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MENAS' RESULT IS BEST POSSIBLE

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ABSTRACT. Generalizing some earlier techniques due to the second author, we show that Menas' theorem which states that the least cardinal κ which is a measurable limit of supercompact or strongly compact cardinals is strongly compact but not 2^κ supercompact is best possible. Using these same techniques, we also extend and give a new proof of a theorem of Woodin and extend and give a new proof of an unpublished theorem due to the first author.

0. INTRODUCTION AND PRELIMINARIES

It is well known that if κ is 2^κ supercompact, then κ is quite large in both size and consistency strength. As an example of the former, if κ is 2^κ supercompact, then κ has a normal measure concentrating on measurable cardinals. The key to the proof of this fact and many other similar ones is the existence of an elementary embedding $j : V \to M$ with critical point κ so that $M^{2^\kappa} \subseteq M$. Thus, if $2^\kappa > \kappa^+$, one can ask whether κ must be large in size if κ is merely δ supercompact for some $\kappa < \delta < 2^\kappa$.

A natural question of the above venue to ask is whether a cardinal κ can be both the least measurable cardinal and δ supercompact for some $\kappa < \delta < 2^\kappa$ if $2^\kappa > \kappa^+$. Indeed, the first author posed this very question to Woodin in the spring of 1983. In response, using Radin forcing, Woodin (see [CW]) proved the following

Theorem. *Suppose $V \models$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is λ^+ supercompact and λ is regular". There is then a generic extension $V[G]$ so that $V[G] \models$ "ZFC + $2^\kappa = \lambda$ + κ is δ supercompact for all regular $\delta < \lambda$ + κ is the least measurable cardinal".*

The purpose of this paper is to extend the techniques of [AS] and show that they can be used to demonstrate that Menas' result of [Me] which says that the least measurable cardinal κ which is a limit of supercompact or strongly compact cardinals is strongly compact but not 2^κ supercompact is best possible. Along the way, we generalize and strengthen Woodin's result above, and we also produce a model in which, on a proper class, the notions of measurability, δ supercompactness,

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and δ strong compactness are all the same. Specifically, we prove the following theorems.

Theorem 1. *Suppose $V \models$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is $< \lambda$ supercompact, $\lambda > \kappa^+$ is a regular cardinal which is either inaccessible or is the successor of a cardinal of cofinality $> \kappa$, and $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$". There is then a cardinal and cofinality preserving generic extension $V[G] \models$ "ZFC + For every inaccessible $\delta < \kappa$ and every cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + For every cardinal $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$ + κ is $< \lambda$ supercompact + κ is the least measurable cardinal".*

Theorem 2. *Let λ be a (class) function such that for any infinite cardinal δ, $\lambda(\delta) > \delta^+$ is a regular cardinal which is either inaccessible or is the successor of a cardinal of cofinality $> \delta$, $\lambda(0) = 0$, and $\lambda(\delta)$ is below the least inaccessible $> \delta$ if δ is singular. Suppose $V \models$ "ZFC + GCH + A is a proper class of cardinals so that for each $\kappa \in A$, $h_\kappa : \kappa \to \kappa$ is a function and $j_\kappa : V \to M$ is an elementary embedding witnessing the $< \lambda(\kappa)$ supercompactness of κ with $j_\kappa(h_\kappa)(\kappa) = \lambda(\kappa) < \kappa^*$ for κ^* the least element of $A > \kappa$". There is then a cardinal and cofinality preserving generic extension $V[G] \models$ "ZFC + $2^\gamma = \lambda(\kappa)$ if $\kappa \in A$ and $\gamma \in [\kappa, \lambda(\kappa))$ is a cardinal + There is a proper class of measurable cardinals + $\forall\kappa[\kappa$ is measurable iff κ is $< \lambda(\kappa)$ strongly compact iff κ is $< \lambda(\kappa)$ supercompact$]$ + No cardinal κ is $\lambda(\kappa)$ strongly compact".*

Theorem 3. *Suppose $V \models$ "ZFC + GCH + κ is the least supercompact limit of supercompact cardinals + $\lambda > \kappa^+$ is a regular cardinal which is either inaccessible or is the successor of a cardinal of cofinality $> \kappa$ and $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$". There is then a generic extension $V[G] \models$ "ZFC + For every cardinal $\delta < \kappa$ which is an inaccessible limit of supercompact cardinals and every cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + For every cardinal $\gamma \in [\kappa, \lambda)$, $2^\kappa = \lambda$ + κ is $< \lambda$ supercompact + $\forall\delta < \kappa[\delta$ is strongly compact iff δ is supercompact$]$ + κ is the least measurable limit of strongly compact or supercompact cardinals".*

Let us take this opportunity to make several remarks concerning Theorems 1, 2, and 3. Note that we use a weaker supercompactness hypothesis in the proof of Theorem 1 than Woodin does in the proof of his Theorem. Also, since Woodin uses Radin forcing in the proof of his Theorem, cofinalities are not preserved in his generic extension (cardinals may or may not be preserved in Woodin's Theorem, depending upon the proof used), although they are in our Theorem 1 when the appropriate forcing conditions are used. Further, in Theorem 2, the model constructed will be so that on the proper class composed of all cardinals possessing some nontrivial degree of strong compactness or supercompactness, κ is γ strongly compact iff κ is γ supercompact, although there won't be any (fully) strongly compact or (fully) supercompact cardinals in this model. (This is the generalized version of the theorem that inspired the work of this paper and of [AS]. The original theorem was initially proven using an iteration of Woodin's version of Radin forcing used to prove his above mentioned Theorem.) Finally, Theorem 3 illustrates the flexibility of our forcing as compared to Radin forcing. Since iterating a Radin, Prikry, or

Magidor [Ma1] forcing (for changing the cofinality of κ to some uncountable $\delta < \kappa$) above a strongly compact or supercompact cardinal κ destroys the strong compactness or supercompactness of κ, it is impossible to use any of these forcings in the proof of Theorem 3. Our forcing for Theorem 3, however, has been designed so that, if κ is a supercompact cardinal which is Laver [L] indestructible, then we can force above κ, destroy measurability, yet preserve the supercompactness of κ.

The structure of this paper is as follows. Section 0 contains our introductory comments and preliminary material concerning notation, terminology, etc. Section 1 defines and discusses the basic properties of the forcing notion used in the iterations we employ to construct our models. Section 2 gives a proof of Theorem 1. Section 3 contains a proof of Theorems 2 and 3. Section 4 concludes the paper by giving an alternate forcing that can be used in the proofs of Theorems 1 and 2.

Before beginning the material of Section 1, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha, \beta), (\alpha, \beta]$, and (α, β) are as in standard interval notation. If f is the characteristic function of a set $x \subseteq \alpha$, then $x = \{\beta < \alpha : f(\beta) = 1\}$. If $\alpha < \alpha'$, f is a characteristic function having domain α, and f' is a characteristic function having domain α', we will when the context is clear abuse notation somewhat and write $f \subseteq f'$, $f = f'$, and $f \neq f'$ when we actually mean that the sets defined by these functions satisfy these properties.

When forcing, $q \geq p$ will mean that q is stronger than p. For P a partial ordering, φ a formula in the forcing language with respect to P, and $p \in P$, $p \| \varphi$ will mean p decides φ. For G V-generic over P, we will use both $V[G]$ and V^P to indicate the universe obtained by forcing with P. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$, especially when x is some variant of the generic set G, or x is in the ground model V.

If κ is a cardinal and P is a partial ordering, P is κ-closed if given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of elements of P so that $\beta < \gamma < \kappa$ implies $p_\beta \leq p_\gamma$ (an increasing chain of length κ), then there is some $p \in P$ (an upper bound to this chain) so that $p_\alpha \leq p$ for all $\alpha < \kappa$. P is $< \kappa$-closed if P is δ-closed for all cardinals $\delta < \kappa$. P is (κ, ∞)-distributive if for any sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of P, $D = \bigcap_{\alpha < \kappa} D_\alpha$ is a dense open subset of P. P is $(< \kappa, \infty)$-distributive if P is (δ, ∞)-distributive for all cardinals $\delta < \kappa$. P is κ-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of P (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for every two distinct elements $p_\rho, p_\nu \in \langle p_\alpha : \alpha < \delta \rangle$, p_ρ and p_ν have a common upper bound) there is an upper bound $p \in P$. P is κ-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. P is $< \kappa$-strategically closed if P is δ-strategically closed for all cardinals $\delta < \kappa$. P is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if P is $< \kappa$-closed, then P is $< \kappa$-strategically closed and $\prec \kappa$-strategically closed. The converse of both of these facts is false.

For $\kappa \le \lambda$ regular cardinals, two partial orderings to which we will refer quite a bit are the standard partial orderings $\mathcal{C}(\lambda)$ for adding a Cohen subset to λ using conditions having support $< \lambda$ and $\mathcal{C}(\kappa, \lambda)$ for adding λ many Cohen subsets to κ using conditions having support $< \kappa$. The basic properties and explicit definitions of these partial orderings may be found in [J].

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [SRK], [Ka], or [KaM] for further details. We note only that all elementary embeddings witnessing the λ supercompactness of κ are presumed to come from some fine, κ-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$, and all elementary embeddings witnessing the $< \lambda$ supercompactness of κ for λ a limit cardinal are presumed to be generated by the appropriate system $\langle \mathcal{U}_\delta : \delta < \lambda \rangle$ of ultrafilters over $P_\kappa(\delta)$ for $\delta \in [\kappa, \lambda)$ a cardinal. Also, where appropriate, all ultrapowers will be confused with their transitive isomorphs.

Finally, we remark that a good deal of the notions and techniques used in this paper are quite similar to those used in [AS]. Since we desire this paper to be as comprehensible as possible, regardless if readers have read [AS], many of the arguments of [AS] will be repeated here in the appropriately modified form.

1. The forcing conditions

In this section, we describe and prove the basic properties of the forcing conditions we shall use in our later iteration. Let $\delta < \lambda$, $\lambda > \delta^+$ be regular cardinals in our ground model V, with δ inaccessible and λ either inaccessible or the successor of a cardinal of cofinality $> \delta$. We assume throughout this section that GCH holds for all cardinals $\kappa \ge \delta$ in V, and we define three notions of forcing. Our first notion of forcing $P^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals of cofinality δ to λ. Specifically, $P^0_{\delta,\lambda} = \{p :$ For some $\alpha < \lambda$, $p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum and not having any initial segment which is stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \delta$ and $\mathrm{cof}(\beta) = \delta\}$, ordered by $q \ge p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$; i.e., S_q is an end extension of S_p. It is well-known that for G V-generic over $P^0_{\delta,\lambda}$ (see [Bu] or [KiM]), in $V[G]$, since GCH holds in V for all cardinals $\kappa \ge \delta$, a non-reflecting stationary set $S = S[G] = \bigcup\{S_p : p \in G\} \subseteq \lambda$ of ordinals of cofinality δ has been introduced, the bounded subsets of λ are the same as those in V, and cardinals, cofinalities, and GCH at cardinals $\kappa \ge \delta$ have been preserved. It is also virtually immediate that $P^0_{\delta,\lambda}$ is δ-directed closed.

Work now in $V_1 = V^{P^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote the above set S. $P^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C which is disjoint to S (and therefore makes S non-stationary). Specifically, $P^2_{\delta,\lambda}[S] = \{p :$ For some successor ordinal $\alpha < \lambda$, $p : \alpha \to \{0, 1\}$ is a characteristic function of C_p, a club subset of α, so that $C_p \cap S = \emptyset\}$, ordered by $q \ge p$ iff C_q is an end extension of C_p. It is again well-known (see [MS]) that for H V_1-generic over $P^2_{\delta,\lambda}[S]$, a club set $C = C[H] = \bigcup\{C_p : p \in H\} \subseteq \lambda$ which is disjoint to S has been introduced, the bounded subsets of λ are the same as those in V_1, and cardinals, cofinalities, and GCH for cardinals $\kappa \ge \delta$ have been preserved.

More will be said about $P^0_{\delta,\lambda}$ and $P^2_{\delta,\lambda}[S]$ in Lemmas 4, 6, and 7. In the meantime, before defining in V_1 the partial ordering $P^1_{\delta,\lambda}[S]$ which will be used to destroy measurability, we first prove two preliminary lemmas.

Lemma 1. $\Vdash_{P^0_{\delta,\lambda}}$ "$\clubsuit(\dot{S})$"*; i.e.,* $V_1 \models$ "*There is a sequence* $\langle x_\alpha : \alpha \in S\rangle$ *so that for each* $\alpha \in S$, $x_\alpha \subseteq \alpha$ *is cofinal in* α, *and for any* $A \in [\lambda]^\lambda$, $\{\alpha \in S : x_\alpha \subseteq A\}$ *is stationary".*

Proof of Lemma 1. Since GCH holds in V for cardinals $\kappa \geq \delta$ and V and V_1 contain the same bounded subsets of λ, we can let $\langle y_\alpha : \alpha < \lambda\rangle \in V$ be a listing of all elements $x \in ([\lambda]^\delta)^V = ([\lambda]^\delta)^{V_1}$ so that each $x \in [\lambda]^\delta$ appears on this list λ times at ordinals of cofinality δ; i.e., for any $x \in [\lambda]^\delta$, $\lambda = \sup\{\alpha < \lambda : \mathrm{cof}(\alpha) = \delta$ and $y_\alpha = x\}$. This then allows us to define $\langle x_\alpha : \alpha \in S\rangle$ by letting x_α be y_β for the least $\beta \in S - (\alpha + 1)$ so that $y_\beta \subseteq \alpha$ and y_β is unbounded in α. By genericity, each x_α is well-defined.

Now let $p \in P^0_{\delta,\lambda}$ be so that $p \Vdash$ "$\dot{A} \in [\lambda]^\lambda$ and $\dot{K} \subseteq \lambda$ is club". We show that for some $r \geq p$ and some $\zeta < \lambda$, $r \Vdash$ "$\zeta \in \dot{K} \cap \dot{S}$ and $\dot{x}_\zeta \subseteq \dot{A}$". To do this, we inductively define an increasing sequence $\langle p_\alpha : \alpha < \delta\rangle$ of elements of $P^0_{\delta,\lambda}$ and increasing sequences $\langle \beta_\alpha : \alpha < \delta\rangle$ and $\langle \gamma_\alpha : \alpha < \delta\rangle$ of ordinals $< \lambda$ so that $\beta_0 \leq \gamma_0 \leq \beta_1 \leq \gamma_1 \leq \cdots \leq \beta_\alpha \leq \gamma_\alpha \leq \cdots$ $(\alpha < \delta)$. We begin by letting $p_0 = p$ and $\beta_0 = \gamma_0 = 0$. For $\eta = \alpha + 1 < \delta$ a successor, let $p_\eta \geq p_\alpha$ and $\beta_\eta \leq \gamma_\eta$, $\beta_\eta \geq \max(\beta_\alpha, \gamma_\alpha, \sup(\mathrm{dom}(p_\alpha))) + 1$ be so that $p_\eta \Vdash$ "$\beta_\eta \in \dot{A}$ and $\gamma_\eta \in \dot{K}$". For $\rho < \delta$ a limit, let $p_\rho = \bigcup_{\alpha<\rho} p_\alpha$, $\beta_\rho = \bigcup_{\alpha<\rho} \beta_\alpha$, and $\gamma_\rho = \bigcup_{\alpha<\rho} \gamma_\alpha$. Note that since $\rho < \delta$, p_ρ is well-defined, and since $\delta < \lambda$, $\beta_\rho, \gamma_\rho < \lambda$. Also, by construction, $\bigcup_{\alpha<\delta} \beta_\alpha = \bigcup_{\alpha<\delta} \gamma_\alpha = \bigcup_{\alpha<\delta} \sup(\mathrm{dom}(p_\alpha)) < \lambda$. Call ζ this common sup. We thus have that $q = \bigcup_{\alpha<\delta} p_\alpha \cup \{\zeta\}$ is a well-defined condition so that $q \Vdash$ "$\{\beta_\alpha : \alpha \in \delta - \{0\}\} \subseteq \dot{A}$ and $\zeta \in \dot{K} \cap \dot{S}$".

