f\theta(N-2) \wedge DF + F\eta(N-2) \wedge DF = 0$$

$$\Longleftrightarrow \theta(N-2) \wedge DF = (-1)^{N-1}F\widetilde{\omega}(N-1) - \sigma(N-1),$$

$$\eta(N-2) \wedge DF = (-1)^{N-2}f\widetilde{\omega}(N-1) + \sigma(N-1),\ \ \Pi_{k-N+2}\sigma(N-1) = 0.$$

$$\Longrightarrow F\Pi_{k-N+2}\widetilde{\omega}(N-1) \wedge DF = 0.$$

$$\Longrightarrow \Pi_{k-N+2}\widetilde{\omega}(N-1) \wedge DF = 0.$$

$$\Longrightarrow \Pi_{k-N+2}\widetilde{\omega}(N-1) = \Pi_{k-N+2}\widetilde{\omega}(N-2) \wedge DF.$$

$$\Longrightarrow \widetilde{\omega}(N-1) = \widetilde{\omega}(N-2) \wedge DF + \tau(N-1). \quad \Pi_{k-N+2}\tau(N-1) = 0.$$

$$\begin{aligned}\Longrightarrow \theta(N-2) \wedge DF &= (-1)^{N-1}F\widetilde{\omega}(N-2) \wedge DF \\ &\quad + (-1)^{N-1}F\tau(N-1) - \sigma(N-1), \\ \eta(N-2) \wedge DF &= (-1)^{N-2}f\widetilde{\omega}(N-2) \wedge DF \\ &\quad + (-1)^{N-2}f\tau(N-1) + \sigma(N-1).\end{aligned}$$

$$\begin{aligned}\Longrightarrow \quad \Pi_{k-N+2}\theta(N-2) &= (-1)^{N-1}F\Pi_{k-N+2}\widetilde{\omega}(N-2) \\ &\quad + (-1)^{N-2}\Pi_{k-N+2}\widetilde{\theta}(N-3) \wedge DF, \\ \Pi_{k-N+2}\eta(N-2) &= (-1)^{N-2}f\Pi_{k-N+2}\widetilde{\omega}(N-2) \\ &\quad + (-1)^{N-2}\Pi_{k-N+2}\widetilde{\eta}(N-3) \wedge DF, \\ \Longrightarrow \quad \theta(N-2) &= (-1)^{N-1}F\widetilde{\omega}(N-2) \\ &\quad + (-1)^{N-2}\widetilde{\theta}(N-3) \wedge DF - \xi(N-2), \\ \eta(N-2) &= (-1)^{N-2}f\widetilde{\omega}(N-2) \\ &\quad + (-1)^{N-2}\widetilde{\eta}(N-3) \wedge DF - \overline{\xi}(N-2),\end{aligned}$$

where $\Pi_{k-N+2}\xi(N-2) = 0$, $\Pi_{k-N+2}\overline{\xi}(N-1)$.

Substitute them into (11),

$$\Longrightarrow \qquad f(-\xi(N-2) + \overline{\xi}(N-2)) \wedge DF = 0.$$

$$\Longrightarrow \qquad \overline{\xi}(N-2) = \xi(N-2) + \nu(N-3) \wedge DF.$$

Adjust $\widetilde{\eta}(N-3)$, we get

$$\theta(N-2) = (-1)^{N-1}F\widetilde{\omega}(N-2) + (-1)^{N-2}\widetilde{\theta}(N-3) \wedge DF - \xi(N-2),$$

$$\eta(N-2) = (-1)^{N-2}f\widetilde{\omega}(N-2) + (-1)^{N-2}\widetilde{\eta}(N-3) \wedge DF + \xi(N-2).$$

Substitute them into (10),

$$\Longrightarrow ((-1)^{N-2}f\widetilde{\theta}(N-3) + (-1)^{N-2}F\widetilde{\eta}(N-3) + \zeta(N-3)) \wedge DF = 0,$$

$$\Longrightarrow \zeta(N-3) = (-1)^{N-3}(f\widetilde{\theta}(N-3) + F\widetilde{\eta}(N-3) + 2\widetilde{\zeta}(N-4) \wedge DF).$$

Let $b^{N-2,-2} = \widetilde{\omega}(N-2)AB + \widetilde{\theta}(N-3)ADB + \widetilde{\eta}(N-3)BDB + \widetilde{\zeta}(N-4)(DB)^2$.

$$\Longrightarrow \qquad a^{N-2,-1} = \xi(N-2)(B-A) + \partial b^{N-2,-2}.$$

It is clear $\pi^{N-2,-1}(\xi(N-2)(B-A))=0$. Hence $\pi^{N-2,-1}Z_1^{N-2,-1}=0$.

Suppose $a^{N-3,0}+a^{N-2,-1}\in Z_2^{N-3,0}$, such that $\pi^{N-2,-1}(a^{N-3,0}+a^{N-2,-1})=0$.

$$\begin{aligned}\iff\quad &\Pi_{k-N+2}\omega(N-1)\\ &\qquad = D\Pi_{k-N+2}\lambda(N-2)+(-1)^{N-1}\Pi_{k-N+2}\widetilde{\omega}(N-2)\wedge DF.\end{aligned}$$

$$\Longrightarrow\ \omega(N-1)=D\lambda(N-2)+(-1)^{N-1}\widetilde{\omega}(N-2)\wedge DF+\sigma(N-1),$$
$$\Pi_{k-N+2}\sigma(N-1)=0.$$

Substitute into (9),

$$\begin{aligned} D\theta(N-2)\wedge DF &= (-1)^{N-1}D(F\widetilde{\omega}(N-2))\wedge DF+D(f\sigma(N-1)-\xi(N-1)),\\ D\eta(N-2)\wedge DF &= (-1)^{N-2}D(f\widetilde{\omega}(N-2))\wedge DF-D(f\sigma(N-1)-\xi(N-1)),\\ \Rightarrow\ D\Pi_{k-N+2}\theta(N-2) &= (-1)^{N-1}D(F\Pi_{k-N+2}\widetilde{\omega}(N-2))+\alpha(N-2)\wedge DF,\\ D\Pi_{k-N+2}\eta(N-2) &= (-1)^{N-2}D(f\Pi_{k-N+2}\widetilde{\omega}(N-2))+\beta(N-2)\wedge DF,\end{aligned}$$

By $H^{N-2}(\Omega^{\cdot}_{f,k-\cdot},0)=0$,

$$\begin{aligned}\Pi_{k-N+2}\theta(N-2) &= (-1)^{N-1}F\Pi_{k-N+2}\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\Pi_{k-N+2}\widetilde{\theta}(N-3)\wedge DF+D\Pi_{k-N+3}\overset{*}{\theta}(N-3),\\ \Pi_{k-N+2}\eta(N-2) &= (-1)^{N-2}f\Pi_{k-N+2}\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\Pi_{k-N+2}\widetilde{\eta}(N-3)\wedge DF+D\Pi_{k-N+3}\overset{*}{\eta}(N-3).\end{aligned}$$

$$\begin{aligned}\Longrightarrow\quad \theta(N-2) &= (-1)^{N-1}F\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\widetilde{\theta}(N-3)\wedge DF+D\overset{*}{\theta}(N-3)+\tau(N-2),\\ \eta(N-2) &= (-1)^{N-2}f\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\widetilde{\eta}(N-3)\wedge DF+D\overset{*}{\eta}(N-3)+\nu(N-2)\end{aligned}$$

where $\Pi_{k-N+2}\tau(N-2)=0,\ \ \Pi_{k-N+2}\nu(N-2)=0$.

Substitute into (8),

$$\Longrightarrow\quad f(D\tau(N-2)\wedge DF+D\nu(N-2)\wedge DF)=0.$$

$$\Longrightarrow\quad D(\tau(N-2)+\nu(N-2))=\gamma(N-2)\wedge DF.$$

By $H^{N-2}(\Omega^{\cdot}_{f,k-\cdot,0})=0$,

$$\Longrightarrow\quad \tau(N-2)+\nu(N-2)=D\varphi(N-3)+\psi(N-3)\wedge DF.$$

Adjust $\widetilde{\theta}(N-3)$ and $\overset{*}{\theta}(n-3)$,

$$\begin{aligned}\Longrightarrow\quad \theta(N-2) &= (-1)^{N-1}F\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\widetilde{\theta}(N-3)\wedge DF+D\overset{*}{\theta}(N-3)-\nu(N-2),\\ \eta(N-2) &= (-1)^{N-2}f\widetilde{\omega}(N-2)\\ &\quad+(-1)^{N-2}\widetilde{\eta}(N-3)\wedge DF+D\overset{*}{\eta}(N-3)+\nu(N-2).\end{aligned}$$

