PARABOLIC HIGGS BUNDLES AND TEICHMÜLLER SPACES FOR PUNCTURED SURFACES

INDRANIL BISWAS, PABLO ARÉS-GASTESI, AND SURESH GOVINDARAJAN

ABSTRACT. In this paper we study the relation between parabolic Higgs vector bundles and irreducible representations of the fundamental group of punctured Riemann surfaces established by Simpson. We generalize a result of Hitchin, identifying those parabolic Higgs bundles that correspond to Fuchsian representations. We also study the Higgs bundles that give representations whose image is contained, after conjugation, in $SL(k, \mathbb{R})$. We compute the real dimension of one of the components of this space of representations, which in the absence of punctures is the generalized Teichmüller space introduced by Hitchin, and which in the case of $k = 2$ is the usual Teichmüller space of the punctured surface.

1. INTRODUCTION

In the well-known paper [3], Hitchin introduced Higgs bundles, and established a one-to-one correspondence between equivalence classes of irreducible $GL(2, \mathbb{C})$ representations of the fundamental group of a compact Riemann surface and isomorphism classes of rank two stable Higgs of degree zero. In [8], Simpson defined parabolic Higgs bundles, which generalized Hitchin's correspondence to the case of punctured Riemann surfaces (see also [7]). Here, by a punctured Riemann surface we mean the complement of finitely many points in a compact surface. More precisely, Simpson identified what he calls filtered local systems with parabolic Higgs bundles.

In [4], Hitchin identified the Higgs bundles corresponding to the Fuchsian representations. Our main aim here is to generalize his results to the case of punctured Riemann surfaces.

Before giving more details, we describe the result of Hitchin on Fuchsian representations. Let X be a compact Riemann surface of genus $g \geq 2$, and let L be a line bundle on X such that $L^2 = K_X$, that is L is a square root of the canonical bundle of X. Define

$$E := L^* \oplus L$$

which is a rank 2 vector bundle on X. For $a \in H^0(X, K^2)$, let

$$\theta(a) := \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \in H^0\big(X, End(E) \otimes K\big)$$

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be the Higgs field. Hitchin proved that the conjugacy classes of Fuchsian representations of $\pi_1(X)$ (homomorphisms of $\pi_1(X)$ into $\mathrm{PSL}(2,\mathbb{R})$ such that the quotient of the action on the upper half plane is a compact Riemann surface of genus g) correspond to the Higgs bundles of the form $(E, \theta(a))$ defined above. Moreover, the Higgs bundle $(E, \theta(0))$ corresponds to the Fuchsian representation for the Riemann surface X itself.

Consider now a punctured Riemann surface $X = \bar{X} - \{p_1, \ldots, p_n\}$, where $\bar{X}$ is a compact surface of genus g, and $p_1, \ldots, p_n$ are $n > 0$ distinct points of $\bar{X}$. Let D denote the divisor given by these points, i.e. $D = \{p_1, \ldots, p_n\}$. We will further assume that $2g - 2 + n > 0$, which is equivalent to the condition that the universal covering space of X is (conformally equivalent to) the upper half plane. Consider the vector bundle

$$E := (L \otimes \mathcal{O}_{\bar{X}}(D))^* \oplus L$$

and give parabolic weight $1/2$ to the fiber E_{p_i}, $1 \leq i \leq n$. (The line bundle L, as before, satisfies $L^2 = K_{\bar{X}}$.) For any $a \in H^0(\bar{X}, K_{\bar{X}}^2 \otimes \mathcal{O}_{\bar{X}}(D))$ let

$$\theta(a) := \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \in H^0\big(\bar{X}, End(E) \otimes K_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(D)\big)$$

be the parabolic Higgs field on the parabolic bundle E.

We prove that under the identification between filtered local systems and parabolic Higgs bundles, Fuchsian representations of n-punctured Riemann surfaces are in one-one correspondence with parabolic Higgs bundles of the type $(E, \theta(a))$ defined above. Moreover, the parabolic Higgs bundle $(E, \theta(0))$ corresponds to the Fuchsian representation of the punctured surface X itself. Thus this is a direct generalization of the result of Hitchin on Fuchsian representations of compact Riemann surfaces to the punctured case.