To complete the proof of Lemma 1, we know that as $\langle \beta_\alpha : \alpha \in \delta - \{0\}\rangle \in V$ and as each $y \in \langle y_\alpha : \alpha < \lambda\rangle$ must appear λ times at ordinals of cofinality δ, we can find some $\eta \in (\zeta, \lambda)$ so that $\mathrm{cof}(\eta) = \delta$ and $\langle \beta_\alpha : \alpha \in \delta - \{0\}\rangle = y_\eta$. If we let $r \geq q$ be so that $r \Vdash$ "$\dot{S} \cap [\zeta, \eta] = \{\zeta, \eta\}$", then $r \Vdash$ "$\dot{x}_\zeta = y_\eta = \langle \beta_\alpha : \alpha \in \delta - \{0\}\rangle$". This proves Lemma 1. $\square$

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha : \alpha \in S\rangle$.

Lemma 2. *Let* S' *be an initial segment of* S *so that* S' *is not stationary at its supremum nor has any initial segment which is stationary at its supremum. There is then a sequence* $\langle y_\alpha : \alpha \in S'\rangle$ *so that for every* $\alpha \in S'$, $y_\alpha \subseteq x_\alpha$, $x_\alpha - y_\alpha$ *is bounded in* α, *and if* $\alpha_1 \neq \alpha_2 \in S'$, *then* $y_{\alpha_1} \cap y_{\alpha_2} = \emptyset$.

Proof of Lemma 2. We define by induction on $\alpha \leq \alpha_0 = \sup S' + 1$ a function h_α so that $\mathrm{dom}(h_\alpha) = S' \cap \alpha$, $h_\alpha(\beta) < \beta$, and $\langle x_\beta - h_\alpha(\beta) : \beta \in S' \cap \alpha\rangle$ is pairwise disjoint. The sequence $\langle x_\beta - h_{\alpha_0}(\beta) : \beta \in S'\rangle$ will be our desired sequence.

If $\alpha = 0$, then we take h_α to be the empty function. If $\alpha = \beta + 1$ and $\beta \notin S'$, then we take $h_\alpha = h_\beta$. If $\alpha = \beta + 1$ and $\beta \in S'$, then we notice that since each $x_\gamma \in X$ has order type δ and is cofinal in γ, for all $\gamma \in S' \cap \beta$, $x_\beta \cap \gamma$ is bounded in γ. This allows us to define a function h_α having domain $S' \cap \alpha$ by $h_\alpha(\beta) = 0$, and for $\gamma \in S' \cap \beta$, $h_\alpha(\gamma) = \min(\{\rho : \rho < \gamma,\ \rho \geq h_\beta(\gamma)$, and $x_\beta \cap \gamma \subseteq \rho\})$. By the next to last sentence and the induction hypothesis on h_β, $h_\alpha(\gamma) < \gamma$. And, if $\gamma_1 < \gamma_2 \in S' \cap \alpha$, then if $\gamma_2 < \beta$, $(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) \subseteq (x_{\gamma_1} - h_\beta(\gamma_1)) \cap (x_{\gamma_2} - h_\beta(\gamma_2)) = \emptyset$ by the induction hypothesis on h_β. If $\gamma_2 = \beta$, then $(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) = (x_{\gamma_1} - h_\alpha(\gamma_1)) \cap x_{\gamma_2} = \emptyset$ by the definition of $h_\alpha(\gamma_1)$. The sequence $\langle x_\gamma - h_\alpha(\gamma) : \gamma \in S' \cap \alpha\rangle$ is thus as desired.

If α is a limit ordinal, then as S' is non-stationary at its supremum nor has any initial segment which is stationary at its supremum, we can let $\langle \beta_\gamma : \gamma < \text{cof}(\alpha)\rangle$ be a strictly increasing, continuous sequence having sup α so that for all $\gamma < \text{cof}(\alpha)$, $\beta_\gamma \notin S'$. Thus, if $\rho \in S' \cap \alpha$, then $\{\beta_\gamma : \beta_\gamma < \rho\}$ is bounded in ρ, meaning we can find some largest γ so that $\beta_\gamma < \rho$. It is also the case that $\rho < \beta_{\gamma+1}$. This allows us to define $h_\alpha(\rho) = \max(\{h_{\beta_{\gamma+1}}(\rho), \beta_\gamma\})$ for the γ just described. It is still the case that $h_\alpha(\rho) < \rho$. And, if $\rho_1, \rho_2 \in (\beta_\gamma, \beta_{\gamma+1})$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq (x_{\rho_1} - h_{\beta_{\gamma+1}}(\rho_1)) \cap (x_{\rho_2} - h_{\beta_{\gamma+1}}(\rho_2)) = \emptyset$ by the definition of $h_{\beta_{\gamma+1}}$. If $\rho_1 \in (\beta_\gamma, \beta_{\gamma+1})$, $\rho_2 \in (\beta_\sigma, \beta_{\sigma+1})$ with $\gamma < \sigma$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq x_{\rho_1} \cap (x_{\rho_2} - \beta_\sigma) \subseteq \rho_1 - \beta_\sigma \subseteq \rho_1 - \beta_{\gamma+1} = \emptyset$. Thus, the sequence $\langle x_\rho - h_\alpha(\rho) : \rho \in S' \cap \alpha\rangle$ is again as desired. This proves Lemma 2. □

At this point, we are in a position to define in V_1 the partial ordering $P^1_{\delta,\lambda}[S]$ which will be used to destroy measurability. $P^1_{\delta,\lambda}[S]$ is the set of all 5-tuples $\langle w, \alpha, \bar{r}, Z, \Gamma\rangle$ satisfying the following properties.

1. $w \subseteq \lambda$ is so that $|w| = \delta$.
2. $\alpha < \delta$.
3. $\bar{r} = \langle r_i : i \in w\rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. Z is a function so that:
a) $\text{dom}(Z) \subseteq \{x_\beta : \beta \in S\}$ and $\text{range}(Z) \subseteq \{0, 1\}$.
b) If $z \in \text{dom}(Z)$, then for some $y \in [w]^\delta$, $y \subseteq z$ and $z - y$ is bounded in the β so that $z = x_\beta$.
5. Γ is a function so that:
a) $\text{dom}(\Gamma) = \text{dom}(Z)$.
b) If $z \in \text{dom}(\Gamma)$, then $\Gamma(z)$ is a closed, bounded subset of α such that if γ is inaccessible, $\gamma \in \Gamma(z)$, and β is the γth element of z, then $\beta \in w$, and for some $\beta' \in \beta \cap w \cap z$, $r_{\beta'}(\gamma) = Z(z)$.

Note that the definitions of Z and Γ imply $|\text{dom}(Z)| = |\text{dom}(\Gamma)| \leq \delta$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1, \Gamma^1\rangle \leq \langle w^2, \alpha^2, \bar{r}^2, Z^2, \Gamma^2\rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \leq \alpha^2$.
3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$ and $|\{i \in w^1 : r^2_i | (\alpha_2 - \alpha_1) \text{ is not constantly } 0\}| < \delta$.
4. $Z^1 \subseteq Z^2$.
5. $\text{dom}(\Gamma^1) \subseteq \text{dom}(\Gamma^2)$.
6. If $z \in \text{dom}(\Gamma^1)$, then $\Gamma^1(z)$ is an initial segment of $\Gamma^2(z)$ and $|\{z \in \text{dom}(\Gamma^1) : \Gamma^1(z) \neq \Gamma^2(z)\}| < \delta$.

At this point, a few intuitive remarks are in order. If δ is measurable, then δ must carry a normal measure. The forcing $P^1_{\delta,\lambda}[S]$ has specifically been designed to destroy this fact. It has been designed, however, to destroy the measurability of δ "as lightly as possible", making little damage, assuming δ is $< \lambda$ supercompact. Specifically, if δ is $< \lambda$ supercompact, then the non-reflecting stationary set S, having been added to λ, does not kill the $< \lambda$ supercompactness of δ by itself. The additional forcing $P^1_{\delta,\lambda}[S]$ is necessary to do the job and has been designed so as not only to destroy the $<\lambda$ supercompactness of δ but to destroy the measurability of δ as well. The forcing $P^1_{\delta,\lambda}[S]$, however, has been designed so that, if necessary, we can resurrect the $<\lambda$ supercompactness of δ by forcing further with $P^2_{\delta,\lambda}[S]$.

Lemma 3. $V_1^{P^1_{\delta,\lambda}[S]} \models$ "*δ is not measurable*".

Proof of Lemma 3. Assume to the contrary that $V_1^{P^1_{\delta,\lambda}[S]} \models$ "δ is measurable". Let $p \Vdash$ "$\dot{\mathcal{D}}$ is a normal measure over δ". We show that p can be extended to a condition q so that $q \Vdash$ "$\dot{\mathcal{D}}$ is non-normal", an immediate contradiction.

We use a Δ-system argument to establish this. First, for G_1 V_1-generic over $P^1_{\delta,\lambda}[S]$ and $i < \lambda$, let $r'_i = \cup\{r^p_i : \exists p = \langle w^p, \alpha^p, \bar{r}^p, Z^p, \Gamma^p\rangle \in G_1[r^p_i \in \bar{r}^p]\}$. An easy density argument shows $\Vdash_{P^1_{\delta,\lambda}[S]}$ "$\dot{r}'_i : \delta \to \{0,1\}$ is a function whose domain is all of δ". Thus, we can let $r^\ell_i = \{\alpha < \delta : r'_i(\alpha) = \ell\}$ for $\ell \in \{0,1\}$.

For each $i < \lambda$, pick $p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, Z^{p_i}, \Gamma^{p_i}\rangle \geq p$ so that $p_i \Vdash$ "$\dot{r}^{\ell(i)}_i \notin \dot{\mathcal{D}}$" for some $\ell(i) \in \{0,1\}$. This is possible since $\Vdash_{P^1_{\delta,\lambda}[S]}$ "For each $i < \lambda$, $\dot{r}^0_i \cap \dot{r}^1_i = \emptyset$ and $\dot{r}^0_i \cup \dot{r}^1_i = \delta$". Without loss of generality, by extending p_i if necessary, since clause 4b) of the definition of the forcing implies $|\operatorname{dom}(Z^{p_i})| \leq \delta$, we can assume that $i \in w^{p_i}$ and $z \subseteq w^{p_i}$ for every $z \in \operatorname{dom}(Z^{p_i})$. Thus, since each $w^{p_i} \in [\lambda]^{<\delta^+}$, $\lambda > \delta^+$, λ is either inaccessible or is the successor of a cardinal of cofinality $> \delta$, and GCH holds in V_1 for cardinals $\kappa \geq \delta$, we can find some $A \in [\lambda]^\lambda$ so that $\{w^{p_i} : i \in A\}$ forms a Δ-system, i.e., so that for $i \neq j \in A$, $w^{p_i} \cap w^{p_j}$ is some constant value w which is an initial segment of both. (Note we can assume that for $i \in A$, $w_i \cap i = w$, and for some fixed $\ell(*) \in \{0,1\}$, for every $i \in A$, $p_i \Vdash$ "$\dot{r}^{\ell(*)}_i \notin \dot{\mathcal{D}}$".) Also, by GCH in V_1 for cardinals $\kappa \geq \delta$, $|[\mathcal{P}(w)]^\delta| = |[\delta^+]^\delta| = \delta^+$. Therefore, since $|\operatorname{dom}(Z^{p_i})| \leq \delta$ for each $i < \lambda$ and $\lambda > \delta^+$, we can assume in addition that for all $i \in A$, $\operatorname{dom}(Z^{p_i}) \cap \mathcal{P}(w) = \operatorname{dom}(\Gamma^{p_i}) \cap \mathcal{P}(w)$ is some constant value Z. Hence, since each Z^{p_i} is a function from a set of cardinality δ into $\{0,1\}$, each Γ^{p_i} is a function from a set of cardinality δ into $[\delta]^{<\delta}$ which has cardinality δ, and $\lambda > \delta^+$, GCH in V_1 for cardinals $\kappa \geq \delta$ allows us to assume that for $i \neq j \in A$, $Z^{p_i}|Z = Z^{p_j}|Z$ and $\Gamma^{p_i}|Z = \Gamma^{p_j}|Z$. Further, since each $\alpha^{p_i} < \delta$, we can assume that α^{p_i} is some constant α^0 for $i \in A$. Then, since any $\bar{r}^{p_i} = \langle r^{p_i}_j : j \in w^{p_i}\rangle$ for $i \in A$ is composed of a sequence of functions from α_0 to 2, $\alpha_0 < \delta$, and $|w| \leq \delta$, GCH in V_1 for cardinals $\kappa \geq \delta$ again allows us to assume that for $i \neq j \in A$, $\bar{r}^{p_i}|w = \bar{r}^{p_j}|w$. And, since $i \in w^{p_i}$, we know that we can also assume (by thinning A if necessary) that $B = \{\sup(w^{p_i}) : i \in A\}$ is so that $i < j \in A$ implies $i \leq \sup(w^{p_i}) < \min(w^{p_j} - w) \leq \sup(w^{p_j})$. We know in addition by the choice of $X = \langle x_\beta : \beta \in S\rangle$ that for some $\gamma \in S$, $x_\gamma \subseteq A$. Let $x_\gamma = \{i_\beta : \beta < \delta\}$.

We are now in a position to define the condition q referred to earlier. We proceed by defining each of the five coordinates of q. First, let $w^q = \bigcup_{\beta<\delta} w^{p_{i_\beta}}$. As δ is regular, $\delta < \lambda$, and each $w^{p_{i_\beta}} \in [\lambda]^{<\delta^+}$, w^q is well-defined and in $[\lambda]^{<\delta^+}$. Second, let $\alpha^q = \alpha^0$. Third, let $\bar{r}^q = \langle r^q_i : i \in w^q\rangle$ be defined by $r^q_i = r^{p_{i_\beta}}_i$ if $i \in w^{p_{i_\beta}}$. The property of the Δ-system that $i \neq j \in A$ implies $\bar{r}^{p_i}|w = \bar{r}^{p_j}|w$ tells us $\bar{r}^q$ is well-defined. Finally, to define Z^q and Γ^q, let $Z^q = \bigcup_{\beta<\delta} Z^{i_\beta} \cup \{\langle\{i_\beta : \beta < \delta\}, \ell(*)\rangle\}$ and $\Gamma^q = \bigcup_{\beta<\delta} \Gamma^{i_\beta} \cup \{\langle\{i_\beta : \beta < \delta\}, \emptyset\rangle\}$. By the preceding paragraph and our construction, $\{i_\beta : \beta < \delta\}$ generates a new set which can be included in $\operatorname{dom}(Z^q)$

and $\mathrm{dom}(\Gamma^q)$. Therefore, since $Z^{p_i}|Z = Z^{p_j}|Z$ and $\Gamma^{p_i}|Z = \Gamma^{p_j}|Z$ for $i \neq j \in A$, Z^q and Γ^q are well-defined.