Substitute into (7),

$$D((-1)^{N-2}(f\widetilde{\theta}(N-3)+F\widetilde{\eta}(N-3)) + \zeta(N-3)+(-1)^{N-2}\Pi_{k-N+3}\overset{*}{\eta}(N-3))\wedge DF=0,$$

$$D((-1)^{N-2}(f\widetilde{\theta}(N-3)+F\widetilde{\eta}(N-3)) + \zeta(N-3)+(-1)^{N-2}\Pi_{k-N+3}\overset{*}{\eta}(N-3))=\chi(N-3)\wedge DF. \tag{12}$$

If $N\geq 4$, by $H^{N-3}(\Omega^{\cdot}_{f,k-1-\cdot,0})=0$

$$\zeta(N-3)=(-1)^{N-3}(f\widetilde{\theta}(N-3)+F\widetilde{\eta}(N-3)+2\widetilde{\zeta}(N-4)\wedge DF) + (-1)^{N-3}\Pi_{k-N+3}\overset{*}{\eta}(N-3)+D\overset{*}{\zeta}(N-4).$$

Let

$$b^{N-3,-1}=\overset{*}{\theta}(N-3)A+\overset{*}{\eta}(N-3)B+\overset{*}{\zeta}(N-4)DB,$$

$$b^{N-2,-2}=\widetilde{\omega}(N-2)AB+\widetilde{\theta}(N-3)ADB+\widetilde{\eta}(N-3)BDB+\widetilde{\zeta}(N-4)(DB)^2.$$

$$\Longrightarrow \quad a^{N-2,-1}=\nu(N-2)(B-A)+Db^{N-3,-1}+\partial b^{N-2,-2}.$$

$$Da^{N-3,0}=-\partial a^{N-2,-1}=D\partial b^{N-3,-1}.$$

$$\Longrightarrow \quad a^{N-3,0}=\partial b^{N-3,-1}+Db^{N-4,0},\ \ b^{N-4,0}\in K^{N-4,0}.$$

$$\Longrightarrow \quad a^{N-3,0}+a^{N-2,-1}\in Z_1^{N-2,-1}+B_1^{N-3,0}.$$

$$\Longrightarrow \qquad ker\pi^{N-2,-1}=Z_1^{N-2,-1}+B_1^{N-3,0}.$$

If $N=3$, by $H^0(\Omega^{\cdot}_{f,k-1-\cdot,0})=\sum_{l=0}^{k-1}O_{U,0}F^l$ and (12),

$$\zeta(N-3)=f\widetilde{\theta}(N-3)+F\widetilde{\eta}(N-3)+\Pi_k\overset{*}{\eta}(N-3)+\sum_{l=0}^{k-1}a_lF^l,$$

where $a_l\in O_{U,0}$.

Let $e^{N-2,-2}=\sum_{i=0}^{k-2}\frac{i+1}{i+2}a_{i+1}F^iBDB$ and $e^{N-3,-1}=\sum_{j=0}^{k-1}\frac{1}{j+1}a_jF^jB$.

$$\Longrightarrow \quad \partial e^{N-2,-2}+De^{N-3,-1}=\sum_{l=0}^{k-1}a_lF^lDB.$$

$$\Longrightarrow \quad a^{1,-1}=\nu(1)(B-A)+\partial(b^{1,-2}+e^{1,-2})+D(b^{0,-1}+e^{0,-1}).$$

$$\Longrightarrow Da^{0,0}=-\partial a^{1,-1}=D\partial(b^{0,-1}+e^{0,-1}).$$

$$\Longrightarrow \quad a^{0,0}=\partial(b^{0,-1}+e^{0,-1})+\sum_{l=0}^{k}b_lF^l, \quad b_l\in O_{U,0},$$

$$=\partial(b^{0,-1}+e^{0,-1})+\partial((\sum_{l=0}^{k-1}b_{l+1}F^l)B)+b_0.$$

On the other hand, $\partial(O_{U,0}A) = fO_{U,0}$.

$$\Longrightarrow \qquad \frac{ker\pi^{1,-1}}{B_1^{2,0}+Z_1^{1,-1}} \cong \frac{O_{U,0}}{fO_{U,0}}.$$

The conclusion of (b) is true.

(b) $p = N-2, \ \ N \geq 2.$

$$K^{N-2,0} = \Omega^{N-2}_{U,k-N+2,0}.$$

$$K^{N-1,-1} = \Omega^{N-1}_{U,k-N+1,0}A + \Omega^{N-1}_{U,k-N+1,0}B + \Omega^{N-2}_{U,k-N+1,0}DB.$$

Suppose $a^{N-2,0} + a^{N-1,-1} \in Z_2^{N-2,0}$, $\quad a^{N-2,0} = \widetilde{\zeta}(N-2) \in K^{N-2,0}$, $a^{N-1,-1} = \theta(N-1)A + \eta(N-1)B + \zeta(N-2)DB \in K^{N-1,-1}$.

$$\begin{aligned} 0 &= Da^{N-2,0} + \partial a^{N-1,-1} \\ &= D\widetilde{\zeta}(N-2) + (-1)^{N-1}(\theta(N-1)f + \eta(N-1)F + \zeta(N-2)\wedge DF). \end{aligned} \tag{13}$$

$\Longrightarrow$ in $W^{N-1}_{f,k-N+1}$, $[\theta(N-1)]f + [\eta(N-1)]F = 0$, where $[\theta(N-1)]$ and $[\eta(N-1)]$ are the cosets of $\theta(N-1)$ and $\eta(N-1)$ in $W^{N-1}_{f,k-N+1}$ respectively.

Define a map

$$\pi^{N-1,-1} : Z_2^{N-2,0} \longrightarrow W^{N-1}_{f,k-N+1} \oplus W^{N-1}_{f,k-N+1},$$

$\pi^{N-1,-1}(a^{N-2,0} + a^{N-1,-1}) = [\theta(N-1)] \oplus [\eta(N-1)]$. Then

$$\pi^{N-1,-1}(Z_2^{N-2,0}) = R^{N-1}_{f,k-N+1}.$$

It is clear $H\Omega^{N-1}_{U,k-N+1,0}(B-A) \subset Z_1^{N-1,-1}$.

$$'E_2^{N-2,0} = \frac{\frac{Z_2^{N-2,0}}{B_1^{N-2,0}+H\Omega^{N-1}_{U,k-N+1,0}(B-A)}}{\frac{Z_1^{N-1,-1}}{Z_1^{N-1,-1}\cap B_1^{N-2,0}+H\Omega^{N-1}_{U,k-N+1,0}(B-A)}}.$$

Suppose $(Db^{N-3,0} + \partial b^{N-2,-1}) + (Db^{N-2,-1} + \partial b^{N-1,-2}) \in B_1^{N-2,0}$, where $b^{N-2,-1} = \overset{*}{\theta}(N-2)A + \overset{*}{\eta}(N-2)B + \overset{*}{\zeta}(N-3)DB \in K^{N-2,-1}$, $b^{N-1,-2} = \widetilde{\omega}(N-1)AB + \widetilde{\theta}(N-2)ADB + \widetilde{\eta}(N-2)BDB + \widetilde{\zeta}(N-3)(DB)^2 \in K^{N-1,-2}$.

$$\begin{aligned} &Db^{N-2,-1} + \partial b^{N-1,-2} \\ &\quad = (D\overset{*}{\theta}(N-2) + (-1)^N F\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\theta}(N-2)\wedge DF)A \\ &(D\overset{*}{\eta}(N-2) + (-1)^{N-1}f\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\eta}(N-2)\wedge DF)B \\ &+ ((-1)^{N-2}\Pi_{k-N+2}\overset{*}{\eta}(N-2) + D\overset{*}{\zeta}.(N-3) \\ &+ (-1)^{N-2}(\widetilde{\theta}(N-2)f + \widetilde{\eta}(N-2)F + 2\widetilde{\zeta}(N-3)\wedge DF))DB \end{aligned}$$

Hence

$$\begin{aligned} &\pi^{N-1,-1}((Db^{N-3,0} + \partial b^{N-2,-1}) + (Db^{N-2,-1} + \partial b^{N-1,-2})) \\ &\qquad = ((-1)^N[\widetilde{\omega}(N-1)]F, (-1)^{N-1}[\widetilde{\omega}(N-1)]f). \end{aligned}$$

$$\pi^{N-1,-1}(B_1^{N-2,0}) = T^{N-1}_{f,k-N+1}.$$

Clearly

$$\pi^{N-1,-1}(H\Omega^{N-1}_{U,k-N+1,0}(B-A)) = S^{N-1}_{f,k-N+1}.$$

Suppose $a^{N-2,0} + a^{N-1,-1} \in Z_2^{N-2,0}$, $a^{N-2,0} = \widetilde{\zeta}(N-2)$, $a^{N-1,-1} = \theta(N-1)A + \eta(N-1)B + \zeta(N-2)DB$, such that $\pi^{N-1,-1}(a^{N-2,0} + a^{N-1,-1}) = 0$.

$$\begin{aligned} \Longrightarrow \quad & \theta(N-1) = (-1)^{N-1}\widetilde{\theta}(N-2) \wedge DF + D\overset{*}{\theta}(N-2), \\ & \eta(N-1) = (-1)^{N-1}\widetilde{\eta}(N-2) \wedge DF + D\overset{*}{\eta}(N-2). \end{aligned}$$