In section 3, we generalize the above results to the case of representations of the fundamental group of the surface X into $\mathrm{PSL}(k, \mathbb{R})$, for $k > 2$. More precisely, we consider a parabolic bundle W_k, obtained by tensoring the $(k-1)$th symmetric product of the vector bundle E defined above with an appropriate power of $\mathcal{O}_{\bar{X}}(D)$. The Higgs fields we consider are generalizations of the 2-dimensional case, namely they are of the form

$$\theta(a_2, \ldots, a_{k-1}) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & 1 \\ a_k & \cdots & a_2 & 0 \end{pmatrix},$$

where a_j is a section of the line bundle $K^j \otimes \mathcal{O}_{\bar{X}}((j-1)D)$. As in section 2, we have that the pair $(W_k, \theta(a_2, \ldots, a_k))$ is a stable parabolic Higgs bundle of parabolic degree 0. It is not difficult to see that the parabolic dual of W_k is naturally isomorphic to the parabolic bundle W_k itself. Moreover, the identification between W_k and its parabolic dual takes $\theta(a_2, \ldots, a_k))$ to itself. This implies that the holonomy of the flat connections corresponding to these parabolic Higgs bundles are contained in a split real form of $\mathrm{SL}(k, \mathbb{C})$, which is isomorphic to $\mathrm{SL}(k, \mathbb{R})$. We prove that one particular component of the space of representations of $\pi_1(X)$ into $\mathrm{SL}(k, \mathbb{R})$, with fixed conjugacy class of monodromy around the punctures, has real dimension equal to $2(k^2 - 1)(g - 1) + k(k - 1)n$. Observe that for $k = 2$,

this dimension is $2(3g - 3 + n)$, which is precisely the real dimension of $\mathcal{T}_g^n$, the Teichmüller space of compact surfaces of genus g with n punctures. It is therefore natural, following [4], to call the above component the *Teichmüller component* of the corresponding space of representations. Further study of this space is perhaps worthwhile.

2. Higgs bundles for Fuchsian representations

Let $\bar{X}$ be a compact Riemann surface of genus g, and let

$$D := \{p_1, p_2, \ldots, p_n\}$$

be n distinct points on $\bar{X}$. Define $X := \bar{X} - D$ to be the punctured Riemann surface given by the complement of the divisor D. We will assume that $2g - 2 + n > 0$, that is, the surface X supports a metric of constant curvature (-4).

The degree of the holomorphic cotangent bundle K, of $\bar{X}$ is $2g - 2$. Therefore, there is a line bundle L on $\bar{X}$ such that $L^2 = K$. Fix such a line bundle L. Note that any two of the 4^g possible choices of L differ by a line bundle of order 2.

Using L we will construct a parabolic Higgs bundle on $\bar{X}$, as follows. Let $\xi = \mathcal{O}_{\bar{X}}(D)$ denote the line bundle on $\bar{X}$ given by the divisor D. Define

$$(2.1) \qquad E := (L \otimes \xi)^* \oplus L$$

to be the rank 2 vector bundle on $\bar{X}$. To define a parabolic structure on E (we will follow the definition of parabolic Higgs bundle given in [8]), on each point $p_i \in D$, $1 \leq i \leq n$, we consider the trivial flag

$$E_{p_i} \supset 0,$$

and give parabolic weight $1/2$ to E_{p_i}. This gives a parabolic structure on E.

Note that

$$(2.2) \qquad Hom(L, L^* \otimes \xi^*) \otimes K \otimes \xi = \mathcal{O} \subset End(E) \otimes K \otimes \xi.$$

Let 1 denote the section of $\mathcal{O}$ given by the constant function 1. So from (2.2) we have

$$(2.3) \qquad \theta := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in H^0(\bar{X}, End(E) \otimes K \otimes \xi).$$

Lemma 2.1. *The parabolic Higgs bundle (E, θ) is a parabolic stable Higgs bundle of parabolic degree zero.*

Proof. From the definition of parabolic degree (see [6, definition 1.11] or [8]) we immediately conclude that the parabolic degree of E is zero.