We claim now that $q \geq p$ is so that $q \Vdash$ "$\dot{\mathcal{D}}$ is non-normal". To see this, assume the claim fails. Since $p \Vdash$ "$\dot{\mathcal{D}}$ is a normal ultrafilter over δ" and by construction $\forall \beta < \delta[q \geq p_{i_\beta} \geq p]$, $q \Vdash$ "$\dot{r}^{\ell(*)}_{i_\beta} \notin \dot{\mathcal{D}}$" for $\beta < \delta$. It must thus be the case that $q \Vdash$ "$\dot{F}_0 = \{\gamma < \delta : \gamma \in \underset{\beta<\gamma}{\cup} \dot{r}^{\ell(*)}_{i_\beta}\} \notin \dot{\mathcal{D}}$ and $\dot{F}_1 = \{\gamma < \delta : \gamma \text{ is not inaccessible}\} \notin \dot{\mathcal{D}}$". As $q \Vdash$ "$\dot{K}^q = \bigcup\{\Gamma^s(\{i_\beta : \beta < \delta\}) : \exists s = \langle w^s, \alpha^s, \bar{r}^s, Z^s, \Gamma^s\rangle \geq q[s \in \dot{G}_1]\}$ is club in δ", $q \Vdash$ "$\dot{F}_2 = \{\gamma < \delta : \gamma \notin \dot{K}^q\} \notin \dot{\mathcal{D}}$", so $q \Vdash$ "$\dot{F} = \dot{F}_0 \cup \dot{F}_1 \cup \dot{F}_2 \notin \dot{\mathcal{D}}$".

We show that $q \Vdash$ "$\dot{F} = \delta$". If $\gamma < \delta$ is an arbitrary inaccessible, then by the definition of $\dot{F}$, it suffices to show that for some $s \geq q$ so that $s \|$ "$\gamma \in \dot{F}$", $s \Vdash$ "$\gamma \in \dot{F}$". If $s \geq q$ is so that $s \Vdash$ "$\gamma \in \dot{F}_2$", then we're done, so assume $s \Vdash$ "$\gamma \notin \dot{F}_2$", i.e., $s \Vdash$ "$\gamma \in \dot{K}^q$". But then, by the definition of $\leq$ and clause 5b) of the definition of the forcing, $s \Vdash$ "$\gamma \in \dot{F}_0$". Thus, $q \Vdash$ "$\dot{F} = \delta$", i.e., $q \Vdash$ "$\dot{F} \in \dot{\mathcal{D}}$", meaning $q \geq p$ is so that $q \Vdash$ "$\dot{\mathcal{D}}$ is both a normal and non-normal ultrafilter over δ". This proves Lemma 3. $\square$

It is clear from the proof of Lemma 3 that since forcing with $P^1_{\delta,\lambda}[S]$ destroys the measurability of δ, $P^1_{\delta,\lambda}[S]$ can't be δ-directed closed. (Otherwise, since $P^0_{\delta,\lambda}$ is δ-directed closed, if δ were supercompact and Laver [L] indestructible and $P^1_{\delta,\lambda}[S]$ were δ-directed closed, then the forcing $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[S]$ would be δ-directed closed and hence would preserve the supercompactness of δ.) Note, however, that if $\gamma < \delta$ and $\langle p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, Z^{p_i}, \Gamma^{p_i}\rangle : i < \gamma\rangle$ is a directed sequence of conditions in $P^1_{\delta,\lambda}[S]$, it is possible to define $\Gamma^0 = \bigcup_{i<\gamma} \Gamma^{p_i}$, where $z \in \mathrm{dom}(\Gamma^0)$ if $z \in \mathrm{dom}(\Gamma^{p_i})$ for some $i < \gamma$ and $\Gamma^0(z)$ is the closure of $\bigcup_{i<\gamma} \Gamma^{p_i}(z)$. ($\Gamma^{p_i}(z) = \emptyset$ if $z \notin \mathrm{dom}(\Gamma^{p_i})$.) Then $p = \langle \bigcup_{i<\gamma} w^{p_i}, \bigcup_{i<\gamma} \alpha^{p_i}, \bigcup_{i<\gamma} \bar{r}^{p_i}, \bigcup_{i<\gamma} Z^{p_i}, \Gamma^0\rangle$, where if $\bar{r}^{p_i} = \langle r^{p_i}_j : j \in w^{p_i}\rangle$, then $r_j \in \bigcup_{i<\gamma} \bar{r}^{p_i}$ if $j \in \bigcup_{i<\gamma} w^{p_i}$ and $r_j = \bigcup_{i<\gamma} r^{p_i}_j$ ($r^{p_i}_j = \emptyset$ if $j \notin w^{p_i}$), is almost a condition. The trouble occurs when for $z \in \mathrm{dom}(\Gamma^0)$, $\Gamma^0(z)$ contains a new element which is inaccessible. If, however, we can guarantee that for any $z \in \mathrm{dom}(\Gamma^0)$, $\Gamma^0(z)$ contains no new element which is inaccessible, then p as just defined is a condition. Therefore, we define a new partial ordering $\leq^\gamma$ on $P^1_{\delta,\lambda}[S]$ by $p_1 = \langle w^{p_1}, \alpha^{p_1}, \bar{r}^{p_1}, Z^{p_1}, \Gamma^{p_1}\rangle \leq^\gamma p_2 = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, Z^{p_2}, \Gamma^{p_2}\rangle$ iff $p_1 = p_2$ or $p_1 < p_2$ and for $z \in \mathrm{dom}(\Gamma^{p_1})$, if $\Gamma^{p_1}(z) \neq \Gamma^{p_2}(z)$, then $\gamma < \max(\Gamma^{p_2}(z))$. If the sequence $\langle p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, Z^{p_i}, \Gamma^{p_i}\rangle : i < \gamma\rangle$ is a directed sequence of conditions in $P^1_{\delta,\lambda}[S]$ with respect to $\leq^\gamma$, then since $\sup(\bigcup_{i<\gamma} \Gamma^{p_i}(z))$ must be an ordinal $> \gamma$ of cofinality γ, the upper bound p as defined earlier exists. Further, if $p_1 = \langle w^{p_1}, \alpha^{p_1}, \bar{r}^{p_1}, Z^{p_1}, \Gamma^{p_1}\rangle \leq p_2 = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, Z^{p_2}, \Gamma^{p_2}\rangle$, for any $z \in \mathrm{dom}(\Gamma^{p_1})$ so that $\Gamma^{p_1}(z) \neq \Gamma^{p_2}(z)$, we can define a function Γ having domain Γ^{p_2} so that $\Gamma|(\mathrm{dom}(\Gamma^{p_2}) - \mathrm{dom}(\Gamma^{p_1})) = \Gamma^{p_2}|(\mathrm{dom}(\Gamma^{p_2}) - \mathrm{dom}(\Gamma^{p_1}))$ and such that for $z \in \mathrm{dom}(\Gamma^{p_1})$, $\Gamma(z) = \Gamma^{p_2}(z) \cup \{\eta_z\}$, where $\eta_z < \delta$ is the least cardinal above $\max(\max(\Gamma^{p_2}(z)), \gamma)$. If $q = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, Z^{p_2}, \Gamma\rangle$, then $q \in P^1_{\delta,\lambda}[S]$ is a valid condition so that $p_1 \leq^\gamma q$ and $p_2 \leq^\gamma q$. This easily implies that G is (appropriately) generic with respect to $\langle P^1_{\delta,\lambda}[S], \leq^\gamma\rangle$, an ordering that is γ^+-directed closed, iff G is (appropriately) generic with respect to $\langle P^1_{\delta,\lambda}[S], \leq\rangle$; i.e., forcing with $\langle P^1_{\delta,\lambda}[S], \leq^\gamma\rangle$

and $\langle P^1_{\delta,\lambda}[S], \leq\rangle$ are equivalent. This key observation will be critical in the proof of Theorem 3.

Recall we mentioned prior to the proof of Lemma 3 that $P^1_{\delta,\lambda}[S]$ is designed so that a further forcing with $P^2_{\delta,\lambda}[S]$ will resurrect the $< \lambda$ supercompactness of δ, assuming the correct iteration has been done. That this is so will be shown in the next section. In the meantime, we give an idea of why this will happen by showing that the forcing $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is rather nice. First, for $\kappa_0 \leq \kappa_1$ regular cardinals, let $\mathcal{K}(\kappa_0, \kappa_1) = \{\langle w, \alpha, \bar{r}\rangle : w \subseteq \kappa_1$ is so that $|w| = \kappa_0$, $\alpha < \kappa_0$, and $\bar{r} = \langle r_i : i \in w\rangle$ is a sequence of functions from α to $\{0, 1\}\}$, ordered by $\langle w^1, \alpha^1, \bar{r}^1\rangle \leq \langle w^2, \alpha^2, \bar{r}^2\rangle$ iff $w^1 \subseteq w^2$, $\alpha^1 \leq \alpha^2$, $r^1_i \subseteq r^2_i$ if $i \in w^1$, and $|\{i \in w^1 : r^2_i|(\alpha_2 - \alpha_1)$ is not constantly $0\}| < \kappa_0$. Given this definition, we now have the following lemma.

Lemma 4. *$P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is equivalent to $\mathcal{C}(\lambda) * \dot{\mathcal{C}}(\delta^+, \lambda) * \dot{\mathcal{K}}(\delta, \lambda)$.*

Proof of Lemma 4. Let G be V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, with $G^0_{\delta,\lambda}$, $G^1_{\delta,\lambda}$, and $G^2_{\delta,\lambda}$ the projections onto $P^0_{\delta,\lambda}$, $P^1_{\delta,\lambda}[S]$, and $P^2_{\delta,\lambda}[S]$ respectively. Each $G^i_{\delta,\lambda}$ is appropriately generic. So, since $P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]$ is a product in $V[G^0_{\delta,\lambda}]$, we can rewrite the forcing in $V[G^0_{\delta,\lambda}]$ as $P^2_{\delta,\lambda}[S] \times P^1_{\delta,\lambda}[S]$ and rewrite $V[G]$ as $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][G^1_{\delta,\lambda}]$.

It is well-known (see [MS]) that the forcing $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is equivalent to $\mathcal{C}(\lambda)$. That this is so can be seen from the fact that $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is non-trivial, has cardinality λ, and is such that $D = \{\langle p, q\rangle \in P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}] :$ For some α, $\mathrm{dom}(p) = \mathrm{dom}(q) = \alpha + 1$, $p \Vdash$ "$\alpha \notin \dot{S}$", and $q \Vdash$ "$\alpha \in \dot{C}$"$\}$ is dense in $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ and is $< \lambda$-closed. This easily implies the desired equivalence. Thus, V and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ have the same cardinals and cofinalities, and the proof of Lemma 4 will be complete once we show that in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, $P^1_{\delta,\lambda}[S]$ is equivalent to $\mathcal{C}(\delta^+, \lambda) * \dot{\mathcal{K}}(\delta, \lambda)$.

To this end, working in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, let $R = \{Z : Z$ is a function from $\{x_\beta : \beta \in S\}$ into $\{0, 1\}$ so that $|\mathrm{dom}(Z)| \leq \delta\}$, ordered by inclusion. Since $|\{x_\beta : \beta \in S\}| = \lambda$ and R is δ-closed, it is clear R is equivalent to $\mathcal{C}(\delta^+, \lambda)$. Further, the following facts are easy to see.

1. If $p = \langle w^p, \alpha^p, \bar{r}^p, Z^p, \Gamma^p\rangle \in P^1_{\delta,\lambda}[S]$, then $Z^p \in R$.
2. If $p_1 = \langle w^{p_1}, \alpha^{p_1}, \bar{r}^{p_1}, Z^{p_1}, \Gamma^{p_1}\rangle$, $p_2 = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, Z^{p_2}, \Gamma^{p_2}\rangle$ are so that $p_1, p_2 \in P^1_{\delta,\lambda}[S]$ and $p_1 \leq p_2$, then $Z^{p_1} \subseteq Z^{p_2}$.
3. If $p_1 = \langle w^{p_1}, \alpha^{p_1}, \bar{r}^{p_1}, Z^{p_1}, \Gamma^{p_1}\rangle \in P^1_{\delta,\lambda}[S]$ is so that $Z^{p_1} \subseteq Z^{p_2}$ for some $Z^{p_2} \in R$, then there exists $p_2 \in P^1_{\delta,\lambda}[S]$ with $p_1 \leq p_2$, $p_2 = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, Z^{p_2}, \Gamma^{p_2}\rangle$.

From these three facts, it then easily follows that $H = \{Z \in R : \exists p \in G^1_{\delta,\lambda}[Z = Z^p]\}$ is $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$-generic over R. This means we can rewrite $P^1_{\delta,\lambda}[S]$ in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ as $R * (\dot{P}^1_{\delta,\lambda}[S]/R)$, which is isomorphic to $\mathcal{C}(\delta^+, \lambda) * (\dot{P}^1_{\delta,\lambda}[S]/R)$. We will thus be done if we can show in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$ (which has the same cardinals and cofinalities as V and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$) that $P^1_{\delta,\lambda}[S]/R$ is equivalent to $\mathcal{K}(\delta, \lambda)$.

Working now in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$, we first note that as $S \subseteq \lambda$ is in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ a non-stationary set all of whose initial segments are non-stationary, by Lemma 2, for the sequence $\langle x_\beta : \beta \in S\rangle$, there must be a sequence $\langle y_\beta : \beta \in S\rangle \in V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}] \subseteq V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$ so that for every $\beta \in S$, $y_\beta \subseteq x_\beta$, $x_\beta - y_\beta$ is bounded in β, and if $\beta_1 \neq \beta_2 \in S$, then $y_{\beta_1} \cap y_{\beta_2} = \emptyset$. Given this fact, it is easy to observe that

$P^1 = \{\langle w, \alpha, \bar{r}, \Gamma\rangle \in P^1_{\delta,\lambda}[S]/R :$ For every $\beta \in S$, either $y_\beta \subseteq w$ or $y_\beta \cap w = \emptyset\}$ is dense in $P^1_{\delta,\lambda}[S]/R$. To show this, given $\langle w, \alpha, \bar{r}, \Gamma\rangle \in P^1_{\delta,\lambda}[S]/R$, $\bar{r} = \langle r_i : i \in w\rangle$, let $Y_w = \{y \in \langle y_\beta : \beta \in S\rangle : y \cap w \neq \emptyset\}$. As $|w| \leq \delta$ and $y_{\beta_1} \cap y_{\beta_2} = \emptyset$ for $\beta_1 \neq \beta_2 \in S$, $|Y_w| \leq \delta$. Hence, as $|y| = \delta < \lambda$ for $y \in Y_w$, $|w'| \leq \delta$ for $w' = w \cup (\cup Y_w)$. This means $\langle w', \alpha, \bar{r}', \Gamma\rangle$ for $\bar{r}' = \langle r'_i : i \in w'\rangle$ defined by $r'_i = r_i$ if $i \in w$ and r'_i is the empty function if $i \in w' - w$ is a well-defined condition extending $\langle w, \alpha, \bar{r}, \Gamma\rangle$. Thus, P^1 is dense in $P^1_{\delta,\lambda}[S]/R$, so to analyze the forcing properties of $P^1_{\delta,\lambda}[S]/R$, it suffices to analyze the forcing properties of P^1.