Substitute into (13),

$$\begin{aligned} \Longrightarrow \quad & D\widetilde{\zeta}(N-2) + (-1)^{N-1}(fD\overset{*}{\theta}(N-2) + FD\overset{*}{\eta}(N-2)) \\ & \quad + (f\widetilde{\theta}(N-2) + F\widetilde{\eta}(N-2) + (-1)^{N-1}\zeta(N-2)) \wedge DF = 0. \\ \Longrightarrow \quad & D(\widetilde{\zeta}(N-2) + (-1)^{N-1}(f\overset{*}{\theta}(N-2) + F\overset{*}{\eta}(N-2))) \\ & \quad + (\Pi_{k-N+2}\overset{*}{\eta}(N-2) + f\widetilde{\theta}(N-2) + F\widetilde{\eta}(N-2) \\ & \qquad + (-1)^{N-1}\zeta(N-2)) \wedge DF = 0. \end{aligned} \tag{14}$$

If $N \geq 3$, by $H^{N-2}(\Omega^{\cdot}_{f,k-\cdot,0}) = 0$,

$$\Longrightarrow \widetilde{\zeta}(N-2) = (-1)^{N-2}(f\overset{*}{\theta}(N-2) + F\overset{*}{\eta}(N-2) + \overset{*}{\zeta}(N-3) \wedge DF) + D\overline{\zeta}(N-3).$$

Substitute into (14),

$$\begin{aligned} \Longrightarrow \quad & (-1)^{N-2}D\overset{*}{\zeta}(N-3) \wedge DF + (\Pi_{k-N+2}\overset{*}{\eta}(N-2) \\ & \qquad + f\widetilde{\theta}(N-2) + F\widetilde{\eta}(N-2) + (-1)^{N-1}\zeta(N-2)) \wedge DF = 0 \\ \Longrightarrow \quad & \zeta(N-2) = (-1)^{N-2}\Pi_{k-N+2}\overset{*}{\eta}(N-2) \\ & \qquad + D\overset{*}{\zeta}(N-3) + (-1)^{N-2}(\widetilde{\theta}(N-2)f \\ & \qquad\qquad + \widetilde{\eta}(N-2)F + 2\widetilde{\zeta}(N-3) \wedge DF). \end{aligned}$$

Let $b^{N-3,0} = \overline{\zeta}(N-3)$, $b^{N-2,-1} = \overset{*}{\theta}(N-2)A + \overset{*}{\eta}(N-2)B + \overset{*}{\zeta}(N-3)DB$, $b^{N-1,-2} = \widetilde{\theta}(N-2)ADB + \widetilde{\eta}(N-2)BDB + \widetilde{\zeta}(N-3)(DB)^2$.

$$\Longrightarrow \quad a^{N-2,0} + a^{N-1,-1} = (Db^{N-3,0} + \partial b^{N-2,-1}) + (Db^{N-2,-1} + \partial b^{N-1,-2}).$$

Hence if $N \geq 3$,

$$\frac{Z_2^{N-2,0}}{B_1^{N-2,0} + H\Omega^{N-1}_{U,k-N+1,0}(B-A)} = \frac{R^{N-1}_{f,k-N+1}}{S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1}}.$$

If $a^{N-1,-1} \in Z_1^{N-1,-1}$, $\quad a^{N-1,-1} = \theta(N-1)A + \eta(N-1)B + \zeta(N-2)DB$.

$$\begin{aligned}
&0 = \partial a^{N-1,-1}\\
(15)\qquad &\quad = (-1)^{N-1}(\theta(N-1)f + \eta(N-1)F + \zeta(N-2) \wedge DF),\\
\iff\quad & f\theta(N-1) \wedge DF + F\eta(N-1) \wedge DF = 0,\\
\iff\quad & \theta(N-1) \wedge DF = \omega(N)F - \xi(N), \quad \Pi_{k-N+1}\xi(N) = 0,\\
& \eta(N-1) \wedge DF = -\omega(N)f + \xi(N),\\
\iff\quad & (F-f)\omega(N) = (F-f)\Pi_{k-N+1}\omega(N) = (\theta(N-1) + \eta(N-1)) \wedge DF,\\
& f\Pi_{k-N+1}\omega(N) = \Pi_{k-N+1}\eta(N-1) \wedge DF.
\end{aligned}$$

Define a map

$$\widetilde{\pi}^{N-1,-1} : Z_1^{N-1,-1} \longrightarrow \Gamma^N_{f,k-N},$$

$\widetilde{\pi}^{N-1,-1}(a^{N-1,-1}) = [\Pi_{k-N+1}\omega(N)]$. Clearly it is a surjective.

If

$$\begin{aligned}
& \widetilde{\pi}^{N-1,-1}(a^{N-1,-1}) = [\Pi_{k-N+1}\omega(N)] = 0\\
\iff\quad & \Pi_{k-N+1}\omega(N) = (-1)^N \Pi_{k-N+1}\widetilde{\omega}(N-1) \wedge DF,\\
\iff\quad & \omega(N) = (-1)^N \widetilde{\omega}(N-1) \wedge DF + \tau(N), \quad \Pi_{k-N+1}\tau(N) = 0.
\end{aligned}$$

(16)

$$\begin{aligned}
\Longrightarrow\quad & \begin{cases}\theta(N-1) \wedge DF = (-1)^N F\widetilde{\omega}(N-1) \wedge DF + (f\tau(N) - \xi(N)),\\ \eta(N-1) \wedge DF = (-1)^{N-1} f\widetilde{\omega}(N-1) \wedge DF - (f\tau(N) - \xi(N)).\end{cases}\\
\Longrightarrow\quad & \Pi_{k-N+1}\theta(N-1) \wedge DF = (-1)^N F\Pi_{k-N+1}\widetilde{\omega}(N-1) \wedge DF,\\
& \Pi_{k-N+1}\eta(N-1) \wedge DF = (-1)^{N-1} f\Pi_{k-N+1}\widetilde{\omega}(N-1) \wedge DF,\\
\Longrightarrow\quad & \theta(N-1) = (-1)^N F\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\theta}(N-2) \wedge DF - \sigma(N-1),\\
& \eta(N-1) = (-1)^{N-1} f\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\eta}(N-2) \wedge DF + \nu(N-1),
\end{aligned}$$

where $\Pi_{k-N+1}\sigma(N-1) = 0$ and $\Pi_{k-N+1}\nu(N-1) = 0$.

Substitute into (16),

$$\Longrightarrow \quad \sigma(N-1) \wedge DF = \xi(N) - f\tau(N) = \nu(N-1) \wedge DF.$$

$$\Longrightarrow \qquad \nu(N-1) = \sigma(N-1) + \mu(N-2) \wedge DF.$$

Adjust $\widetilde{\eta}(N-2)$,

$$\begin{aligned}
\Longrightarrow & \theta(N-1) = (-1)^N F\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\theta}(N-2) \wedge DF - \sigma(N-1),\\
& \eta(N-1) = (-1)^{N-1} f\widetilde{\omega}(N-1) + (-1)^{N-1}\widetilde{\eta}(N-2) \wedge DF + \sigma(N-1).
\end{aligned}$$

Subatitute into (15),

$$\begin{aligned}
\Longrightarrow\quad & \zeta(N-2) \wedge DF + (-1)^{N-1}(\widetilde{\theta}(N-2)f + \widetilde{\eta}(N-2)F) \wedge DF = 0,\\
& \zeta(N-2) = (-1)^{N-2}(\widetilde{\theta}(N-2)f + \widetilde{\eta}(N-2)F + 2\widetilde{\zeta}(N-3) \wedge DF).
\end{aligned}$$

Let $b^{N-1,-2} = \widetilde{\omega}(N-1)AB + \widetilde{\theta}(N-2)ADB + \widetilde{\eta}(N-2)BDB + \widetilde{\zeta}(N-3)(DB)^2$.

$$\Longrightarrow \qquad a^{N-1,-1} = \sigma(N-1)(B-A) + \partial b^{N-1,-2}.$$

Clearly $\widetilde{\pi}^{N-1,-1}(\partial K^{N-1,-2}) = 0$ and $\widetilde{\pi}^{N-1,-1}(H\Omega^{N-1}_{U,k-N+1,0}(B-A)) = 0$. Hence

$$ker\widetilde{\pi}^{N-1,-1} = \partial K^{N-1,-2} + H\Omega^{N-1}_{U,k-N+1,0}(B-A).$$

$$\frac{Z_1^{N-1,-1}}{\partial K^{N-1,-2} + H\Omega^{N-1}_{U,k-N+1,0}(B-A)} \overset{\widetilde{\pi}^{N-1,-1}}{\cong} \Gamma^N_{f,k-N}.$$