To see that (E, θ) is stable, first note that there is only one subbundle of E which is invariant under θ, namely the summand $(L \otimes \xi)^*$ in (2.1). (A subbundle $F \subset E$ is called *invariant* under θ if $\theta(F) \subset F \otimes K \otimes \xi$.) The degree of $(L \otimes \xi)^*$ is $1 - g - n$. So the parabolic degree of $(L \otimes \xi)^*$, for the induced parabolic structure, is $1 - g - n/2$.

But, from our assumption, namely that $2g - 2 + n > 0$, we have $1 - g - n/2 < 0$. So (E, θ) is parabolic stable. ☐

From the proof of Lemma 2.1 it follows that (E, θ) constructed above is parabolic stable if and only if $2g - 2 + n > 0$. We will show later that this corresponds to the fact that X admits a complete metric of constant negative curvature if and only if $2g - 2 + n > 0$.

From the main theorem of [8, pg. 755] we know that there is a tame harmonic metric on the vector bundle E. (See the Synopsis of that paper for the definition of tame harmonic metric.)

It is well-known that there is a unique complete Kähler metric on X, known as the *Poincaré metric*, such that its curvature is (-4).

Both the line bundles L and $(L \otimes \xi)^*$ are equipped with metrics induced by the tame harmonic metric on E. So

$$Hom(L, (L \otimes \xi)^*) = L^{-2} \otimes \xi^* = T_{\bar{X}} \otimes \xi^*$$

is equipped with an induced metric. The restriction to X of the line bundle ξ, and hence ξ^*, on $\bar{X}$ has a canonical trivialization. Therefore we have a hermitian metric on T_X the tangent bundle of X. We will denote this hermitian metric on T_X by H. Note that H is singular at D, *i.e.* it does not induce a hermitian metric on $T_{\bar{X}}$.

Lemma 2.2. *The hermitian metric H on the holomorphic tangent bundle on X obtained above is the Poincaré metric.*

Proof. We recall the Hermitian-Yang-Mills equation which gives the harmonic metric on E [8]. This equation was first introduced in [3].

Let ∇ denote the holomorphic hermitian connection on the restriction of E to X for the harmonic metric. Then the Hermitian-Yang-Mills equation of the curvature of ∇ is the following:

$$(2.4) \qquad K(\nabla) := \nabla^2 = -[\theta, \theta^*].$$

If the decomposition (2.1) is orthogonal with respect to the metric, then $[\theta, \theta^*]$ is a 2-form with values in the diagonal endomorphisms of E (diagonal for the decomposition (2.1)). Using this, the equation (2.4) reduces to the following equation on X

$$(2.5) \qquad F_H = -2\bar{H},$$

where H is a hermitian metric on T_X and $\bar{H}$ is the $(1,1)$-form on X given by H, *i.e.* the Kähler 2-form for the metric H.

A metric H' on T_X induces a metric on L. Since the line bundle ξ has a natural trivialization over X, the metric H' also induces a metric on $(L \otimes \xi)^*$, and therefore also on E. If H' satisfies the equation (2.5) then the metric on E obtained this way satisfies (2.4). Now from the uniqueness of the solution of (2.4) ([8]) we have that any such metric on E is obtained from the solution of (2.5) in the above fashion.

From the computation in Example (1.5) of [3, pg. 66], we conclude that the Kähler metric H on X has Gaussian curvature (-4).

So in order to complete the proof of the lemma we must show that the Kähler metric on X is complete.