For $\beta \in S$, let $Q_\beta = \{\langle w, \alpha, \bar{r}, \Gamma\rangle \in P^1 : w = y_\beta\}$, and let $Q^* = \{\langle w, \alpha, \bar{r}, \Gamma\rangle \in P^1 : w \subseteq \lambda - \bigcup_{\beta \in S} y_\beta\}$. Let Q be those elements of $\prod_{\beta \in S} Q_\beta \times Q^*$ of support δ so that for $p = \langle\langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, \Gamma^{p_i}\rangle_{i<\delta}, \langle w^p, \alpha^p, \bar{r}^p, \Gamma^p\rangle\rangle \in Q$, $\alpha^{p_i} = \alpha^{p_j} = \alpha^p$ for $i < j < \delta$. Let $\leq_Q$ on Q be defined by $p = \langle\langle w^{p_i}, \alpha, \bar{r}^{p_i}, \Gamma^{p_i}\rangle_{i<\delta}, \langle w^p, \alpha, \bar{r}^p, \Gamma^p\rangle\rangle \leq_Q q = \langle\langle w^{q_i}, \beta, \bar{r}^{q_i}, \Gamma^{q_i}\rangle_{i<\delta}, \langle w^q, \beta, \bar{r}^q, \Gamma^q\rangle\rangle$ iff the following hold.

1. $\langle w^p, \alpha, \bar{r}^p, \Gamma^p\rangle \leq \langle w^q, \beta, \bar{r}^q, \Gamma^q\rangle$.
2. q can be written in the form
$$\langle\langle w^{q_i}, \beta, \bar{r}^{q_i}, \Gamma^{q_i}\rangle_{i<\delta}, \langle u^{q_i}, \beta, \bar{s}^{q_i}, \Delta^{q_i}\rangle_{i<i_0\leq\delta}, \langle w^q, \beta, \bar{r}^q, \Gamma^q\rangle\rangle$$
so that $\forall i < \delta[w^{p_i} = w^{q_i}$ and $\langle w^{p_i}, \alpha, \bar{r}^{p_i}, \Gamma^{p_i}\rangle \leq \langle w^{q_i}, \beta, \bar{r}^{q_i}, \Gamma^{q_i}\rangle]$.
3. $|\{j \in \bigcup_{i<\delta} w^{p_i} :$ For the unique i so that $j \in w^{p_i} = w^{q_i}$, $r^{p_i}_j \neq r^{q_i}_j\}| < \delta$, where $\bar{r}^{p_i} = \langle r^{p_i}_j : j \in w^{p_i}\rangle$ and $\bar{r}^{q_i} = \langle r^{q_i}_j : j \in w^{q_i}\rangle$.
4. $|\{z \in \bigcup_{i<\delta} \mathrm{dom}(\Gamma^{p_i}) : \Gamma^{p_i}(z) \neq \Gamma^{q_i}(z)\}| < \delta$.

Then, for $p = \langle\langle w^{p_i}, \alpha, \bar{r}^{p_i}, \Gamma^{p_i}\rangle_{i<\delta}, \langle w^p, \alpha, \bar{r}^p, \Gamma^p\rangle\rangle \in Q$, as $w^{p_i} \cap w^{p_j} = \emptyset$ for $i < j < \delta$ (each $w^{p_i} = y_{\beta_i}$ for some unique $\beta_i \in S$ and $y_{\beta_i} \cap y_{\beta_j} = \emptyset$ for $\beta_i \neq \beta_j$), $w^{p_i} \cap w^p = \emptyset$ for $i < \delta$, $\mathrm{dom}(\bar{r}^{p_i}) \cap \mathrm{dom}(\bar{r}^{p_j}) = \emptyset$ for $i < j < \delta$, $\mathrm{dom}(\bar{r}^{p_i}) \cap \mathrm{dom}(\bar{r}^p) = \emptyset$ for $i < \delta$, $\mathrm{dom}(\Gamma^{p_i}) \cap \mathrm{dom}(\Gamma^{p_j}) = \emptyset$ for $i < j < \delta$ (since if $z \in \mathrm{dom}(\Gamma^{p_i})$, $z = x_\beta$ for some $\beta \in S$, meaning $w^{p_i} = y_\beta$ by the definitions of $P^1_{\delta,\lambda}[S]$, $P^1_{\delta,\lambda}[S]/R$, and Q), and $\mathrm{dom}(\Gamma^p) = \emptyset$ (since for every $\beta \in S$, $w^p \cap y_\beta = \emptyset$, $y_\beta \subseteq x_\beta$, and $x_\beta - y_\beta$ is bounded in β), conditions 3) and 4) above on the definition of $\leq_Q$ show the function $F(p) = \langle \bigcup_{i<\delta} w^{p_i} \cup w^p, \alpha, \bigcup_{i<\delta} \bar{r}^{p_i} \cup \bar{r}^p, \bigcup_{i<\delta} \Gamma^{p_i}\rangle$ yields an isomorphism between Q and P^1. Thus, over $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$, forcing with P^1, $P^1_{\delta,\lambda}[S]/R$, and Q are all equivalent.

We examine next in more detail the exact nature of $\langle Q, \leq_Q\rangle$. For $\beta \in S$, note that if $p = \langle w^p, \alpha^p, \bar{r}^p, \Gamma^p\rangle \in Q_\beta$ and $\mathrm{dom}(\Gamma^p) \neq \emptyset$, then $\mathrm{dom}(\Gamma^p) = \{x_\beta\}$. We can therefore define an ordering $\leq_\beta$ on Q_β by $p_1 = \langle w^{p_1}, \alpha^{p_1}, \bar{r}^{p_1}, \Gamma^{p_1}\rangle \leq_\beta p_2 = \langle w^{p_2}, \alpha^{p_2}, \bar{r}^{p_2}, \Gamma^{p_2}\rangle$ iff $p_1 = p_2$ or $p_1 < p_2$ and $\alpha^{p_1} \leq \max(\Gamma^{p_2}(x_\beta))$, and we can reorder Q by replacing each occurrence of $\leq$ on Q_β by $\leq_\beta$. If we call the new ordering on Q thus obtained $\leq'_Q$, then by an argument virtually identical to the one given in the remark following the proof of Lemma 3, if $p_1 \leq_Q p_2$, $p_1, p_2 \in Q$, there is some condition $q \in Q$ so that $p_1 \leq'_Q q$ and $p_2 \leq'_Q q$. It is hence once more the case that $\langle Q, \leq_Q\rangle$ and $\langle Q, \leq'_Q\rangle$ are forcing equivalent, i.e., I is (appropriately) generic with respect to $\langle Q, \leq_Q\rangle$ iff I is (appropriately) generic with respect to $\langle Q, \leq'_Q\rangle$, so without loss of generality, we analyze the forcing properties of $\langle Q, \leq'_Q\rangle$.

We examine now $\langle Q_\beta, \leq_\beta\rangle$ for $\beta \in S$. We first note that by the definition of $\leq_\beta$, for $\gamma < \delta$ any fixed but arbitrary cardinal, if $\langle p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, \Gamma^{p_i}\rangle : i < \gamma\rangle$ is a directed sequence of conditions with respect to $\leq_\beta$, then (using the notation in the remark immediately following the proof of Lemma 3) the 4-tuple $p' = \langle \bigcup_{i<\gamma} w^{p_i}, \bigcup_{i<\gamma} \alpha^{p_i}, \bigcup_{i<\gamma} \bar{r}^{p_i}, \bigcup_{i<\gamma} \Gamma^{p_i} \rangle$ can be extended to a condition $p \in Q_\beta$. This is since the definition of $\leq_\beta$ ensures $\cup_{i<\gamma} \alpha^{p_i} = \alpha' \leq \eta$ for $\eta = \max(\bigcup_{i<\gamma} \Gamma^{p_i})$. Thus, if we let $\alpha'' = \max(\alpha', \eta) + 1$, we can extend each $r \in \bigcup_{i<\gamma} \bar{r}^{p_i}$ to a function s having domain α'' by letting $s|\alpha' = r|\alpha'$, and for $\alpha \in [\alpha', \alpha'')$, $s(\alpha) = (\bigcup H)(x_\beta)$. If we call the sequence of all such extensions $\bar{r}'$, $p = \langle \bigcup_{i<\gamma} w^{p_i}, \alpha'', \bar{r}', \bigcup_{i<\gamma} \Gamma^{p_i} \rangle$ is a well-defined element of Q_β so that $p_i \leq_\beta p$ for all $i < \gamma$. This just means $\langle Q_\beta, \leq_\beta \rangle$ is δ-directed closed.

Now, let $R_\beta = \{\langle y_\beta, \alpha, \bar{r} \rangle : \alpha < \delta$ and $\bar{r} = \langle r_i : i \in y_\beta \rangle$ is a sequence of functions from α to $\{0,1\}\}$, ordered by $p_1 = \langle y_\beta, \alpha^1, \bar{r}^1 \rangle \leq_{R_\beta} p_2 = \langle y_\beta, \alpha^2, \bar{r}^2 \rangle$ iff $\alpha^1 \leq \alpha^2$, $r^1_i \subseteq r^2_i$ for $r^1_i \in \bar{r}^1$, $r^2_i \in \bar{r}^2$, and $|\{i \in y_\beta : r^1_i \neq r^2_i\}| < \delta$. Further, if I_β is $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$-generic over R_β, define in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H][I_\beta]$ an ordering T_β having field $\{\Gamma : \Gamma$ is a function having domain $\{x_\beta\}$ and range $\{C \subseteq \delta : C$ is closed and bounded$\}\}$, ordered by $\Gamma^1 \leq_{T_\beta} \Gamma^2$ iff $\exists \langle y_\beta, \alpha^1, \bar{r}^1 \rangle \in I_\beta \exists \langle y_\beta, \alpha^2, \bar{r}^2 \rangle \in I_\beta [\langle y_\beta, \alpha^1, \bar{r}^1, \Gamma^1 \rangle \leq_\beta \langle y_\beta, \alpha^2, \bar{r}^2, \Gamma^2 \rangle]$. Since R_β is δ-directed closed, $\{C \subseteq \delta : C$ is closed and bounded$\}$ is the same in either $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$ or $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H][I_\beta]$. This means $R_\beta * \dot{T}_\beta$ is isomorphic to $\langle Q_\beta, \leq_\beta \rangle$.

It is easy to see that the definition of R_β implies R_β is isomorphic to $\mathcal{K}(\delta, \delta)$. Further, since $\langle Q_\beta, \leq_\beta \rangle$ is δ-directed closed and $\langle Q_\beta, \leq_\beta \rangle$ is isomorphic to $R_\beta * \dot{T}_\beta$, T_β is δ-directed closed in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H][I_\beta]$. Also, by its definition, T_β has cardinality δ in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H][I_\beta]$; i.e., since T_β is δ-directed closed, $R_\beta * \dot{T}_\beta$ is isomorphic to $\mathcal{K}(\delta, \delta) * \dot{\mathcal{C}}(\delta)$; i.e., $\langle Q_\beta, \leq_\beta \rangle$ is isomorphic to $\mathcal{K}(\delta, \delta)$. Since $|\{x_\beta : \beta \in S\}| = \lambda$, conditions 3) and 4) on the definition of $\leq_Q$ ensure the ordering composed of those elements of $\prod_{\beta \in S} Q_\beta$ having support δ ordered by $\leq'_Q | \prod_{\beta \in S} Q_\beta$ is isomorphic to $\mathcal{K}(\delta, \lambda)$. Then, if $\langle w, \alpha, \bar{r}, \Gamma \rangle \in Q^*$, since we have already observed $\mathrm{dom}(\Gamma) = \emptyset$, Q^* can easily be seen to be isomorphic to $\mathcal{K}(\delta, \lambda)$. Putting all of this together yields Q ordered by $\leq'_Q$ is isomorphic to $\mathcal{K}(\delta, \lambda)$. Thus, since $\langle Q, \leq_Q \rangle$ and $\langle Q, \leq'_Q \rangle$ are forcing equivalent, this proves Lemma 4. □

We remark that, in what follows, it will frequently be the case that a partial ordering P is forcing equivalent to a partial ordering P' in the sense that a generic object for one generates a generic object for the other. Under these circumstances, we will often abuse terminology somewhat by saying that either P or P' satisfies a certain chain condition, a certain amount of closure, etc. when this is true of at least one of these partial orderings. We will then as appropriate further compound the abuse by using this property interchangeably for either partial ordering.

As the definition of $\mathcal{K}(\kappa_0, \kappa_1)$ indicates, without the last coordinates Z^p and Γ^p of a condition $p \in P^1_{\delta,\lambda}[S]$ and the associated restrictions on the ordering, $P^1_{\delta,\lambda}[S]$ is essentially $\mathcal{K}(\delta, \lambda)$. These last coordinates and change in the ordering are necessary to destroy the measurability of δ when forcing with $P^1_{\delta,\lambda}[S]$. Once the fact S is stationary has been destroyed by forcing with $P^2_{\delta,\lambda}[S]$, Lemma 4 shows that these last two coordinates Z^p and Γ^p of a condition $p \in P^1_{\delta,\lambda}[S]$ can be factored out to produce the ordering $\mathcal{C}(\delta^+, \lambda) * \dot{\mathcal{K}}(\delta, \lambda)$.

$\mathcal{K}(\delta, \lambda)$, although somewhat similar in nature to $\mathcal{C}(\delta, \lambda)$ (e.g., $\mathcal{K}(\delta, \lambda)$ is δ-directed closed), differs from $\mathcal{C}(\delta, \lambda)$ in a few very important aspects. In particular, as we shall see presently, forcing with $\mathcal{K}(\delta, \lambda)$ will collapse δ^+. An indication that this occurs is provided by the next lemma.

Lemma 5. *$\mathcal{K}(\delta, \lambda)$ satisfies δ^{++}-c.c. whenever $2^\delta = \delta^+$.*

Proof of Lemma 5. Suppose $\langle \langle w^\beta, \alpha^\beta, \bar{r}^\beta \rangle : \beta < \delta^{++} \rangle$ is a sequence of δ^{++} many incompatible elements of $\mathcal{K}(\delta, \lambda)$. Since $\lambda > \delta^+$, each $w^\beta \in [\lambda]^{<\delta^+}$, $2^\delta = \delta^+$, and λ is either inaccessible or the successor of a cardinal of cofinality $> \delta$, we can find some $A \subseteq \delta^{++}$, $|A| = \delta^{++}$ so that $\{w^\beta : \beta \in A\}$ forms a Δ-system; i.e., $\{w^\beta : \beta \in A\}$ is so that for $\beta_1 \neq \beta_2 \in A$, $w^{\beta_1} \cap w^{\beta_2}$ is some constant value w. Since each $\alpha^\beta < \delta$, let $B \subseteq A$, $|B| = \delta^{++}$ be so that for $\beta_1 \neq \beta_2 \in B$, $\alpha^{\beta_1} = \alpha^{\beta_2} = \alpha$. Since for $\beta \in B$, $\bar{r}^\beta | w$ is a sequence of functions from $\alpha < \delta$ into $\{0, 1\}$, the facts $|w| \leq \delta$ and $2^\delta = \delta^+$ together imply there is $C \subseteq B$, $|C| = \delta^{++}$ so that for $\beta_1 \neq \beta_2 \in C$, $\bar{r}^{\beta_1} | w = \bar{r}^{\beta_2} | w$. It is thus the case that $\langle \langle w^\beta, \alpha^\beta, \bar{r}^\beta \rangle : \beta \in C \rangle$ is now a sequence of δ^{++} many compatible elements of $\mathcal{K}(\delta, \lambda)$, a contradiction. This proves Lemma 5. $\square$

It is clear from Lemmas 4 and 5 and the definition of $\mathcal{K}(\delta, \lambda)$ that since GCH holds in V for cardinals $\kappa \geq \delta$, $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, being equivalent to $\mathcal{C}(\lambda) * \dot{C}(\delta^+, \lambda) * \dot{K}(\delta, \lambda)$, preserves cardinals and cofinalities $\leq \delta$ and $\geq \delta^{++}$, has a dense subset which is δ-directed closed, satisfies λ^+-c.c., and is so that $V^{P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])} \models$ "For every cardinal $\kappa \in [\delta, \lambda)$, $2^\kappa = \lambda$". Our next lemma shows that the forcing $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ is also rather nice, with the exception that it collapses δ^+. By Lemma 4, this has as an immediate consequence that the forcing $\mathcal{K}(\delta, \lambda)$ also collapses δ^+.