Now we compute

$$\frac{Z_1^{N-1,-1} \cap B_1^{N-2,0} + H\Omega^{N-1}_{U,k-N+1,0}(B-A)}{\partial K^{N-1,-2} + H\Omega^{N-1}_{U,k-N+1,0}(B-A)}.$$

$$\begin{aligned} & (Db^{N-3,0} + \partial b^{N-2,-1}) + (Db^{N-2,-1} + \partial b^{N-1,-2}) \in Z_1^{N-1,-1} \\ \Longleftrightarrow \quad & Db^{N-3,0} + \partial b^{N-2,-1} = 0, \\ \Longleftrightarrow \quad & \partial Db^{N-2,-1} = 0. \end{aligned}$$

Suppose $b^{N-2,-1} = \overset{*}{\theta}(N-2)A + \overset{*}{\eta}(N-2)B + \overset{*}{\zeta}(N-3)DB$.

$$\begin{aligned} \partial Db^{N-2,-1} = {} & (-1)^{N-1}(fD\overset{*}{\theta}(N-2) + FD\overset{*}{\eta}(N-2) \\ & + ((-1)^{N-2}\Pi_{k-N+2}\overset{*}{\eta}(N-2) + D\overset{*}{\zeta}(N-3)) \wedge DF) = 0 \end{aligned}$$

By the proof of (b), it is equivalent to

$$\begin{aligned} & D\overset{*}{\theta}(N-2) \wedge DF = FD\overset{*}{\omega}(N-1) - D\overset{*}{\xi}(N-1), \quad \Pi_{k-N+2}\overset{*}{\xi}(N-1) = 0, \\ & D\overset{*}{\eta}(N-2) \wedge DF = -fD\overset{*}{\omega}(N-1) + D\overset{*}{\xi}(N-1). \\ \Longleftrightarrow & (\delta^{-1}_{k-N+2} + f - F)[\Pi_{k-N+2}\overset{*}{\omega}(N-1)] = 0, \\ & f[\Pi_{k-N+2}\overset{*}{\omega}(N-1)] = 0. \end{aligned}$$

The map

$$\overset{*}{\pi}^{N-1,-1} : Z_1^{N-1,-1} \cap B_1^{N-2,0} \longrightarrow U^{N-1}_{f,k-N+1}$$

defined by

$$\overset{*}{\pi}^{N-1,-1}(Db^{N-2,-1} + \partial b^{N-1,-2}) = [\Pi_{k-N+2}\overset{*}{\omega}(N-1)]$$

(with $\partial Db^{N-2,-1} = 0$) is surjective. It is clear

$$\overset{*}{\pi}^{N-1,-1}(\partial K^{N-1,-2}) = 0,$$

for the corresponding $b^{N-2,-1} = 0$. Because

$$\delta^{-1}_{k-N+2} W^{N-1}_{f,k-N+1} = H^{N-1}(\Omega^{\cdot}_{f,k+1-\cdot,0}),$$

$$D\delta^{-1}_{k-N+2} W^{N-1}_{f,k-N+1} = 0.$$

Hence

$$DU^{N-1}_{f,k-N+1} \subset \Gamma^N_{f,k-N}.$$

Clearly

$$D\overset{*}{\pi}^{N-1,-1} = \widetilde{\pi}^{N-1,-1}|_{Z_1^{N-1,-1} \cap B_1^{N-2,0}}.$$

If $D\overset{*}{\pi}^{N-1,-1}(Db^{N-2,-1}+\partial b^{N-1,-2})=0$, where $\partial Db^{N-2,-1}=0$. Then

$$\begin{aligned}
&D[\Pi_{k-N+2}\overset{*}{\omega}(N-1)]=0.\\
\Longleftrightarrow\, &D\Pi_{k-N+2}\overset{*}{\omega}(N-1)=(-1)^N\Pi_{k-N+1}\overline{\omega}(N-1)\wedge DF.\\
\Longrightarrow\, &D\Pi_{k-N+2}\overset{*}{\theta}(N-2)\wedge DF=(-1)^N F\Pi_{k-N+1}\overline{\omega}(N-1)\wedge DF.\\
&D\Pi_{k-N+2}\overset{*}{\eta}(N-2)\wedge DF=(-1)^{N-1} f\Pi_{k-N+1}\overline{\omega}(N-1)\wedge DF.
\end{aligned}$$

Adjust $\overline{\eta}(N-1)$ suitably

$$\begin{aligned}
\Rightarrow\quad &D\overset{*}{\theta}(N-2)=(-1)^N F\overline{\omega}(N-1)+(-1)^{N-1}\overline{\theta}(N-2)\wedge DF-\overline{\xi}(N-1),\\
&\qquad\qquad \Pi_{k-N+1}\overline{\xi}(N-1)=0,\\
&D\overset{*}{\eta}(N-2)=(-1)^{N-1} f\overline{\omega}(N-1)+(-1)^{N-1}\overline{\eta}(N-2)\wedge DF+\overline{\xi}(N-1).
\end{aligned}$$

Substitute into (17),

$$\begin{aligned}
\Longrightarrow\quad &(-1)^{N-2}\Pi_{k-N+2}\overset{*}{\eta}(N-2)+D\overset{*}{\zeta}(N-3)\\
&\qquad =(-1)^{N-2}(f\overline{\theta}(N-2)+F\overline{\eta}(N-2)+2\overline{\zeta}(N-3)\wedge DF).\\
\Longrightarrow\quad &Db^{N-2,-1}=\overline{\xi}(N-1)(B-A)+\partial\overline{b}^{N-1,-2}
\end{aligned}$$

where $\overline{b}^{N-1,-2}=\overline{\omega}(N-1)AB+\overline{\theta}(N-2)ADB+\overline{\eta}(N-2)BDB+\overline{\zeta}(N-3)(DB)^2$. Hence

$$\frac{Z_1^{N-1,-1}\cap B_1^{N-2,0}+H\Omega^{N-1}_{U,k-N+1,0}(B-A)}{\partial K^{N-1,-2}+H\Omega^{N-1}_{U,k-N+1,0}(B-A)}\overset{\overset{*}{\pi}^{N-1,-1}}{\cong} DU^{N-1}_{f,k-N+1}.$$

If $N=2$. We have the map

$$\pi^{1,-1}:Z_2^{0,0}\longrightarrow W^1_{f,k-1}\oplus W^1_{f,k-1}.$$

Suppose $a^{0,0}=\widetilde{\zeta}(0)\in K^{0,0}$, $a^{1,-1}=\theta(1)A+\eta(1)B+\zeta(0)DB\in K^{1,-1}$. $a^{0,0}+a^{1,-1}\in Z_2^{0,0}$, such that $\pi^{1,-1}(a^{0,0}+a^{1,-1})=0$.

$$\begin{aligned}
\Longleftrightarrow\quad &\theta(1)=-\widetilde{\theta}(0)DF+D\overset{*}{\theta}(0),\\
&\eta(1)=-\widetilde{\eta}(0)DF+D\overset{*}{\eta}(0).
\end{aligned}$$

Substitute into (13)

$$\begin{aligned}
\Longrightarrow\quad &D(\widetilde{\zeta}(0)-f\overset{*}{\theta}(0)-F\overset{*}{\eta}(0))\\
&\quad+(\Pi_k\overset{*}{\eta}(0)+f\widetilde{\theta}(0)+F\widetilde{\eta}(0)-\zeta(0))DF=0.\\
\Longrightarrow\quad &\widetilde{\zeta}(0)=f\overset{*}{\theta}(0)+F\overset{*}{\eta}(0))+\sum_{l=0}^{k}a_lF^l,\quad a_l\in O_{U,0}.\\
\Longrightarrow\quad &\zeta(0)=\Pi_k\overset{*}{\eta}(0)+f\widetilde{\theta}(0)+F\widetilde{\eta}(0)+\sum_{l=0}^{k-1}(l+1)a_{l+1}F^l.
\end{aligned}$$

Let

$$b^{0,-1} = \overset{*}{\theta}(0)A + (\overset{*}{\eta}(0) + \sum_{l=0}^{k-1} a_{l+1}F^l)B,$$

$$b^{1,-2} = \widetilde{\theta}(0)ADB + (\widetilde{\eta}(0) + \sum_{l=0}^{k-2}(l+1)a_{l+2}F^l)BDB.$$

$$\Longrightarrow \quad a^{0,0} + a^{1,-1} = \partial b^{0,-1} + (Db^{0,-1} + \partial b^{1,-2}) + a_0 \in B^{0,0} + O_{U,0}.$$

where the coefficient of AB in $b^{1,-2}$ is zero. Hence

$$ker\pi^{1,-1} = \overline{B}_1^{0,0} + O_{U,0},$$

where $\overline{B}_1^{0,0} = B_1^{0,0} \cap ker\pi^{1,-1}$. The sequence

$$0 \longrightarrow \frac{O_{U,0}}{fO_{U,0}} \longrightarrow \frac{Z_2^{0,0}}{\overline{B}_1^{0,0}} \longrightarrow R^1_{f,k-1} \longrightarrow 0$$

is split exact.