Recall the asymptotic behavior of the harmonic metric near the punctures given in Section 7 of [8]. First of all, observe that the fiber of $K \otimes \xi$ at any $p_i \in D$ is canonically isomorphic to $\mathbb{C}$. So the fiber $(End(E) \otimes K \otimes \xi)_{p_i}$ is actually $End(E_{p_i})$. The evaluation of the section θ at p_i as an element of $End(E_{p_i})$ is defined to be the *residue* of θ at p_i. Thus for the Higgs field θ, the residue at each p_i is

$$N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In [8, pg. 755], Simpson studies parabolic Higgs bundles with residue N as above. Consider the displayed equation on page 758 of [8], which describes the asymptotic

behavior of the corresponding harmonic metric. Using the fact that the parabolic weight of E_{p_i} is $1/2$ we conclude that for the metric on L induced by the tame harmonic metric on E, both a_i and n_i in the equation on page 758 of [8] are $1/2$. (We also use the fact that, in the notation of [8, pg. 755], $L \subset W_1$ and L is not contained in W_0.) In other words, in a suitable trivialization of L on an open set containing a puncture $p_i \in D$, and with holomorphic coordinate z around p_i, the hermitian metric on L obtained by restricting the harmonic metric on E is

$$r^{1/2}|\log(r)|^{1/2},$$

where $r = |z|$.

Similarly, for $(L \otimes \xi)^*$, the a_i and n_i in the equation [8, pg. 758] are $1/2$ and $-1/2$ respectively.

So the metric on $Hom(L, (L \otimes \xi)^*)$ is $(\log|(r)|)^{-1}$. Recall the earlier remark that ξ^* has a natural trivialization on X. The section of ξ^* on X has a pole of order 1 at the points of D, when it is considered as a meromorphic section of ξ^* on $\bar{X}$. This implies that the hermitian metric on $T = L^{-2}$ is

$$(2.6) \qquad r^{-1}|\log(r)|^{-1}.$$

But this is the expression of the Poincaré metric of the punctured disk in $\mathbb{C}$. Thus, from the completeness of the Poincaré metric we conclude that the Kähler metric on X induced by H is indeed complete. This completes the proof of the lemma. □

From the decomposition (2.1) it follows that

$$(2.7) \qquad Hom(L^* \otimes \xi^*, L) \otimes K \otimes \xi \ = \ K^2 \otimes \xi^2 \ \subset \ End(E) \otimes K \otimes \xi.$$

Note that the line bundle ξ has a natural section given by the constant function 1, which we will denote by 1_ξ. We may embed $H^0(\bar{X}, K^2 \otimes \xi)$ into $H^0(\bar{X}, K^2 \otimes \xi^2)$ by the homomorphism $s \longmapsto s \otimes 1_\xi$. So using (2.7) we have a natural homomorphism

$$(2.8) \qquad \rho \ : \ H^0(\bar{X}, K^2 \otimes \xi) \ \longrightarrow \ H^0(\bar{X}, End(E) \otimes K \otimes \xi).$$

Note that the image of ρ is contained in the image of the inclusion

$$H^0(\bar{X}, End(E) \otimes K) \ \longrightarrow \ H^0(\bar{X}, End(E) \otimes K \otimes \xi).$$

With a slight abuse of notation, for any $a \in H^0(\bar{X}, K^2 \otimes \xi)$, the corresponding element in $H^0(\bar{X}, End(E) \otimes K)$ will also be denoted by $\rho(a)$.

The following theorem is a generalization of theorem (11.2) of [3] to the case of punctured Riemann surfaces.

Theorem 2.3. *For any $a \in H^0(\bar{X}, K^2 \otimes \xi)$, the Higgs structure*

$$\theta_a \ := \ \theta + \rho(a) \ = \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \rho(a)$$

on the parabolic bundles E (defined in (2.1)) makes (E, θ_a) a parabolic stable Higgs bundle of parabolic degree zero.

Let H_a denote the harmonic metric (given by the main theorem of [8]) on the restriction of E to X, and let h denote the Kähler metric on X induced by the tame harmonic metric H_a as in Lemma 2.2. Then the following holds :

1. *The section of the 2nd symmetric power of the complex tangent bundle*

$$h_a \ := \ a + h + \bar{a} + a\bar{a}/h \ \in \ \Omega^0(X, S^2(T^*_{\mathbb{R}} \otimes \mathbb{C}))$$

   is a Riemannian metric on X.