Lemma 6. *$P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ preserves cardinals and cofinalities $\leq \delta$ and $\geq \delta^{++}$, collapses δ^+, is $< \delta$-strategically closed, satisfies λ^+-c.c., and is so that $V^{P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\kappa = \lambda$ for all cardinals $\kappa \in [\delta, \lambda)$".*

Proof of Lemma 6. Let $G' = G^0_{\delta,\lambda} * G^1_{\delta,\lambda}$ be V-generic over $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$, and let $G^2_{\delta,\lambda}$ be $V[G']$-generic over $P^2_{\delta,\lambda}[S]$. Thus, $G' * G^2_{\delta,\lambda} = G$ is V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] * P^2_{\delta,\lambda}[\dot{S}]) = P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$.

By Lemmas 4 and 5 and the remarks immediately following, since GCH holds in V for cardinals $\kappa \geq \delta$, $V[G] \models$ "$2^\kappa = \lambda$ for all cardinals $\kappa \in [\delta, \lambda)$" and has the same cardinals and cofinalities as $V \leq \delta$ and $\geq \delta^{++}$. Hence, since $V[G'] \subseteq V[G]$, forcing with $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ over V preserves cardinals and cofinalities $\leq \delta$ and $\geq \delta^{++}$ and is so that $V^{P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\kappa = \lambda$ for all cardinals $\kappa \in [\delta, \lambda)$".

We now show forcing with $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ over V collapses δ^+. Since forcing with $P^0_{\delta,\lambda}$ over V collapses no cardinals and preserves GCH for cardinals $\kappa \geq \delta$, we assume without loss of generality our ground model is $V[G^0_{\delta,\lambda}] = V_1$.

Using the notation of Lemma 3, i.e., that for $i < \lambda$, $\dot{r}'_i$ is a term for $\cup\{r^p_i : \exists p = \langle w^p, \alpha^p, \bar{r}^p, Z^p, \Gamma^p \rangle \in G^1_{\delta,\lambda}[r^p_i \in \bar{r}^p]\}$, we can now define a term $\dot{\beta}_\zeta$ for $\zeta < \delta$ by $\dot{\beta}_\zeta = \min(\{\beta : \beta < \delta^+$ and δ is the order type of $\{i < \beta : \dot{r}'_i(\zeta) = 1\}\})$. To see that $\dot{\beta}_\zeta$ is well-defined, let $p = \langle w^p, \alpha^p, \bar{r}^p, Z^p, \Gamma^p \rangle \in P^1_{\delta,\lambda}[S]$ be a condition. Without loss of generality, we can assume that $\alpha^p > \zeta$. Further, since $|w^p| = \delta$, $\sup(w^p \cap \delta^+) < \delta^+$, so we can let $\gamma < \delta^+$ be so that $\gamma > \sup(w^p \cap \delta^+)$. We can then define $q \geq p$, $q = \langle w^q, \alpha^q, \bar{r}^q, Z^q, \Gamma^q \rangle$ by letting $\alpha^q = \alpha^p$, $Z^q = Z^p$, $\Gamma^q = \Gamma^p$, $w^q = w^p \cup [\gamma, \gamma + \delta)$, and $\bar{r}^q$ by $\bar{r}^q = \langle r^q_i : i \in w^q \rangle$ where r^q_i is r^p_i if $i \in w^p$, and r^q_i is the function having domain α^q which is constantly 1 if $i \in w^q - w^p$.

Clearly, q is well-defined, and $p \le q$. Also, by the definition of q, since $\zeta < \alpha^q$, $q \Vdash$ "$\forall i \in [\gamma, \gamma + \delta)[\dot{r}'_i(\zeta) = 1]$". This means $q \Vdash$ "$\dot{\beta}_\zeta \le \gamma + \delta$", so $\dot{\beta}_\zeta$ is well-defined.

We will be done if we can show $\Vdash_{P^1_{\delta,\lambda}[S]}$ "$\langle \dot{\beta}_\zeta : \zeta < \delta \rangle$ is unbounded in δ^+". Assume now towards a contradiction that $p = \langle w^p, \alpha^p, \bar{r}^p, Z^p, \Gamma^p \rangle$ is so that $p \Vdash$ "$\sup(\langle \dot{\beta}_\zeta : \zeta < \delta \rangle) = \sigma < \delta^+$". If we define $q = \langle w^q, \alpha^q, \bar{r}^q, Z^q, \Gamma^q \rangle$ by $w^q = w^p \cup \{i < \sigma : i \in \sigma - w^p\}$, $\alpha^q = \alpha^p$, $\bar{r}^q = \langle r^q_i : i \in w^q \rangle$ where $r^q_i = r^p_i$ if $i \in w^p$ and r^q_i is the function having domain α^q which is constantly 0 if $i \in w^q - w^p$, $Z^q = Z^p$, and $\Gamma^q = \Gamma^p$, then as above, q is well-defined, and $p \le q$. We claim that for $\zeta = \alpha^q$, $q \Vdash$ "$\dot{\beta}_\zeta > \sigma$".

To see that the claim is true, let $s \ge q$, $s = \langle w^s, \alpha^s, \bar{r}^s, Z^s, \Gamma^s \rangle$ be so that $\alpha^s > \alpha^q$. By clause 3 in the definition of $\le$ on $P^1_{\delta,\lambda}[S]$, $s \Vdash$ "$|\{i < \sigma : \dot{r}'_i(\zeta) = 1\}| < \delta$". This means $q \Vdash$ "$\dot{\beta}_\zeta > \sigma$", thus proving our claim and showing that δ^+ is collapsed.

We next show the $< \delta$-strategic closure of $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$. We first note that as $(P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]) * P^2_{\delta,\lambda}[\dot{S}] = P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] * P^2_{\delta,\lambda}[\dot{S}])$ has by Lemma 4 a dense subset which is $< \delta$-closed, the desired fact follows from the more general fact that if $P * \dot{Q}$ is a partial ordering with a dense subset R so that R is $< \delta$-closed, then P is $< \delta$-strategically closed. To show this more general fact, let $\gamma < \lambda$ be a cardinal. Suppose I and II play to build an increasing chain of elements of P, with $\langle p_\beta : \beta \le \alpha + 1 \rangle$ enumerating all plays by I and II through an odd stage $\alpha + 1$ and $\langle \dot{q_\beta} : \beta < \alpha + 1$ and β is even or a limit ordinal$\rangle$ enumerating a set of auxiliary plays by II which have been chosen so that $\langle \langle p_\beta, \dot{q_\beta} \rangle : \beta < \alpha + 1$ and β is even or a limit ordinal$\rangle$ enumerates an increasing chain of elements of the dense subset $R \subseteq P * \dot{Q}$. At stage $\alpha + 2$, II chooses $\langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle$ so that $\langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle \in R$ and so that $\langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle \ge \langle p_{\alpha+1}, \dot{q_\alpha} \rangle$; this makes sense, since inductively, $\langle p_\alpha, \dot{q}_\alpha \rangle \in R \subseteq P * \dot{Q}$, so as I has chosen $p_{\alpha+1} \ge p_\alpha$, $\langle p_{\alpha+1}, \dot{q}_\alpha \rangle \in P * \dot{Q}$. By the $< \delta$-closure of R, at any limit stage $\eta \le \gamma$, II can choose $\langle p_\eta, \dot{q_\eta} \rangle$ so that $\langle p_\eta, \dot{q_\eta} \rangle$ is an upper bound to $\langle \langle p_\beta, \dot{q_\beta} \rangle : \beta < \eta$ and β is even or a limit ordinal$\rangle$. The preceding yields a winning strategy for II, so P is $< \delta$-strategically closed.

Finally, to show $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ satisfies λ^+-c.c., we simply note that this follows from the general fact about iterated forcing (see [Ba]) that if $P * \dot{Q}$ satisfies λ^+-c.c., then P satisfies λ^+-c.c. (Here, $P = P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ and $Q = P^2_{\delta,\lambda}[\dot{S}]$.) This proves Lemma 6. $\square$

We remark that $\Vdash_{P^0_{\delta,\lambda}}$ "$P^1_{\delta,\lambda}[\dot{S}]$ is δ^{++}-c.c.". Otherwise, if $\mathcal{A} = \langle p_\alpha : \alpha < \delta^{++} \rangle$ were a size δ^{++} antichain of elements of $P^1_{\delta,\lambda}[S]$ in $V[G^0_{\delta,\lambda}]$, then (using the notation of Lemma 4) since $P^1_{\delta,\lambda}[S]$ is isomorphic to $\mathcal{C}(\delta^+, \lambda) * (\dot{P}^1_{\delta,\lambda}[S]/R)$ and $P^1_{\delta,\lambda}[S]/R$ has a dense subset which is isomorphic to $\langle Q, \le_Q \rangle$, without loss of generality, $\mathcal{A}$ can be taken as an antichain in $\langle Q, \le_Q \rangle$. Since $\le'_Q \subseteq \le_Q$, $\mathcal{A}$ must also be an antichain with respect to $\le'_Q$, and as $V[G^0_{\delta,\lambda}]$, $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$ all have the same cardinals, $\mathcal{A}$ must be a size δ^{++} antichain with respect to $\le'_Q$ in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$. The fact that $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H] \models$ "$2^\delta = \delta^+$ and $\langle Q, \le'_Q \rangle$ is isomorphic to $\mathcal{K}(\delta, \lambda)$" then tells us that $\mathcal{A}$ is isomorphic to a size δ^{++} antichain with respect to $\mathcal{K}(\delta, \lambda)$ in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][H]$. Lemma 5, which says that $\mathcal{K}(\delta, \lambda)$ is δ^{++}-c.c. in any model in which $2^\delta = \delta^+$, now yields an immediate contradiction.

We conclude this section with a lemma that will be used later in showing that it is possible to extend certain elementary embeddings witnessing the appropriate degree of supercompactness.

Lemma 7. *For $V_1 = V^{P^0_{\delta,\lambda}}$, the models $V_1^{P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]}$ and $V_1^{P^1_{\delta,\lambda}[S]}$ contain the same $< \lambda$ sequences of elements of V_1.*

Proof of Lemma 7. By Lemma 4, since $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is equivalent to the forcing $\mathcal{C}(\lambda)$ and $V \subseteq V^{P^0_{\delta,\lambda}} \subseteq V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$, the models V, $V^{P^0_{\delta,\lambda}}$, and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ all contain the same $< \lambda$ sequences of elements of V. Thus, since a $< \lambda$ sequence of elements of $V_1 = V^{P^0_{\delta,\lambda}}$ can be represented by a V-term which is actually a function $h : \gamma \to V$ for some $\gamma < \lambda$, it immediately follows that $V^{P^0_{\delta,\lambda}}$ and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ contain the same $< \lambda$ sequences of elements of $V^{P^0_{\delta,\lambda}}$.

Let $f : \gamma \to V_1$ for $\gamma < \lambda$ be so that $f \in (V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]})^{P^1_{\delta,\lambda}[S]} = V_1^{P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]}$, and let $g : \gamma \to V_1$, $g \in V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ be a term for f. By the previous paragraph, $g \in V^{P^0_{\delta,\lambda}}$. Since Lemma 5 shows that $P^1_{\delta,\lambda}[S]$ is δ^{++}-c.c. in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ and $\delta^{++} \le \lambda$, for each $\alpha < \gamma$, the antichain $\mathcal{A}_\alpha$ defined in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ by $\{p \in P^1_{\delta,\lambda}[S] : p$ decides a value for $g(\alpha)\}$ is so that $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]} \models$ "$|\mathcal{A}_\alpha| < \lambda$". Hence, by the preceding paragraph, since $\mathcal{A}_\alpha$ is a set of elements of $V^{P^0_{\delta,\lambda}}$, $\mathcal{A}_\alpha \in V^{P^0_{\delta,\lambda}}$ for each $\alpha < \gamma$. Therefore, again by the preceding paragraph, the sequence $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle \in V^{P^0_{\delta,\lambda}}$. This just means that the term $g \in V^{P^0_{\delta,\lambda}}$ can be evaluated in $V_1^{P^1_{\delta,\lambda}[S]}$, i.e., $f \in V_1^{P^1_{\delta,\lambda}[S]}$. This proves Lemma 7. □

2. The proof of Theorem 1

We turn now to the proof of Theorem 1. Recall that we are assuming our ground model $V \models$ "ZFC + GCH + κ is $< \lambda$ supercompact + $\lambda > \kappa^+$ is regular and is either inaccessible or is the successor of a cardinal of cofinality $> \kappa$ + $h : \kappa \to \kappa$ is so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$". By reflection, we can assume without loss of generality that for every inaccessible $\delta < \kappa$, $h(\delta) > \delta^+$ and $h(\delta)$ is regular. Given this, we are now in a position to define the partial ordering P used in the proof of Theorem 1. We define a κ stage Easton support iteration $P_\kappa = \langle \langle P_\alpha, \dot{Q}_\alpha \rangle : \alpha < \kappa \rangle$, and then define $P = P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$ for a certain partial ordering Q_κ definable in V^{P_κ}. The definition is as follows:

1. P_0 is trivial.
2. Assuming P_α has been defined for $\alpha < \kappa$, let δ_α be so that δ_α is the least cardinal $\ge \bigcup_{\beta<\alpha} \delta_\beta$ such that $\Vdash_{P_\alpha}$ "δ_α is inaccessible", where $\delta_{-1} = 0$. Then $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, with $\dot{Q}_\alpha$ a term for $P^0_{\delta_\alpha, h(\delta_\alpha)} * P^1_{\delta_\alpha, h(\delta_\alpha)}[\dot{S}_{h(\delta_\alpha)}]$, where $\dot{S}_{h(\delta_\alpha)}$ is a term for the non-reflecting stationary subset of $h(\delta_\alpha)$ introduced by $P^0_{\delta_\alpha, h(\delta_\alpha)}$.
3. $\dot{Q}_\kappa$ is a term for $P^0_{\kappa,\lambda} * (P^1_{\kappa,\lambda}[\dot{S}_\lambda] \times P^2_{\kappa,\lambda}[\dot{S}_\lambda])$, where again, $\dot{S}_\lambda$ is a term for the non-reflecting stationary subset of λ introduced by $P^0_{\kappa,\lambda}$.

The intuitive motivation behind the above definition is that below κ at any inaccessible δ, we must force to ensure that δ becomes non-measurable and is so that $2^\delta = h(\delta)$. At κ, however, we must force so as simultaneously to make $2^\kappa = \lambda$ while first destroying and then resurrecting the δ supercompactness of κ for all regular $\delta \in [\kappa, \lambda)$.