We have the surjective map

$$\widetilde{\pi}^{1,-1} : Z_1^{1,-1} \longrightarrow \Gamma^2_{f,k-2}.$$

Suppose $\partial b^{0,-1} + (Db^{0,-1} + \partial b^{1,-2}) \in Z_1^{1,-1} \cap B_1^{0,0}, \Longleftrightarrow \partial b^{0,-1} = 0$. We have proved $\widetilde{\pi}^{1,-1}(\partial b^{0,-1}) = 0$. Let $b^{0,-1} = \overset{*}{\theta}(0)A + \overset{*}{\eta}(0)B$.

$$\begin{aligned}
& \partial b^{0,-1} = \overset{*}{\theta}(0)f + \overset{*}{\eta}(0)F = 0. \\
\Longleftrightarrow \quad & \overset{*}{\theta}(0) = F\omega(0) - \xi(0), \quad \Pi_k \xi(0) = 0. \\
& \overset{*}{\eta}(0) = -f\omega(0) + \xi(0). \\
\Longrightarrow \quad & \widetilde{\pi}^{1,-1}(\partial b^{0,-1} + (Db^{0,-1} + \partial b^{1,-2})) = 0. \\
\Longrightarrow \quad & \frac{Z_1^{1,-1} + B_1^{0,0}}{B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)} \\
& = \frac{Z_1^{1,-1}}{Z_1^{1,-1} \cap B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)} \overset{\widetilde{\pi}^{1,-1}}{\cong} \Gamma^2_{f,k-2}.
\end{aligned}$$

Hence we get the exact sequences

$$0 \longrightarrow \frac{B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)}{\overline{B}_1^{0,0}}$$

$$\longrightarrow \frac{Z_1^{1,-1} + B_1^{0,0}}{\overline{B}_1^{0,0}} \longrightarrow \Gamma^2_{f,k-2} \longrightarrow 0.$$

$$0 \longrightarrow \frac{(B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)) \cap (\overline{B}_1^{0,0} + O_{U,0})}{\overline{B}_1^{0,0}}$$

$$\longrightarrow \frac{B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)}{\overline{B}_1^{0,0}} \overset{\pi^{1,-1}}{\longrightarrow} S^1_{f,k-1} + T^1_{f,k-1} \longrightarrow 0.$$

Clearly

$$\begin{aligned}&(B_1^{0,0}+H\Omega^1_{U,k-1,0}(B-A))\cap(\overline{B}_1^{0,0}+O_{U,0})\\&\quad=\{\overline{b}+a:\overline{b}\in\overline{B}_1^{0,0},a\in O_{U,0},\ \text{s.t. } a=\partial b^{0,-1},\\&\qquad\qquad Db^{0,-1}+\partial b^{1,-2}\in H\Omega^1_{U,k-1,0}(B-A)\}.\end{aligned}$$

Now we prove

$$\frac{(B_1^{0,0}+H\Omega^1_{U,k-1,0}(B-A))\cap(\overline{B}_1^{0,0}+O_{U,0})}{\overline{B}_1^{0,0}}\cong U^1_{f,k-1}\cap H^1(\Omega^{\cdot}_{f,k-\cdot,0}).$$

Let

$$\begin{aligned}\nabla=&\{b^{0,-1}+b^{1,-2}\in K^{0,-1}+K^{1,-2}:\\&\partial b^{0,-1}+Db^{0,-1}+\partial b^{1,-2}\in O_{U,0}+H\Omega^1_{U,k-1,0}(B-A)\}.\end{aligned}$$

Suppose $b^{0,-1}=\overset{*}{\theta}(0)A+\overset{*}{\eta}(0)B$, $b^{1,-2}=\widetilde{\omega}(1)AB+\widetilde{\theta}(0)ADB+\widetilde{\eta}(0)BDB$.

$$\begin{aligned}&b^{0,-1}+b^{1,-2}\in\nabla,\\ \Longleftrightarrow\quad& a=\overset{*}{\theta}(0)f+\overset{*}{\eta}(0)F.\\&\begin{cases}D\overset{*}{\theta}(0)+\widetilde{\omega}(1)F-\widetilde{\theta}(0)DF=-\tau(1),\quad \Pi_k\tau(1)=0,\\ D\overset{*}{\eta}(0)-\widetilde{\omega}(1)f-\widetilde{\eta}(0)DF=\tau(1),\\ \Pi_k\overset{*}{\eta}(0)+(\widetilde{\theta}(0)f+\widetilde{\eta}(0)F)=0.\end{cases}\\ \Longrightarrow\quad& 0=D(\overset{*}{\theta}(0)f+\overset{*}{\eta}(0)F)\\&\ =fD\overset{*}{\theta}(0)+FD\overset{*}{\eta}(0)+DF\wedge\Pi_k\overset{*}{\eta}(0).\\ \Longrightarrow\quad&(fD\overset{*}{\theta}(0)+FD\overset{*}{\eta}(0))\wedge DF=0.\\ \Longleftrightarrow\quad& D\overset{*}{\theta}(0)\wedge DF=\omega(2)F-\sigma(2),\\&D\overset{*}{\eta}(0)\wedge DF=\omega(2)f+\sigma(2)\end{aligned}\tag{12}$$

where $\Pi_{k-1}\sigma(1)=0$.

Let $\omega(2)=D\omega(1)$, $\sigma(2)=D\sigma(1)$, $\Pi_k\omega(1)=0$, $\Pi_k\sigma(1)=0$.

$$\begin{aligned}&\Longleftrightarrow\quad D(F\omega(1)-\sigma(1))=D\overset{*}{\theta}(0)\wedge DF-\Pi_k\omega(1)\wedge DF,\\&\qquad\quad D(f\omega(1)-\sigma(1))=-D\overset{*}{\eta}(0)\wedge DF.\\ \Longleftrightarrow\quad \text{in}\quad W^1_{f,k-1}\quad&\delta_k([F\omega(1)-\sigma(1)])=[\Pi_k\omega(1)],\\&f[\omega(1)]-[\sigma(1)]=0.\\&\Longleftrightarrow\quad(\delta_k^{-1}+f-F)[\Pi_k\omega(1)]=0\\&\qquad\quad f[\Pi_k\omega(1)]=0.\end{aligned}$$

On the other hand

$$\begin{aligned} & D\overset{*}{\theta}(0) \wedge DF = -F\widetilde{\omega}(1) \wedge DF - \tau(1) \wedge DF, \\ & D\overset{*}{\eta}(0) \wedge DF = f\widetilde{\omega}(1) \wedge DF + \tau(1) \wedge DF, \\ \Longrightarrow \quad & D\Pi_k\omega(1) = \Pi_{k-1}\widetilde{\omega}(1) \wedge DF. \\ \Longrightarrow \quad & [\Pi_k\omega(1)] \in H^1(\Omega^{\cdot}_{f,k-\cdot,0}). \end{aligned}$$

Define a map

$$\pi^{0,-1} : \nabla \longrightarrow U^1_{f,k-1} \cap H^1(\Omega^{\cdot}_{f,k-\cdot,0}),$$

$\pi^{0,-1}(b^{0,-1} + b^{1,-2}) = [\Pi_k\omega(1)]$.

Conversly, suppose $\omega \in U^1_{f,k-1} \cap H^1(\Omega^{\cdot}_{f,k-\cdot,0})$, $\omega = [\Pi_k\omega(1)]$. By the above inference (18) holds. Hence

$$\begin{aligned} & D(f\overset{*}{\theta}(0) + F\overset{*}{\eta}(0)) \wedge DF = 0, \\ \Longrightarrow \quad & (f\overset{*}{\theta}(0) + F\overset{*}{\eta}(0)) \wedge DF = D\mu(0). \\ \Longrightarrow \quad & \mu(0) = \sum_{l=0}^{k+1} c_l F^l, \quad c_l \in O_{U,0}. \\ \Longrightarrow \quad & f\overset{*}{\theta}(0) + F\overset{*}{\eta}(0) = \sum_{l=0}^{k} (l+1)c_{l+1}F^l. \end{aligned}$$

But

$$\begin{aligned} & D\Pi_k\omega(1) = DF \wedge \Pi_{k-1}\widetilde{\omega}(1). \\ \Longrightarrow \quad & \omega(2) = D\omega(1) = DF \wedge \widetilde{\omega}(1) + \xi(2), \quad \Pi_{k-1}\xi(2) = 0. \\ \Longrightarrow \quad & D\overset{*}{\theta}(0) = -F\widetilde{\omega}(1) + \widetilde{\theta}(0)DF - \tau(1), \\ & D\overset{*}{\eta}(0) = f\widetilde{\omega}(1) + \widetilde{\eta}(0)DF + \tau(1). \end{aligned}$$

where $\Pi_{k-1}\tau(1) = 0$. Let

$$\begin{aligned} & b^{0,-1} = \overset{*}{\theta}(0)A + (\overset{*}{\eta}(0) - \sum_{l=0}^{k-1} a_{l+1}F^l)B, \\ & b^{1,-2} = \widetilde{\omega}(1)AB + \widetilde{\theta}(0)ADB + (\widetilde{\eta}(0) - \sum_{l=0}^{k-2} (l+1)a_{l+2}F^l)BDB. \\ \Longrightarrow \quad & \pi^{0,-1}(b^{0,-1} + b^{1,-2}) = \omega. \end{aligned}$$

Hence $\pi^{0,-1}$ is surjective.