2. *The metric h_a is a complete Riemannian metric of constant Gaussian curvature (-4). The Riemann surface structure on X given by the metric h_a is a Riemann surface with punctures, i.e. there are no holes. (A Riemann surface with a hole is a complement of a disk in a compact Riemann surface.)*
3. *Associating to any $a \in H^0(\bar{X}, K^2 \otimes \xi)$ the complex structure on the C^∞ surface X given by the metric h_a, the map obtained from $H^0(\bar{X}, K^2 \otimes \xi)$ to the Teichmüller space $\mathcal{T}_g^n$ of surfaces of genus g and n punctures, is actually a bijection.*

Proof. To prove that (E, θ_a) is stable we use a trick of [4]. For $\mu > 0$, define an automorphism of E by

$$T := \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}.$$

The parabolic Higgs bundle (E, θ_a) is isomorphic to $(E, T^{-1} \circ \theta_a \circ T)$, and hence $(E, T^{-1} \circ \theta_a \circ T)$ is parabolic stable if and only if (E, θ_a) is so. Since $\mu \neq 0$, we have $(E, T^{-1} \circ \theta_a \circ T)$ is parabolic stable if and only if $(E, \frac{1}{\mu} T^{-1} \circ \theta_a \circ T)$ is parabolic stable. Now

$$\frac{1}{\mu} T^{-1} \circ \theta_a \circ T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \rho(a)/\mu = \theta_{a/\mu}.$$

But from the openness of the stability condition we have that, since (E, θ) is stable [Lemma 2.1], there is a non-empty open set U in $H^0(\bar{X}, K^2 \otimes \xi)$ containing the origin such that for any $a \in U$, the parabolic Higgs bundle (E, θ_a) is parabolic stable. Taking μ to be sufficiently large so that $\theta_{a/\mu} \in U$, we conclude that any (E, θ_a) is parabolic stable.

The vector bundle E is equipped with the harmonic metric H_a, and K has a metric induced by h_a. Using these metrics we construct a hermitian metric on $End(E) \otimes K$. Since $\rho(a) \in H^0(\bar{X}, End(E) \otimes K)$, we may take its pointwise norm with respect to this metric.

As the first step to prove the statement (1) we will calculate the behavior of $||\rho(a)||$ near the punctures. Since $\rho(a) \in H^0(\bar{X}, End(E) \otimes K)$, we have

$$\text{residue}\,(\theta_a) = \text{residue}\,(\theta) = N.$$

So the two hermitian metrics H_0 (corresponding to $a = 0$) and H_a on E are mutually bounded, *i.e.* $C_1.H_0 \leq H_a \leq C_2.H_0$ for some constants C_1 and C_2. (Recall that the metric in Lemma 2.2 was induced by H_0.) From this it is easy to check that around any puncture p_i, the norm $||\rho(a)||$ is bounded by $r|\log(r)|^{3/2}$. This implies that $||\rho(a)||$ converges to zero as we approach a puncture.

Arguing as in (11.2) of [3], if h_a is not a metric then

$$1 - ||\rho(a)|| \leq 0$$

at some point $x \in X$. Since $||\rho(a)||$ converges to zero as we approach a puncture, the infimum of the function $1 - ||\rho(a)||$ on X must be attained somewhere, say at $x_0 \in X$.

Let Δ denote the Laplacian operator acting on smooth functions on X. Since the operator $\mathcal{L} := -\Delta - 4||\rho(a)||^2$ is uniformly elliptic on X, we may apply [5, Section VI.3., Proposition 3.3] for the operator $\mathcal{L}$ and the point x_0. We conclude

that either $1 - ||\rho(a)|| > 0$ or $1 - ||\rho(a)||$ is a constant function. This proves that h_a is a Riemannian metric on X.

From the computation in the proof of Theorem (11.3)(ii) of [3, pg. 120], we conclude that h_a is a metric of curvature (-4).

To complete the proof of the statement (2) we must show that h_a is complete and it has finite volume. (If the volume of the Poincaré metric on a Riemann surface is finite then the Riemann surface is a complement of finite number of points in a compact Riemann surface. In particular, the Riemann surface can not have any holes.)