Lemma 8. $V^P \models$ "*For all inaccessible* $\delta < \kappa$ *and all cardinals* $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, *for all cardinals* $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$, *and no cardinal* $\delta < \kappa$ *is measurable*".

Proof of Lemma 8. We first note that Easton support iterations of δ-strategically closed partial orderings are δ-strategically closed for δ any regular cardinal. The proof is via induction. If R_1 is δ-strategically closed and $\Vdash_{R_1}$ "$\dot{R}_2$ is δ-strategically closed", then let $p \in R_1$ be so that $p \Vdash$ "$\dot{g}$ is a strategy for player II ensuring that the game which produces an increasing chain of elements of $\dot{R}_2$ of length δ can always be continued for $\alpha \leq \delta$". If II begins by picking $r_0 = \langle p_0, \dot{q}_0 \rangle \in R_1 * \dot{R}_2$ so that $p_0 \geq p$ has been chosen according to the strategy f for R_1 and $p_0 \Vdash$ "$\dot{q}_0$ has been chosen according to $\dot{g}$", and at even stages $\alpha + 2$ picks $r_{\alpha+2} = \langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle$ so that $p_{\alpha+2}$ has been chosen according to f and is so that $p_{\alpha+2} \Vdash$ "$\dot{q}_{\alpha+2}$ has been chosen according to $\dot{g}$", then at limit stages $\lambda \leq \delta$, the chain $r_0 = \langle p_0, \dot{q}_0 \rangle \leq r_1 = \langle p_1, \dot{q}_1 \rangle \leq \cdots \leq r_\alpha = \langle p_\alpha, \dot{q}_\alpha \rangle \leq \cdots (\alpha < \lambda)$ is so that II can find an upper bound p_λ for $\langle p_\alpha : \alpha < \lambda \rangle$ using f. By construction, $p_\lambda \Vdash$ "$\langle \dot{q}_\alpha : \alpha < \lambda \rangle$ is so that at limit and even stages, II has played according to $\dot{g}$", so for some $\dot{q}_\lambda$, $p_\lambda \Vdash$ "$\dot{q}_\lambda$ is an upper bound to $\langle \dot{q}_\alpha : \alpha < \lambda \rangle$", meaning the condition $\langle p_\lambda, \dot{q}_\lambda \rangle$ is as desired. These methods, together with the usual proof at limit stages (see [Ba], Theorem 2.5) that the Easton support iteration of δ-closed partial orderings is δ-closed, yield that δ-strategic closure is preserved at limit stages of any Easton support iteration of δ-strategically closed partial orderings. In addition, the ideas of this paragraph will also show that Easton support iterations of $\prec \delta$-strategically closed partial orderings are $\prec \delta$-strategically closed for δ any regular cardinal.

Given this fact, it is now easy to prove by induction that $V^{P_\kappa} \models$ "For all inaccessible $\delta < \kappa$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta < \kappa$ is measurable". Given $\alpha < \kappa$, we assume inductively that $V^{P_\alpha} \models$ "For all inaccessible $\delta < \delta_\alpha$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta < \delta_\alpha$ is measurable", where δ_α is as in the definition of P. By Lemmas 3 and 6 and the definition of P, since inductively $\Vdash_{P_\alpha}$ "GCH holds for all cardinals $\delta \geq \delta_\alpha$", $V^{P_\alpha * \dot{Q}_\alpha} = V^{P_{\alpha+1}} \models$ "For all inaccessible $\delta \leq \delta_\alpha$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta \leq \delta_\alpha$ is measurable".

If now $\beta \leq \kappa$ is a limit ordinal, then we know by induction that for all $\alpha < \beta$, $V^{P_\alpha} \models$ "For all inaccessible $\delta < \delta_\alpha$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta < \delta_\alpha$ is measurable". If we write $P_\beta = P_\alpha * \dot{R}$, then by the definition of P, the proof of Lemma 4, Lemma 6, and the fact contained in the first paragraph of the proof of this lemma, $\Vdash_{P_\alpha}$ "$\dot{R}$ is forcing equivalent to a $< \delta_\alpha$-strategically closed partial ordering", so $V^{P_\alpha * \dot{R}} = V^{P_\beta} \models$ "For all inaccessible $\delta < \delta_\alpha$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta < \delta_\alpha$ is measurable". If we let $\delta_\kappa = \kappa$, since $\alpha < \beta$ is arbitrary in the preceding, it thus follows by the definition of δ_α for $\alpha < \kappa$ that $V^{P_\kappa} \models$ "For all inaccessible $\delta < \delta_\kappa$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, and no cardinal $\delta < \delta_\kappa$ is measurable".

The proof of Lemma 8 will be complete once we show $V^{P_\kappa * \dot{Q}_\kappa} = V^P$ is so that $V^P \models$ "For all inaccessible $\delta < \kappa$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, for all cardinals $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$, and no cardinal $\delta < \kappa$ is measurable". By the last paragraph, $\Vdash_{P_\kappa}$ "κ is inaccessible", and by the definition of P_κ, $|P_\kappa| = \kappa$, meaning $\Vdash_{P_\kappa}$ "GCH holds for all cardinals $\delta \geq \kappa$". Therefore, by Lemma 4 and the definition of $\dot{Q}_\kappa$, $V^{P_\kappa} \models$ "Q_κ is equivalent to $\mathcal{C}(\lambda) * \dot{\mathcal{C}}(\kappa^+, \lambda) * \dot{\mathcal{K}}(\kappa, \lambda)$", so $V^P \models$ "For all inaccessible $\delta < \kappa$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, for all cardinals $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$, and no cardinal $\delta < \kappa$ is measurable". This proves Lemma 8. $\square$

We now show that the intuitive motivation for the definition of P as set forth in the paragraph immediately preceding the statement of Lemma 8 actually works.

Lemma 9. *For G V-generic over P, $V[G] \models$ "κ is $< \lambda$ supercompact".*

Proof of Lemma 9. Let $j : V \to M$ be an elementary embedding witnessing the $< \lambda$ supercompactness of κ so that $j(h)(\kappa) = \lambda$. We will actually show that for $G = G_\kappa * G'_\kappa$ our V-generic object over $P = P_\kappa * \dot{Q}_\kappa$, the embedding j extends to $k : V[G_\kappa * G'_\kappa] \to M[H]$ for some $H \subseteq j(P)$. As $\langle j(\alpha) : \alpha < \gamma\rangle \in M$ for every $\gamma < \lambda$, this will be enough to allow for every $\gamma < \lambda$ the definition of the ultrafilter $x \in \mathcal{U}_\gamma$ iff $\langle j(\alpha) : \alpha < \gamma\rangle \in k(x)$ to be given in $V[G_\kappa * G'_\kappa]$, thereby showing $V[G] \models$ "κ is $< \lambda$ supercompact".

We construct H in stages. In M, as κ is the critical point of j, $j(P_\kappa * \dot{Q}_\kappa) = P_\kappa * \dot{R}'_\kappa * \dot{R}''_\kappa * \dot{R}'''_\kappa$, where $\dot{R}'_\kappa$ will be a term for $P^0_{\kappa,\lambda} * P^1_{\kappa,\lambda}[\dot{S}_\lambda]$ (note that as $M^{<\lambda} \subseteq M$, $M \models$ "$\delta_\kappa = \kappa$", $j(\kappa) > \kappa$, and $j(h)(\kappa) = \lambda$, $\dot{R}'_\kappa$ is indeed as just stated), $\dot{R}''_\kappa$ will be a term for the rest of the portion of $j(P_\kappa)$ defined below $j(\kappa)$, and $\dot{R}'''_\kappa$ will be a term for $j(\dot{Q}_\kappa)$. This will allow us to define H as $H_\kappa * H'_\kappa * H''_\kappa * H'''_\kappa$. Factoring G'_κ as $G^0_{\kappa,\lambda} * (G^1_{\kappa,\lambda} \times G^2_{\kappa,\lambda})$, we let $H_\kappa = G_\kappa$ and $H'_\kappa = G^0_{\kappa,\lambda} * G^1_{\kappa,\lambda}$. Thus, H'_κ is the same as G'_κ, except, since $M \models$ "$\kappa < j(\kappa)$ and $j(h)(\kappa) = \lambda$", we omit the generic object $G^2_{\kappa,\lambda}$.

To construct H''_κ, we first note that the definition of P ensures $|P_\kappa| = \kappa$ and, since κ is necessarily Mahlo, P_κ is κ-c.c. As $V[G_\kappa]$ and $M[G_\kappa]$ are both models of GCH for cardinals $\gamma \geq \kappa$, the definition of R'_κ in $M[H_\kappa]$ and the remark following Lemma 6 then ensure that $M[H_\kappa] \models$ "R'_κ is a $< \lambda$-strategically closed partial ordering followed by a κ^{++}-c.c. partial ordering and $\kappa^{++} \leq \lambda$". Since $M^{<\lambda} \subseteq M$ implies cardinals in $V \leq \lambda$ are the same as cardinals in $M \leq \lambda$ and P_κ is κ-c.c., Lemma 6.4 of [Ba] shows $V[G_\kappa]$ satisfies these facts as well. This means $< \lambda$-strategic closure and the argument of Lemma 6.4 of [Ba] can be applied to show $M[H_\kappa * H'_\kappa] = M[G_\kappa * H'_\kappa]$ is closed under $< \lambda$ sequences with respect to $V[G_\kappa * H'_\kappa]$, i.e., if $\gamma < \lambda$ is a cardinal, $f : \gamma \to M[H_\kappa * H'_\kappa]$, $f \in V[G_\kappa * H'_\kappa]$, then $f \in M[H_\kappa * H'_\kappa]$. Therefore, as Lemma 8 tells us $M[H_\kappa * H'_\kappa] \models$ "R''_κ is forcing equivalent to a $\prec \lambda$-strategically closed partial ordering"; this fact is true in $V[G_\kappa * H'_\kappa]$ as well.

Observe now that GCH in V allows us to assume $\lambda < j(\kappa) < j(\kappa^+) < \lambda^+$. Since $M[H_\kappa * H'_\kappa] \models$ "$|R''_\kappa| = j(\kappa)$ and $|\mathcal{P}(R''_\kappa)| = j(\kappa^+)$" (this last fact follows from GCH in $M[H_\kappa * H'_\kappa]$ for cardinals $\gamma \geq \lambda$), in $V[G_\kappa * H'_\kappa]$, we can let $\langle D_\alpha : \alpha < \lambda\rangle$ be an enumeration of the dense open subsets of R''_κ present in $M[H_\kappa * H'_\kappa]$. The $\prec \lambda$-strategic closure of R''_κ in both $M[H_\kappa * H'_\kappa]$ and $V[G_\kappa * H'_\kappa]$ now allows us to meet all of these dense subsets as follows. Work in $V[G_\kappa * H'_\kappa]$. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta < \alpha\rangle)$ (initially, q_{-1} is the trivial condition), and player II responds by picking $q_\alpha \geq p_\alpha$ (so $q_\alpha \in D_\alpha$). By the $\prec \lambda$-strategic closure of R''_κ in $V[G_\kappa * H'_\kappa]$, player II has a winning strategy for this game, so $\langle q_\alpha : \alpha < \lambda\rangle$ can be taken as an increasing sequence of conditions with $q_\alpha \in D_\alpha$ for $\alpha < \lambda$. Clearly, $H''_\kappa = \{p \in R''_\kappa : \exists\alpha < \lambda[q_\alpha \geq p]\}$ is our $M[H_\kappa * H'_\kappa]$-generic object over R''_κ which has been constructed in $V[G_\kappa * H'_\kappa] \subseteq V[G_\kappa * G'_\kappa]$, so $H''_\kappa \in V[G_\kappa * G'_\kappa]$.

By the above construction, in $V[G_\kappa * G'_\kappa]$, the embedding j extends to an embedding $j^* : V[G_\kappa] \to M[H_\kappa * H'_\kappa * H''_\kappa]$. We will be done once we have constructed in $V[G_\kappa * G'_\kappa]$ the appropriate generic object for $R'''_\kappa = P^0_{j(\kappa),j(\lambda)} * (P^1_{j(\kappa),j(\lambda)}[\dot{S}_{j(\lambda)}] \times$

$P^2_{j(\kappa),j(\lambda)}[\dot{S}_{j(\lambda)}]) = (P^0_{j(\kappa),j(\lambda)} * P^2_{j(\kappa),j(\lambda)}[\dot{S}_{j(\lambda)}]) * P^1_{j(\kappa),j(\lambda)}[\dot{S}_{j(\lambda)}]$. To do this, first rewrite G'_κ as $(G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}) * G^1_{\kappa,\lambda}$. By the nature of the forcings, $G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}$ is $V[G_\kappa]$-generic over a partial ordering which is $(<\lambda,\infty)$-distributive. Thus, by a general fact about transference of generics via elementary embeddings (see [C], Section 1.2, Fact 2, pp. 5-6), since $j^* : V[G_\kappa] \to M[H_\kappa * H'_\kappa * H''_\kappa]$ is so that every element of $M[H_\kappa * H'_\kappa * H''_\kappa]$ can be written $j^*(F)(a)$ with $\mathrm{dom}(F)$ having cardinality $<\lambda$, $j^{*\prime\prime} G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}$ generates an $M[H_\kappa * H'_\kappa * H''_\kappa]$-generic set H^4_κ.

It remains to construct H^5_κ, our $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$-generic object over $P^1_{j(\kappa),j(\lambda)}[S_{j(\lambda)}]$. To do this, first recall that in $M[H_\kappa * H'_\kappa]$, as previously noted, R''_κ is $\prec \lambda$-strategically closed. Since $M[H_\kappa * H'_\kappa]$ has already been observed to be closed under $<\lambda$ sequences with respect to $V[G_\kappa * H'_\kappa]$, and since any γ sequence of elements for $\gamma < \lambda$ a cardinal of $M[H_\kappa * H'_\kappa * H''_\kappa]$ can be represented, in $M[H_\kappa * H'_\kappa]$, by a term which is actually a function $f : \gamma \to M[H_\kappa * H'_\kappa]$, $M[H_\kappa * H'_\kappa * H''_\kappa]$ is closed under $<\lambda$ sequences with respect to $V[G_\kappa * H'_\kappa]$; i.e., if $f : \gamma \to M[H_\kappa * H'_\kappa * H''_\kappa]$ for $\gamma < \lambda$ a cardinal, $f \in V[G_\kappa * H'_\kappa]$, then $f \in M[H_\kappa * H'_\kappa * H''_\kappa]$.