Suppose $\pi^{0,-1}(b^{0,-1} + b^{1,-2}) = 0$.

$$\begin{aligned} \Longrightarrow \quad & \omega(1) = D\lambda(0) - \overline{\omega}(0)DF + \tau(1), \quad \Pi_k\tau(1) = 0. \\ \Longrightarrow \quad & D\overset{*}{\theta}(0) \wedge DF = -D(F\overline{\omega}(0)) \wedge DF + D(f\tau(1) - \sigma(1)), \\ & D\overset{*}{\eta}(0) \wedge DF = D(f\overline{\omega}(0)) \wedge DF - D(f\tau(1) - \sigma(1)). \end{aligned}$$

By Lemma A

$$
\begin{aligned}
& D\Pi_k\overset{*}{\theta}(0) = -D(F\Pi_k\overline{\omega}(0)) + \alpha(0)DF, \\
& D\Pi_k\overset{*}{\eta}(0) = D(f\Pi_k\overline{\omega}(0)) + \beta(0)DF, \\
\Longrightarrow\quad & \Pi_k\overset{*}{\theta}(0) = -F\Pi_k\overline{\omega}(0) + \sum_{l=0}^{k-1} a_l F^l, \quad a_l \in O_{U,0}, \\
& \Pi_k\overset{*}{\eta}(0) = f\Pi_k\overline{\omega}(0) + \sum_{l=0}^{k-1} b_l F^l, \quad b_l \in O_{U,0}, \\
\Longrightarrow\quad & \overset{*}{\theta}(0) = -F\overline{\omega}(0) + \sum_{l=0}^{k-1} a_l F^l + \gamma(0), \quad \Pi_k\gamma(0) = 0. \\
& \overset{*}{\eta}(0) = f\overline{\omega}(0) + \sum_{l=0}^{k-1} b_l F^l - \gamma(0), \\
\Longrightarrow\quad & a = f\overset{*}{\theta}(0) + F\overset{*}{\eta}(0) \\
& \ = fa_0 + \sum_{l=1}^{k-1}(fa_l + b_{l-1})F^l + b_{k-1}F^k.
\end{aligned}
$$

By Theorem D, $H^0(\Omega^{\cdot}_{f,k-1-\cdot,0}) = \bigoplus_{l=0}^{k-1} O_{U,0}F^i$ is direct sum.

$$
\Longrightarrow\quad a = fa_0; \quad b_{k-1} = 0; \quad b_l = -fa_{l+1}, \qquad l = 0, \cdots, k-2.
$$

$$
\begin{aligned}
\Longrightarrow\quad & \overset{*}{\theta}(0) = -F(\overline{\omega}(0) - \sum_{l=0}^{k-2} a_{l+1}F^l) + \gamma(0) + a_0, \\
& \overset{*}{\eta}(0) = f(\overline{\omega}(0) - \sum_{l=0}^{k-2} a_{l+1}F^l) - \gamma(0).
\end{aligned}
$$

Adjust $\overline{\omega}(0)$,

$$
\begin{aligned}
\Longrightarrow\quad & \overset{*}{\theta}(0) = -F\overline{\omega}(0) + \gamma(0) + a_0, \\
& \overset{*}{\eta}(0) = f\overline{\omega}(0) - \gamma(0). \\
\Longrightarrow\quad & (\widetilde{\theta}(0) + \Pi_k\overline{\omega}(0))f + \widetilde{\eta}(0)F = 0, \\
\Longrightarrow\quad & \widetilde{\theta}(0) = -F\omega(0) - \Pi_k\overline{\omega}(0) - \xi(0), \quad \Pi_{k-1}\xi(0) = 0, \\
& \widetilde{\eta}(0) = f\omega(0) + \xi(0) \\
\Longrightarrow\quad & F(\widetilde{\omega}(1) - D\overline{\omega}(0) + \omega(0)DF) + D\alpha(0) + \xi(0)DF + \tau(1) = 0, \\
& -f(\widetilde{\omega}(1) - D\overline{\omega}(0) + \omega(0)DF) - (D\alpha(0) + \xi(0)DF + \tau(1)) = 0. \\
\Longrightarrow\quad & \widetilde{\omega}(1) = D\overline{\omega}(0) - \omega(0)DF + \sigma(1), \quad \Pi_{k-1}\sigma(1) = 0.
\end{aligned}
$$

Let $c^{0,-2} = \overline{\omega}(0)AB \in K^{2,-2}$, $c^{1,-3} = \omega(0)ABDB \in K^{1,-3}$.

$$
\begin{aligned}
\Longrightarrow\quad b^{0,-1} + b^{1,-2} &= \partial c^{0,-2} + (Dc^{0,-2} + \partial c^{1,-3}) \\
&\quad + a_0A + \alpha(0)(A-B) + \sigma(1)AB + \xi(0)(B-A)DB, \\
&\Pi_k\alpha(0) = 0, \quad \Pi_{k-1}\sigma(1) = 0, \quad \Pi_{k-1}\xi(0) = 0.
\end{aligned}
$$

Conversely, if $b^{0,-1}+b^{1,-2}$ satisfies (19), $\pi^{0,-1}(b^{0,-1}+b^{1,-2})=0$ by the definition of $\pi^{0,-1}$ directly. Hence

$$\begin{aligned} ker\pi^{0,-1} = \{ & \partial c^{0,-2} + (Dc^{0,-2} + \partial c^{1,-3}) \\ & a_0 A + \alpha(0)(A-B) + \sigma(1)AB + \xi(0)(B-A)DB : \\ & c^{0,-2} \in K^{0,-2}, \quad c^{1,-3} \in K^{1,-3}, \\ & \Pi_k \alpha(0) = 0, \quad \Pi_{k-1}\sigma(1) = 0, \quad \Pi_{k-1}\xi(0) = 0\}. \end{aligned}$$

Let

$$\begin{aligned} \Delta = \{ & a \in O_{U,0} : \exists b^{0,-1} \in K^{0,-1}, \quad b^{1,-2} \in K^{1,-2} \\ & \text{s.t. } a = \partial b^{0,-1}, \quad Db^{0,-1} + \partial b^{1,-2} \in H\Omega^1_{U,k-1,0}(B-A)\}, \\ \widetilde{\Delta} = \{ & a \in \Delta : \partial b^{0,-1} + (Db^{0,-1} + \partial b^{1,-2}) \in \overline{B}_1^{0,0}\} \end{aligned}$$

It is clear

$$\frac{\Delta}{\widetilde{\Delta}} \cong \frac{(B_1^{0,0} + H\Omega^1_{U,k-1,0}(B-A)) \cap (\overline{B}_1^{0,0} + O_{U,0})}{\overline{B}_1^{0,0}}.$$

Define a map

$$\varphi : \nabla \longrightarrow \frac{\Delta}{\widetilde{\Delta}}$$

by $\varphi(b^{0,-1}+b^{1,-2}) = \partial b^{0,-1}$. It is surjective clearly.

Suppose $b^{0,-1}+b^{1,-2} \in \nabla$, such that

$$\begin{aligned} & \partial b^{0,-1} = \overset{*}{\theta}(0) f + \overset{*}{\eta}(0) F = 0. \\ \Longrightarrow \quad & \overset{*}{\theta}(0) = -F\lambda(0) + \alpha(0), \quad \Pi_k \alpha(0) = 0, \\ & \overset{*}{\eta}(0) = f\lambda(0) - \alpha(0). \\ \Longrightarrow \quad & D\overset{*}{\theta}(0) \wedge DF = -FD(\lambda(0) \wedge DF). \\ \Longrightarrow \quad & \pi^{0,-1}(b^{0,-1} + b^{1,-2}) = 0. \end{aligned}$$

Hence φ is well defined.

By the definition of $\pi^{1,-1}$, clearly

$$\pi^{1,-1}((\partial + D)ker\pi^{0,-1} \mathrm{mod}' F_2 K^{-1}) = 0.$$

Hence $\varphi(ker\pi^{0,-1}) \subset \widetilde{\nabla}$.

Conversely, if $\partial b^{0,-1} + (Db^{0,-1} + \partial b^{1,-2}) = a \in \widetilde{\Delta}$. By the above inference to compute $ker\pi^{1,-1}$, we can choose $b^{1,-2} = \widetilde{\theta}(0)ADB + \widetilde{\eta}(0)BDB$ (i.e. $\widetilde{\omega}(1) = 0$). Therefore $\pi^{0,-1}(b^{0,-1}+b^{1,-2}) = 0$. φ induces the isomorphism

$$\varphi : \frac{\nabla}{ker\pi^{0,-1}} \cong \frac{\Delta}{\widetilde{\Delta}}.$$

The proof for case $N=2$ has been completed.