The above established fact that the metrics H_0 and H_a on E are mutually bounded, together with Lemma 2.2 imply that the Riemannian metric h_a and the Poincaré metric on X are mutually bounded. Since the Poincaré metric is complete and of finite volume, the same must hold for h_a.

To prove the statement (3) we have to show that the map from the vector space $H^0(\bar{X}, K^2 \otimes \xi)$ to the Teichmüller space $\mathcal{T}_g^n$ obtained in (2) is surjective. This will follow from Section 3 where we will prove that the image is both open and closed, and hence it must be surjective as $\mathcal{T}_g^n$ is connected.

However we may also use the argument in [3, Theorem (11.2)(iii)] to prove statement (3). Let h_0 denote the Poincaré metric on X. Indeed, to make the argument work all we need to show is the following generalization of the Eells-Sampson theorem to punctured Riemann surfaces: given a complete Riemannian metric h of constant curvature (-4) and finite volume on the C^∞ surface X, there is a unique diffeomorphism f, of X homotopic to the identity map, such that f is a harmonic map from (X, h_0) to (X, h). This follows from the generalization of the theorem of Corlette, [2], to the non-compact case as mentioned in [8, pg. 754].

Let (V, ∇) be the flat rank two vector bundle given by the Fuchsian representation for the Riemann surface (X, g). Let H be the harmonic metric on V given by the main theorem of [8] (pg. 755) for the flat bundle (V, ∇) on the Riemann surface (X, h_0). In other words, H gives a section, denoted by s, of the associated fiber bundle with fiber $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) = \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. This section s gives the harmonic map f mentioned above. This completes the proof of the theorem. □

The vector space $H^0(\bar{X}, K^2 \otimes \xi)$ has a natural complex structure. So does the Teichmüller space $\mathcal{T}_g^n$. The identification of $H^0(\bar{X}, K^2 \otimes \xi)$ with $\mathcal{T}_g^n$ given by Theorem 2.11 does not preserve the complex structures. Indeed, $\mathcal{T}_g^n$ is known to be biholomorphic to a bounded domain in $\mathbb{C}^{3g-3+n}$. Since any bounded holomorphic function on an affine space must be constant, the identification in Theorem 2.11 is never holomorphic.

Remark. The parabolic dual of the parabolic bundle E is $E^* \otimes \xi^*$ with trivial parabolic flag and parabolic weight $1/2$ at the parabolic points p_i, $1 \leq i \leq n$. So the parabolic dual of E is E itself. Any parabolic Higgs bundle (E, θ_a) (as in Theorem 2.3) is naturally isomorphic to the parabolic Higgs bundle (E^*, θ_a^*), where E^* is the parabolic dual of E. This implies that the holonomy of the flat connection on X corresponding to the Higgs bundle (E, θ_a) is contained (after conjugation) in $\mathrm{SL}(2, \mathbb{R})$. This of course is also implied by Theorem 2.3 since the image of a Fuchsian representation is contained in $\mathrm{PSL}(2, \mathbb{R})$.

3. Higgs bundles for $\mathrm{SL}(k, \mathbb{R})$ representations

Recall the vector bundle E of section 2, which was defined to be $E = (L \otimes \xi)^* \oplus L$, where L is a (fixed) square root of the canonical bundle K, and $\xi = \mathcal{O}_{\bar{X}}(D)$. The $(k-1)$th symmetric product of $\mathbb{C}^2$ produces an embedding of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{SL}(k, \mathbb{R})$, via action on homogeneous polynomials of degree k. Let V_k denote the vector bundle given by the $(k-1)$th symmetric product of E, that is $V_k := S^{k-1}(E)$. At each point $p_i \in D$ we have the trivial flag $(V_k)_{p_i} \supset 0$, $1 \leq p_i \leq n$, with weight equal to $\frac{k-1}{2}$. In order to construct a parabolic bundle, we need to reduce the weight to a number in the interval $[0, 1)$. We do this by tensoring V_k with $\xi^{m(k)}$, where $m(k)$ is equal to $\frac{k}{2} - 1$, if k is even, or $\frac{k-1}{2}$, if k is odd. We will denote the vector bundle $V_k \otimes \xi^{m(k)}$ by W_k. At each point $p_i \in D$, we take the trivial flag $(W_k)_{p_i} \supset 0$ of W_k, with weight equal to $\frac{1}{2}$, if k is even, or 0, if k is odd. (This parabolic bundle is the $(k-1)$th parabolic symmetric power of the parabolic bundle E; see [1] for the general definition of parabolic symmetric power.)