Choose in $V[G_\kappa * G'_\kappa]$ an enumeration $\langle p_\alpha : \alpha < \lambda\rangle$ of $G^1_{\kappa,\lambda}$. Adopting the notation of Lemma 4 and working now in $V[G_\kappa * G'_\kappa]$, first note that by Lemma 4, there is an isomorphism between $P^1_{\kappa,\lambda}[S_\lambda]$ and $R * (\dot{P}^1_{\kappa,\lambda}[S_\lambda]/R)$. Again by Lemma 4, since P^1 is dense in $P^1_{\kappa,\lambda}[S_\lambda]/R$ and $\langle Q, \le_Q\rangle$ and P^1 are isomorphic, there is an isomorphism g_0 between a dense subset of $P^1_{\kappa,\lambda}[S_\lambda]$ and $R * \langle \dot{Q}, \le_Q\rangle$. Therefore, as R is isomorphic to $\mathcal{C}(\kappa^+,\lambda)$ and $\langle Q, \le'_Q\rangle$ is isomorphic to $\mathcal{K}(\kappa,\lambda)$, we can let $g_1 : R * \langle \dot{Q}, \le'_Q\rangle \to \mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda)$ be an isomorphism. It is then the case that $g = g_1 \circ g_0$ is a bijection between a dense subset of $P^1_{\kappa,\lambda}[S_\lambda]$ and $\mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda)$. This gives us a sequence $I = \langle g(p_\alpha) : \alpha < \lambda\rangle$ of λ many compatible elements of $\mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda)$. By Lemma 7,

$$V[G_\kappa * G^0_{\kappa,\lambda} * G^1_{\kappa,\lambda} * G^2_{\kappa,\lambda}] = V[G_\kappa * G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda} * G^1_{\kappa,\lambda}] = V[G_\kappa * G'_\kappa]$$

and

$$V[G_\kappa * G^0_{\kappa,\lambda} * G^1_{\kappa,\lambda}] = V[G_\kappa * H'_\kappa]$$

have the same $<\lambda$ sequences of elements of $V[G_\kappa * G^0_{\kappa,\lambda}]$ and hence of $V[G_\kappa * H'_\kappa]$. Thus, any $<\lambda$ sequence of elements of $M[H_\kappa * H'_\kappa * H''_\kappa]$ present in $V[G_\kappa * G'_\kappa]$ is actually an element of $V[G_\kappa * H'_\kappa]$ (so $M[H_\kappa * H'_\kappa * H''_\kappa]$ is really closed under $<\lambda$ sequences with respect to $V[G_\kappa * G'_\kappa]$).

For $\alpha \in (\kappa^+,\lambda)$, if $p \in \mathcal{C}(\kappa^+,\lambda)$, let $p|\alpha = \{\langle\langle\rho,\sigma\rangle,\eta\rangle \in p : \sigma < \alpha\}$, and if $q \in \mathcal{K}(\kappa,\lambda)$, $q = \langle w,\sigma,\bar{r}\rangle$, let $q|\alpha = \langle w\cap\alpha, \sigma, \bar{r}|(w\cap\alpha)\rangle$. Call w the domain of q. Since $\Vdash_{C(\kappa^+,\lambda)}$ "There are no new κ sequences of ordinals", we can assume without loss of generality that for any condition $p = \langle p^0, p^1\rangle \in \mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda)$, p^1 is an actual condition and not just a term for a condition. Thus, for $\alpha \in (\kappa^+,\lambda)$ and $p = \langle p^0, p^1\rangle \in \mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda)$, the definitions $p|\alpha = \langle p^0|\alpha, p^1|\alpha\rangle$ and $I|\alpha = \{p|\alpha : p \in I\}$ make sense. And, since GCH holds in $V[G_\kappa * G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}]$ for cardinals $\gamma \ge \kappa$, it is clear $V[G_\kappa * G'_\kappa] \models$ "$|I|\alpha| < \lambda$ for all $\alpha \in (\kappa^+,\lambda)$". Thus, since $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda)) \in M[H_\kappa * H'_\kappa * H''_\kappa]$ ($\mathcal{C}(\kappa^+,\lambda) * \dot{\mathcal{K}}(\kappa,\lambda) \in V[G_\kappa]$) and $M[H_\kappa * H'_\kappa * H''_\kappa] \models$ "$\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ is $j(\kappa)$-directed closed", the facts $M[H_\kappa * H'_\kappa * H''_\kappa]$ is closed under $<\lambda$ sequences with respect to $V[G_\kappa * G'_\kappa]$ and I is compatible imply that $q_\alpha = \langle q^0_\alpha, q^1_\alpha\rangle = \bigcup\{j^*(p) : p \in I|\alpha\}$ for $\alpha \in (\kappa^+,\lambda)$ is well-defined and is an element of $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$.

Letting $q^1_\alpha = \langle w^\alpha, \sigma^\alpha, \bar{r}^\alpha\rangle$, the definition of $I|\alpha$ and the elementarity of j^* easily imply that if $\rho \in w^\alpha - \bigcup_{\beta<\alpha} w^\beta = \mathrm{dom}(q^1_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q^1_\beta)$, then $\rho \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. Also, by the fact $M[H_\kappa * H'_\kappa * H''_\kappa]$ is closed under $< \lambda$ sequences with respect to $V[G_\kappa * G'_\kappa]$, $\bigcup_{\beta<\alpha} q^1_\beta \in \mathcal{K}(j(\kappa), j(\lambda))$ and $\bigcup_{\beta<\alpha} q^0_\beta \in \mathcal{C}(j(\kappa^+), j(\lambda))$, i.e., $\bigcup_{\beta<\alpha} q_\beta \in \mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$. Further, if $\langle\rho, \sigma\rangle \in \mathrm{dom}(q^0_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q^0_\beta)$, then again as before, $\sigma \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. This is since if $\sigma < \bigcup_{\beta<\alpha} j(\beta)$, then let β be minimal so that $\sigma < j(\beta)$, and let ρ and σ be so that $\langle\rho, \sigma\rangle \in \mathrm{dom}(q^0_\alpha)$. It follows that for some $r = \langle r^0, r^1\rangle \in I|\alpha$, $\langle\rho, \sigma\rangle \in \mathrm{dom}(j^*(r^0))$. Since by elementarity and the definitions of $I|\beta$ and $I|\alpha$, for $r^0|\beta = s^0$ and $r^1|\beta = s^1$, $\langle s^0, s^1\rangle \in I|\beta$ and $j^*(s^0) = j^*(r^0)|j(\beta) = j^*(r^0|\beta)$, it must be the case that $\langle\rho, \sigma\rangle \in \mathrm{dom}(j^*(s^0))$. This means $\langle\rho, \sigma\rangle \in \mathrm{dom}(q^0_\beta)$, a contradiction.

We define now an $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$-generic object $H^{5,0}_\kappa$ over $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ so that $p \in g''G^1_{\kappa,\lambda}$ implies $j^*(p) \in H^{5,0}_\kappa$. First, define in $M[H_\kappa * H'_\kappa * H''_\kappa]$ for $\beta \in (j(\kappa^+), j(\lambda))$ the partial ordering $\mathcal{C}(j(\kappa^+), \beta) * \dot{\mathcal{K}}(j(\kappa), \beta) = \{p|\beta : p \in \mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))\}$, ordered analogously to $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$. ($p|\beta$ essentially has the same meaning as when $p \in \mathcal{C}(\kappa^+, \lambda) * \dot{\mathcal{K}}(\kappa, \lambda)$ and $\beta \in (\kappa^+, \lambda)$.) Next, note that since GCH holds in $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$ for cardinals $\gamma \geq j(\kappa)$, $j(\kappa^{++}) \leq j(\lambda)$, and $j(\lambda)$ is regular, Lemma 5 implies $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa] \models$ "$\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ is $j(\kappa^{++})$-c.c., $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ has $j(\lambda)$ many antichains, and if $\mathcal{A} \subseteq \mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ is a maximal antichain, then $\mathcal{A} \subseteq \mathcal{C}(j(\kappa^+), \beta) * \dot{\mathcal{K}}(j(\kappa), \beta)$ for some $\beta \in (j(\kappa^+), j(\lambda))$". As $V \models$ "$|j(\lambda)| = \lambda$" and by Lemma 5 and the fact $\kappa^{++} \leq \lambda$, $V[G_\kappa * G'_\kappa] \models$ "λ is a cardinal", we can let $\langle \mathcal{A}_\alpha : \alpha \in (\kappa^+, \lambda)\rangle \in V[G_\kappa * G'_\kappa]$ be an enumeration of the maximal antichains of $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ present in $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$.

Working in $V[G_\kappa * G'_\kappa]$, we define now an increasing sequence $\langle r_\alpha : \alpha \in (\kappa^+, \lambda)\rangle$ so that $\forall\alpha < \lambda[r_\alpha \geq q_\alpha$ and $r_\alpha \in \mathcal{C}(j(\kappa^+), j(\alpha)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha))]$ and so that $\forall\mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\kappa^+, \lambda)\rangle \exists\beta \in (\kappa^+, \lambda)\exists r \in \mathcal{A}[r_\beta \geq r]$. Assuming we have such a sequence, $H^{5,0}_\kappa = \{p \in \mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda)) : \exists r \in \langle r_\alpha : \alpha \in (\kappa^+, \lambda)\rangle[r \geq p]\}$ is our desired generic object. To define $\langle r_\alpha : \alpha \in (\kappa^+, \lambda)\rangle$, if α is a limit and each r_β for $\beta < \alpha$ is written as $\langle r^0_\beta, r^1_\beta\rangle$, we let $r_\alpha = \langle \bigcup_{\beta<\alpha} r^0_\beta, \bigcup_{\beta<\alpha} r^1_\beta\rangle = \bigcup_{\beta<\alpha} r_\beta$. By the facts $\langle q_\beta : \beta \in (\kappa^+, \lambda)\rangle$ is (strictly) increasing and $M[H_\kappa * H'_\kappa * H''_\kappa]$ is closed under $< \lambda$ sequences with respect to $V[G_\kappa * G'_\kappa]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta : \beta < \eta < \lambda\rangle$ be the subsequence of $\langle \mathcal{A}_\beta : \beta \leq \alpha + 1\rangle$ containing each antichain $\mathcal{A}$ so that $\mathcal{A} \subseteq \mathcal{C}(j(\kappa^+), j(\alpha+1)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha+1))$. Since $q_\alpha, r_\alpha \in \mathcal{C}(j(\kappa^+), j(\alpha)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha))$, $q_{\alpha+1} \in \mathcal{C}(j(\kappa^+), j(\alpha+1)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha+1))$, and $j(\alpha) < j(\alpha+1)$, if we write as before r_α, q_α, and $q_{\alpha+1}$ as $r_\alpha = \langle r^0_\alpha, r^1_\alpha\rangle$, $q_\alpha = \langle q^0_\alpha, q^1_\alpha\rangle$, and $q_{\alpha+1} = \langle q^0_{\alpha+1}, q^1_{\alpha+1}\rangle$, then the condition $r'_{\alpha+1} = \langle r^0_\alpha \cup q^0_{\alpha+1}, r^1_\alpha \cup q^1_{\alpha+1}\rangle = r_\alpha \cup q_{\alpha+1}$ is well-defined. This is because, as our earlier observations show, any new elements of $\mathrm{dom}(q^i_{\alpha+1})$ won't be present in either $\mathrm{dom}(q^i_\alpha)$ or $\mathrm{dom}(r^i_\alpha)$ for $i \in \{0, 1\}$. We can thus, using the fact $M[H_\kappa * H'_\kappa * H''_\kappa]$ is closed under $< \lambda$ sequences with respect to $V[G_\kappa * G'_\kappa]$, define by induction an increasing sequence $\langle s_\beta : \beta < \eta\rangle$ of elements of $\mathcal{C}(j(\kappa^+), j(\alpha+1)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha+1))$ so that $s_0 \geq r'_{\alpha+1}$, $s_\rho = \bigcup_{\beta<\rho} s_\beta$ if ρ is a limit, and $s_{\beta+1} \geq s_\beta$ is so that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta<\eta} s_\beta$ is a well-defined condition in $\mathcal{C}(j(\kappa^+), j(\alpha+1)) * \dot{\mathcal{K}}(j(\kappa), j(\alpha+1))$.

In order to show $H^{5,0}_\kappa$ is $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$-generic over $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$, we must show $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\kappa^+, \lambda)\rangle \exists \beta \in (\kappa^+, \lambda) \exists r \in \mathcal{A}[r_\beta \geq r]$. To do this, we first note that $\langle j(\alpha) : \alpha < \lambda\rangle$ is unbounded in $j(\lambda)$. To see this, if $\beta < j(\lambda)$ is an ordinal, then for some $\gamma < \lambda$ and some $f : \gamma \to M$ representing β, we can assume that for $\rho < \gamma$, $f(\rho) < \lambda$. Thus, by the regularity of λ in V, $\beta_0 = \bigcup_{\rho<\gamma} f(\rho) < \lambda$, and $j(\beta_0) > \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha < \lambda\rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\kappa^+, \lambda)$ be so that $\mathcal{A} \subseteq \mathcal{C}(j(\kappa^+), j(\beta)) * \dot{\mathcal{K}}(j(\kappa), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ so that $r_\eta \geq r$. Finally, since any $p \in \mathcal{C}(\kappa^+, \lambda) * \dot{\mathcal{K}}(\kappa, \lambda)$ is so that for some $\alpha \in (\kappa^+, \lambda)$, $p = p|\alpha$, $H^{5,0}_\kappa$ is so that if $p \in g''G^1_{\kappa,\lambda}$, then $j^*(p) \in H^{5,0}_\kappa$.

Note now that our earlier work ensures j^* extends to $j^{**} : V[G_\kappa * G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}] \to M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$. The proof of Lemma 4 can be given in $V[G_\kappa * G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}]$, meaning we can assume without loss of generality that $g = g_1 \circ g_0 \in V[G_\kappa * G^0_{\kappa,\lambda} * G^2_{\kappa,\lambda}]$. Thus, by elementarity, $j^{**}(g_1^{-1})$ is an order isomorphism between $\mathcal{C}(j(\kappa^+), j(\lambda)) * \dot{\mathcal{K}}(j(\kappa), j(\lambda))$ and $j^{**}(R) * j^{**}(\langle Q, \leq'_Q\rangle)$. Our earlier observations on the forcing equivalence between $\langle Q, \leq_Q\rangle$ and $\langle Q, \leq'_Q\rangle$ (a generic for one is a generic for the other) combined with elementarity hence show $H^{5,1}_\kappa = \{j^{**}(g_1^{-1})(p) : p \in H^{5,0}_\kappa\}$ is an $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$-generic object over $j^{**}(R) * j^{**}(\langle \dot{Q}, \leq_Q\rangle)$. Since the elementarity of j^{**} implies $j^{**}(g_0^{-1})$ is an order isomorphism between $j^{**}(R) * j^{**}(\langle \dot{Q}, \leq_Q\rangle)$ and a dense subset of $P^1_{j(\kappa),j(\lambda)}[S_{j(\lambda)}]$, $H^5_\kappa = \{j^{**}(g_0^{-1})(p) : p \in H^{5,1}_\kappa\}$ is an $M[H_\kappa * H'_\kappa * H''_\kappa * H^4_\kappa]$-generic object over a dense subset of $P^1_{j(\kappa),j(\lambda)}[S_{j(\lambda)}]$ so that $p \in$ (a dense subset of) $P^1_{\kappa,\lambda}[S_\lambda]$ implies $j^{**}(p) \in H^5_\kappa$. Therefore, for $H'''_\kappa = H^4_\kappa * H^5_\kappa$ and $H = H_\kappa * H'_\kappa * H''_\kappa * H'''_\kappa$, $j : V \to M$ extends to $k : V[G_\kappa * G'_\kappa] \to M[H]$, so $V[G] \models$ "κ is $< \lambda$ supercompact". This proves Lemma 9. $\square$

Lemmas 1–9 complete the proof of Theorem 1.

We remark that as opposed to the statement of Theorem 1 in Section 0, the proof just given is not so that V and $V[G]$ share the same cardinals and cofinalities. By Lemma 5, at each stage of the iteration, one cardinal is collapsed. We will outline in Section 4 a notion of forcing similar to the one found in [AS] that can be used to give a proof of Theorems 1 and 2 (although not of Theorem 3) preserving cardinals and cofinalities.