(d) $p = N-1, \ N \geq 2$.

$$K^{N-1,0} = \Omega^{N-1}_{U,k-N+1,0},$$

$$K^{N,-1} = \Omega^N_{U,k-N,0} A + \Omega^N_{U,k-N,0} B + \Omega^{N-1}_{U,k-N,0} DB.$$

$a^{N-1,0}+a^{N,-1} \in Z_2^{N-1,0}$, $a^{N-1,0} = \widetilde{\zeta}(N-1) \in K^{N-1,0}$, $a^{N,-1} = \theta(N)A+\eta(N)B+\zeta(N-1)DB$.

$$\begin{aligned}0 &= Da^{N-1,0} + \partial a^{N,-1}\\ &= D\widetilde{\zeta}(N-1) + (-1)^N(\theta(N)f + \eta(N)F + \zeta(N-1)\wedge DF).\end{aligned}$$

Let $\theta(N) = D\theta(N-1)$, $\eta(N) = D\eta(N-1)$. Define a map

$$\pi^{N,-1} : Z_2^{N-1,0} \longrightarrow W_{f,k-N}^{N-1}.$$

$\pi^{N,-1}(a^{N-1,0}+a^{N,-1}) = [(-1)^N\Pi_{k-N+1}\eta(N-1)+\zeta(N-1)]$. Clearly it is surjective.

Suppose

$$\begin{aligned}&\pi^{N,-1}(a^{N-1,0} + a^{N,-1}) = 0.\\ \Leftrightarrow\ &(-1)^N\Pi_{k-N+1}\eta(N-1) + \zeta(N-1)\\ &\qquad = D\overset{*}{\zeta}(N-2) + (-1)^{N-1}2\widetilde{\zeta}(N-2)\wedge DF,\\ \Rightarrow\ &a^{N,-1} = D(\theta(N-1)A + \eta(N-1)B + \overset{*}{\zeta}(N-2)DB)\\ &\qquad + \partial(\widetilde{\zeta}(N-2)(DB)^2).\end{aligned}$$

Let $\overline{b}^{N-1,-1} = \theta(N-1)A + \eta(N-1)B + \overset{*}{\zeta}(N-2)DB \in K^{N-1,-1}$, $\overline{b}^{N,-2} = \widetilde{\zeta}(N-2)(DB)^2 \in K^{N,-2}$. $0 = Da^{N-1,0}+\partial a^{N,-1} = Da^{N-1,0}+\partial D\overline{b}^{N-1,-1}$, $a^{N-1,0} = \partial\overline{b}^{N-1,-1} + D\overline{b}^{N-2,0}$.

Hence $a^{N-1,0} + a^{N,-1} \in B_1^{N-1,0}$. Suppose $b^{N-1,-1} = \overset{*}{\theta}(N-1)A + \overset{*}{\eta}(N-1)B + \overset{*}{\zeta}(N-2)DB \in K^{N-1,-1}$.

$$\begin{aligned}Db^{N-1,-1} &= D\overset{*}{\theta}(N-1)A + D\overset{*}{\eta}(N-1)B\\ &\qquad + ((-1)^{N-1}\Pi_{k-N+1}\overset{*}{\eta}(N-1) + D\overset{*}{\zeta}(N-2))DB\\ \Longrightarrow \pi^{N,-1}&(Db^{N-1,-1}) = [(-1)^N\Pi_{k-N+1}\overset{*}{\eta}(N-1)\\ &\qquad (-1)^{N-1}\Pi_{k-N+1}\overset{*}{\eta}(N-1) + D\overset{*}{\zeta}(N-2)] = 0.\\ \Longrightarrow \pi^{N,-1}&(DK^{N-1,-1}) = 0.\end{aligned}$$

Because $\partial K^{N,-2} \subset Z_1^{N,-1}$, we need only to compute $\pi^{N,-1}(Z_1^{N,-1})$

Suppose $a^{N,-1} = \theta(N)A + \eta(N)B + \zeta(N-1)DB \in Z_1^{N,-1}$, $\theta(N) = D\theta(N-1)$, $\eta(N) = D\eta(N-1)$.

$$
\begin{aligned}
0 &= (-1)^N \partial a^{N,-1} \\
&= \theta(N) f + \eta(N) F + \zeta(N-1) \wedge DF \\
&= D(\theta(N-1) f + \eta(N-1) F) \\
&\quad + ((-1)^N \Pi_{k-N+1} \eta(N-1) + \zeta(N-1)) \wedge DF, \\
\Longrightarrow & \delta_{k-N+1}([\theta(N-1)] f + [\eta(N-1)] F) \\
&= (-1)^N [(-1)^N \Pi_{k-N+1} \eta(N-1) + \zeta(N-1)], \\
\Longrightarrow & \pi^{N,-1}(a^{N,-1}) = (-1)^N \delta_{k-N+1}([\theta(N-1)] f + [\eta(N-1)] F). \\
\Longrightarrow & [\theta(N-1)] f + [\eta(N-1)] F \in H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}). \\
\Longrightarrow & {}' E_2^{N-1,0} = \frac{W^{N-1}_{f,k-N}}{\delta_{k-N+1} V^{N-1}_{f,k-N+1}}.
\end{aligned}
$$

The proof of the theorem has been completed. □

Corollary 1. *If f is quasi-homogeneous,* $H^{N-1}(\Omega^{\cdot}_{V,k-\cdot,0}) = 0$.

Proof. By [7],

$$FW^{N-1}_{f,k-N+1} = H^{N-1}(\Omega^{\cdot}_{V,k-\cdot,0}).$$

By Theorem D

$$\delta_{k-N+1} : H^{N-1}(\Omega^{\cdot}_{V,k-\cdot,0}) \cong W^{N-1}_{f,k-N}.$$

□

Now we prove Theorem 2. Suppose first $N \geq 3$. By the theorem,

$$
\begin{aligned}
\chi_{f,k} = (-1)^{N-3} (& dim_C U^{N-1}_{f,k-N+1} - dim_C DU^{N-1}_{f,k-N+1} \\
& + dim_C \Gamma^N_{f,k-N} + dim_C (S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1}) \\
& - dim_C R^{N-1}_{f,k-N+1} - dim_C V^{N-1}_{f,k-N+1} + dim_C W^{N-1}_{f,k-N}.
\end{aligned}
$$

By the following exact sequences and equality,

$$0 \longrightarrow R^{N-1}_{f,k-N+1} \longrightarrow W^{N-1}_{f,k-N+1} \oplus W^{N-1}_{f,k-N+1} \longrightarrow fW^{N-1}_{f,k-N+1} + FW^{N-1}_{f,k-N+1} \longrightarrow 0.$$

$$0 \longrightarrow V^{N-1}_{f,k-N+1} \longrightarrow fW^{N-1}_{f,k-N+1} + FW^{N-1}_{f,k-N+1} \longrightarrow DV^{N-1}_{f,k-N+1} \longrightarrow 0.$$

$$DV^{N-1}_{f,k-N+1} = f\Omega^N_{f,k-N,0} + F\Omega^N_{f,k-N,0}.$$

$$0 \longrightarrow F\Omega^N_{f,k-N,0} \longrightarrow f\Omega^N_{f,k-N,0} + F\Omega^N_{f,k-N,0} \longrightarrow \frac{f\Omega^N_{f,k-N,0} + F\Omega^N_{f,k-N,0}}{F\Omega^N_{f,k-N,0}} \longrightarrow 0.$$

$$0 \longrightarrow ker(\Lambda^N_{f,k-N,0} \xrightarrow{f} \Lambda^N_{f,k-N,0}) \longrightarrow \Lambda^N_{f,k-N,0} \longrightarrow \frac{f\Omega^N_{f,k-N,0} + F\Omega^N_{f,k-N,0}}{F\Omega^N_{f,k-N,0}} \longrightarrow 0.$$

$$0 \longrightarrow ker(\Omega^N_{f,k-N,0} \xrightarrow{F} \Omega^N_{f,k-N,0}) \longrightarrow \Omega^N_{f,k-N,0} \longrightarrow F\Omega^N_{f,k-N,0} \longrightarrow 0.$$

$$0 \longrightarrow ker(\Omega^N_{f,k-N,0} \xrightarrow{F} \Omega^N_{f,k-N,0}) \longrightarrow \Omega^N_{f,k-N,0} \xrightarrow{F} \Omega^N_{f,k-N,0} \longrightarrow \Lambda^N_{f,k-N,0} \longrightarrow 0.$$

Hence

$$\begin{aligned} 2dim_C W^{N-1}_{f,k-N+1} &= dim_C R^{N-1}_{f,k-N+1} + dim_C V^{N-1}_{f,k-N+1} + dim_C \Omega^N_{f,k-N,0} \\ &\quad - ker(\Lambda^N_{f,k-N,0} \xrightarrow{f} \Lambda^N_{f,k-N,0}). \end{aligned}$$