Considering 1 as the section of $\mathcal{O}$ given by the constant function 1, we can define

$$(3.1) \qquad \theta(0, \dots, 0) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

which represents an element of $H^0\big(\bar{X}, End(W) \otimes K \otimes \xi\big)$.

Lemma 3.1. *The pair $(W_k, \theta(0, \dots, 0))$ is a parabolic stable Higgs bundle of parabolic degree zero.*

Proof. If k is even, we have that the parabolic degree of W_k is

$$\text{degree of } W_k + \text{parabolic weight of } W_k = -\frac{kn}{2} + \frac{kn}{2} = 0.$$

In the case of odd k, it is easy to see that the degree (as a vector bundle) of W_k is 0, and since the weight is equal to 0, we get that the parabolic degree of W_k is zero.

The invariant proper subbundles of 3.1 are of the form

$$L^{1-k} \otimes \xi^{-k/2} \oplus L^{3-k} \otimes \xi^{1-k/2} \oplus \dots \oplus L^{2m+1-k} \otimes \xi^{m-k/2}$$

$(0 \leq m \leq k-1)$ if k is even; or

$$L^{1-k} \otimes \xi^{(1-k)/2} \oplus L^{3-k} \otimes \xi^{1+(1-k)/2} \oplus \dots \oplus L^{2m+1-k} \otimes \xi^{m+(1-k)/2}$$

$(0 \leq m \leq k-1)$ if k is odd. It is not difficult to see that all these subbundles have negative parabolic degree. $\qquad \square$

Using the natural section 1_ξ of ξ, we embed the spaces $H^0(\bar{X}, K^j \otimes \xi^{j-1})$, $j = 2, \dots, k$, into $H^0(\bar{X}, End(W_k) \otimes K \otimes \xi)$. By an abuse of notation, if $a_j \in H^0(\bar{X}, K^j \otimes \xi^{j-1})$, we understand the above embedding as producing an element

$$(3.2) \qquad \theta(a_2, \dots, a_{k-1}) := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & 1 \\ a_k & \cdots & a_2 & 0 \end{pmatrix}$$

of $H^0(\bar{X}, End(W_k) \otimes K \otimes \xi)$. Now, by the arguments of Hitchin, based on the openness of the stability of parabolic Higgs bundles, we get that $(W_k, \theta(a_2, \ldots, a_k))$ is a stable parabolic Higgs bundle of parabolic degree 0. Using these special Higgs bundles, one can obtain some information about the space of representations of the fundamental group of X into $SL(k, \mathbb{R})$. More precisely, our result is as follows.

Proposition 3.2. *The space of representations of the fundamental group of X in $SL(k, \mathbb{R})$, with fixed conjugacy class of monodromy around the punctures, has a component of real dimension $2(k^2 - 1)(g - 1) + k(k - 1)n$.*

Proof. By the work of Simpson [8] and Balaji Srinivasan [9], we have a one-to-one continuous correspondence between the space M of stable parabolic Higgs bundles of degree zero, and the space of representations of the fundamental group of X into $SL(k, \mathbb{C})$. Consider the parabolic dual of W_k, which is constructed as follows. First, take the dual vector bundle W_k^* of W_k. If k is odd, since the weight of the flag is 0, we have that the parabolic dual of W_k is W_k^*, with trivial flag at the points $p_i \in D$, and weight equal to zero. If k is even, we have a weight of $-\frac{1}{2}$ associated to the trivial flag of W_k^*. Tensor W_k^* with ξ to obtain that the parabolic dual of W_k is $W_k^* \otimes \xi$. So we always have that the parabolic dual of the parabolic bundle W_k is W_k itself. This implies that the image of the fundamental group under the representation induced by (W_k, θ) lies in $SL(k, \mathbb{R})$.

Since a_j is a section of $K^j \otimes \xi^{j-1}$, we have that the residue of the Higgs field is invariant, *i.e.*

$$\text{residue}\,(\theta(a_2, \ldots, a_{k-1})) = \text{residue}\,(\theta(0, \ldots, 0)) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This implies that in the above representation, the conjugacy class of the elements corresponding to small loops around the punctures of X is invariant. By the embedding of $SL(2, \mathbb{R})$ into $SL(k, \mathbb{R})$, we have that this is the class of the element

$$(3.3) \qquad\qquad U = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \vdots \\ \vdots & & & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Using a bases, $\{p_1, \ldots, p_{k-1}\}$, for the set of invariant polynomials of the Lie algebra of $SL(k, \mathbb{C})$, we can construct a continuous mapping

$$p : M \to \bigoplus_{j=2}^{k} H^0(\bar{X}, K^j \otimes \xi^j),$$

given by assigning to the Higgs field (W_k, Φ) the elements $(p_1(\Phi), \ldots, p_{k-1}(\Phi))$. The Higgs fields of the form (3.2) produce a section s of p, defined over the closed subspace $\bigoplus_{j=2}^{k} H^0(\bar{X}, K^j \otimes \xi^{j-1})$. Therefore, we have that the image of s is closed. One can easily compute that the dimension (over $\mathbb{R}$) of the space of

sections $\bigoplus_{j=2}^{k} H^0(\bar{X}, K^j \otimes \xi^{j-1})$ is equal to

$$\sum_{j=2}^{k} 2(2j-1)(g-1) + 2\sum_{j=2}^{k}(j-1)n = 2(k^2-1)(g-1) + k(k-1)n.$$

On the other hand, the dimension of the space of representations of the fundamental group of X into $\mathrm{SL}(k, \mathbb{R})$, with the condition that the monodromy around the punctures lies in the above conjugacy class, can be computed as follows. The fundamental group of X can be identified with a group of Möbius transformations (or elements of $\mathrm{SL}(2, \mathbb{R})$), generated by elements $\{c_1, d_1, \ldots, c_g, d_g, e_1, \ldots, e_n\}$, and with one relation of the form $\prod_{j=1}^{g}[c_j, d_j]\prod_{j=1}^{n} e_j = id$, where $[c, d] = cdc^{-1}d^{-1}$ denotes the commutator of the elements c and d. In classical terms, the transformations c_j's and d_j's are hyperbolic, that is conjugate to dilatation, while the e_j's are parabolic, or conjugate to translations. In terms of loops on X, we have that the c_j's and d_j's can be identified with paths around the handles of X, while the e_j's are simple loops around the punctures. The image of the elements c_j and d_j depends on $\dim(\mathrm{SL}(k, \mathbb{R})) = k^2 - 1$ parameters. We will now compute the number of parameters of the image of e_j. Any e_j can be written as $e_j = AUA^{-1}$, where U is as in (3.3), and $A \in SL(k, \mathbb{R})/I$, with I being the commutator subgroup for U. One can check that the dimension of I is $k-1$. Thus the required number of parameters for e_j is $(k^2 - 1) - (k-1) = k^2 - k$. Therefore, we have that the real dimensions of $\bigoplus_{j=2}^{k} H^0(\bar{X}, K^j \otimes \xi^{j-1})$ and the space of representations of $\pi_1(X)$, with fixed conjugacy class for the monodromy elements around the punctures, are equal. Standard arguments using the invariance of domain theorem complete the proof. $\square$

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School of Mathematics, Tata Institute of Fundamental Research, Bombay, India
E-mail address: `indranil@math.tifr.res.in`

E-mail address: `pablo@math.tifr.res.in`

Department of Physics, Indian Institute of Technology, Madras, India
E-mail address: `suresh@imsc.ernet.in`