By the exact sequence

$$0 \longrightarrow U^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) \longrightarrow U^{N-1}_{f,k-N+1} \xrightarrow{D} DU^{N-1}_{f,k-N+1} \longrightarrow 0,$$

$$\begin{aligned} \chi_{f,k} = (-1)^{N-3}(&dim_C U^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) + dim_C \Gamma^N_{f,k-N} \\ &- dm_C ker(\Lambda^N_{f,k-N,0} \xrightarrow{f} \Lambda^N_{f,k-N,0}) - 2dim_C W^{N-1}_{f,k-N+1} \\ &+ dim_C W^{N-1}_{f,k-N} + dim_C \Omega^N_{f,k-N,0} + dim_C (S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1})). \end{aligned}$$

$$U^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})$$

$$= \{\omega \in H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}) : f\omega = 0, \quad (\delta^{-1}_{k-N+2} + f - F)\omega = 0\}.$$

By the canonical basis in [7] it is clear

$$\Pi_{k-N+2} : H^{N-1}(\Omega^{\cdot}_{f,k+1-\cdot,0}) \longrightarrow H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})$$

is surjective.

$$\begin{aligned} &(\delta^{-1}_{k-N+2} + f - F)\omega = 0, \quad f\omega = 0, \\ \Longleftrightarrow \quad &\delta^{-1}_{k-N+2}\Pi_{k-N+2}\widetilde{\omega} = (F-f)\widetilde{\omega}, \quad f\Pi_{k-N+2}\widetilde{\omega} = 0. \end{aligned}$$

where $\omega \in H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0})$, $\widetilde{\omega} \in H^{N-1}(\Omega^{\cdot}_{f,k+1-\cdot,0})$, and $\Pi_{k-N+2}\widetilde{\omega} = \omega$.

$$\begin{aligned} \Longleftrightarrow \quad &\Pi_{k-N+2}\widetilde{\omega} = \delta_{k-N+2}(F-f)\widetilde{\omega}, \quad f\Pi_{k-N+2}\widetilde{\omega} = 0. \\ \Longleftrightarrow \quad &(F-f)\delta_{k-N+2}\widetilde{\omega} = 0, \quad f\Pi_{k-N+2}\widetilde{\omega} = 0. \\ \Longleftrightarrow \quad &(F-f)\overline{\omega} = 0, \quad f\Pi_{k-N+2}\delta^{-1}_{k-N+2}\overline{\omega} = 0, \end{aligned}$$

where $\overline{\omega} = \delta_{k-N+2}\widetilde{\omega} \in W^{N-1}_{f,k-N+1}$. Hence $\omega = \Pi_{k-N+2}\delta^{-1}_{k-N+2}\overline{\omega}$.

$$\begin{aligned} \Longleftrightarrow \quad &(F-f)\overline{\omega} = 0, \quad f\Pi_{k-N+2}\overline{\omega} = 0, \\ \Longleftrightarrow \quad &F\overline{\omega} = [\xi], \quad f\overline{\omega} = [\xi], \quad \xi \in H\Omega^{N-1}_{U,k-N+1,0}. \end{aligned}$$

Let

$$P^{N-1}_{f,k-N+1} = \{\overline{\omega} \in W^{N-1}_{f,k-N+1} : F\overline{\omega} = [\xi], \quad f\overline{\omega} = [\xi], \quad \xi \in H\Omega^{N-1}_{U,k-N+1,0}\}.$$

The sequence

$$0 \longrightarrow P^{N-1}_{f,k-N+1} \longrightarrow W^{N-1}_{f,k-N+1} \stackrel{(-F\times, f\times)}{\longrightarrow} \frac{S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1}}{S^{N-1}_{f,k-N+1}} \longrightarrow 0$$

is clearly exact. Let

$$HW^{N-1}_{f,k-N+1} = \{[\xi] \in W^{N-1}_{f,k-N+1} : \xi \in H\Omega^{N-1}_{U,k-N+1,0}\}.$$

Clearly $HW^{N-1}_{f,k-N+1} \subset P^{N-1}_{f,k-N+1}$ and

$$\Pi_{k-N+2}\delta^{-1}_{k-N+2} : \frac{W^{N-1}_{f,k-N+1}}{HW^{N-1}_{f,k-N+1}} \cong H^{N-1}(\Omega^{\cdot}_{f,k-\cdot,0}).$$

It induces the isomorphism

$$\frac{P^{N-1}_{f,k-N+1}}{HW^{N-1}_{f,k-N+1}} \cong U^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{U,k-\cdot,0}).$$

Hence

$$dim_C U^{N-1}_{f,k-N+1} \cap H^{N-1}(\Omega^{\cdot}_{U,k-\cdot,0}) = dim_C H^{N-1}(\Omega^{\cdot}_{U,k-\cdot,0})$$

$$-dim_C(S^{N-1}_{f,k-N+1} + T^{N-1}_{f,k-N+1}) + dim_C S^{N-1}_{f,k-N+1}.$$

By the canonical basis

$$dim_C S^{N-1}_{f,k-N+1} = \binom{k}{N} dim_C Q_f.$$

By the Theorem D

$$dim_C W^{N-1}_{f,k-N+1} = \binom{k+1}{N+1} dim_C Q_f.$$

$$dim_C H^{N-1}(\Omega^{\cdot}_{U,k-\cdot,0}) = dim_C W^{N-1}_{f,k-N} = \binom{k}{N+1} dim_C Q_f.$$

$$dim_C \Omega^N_{f,k-N,0} = \binom{k}{N+1} dim_C Q_f.$$

We get the conclusion of Theorem 2 for $N \geq 3$. □

Similarly we can prove the case $N = 2$.

Corollary 1. *If $N \geq 3$,*

$$dim_C H^{N-2}(\Omega^{\cdot}_{V,k-\cdot,0}) = dim_C U^{N-1}_{f,k-N+1} + dim_C W^{N-1}_{f,k-N} - dim_C V^{N-1}_{f,k-N+1}$$

$$- dim_C \Gamma^N_{f,k-N} + dim_C ker(\Lambda^N_{f,k-N,0} \xrightarrow{f} \Lambda^N_{f,k-N,0})$$

If $N = 2$,

$$dim_C \widetilde{H}^{N-2}(\Omega^{\cdot}_{V,k-\cdot,0}) = dim_C W^1_{f,k-2} - dim_C V^1_{f,k-1} - dim_C \Gamma^2_{f,k-2}$$

$$+ dim_C ker(\Lambda^2_{f,k-2,0} \xrightarrow{f} \Lambda^2_{f,k-2,0}).$$

Corollary 2. *If $k = N + 1$, then*

$$\chi_{f,N+1} = (-1)^{N-2}\nu$$

$$+ (-1)^{N-3} dim_C\{a \in Q_f(y) : af = 0, a\frac{\partial^2 f(y)}{\partial y_i \partial y_j} = 0,\ i,j = 1, \cdots, N\}.$$

If moreover f is quasi-homogeneous, then

$$\chi_{f,N+1} = (-1)^{N-2}\nu + (-1)^{N-3} dim_C\{a \in Q_f(y) : a\frac{\partial^2 f(y)}{\partial y_i \partial y_j} = 0,\ i,j = 1, \cdots, N\}.$$

Proof. If f is quasi-homogeneous, $\Lambda^N_{f,l} = \Xi^N_{f,l} = \Omega^N_{f,l}$. □

Example 1. $f = x_1^5 + x_2^4$.

$\mu = \nu = 12$. The basis of Q_f is

$$\{x_1^{i_1} x_2^{i_2} : i_1 = 0, \cdots, 3; i_2 = 0, \cdots, 2\}.$$

$\frac{\partial^2 f}{\partial x_1^2} = 20x_1^3$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f}{\partial x_2^2} = 12x_2^2$. Hence

$$\{a \in Q_f : ax_1^3 = 0, ax_2^2 = 0\} = \{x_1^{i_1} x_2^{i_2} : i_1 = 1, 2, 3;\ \ i_2 = 1, 2\} Q_f.$$

If $k = 3$, $\chi_{f,3} = 6$ and $dim_C \widetilde{H}^0(\Omega^{\cdot}_{V,3-\cdot,0}) = 6$.

Example 2. $f = x_1^3 + x_2^7$.

$\mu = \nu = 12$. The basis of Q_f is

$$\{x_1^{i_1} x_2^{i_2} : i_1 = 0, 1; i_2 = 0, \cdots, 5\}.$$

$\frac{\partial^2 f}{\partial x_1^2} = 6x_1$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f}{\partial x_2^2} = 42x_2^5$. Hence

$$\{a \in Q_f : ax_1 = 0, ax_2^5 = 0\} = \{x_1^{i_1} x_2^{i_2} : i_1 = 1;\ \ i_2 = 1, \cdots, 5\} Q_f.$$

If $k = 3$, $\chi_{f,3} = 7$ and $dim_C \widetilde{H}^0(\Omega^{\cdot}_{V,3-\cdot,0}) = 7$.

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