In conclusion to this section, we note that it is tempting to think the above proof of Lemma 9 contains some hidden mistake, i.e., that it can be extended to show κ remains γ supercompact for cardinals $\gamma \geq 2^\kappa$. If we tried to prove Lemma 9 for some $\gamma \geq 2^\kappa$, then we would run into trouble when we tried to construct the generic object H^5_κ. In the above proof, the construction of H^5_κ depends heavily on Lemma 7, more specifically, on the fact that $V[G_\kappa * H'_\kappa]$ and $V[G_\kappa * G'_\kappa]$ have the same $< \lambda$ sequences of elements of $V[G_\kappa * H'_\kappa]$. Since $V[G_\kappa * G'_\kappa]$ contains a subset of λ not present in $V[G_\kappa * H'_\kappa]$, i.e., the generic object $G^2_{\kappa,\lambda}$, if $\gamma \geq 2^\kappa$, it will be false that $V[G_\kappa * H'_\kappa]$ and $V[G_\kappa * G'_\kappa]$ contain the same γ sequences of elements of $V[G_\kappa * H'_\kappa]$.

3. The proofs of Theorems 2 and 3

We turn now to the proof of Theorem 2. Recall that we are assuming the existence of a class function λ so that for any infinite cardinal δ, $\lambda(\delta) > \delta^+$ is a regular cardinal which is either inaccessible or is the successor of a cardinal of cofinality $> \delta$ with $\lambda(\delta)$ below the least inaccessible $> \delta$ if δ is singular and $\lambda(0) = 0$ and that our ground model V is such that $V \models$ "ZFC + GCH + A is a proper class of cardinals so that for each $\kappa \in A$, $h_\kappa : \kappa \to \kappa$ is a function and $j_\kappa : V \to M$ is an elementary embedding witnessing the $< \lambda(\kappa)$ supercompactness of κ with $j_\kappa(h_\kappa)(\kappa) = \lambda(\kappa) < \kappa^*$ for κ^* the least element of $A > \kappa$". Without loss of generality (by "cutting off" V if necessary), we assume that for each $\kappa \in A$, $\sup(\{\delta \in A : \delta < \kappa\}) = \rho_\kappa < \kappa$ and isn't inaccessible if order type$(\{\delta \in A : \delta < \kappa\})$ is a limit ordinal.

For each $\kappa \in A$, let $P(\kappa, \lambda(\kappa))$ be the version of the partial ordering P used in the proof of Theorem 1 which ensures each inaccessible $\delta \in (\lambda(\rho_\kappa), \kappa)$ isn't measurable yet κ is $< \lambda(\kappa)$ supercompact; i.e., using the notation of Section 2, $P(\kappa, \lambda(\kappa))$ is the $\kappa + 1$ stage Easton support iteration $\langle \langle P_\alpha, \dot{Q}_\alpha \rangle : \alpha \leq \kappa \rangle$ defined as follows:

1. P_0 is trivial.
2. Assuming P_α has been defined for $\alpha < \kappa$, let δ_α be so that δ_α is the least cardinal $\geq \bigcup_{\beta<\alpha} \delta_\alpha$ such that $\Vdash_{P_\alpha}$ "δ_α is inaccessible", where $\delta_{-1} = \lambda(\rho_\kappa)$ and δ_0 is the least inaccessible in $(\lambda(\rho_\kappa), \kappa)$. Then $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, with $\dot{Q}_\alpha$ a term for $P^0_{\delta_\alpha, h_\kappa(\delta_\alpha)} * P^1_{\delta_\alpha, h_\kappa(\delta_\alpha)}[\dot{S}_{h_\kappa(\delta_\alpha)}]$.
3. $\dot{Q}_\kappa$ is a term for $P^0_{\kappa,\lambda(\kappa)} * (P^1_{\kappa,\lambda(\kappa)}[\dot{S}_{\lambda(\kappa)}] \times P^2_{\kappa,\lambda(\kappa)}[\dot{S}_{\lambda(\kappa)}])$.

We then define the first partial ordering P used in the proof of Theorem 2 as $P = \{p \in \prod_{\kappa \in A} P(\kappa, \lambda(\kappa)) :$ support(p) is a set$\}$, ordered by componentwise extension.

Note that for each $\kappa \in A$, we can write P as $T_\kappa \times T^\kappa$, where

$$T_\kappa = \prod_{\{\delta \in A : \delta \leq \kappa\}} P(\delta, \lambda(\delta)),$$

$$T^\kappa = \{p \in \prod_{\{\delta \in A : \delta > \kappa\}} P(\delta, \lambda(\delta)) : \text{ support } (p) \text{ is a set}\},$$

and T_κ and T^κ are both ordered componentwise. For each $\kappa \in A$, the definition of the component partial orderings of T_κ and T^κ ensures that T_κ is $\lambda(\kappa)^+$-c.c. and T^κ is $\lambda(\kappa)^+$-strategically closed. This allows us to conclude in the manner of [KiM] that $V^P \models$ ZFC.

Lemma 10. $V^P \models$ "$2^\delta = \lambda(\kappa)$ *if* $\kappa \in A$ *and* $\delta \in [\kappa, \lambda(\kappa))$ + *Every* $\kappa \in A$ *is* $< \lambda(\kappa)$ *supercompact* + $\forall \kappa [\kappa$ *is measurable iff* κ *is* $< \lambda(\kappa)$ *strongly compact iff* κ *is* $< \lambda(\kappa)$ *supercompact*$]$".

Proof of Lemma 10. Using the notation above, for each $\kappa \in A$, write $P = T_\kappa \times T^\kappa$; further, write T_κ as $T_{<\kappa} \times P(\kappa, \lambda(\kappa))$ for $T_{<\kappa} = \prod_{\{\delta \in A : \delta < \kappa\}} P(\delta, \lambda(\delta))$. Since each component partial ordering of T^κ is at least $< \sigma(\kappa)$-strategically closed for $\sigma(\kappa)$ the least inaccessible $> \lambda(\kappa)$, T^κ is $< \sigma(\kappa)$-strategically closed, meaning by the remark immediately following Lemma 5 and Lemma 9 that $V^{T^\kappa \times P(\kappa,\lambda(\kappa))} \models$ "$2^\delta = \lambda(\kappa)$ if $\delta \in [\kappa, \lambda(\kappa))$, κ is $< \lambda(\kappa)$ supercompact, and no cardinal $\delta \in [\lambda(\rho_\kappa), \kappa)$ is measurable". As our assumptions and the definition of $T_{<\kappa}$ ensure $V^{T^\kappa \times P(\kappa,\lambda(\kappa))} \models$

"$|T_{<\kappa}| <$ the least inaccessible $\theta(\kappa)$ above $\lambda(\rho_\kappa)$", by the Lévy-Solovay arguments [LS], $V^{T^\kappa \times P(\kappa,\lambda(\kappa)) \times T_{<\kappa}} = V^P \models$ "$2^\kappa = \lambda(\kappa)$, κ is $< \lambda(\kappa)$ supercompact, and no cardinal $\delta \in [\lambda(\rho_\kappa), \kappa)$ is measurable". Finally, as the definitions of $\lambda(\kappa)$, ρ_κ, $\sigma(\kappa)$, and $\theta(\kappa)$ guarantee every inaccessible cardinal δ in V^P must be so that $\delta \in [\lambda(\rho_\kappa), \kappa]$ for some $\kappa \in A$ and every ordinal $\delta \geq$ the least inaccessible must be so that $\delta \in [\theta(\kappa), \sigma(\kappa)]$ for some $\kappa \in A$, $V^P \models$ "κ is measurable iff κ is $< \lambda(\kappa)$ strongly compact iff κ is $< \lambda(\kappa)$ supercompact". This proves Lemma 10. $\square$

Note it may be the case that $V^P \models$ "Some cardinal κ is $\lambda(\kappa)$ strongly compact". In order to ensure this doesn't happen, we must for each $\kappa \in A$ force with the partial ordering $P^0_{\omega,\lambda(\kappa)}$ of Section 1, i.e., for each $\kappa \in A$, we add a non-reflecting stationary set of ordinals of cofinality ω to $\lambda(\kappa)$. By a theorem of [SRK], this guarantees κ isn't $\lambda(\kappa)$ strongly compact.

Keeping the preceding paragraph in mind, we take as our ground model $V^P = V_1$. Working in V_1, we let $R = \{p \in \prod_{\kappa \in A} P^0_{\omega,\lambda(\kappa)} :$ support(p) is a set$\}$, ordered by componentwise extension. As before, we can write for each $\kappa \in A$ $R = R_\kappa \times R^\kappa$, where $R_\kappa = \prod_{\{\delta \in A : \delta \leq \kappa\}} P^0_{\omega,\lambda(\delta)}$ and $R^\kappa = \{p \in \prod_{\{\delta \in A : \delta > \kappa\}} P^0_{\omega,\lambda(\delta)} :$ support(p) is a set$\}$. We can also write R_κ as $R_{<\kappa} \times P^0_{\omega,\lambda(\kappa)}$, where $R_{<\kappa} = \prod_{\{\delta \in A : \delta < \kappa\}} P^0_{\omega,\lambda(\delta)}$. Again, for each $\kappa \in A$, the fact that $V_1 \models$ "$2^\delta = \lambda(\kappa)$ for each cardinal $\delta \in [\kappa, \lambda(\kappa))$" and the definitions of R_κ and R^κ ensure that R_κ is $\lambda(\kappa)^+$-c.c. and R^κ is $\lambda(\kappa)^+$-strategically closed. Hence, once more, $V_1^R \models$ ZFC.

Lemma 11. *V_1 and V_1^R have the same cardinals and cofinalities and $V_1^R \models$ "$2^\delta = \lambda(\kappa)$ if $\kappa \in A$ and $\delta \in [\kappa, \lambda(\kappa))$ + Every $\kappa \in A$ is $< \lambda(\kappa)$ supercompact + $\forall \kappa[\kappa$ is measurable iff κ is $< \lambda(\kappa)$ strongly compact iff κ is $< \lambda(\kappa)$ supercompact$]$ + No cardinal κ is $\lambda(\kappa)$ strongly compact".*

Proof of Lemma 11. We mimic the proof of Lemma 10. Using the notation of Lemma 10, for $\kappa \in A$, since each component partial ordering of R^κ is at least $\sigma(\kappa)$-strategically closed and $V_1^{R^\kappa} \models$ "$P^0_{\omega,\lambda(\kappa)}$ is $< \lambda(\kappa)$-strategically closed and is $\lambda(\kappa)^+$-c.c.", $V_1^{R^\kappa \times P^0_{\omega,\lambda(\kappa)}} \models$ "$2^\delta = \lambda(\kappa)$ if $\delta \in [\kappa, \lambda(\kappa))$, κ is $< \lambda(\kappa)$ supercompact, no cardinal $\delta \in [\lambda(\rho_\kappa), \kappa)$ is measurable, cardinals and cofinalities for any $\delta \leq \sigma(\kappa)$ are the same as in V_1, and κ isn't $\lambda(\kappa)$ strongly compact". Since analogously to Lemma 10 $V_1^{R^\kappa \times P^0_{\omega,\lambda(\kappa)}} \models$ "$|R_{<\kappa}| < \theta(\kappa)$", as in Lemma 10, $V_1^{R^\kappa \times P^0_{\omega,\lambda(\kappa)} \times R_{<\kappa}} = V_1^R \models$ "$2^\delta = \lambda(\kappa)$ if $\delta \in [\kappa, \lambda(\kappa))$, κ is $< \lambda(\kappa)$ supercompact, no cardinal $\delta \in [\lambda(\rho_\kappa), \kappa)$ is measurable, cardinals and cofinalities for any $\delta \in [\theta(\kappa), \sigma(\kappa)]$ are the same as in V_1, and κ isn't $\lambda(\kappa)$ strongly compact". Once more, every inaccessible δ in V^P must be so that $\delta \in [\lambda(\rho_\kappa), \kappa]$ for some $\kappa \in A$ and every ordinal $\delta \geq$ the least inaccessible must be so that $\delta \in [\theta(\kappa), \sigma(\kappa)]$ for some $\kappa \in A$, so V_1^R and V_1 have the same cardinals and cofinalities, and $V_1^R \models$ "$2^\delta = \lambda(\kappa)$ if $\kappa \in A$ and $\delta \in [\kappa, \lambda(\kappa))$ + Every $\kappa \in A$ is $< \lambda(\kappa)$ supercompact + $\forall \kappa[\kappa$ is measurable iff κ is $< \lambda(\kappa)$ strongly compact iff κ is $< \lambda(\kappa)$ supercompact$]$ + No cardinal κ is $\lambda(\kappa)$ strongly compact". This proves Lemma 11. $\square$

Lemmas 10 and 11 complete the proof of Theorem 2.

We turn now to the proof of Theorem 3. We begin by giving a proof of Menas' theorem that the least measurable limit κ of strongly compact or supercompact cardinals is not 2^κ supercompact.

Lemma 12 (Menas [Me])**.** *If κ is the least measurable limit of either strongly compact or supercompact cardinals, then κ is strongly compact but isn't 2^κ supercompact.*

Proof of Lemma 12. We assume without loss of generality that κ is the least measurable limit of strongly compact cardinals. As readers will easily see, the proof given works equally well if κ is the least measurable limit of supercompact cardinals.

Let $\langle \kappa_\alpha : \alpha < \kappa \rangle$ enumerate in increasing order the strongly compact cardinals below κ. Fix $\lambda > \kappa$ an arbitrary cardinal. Let μ be any measure (normal or non-normal) over κ, and let $\langle \mu_\alpha : \alpha < \kappa \rangle$ be a sequence of fine, κ_α-complete measures over $P_{\kappa_\alpha}(\lambda)$. The set $\mathcal{U}_\lambda$ given by $X \in \mathcal{U}_\lambda$ iff $X \subseteq P_\kappa(\lambda)$ and $\{\alpha < \kappa : X | \kappa_\alpha \in \mu_\alpha\} \in \mu$, where for $X \subseteq P_\kappa(\lambda)$, $\alpha < \kappa$, $X|\kappa_\alpha = \{p \in X : p \in P_{\kappa_\alpha}(\lambda)\}$. It can easily be verified that $\mathcal{U}_\lambda$ is a κ-additive, fine measure over $P_\kappa(\lambda)$. Since $\lambda > \kappa$ is arbitrary, κ is strongly compact.

Assume now that κ is 2^κ supercompact, and let $k : V \to M$ be an elementary embedding with critical point κ so that $M^{2^\kappa} \subseteq M$. By the fact that κ is the critical point of k, if $\delta < \kappa$ is strongly compact, then $M \models$ "$k(\delta) = \delta$ is strongly compact". By the fact $M^{2^\kappa} \subseteq M$, $M \models$ "κ is measurable". Thus, $M \models$ "κ is a measurable limit of strongly compact cardinals", contradicting the fact that $M \models$ "$k(\kappa) > \kappa$ is the least measurable limit of strongly compact cardinals". This proves Lemma 12. $\square$

We return to the proof of Theorem 3. Recall that we are assuming our ground model $V \models$ "ZFC + GCH + κ is the least supercompact limit of supercompact cardinals + $\lambda > \kappa^+$ is a regular cardinal which is either inaccessible or is the successor of a cardinal of cofinality $> \kappa$ and $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V \to M$ witnessing the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$". As in the proof of Theorem 1, we assume without loss of generality that for every inaccessible $\delta < \kappa$, $h(\delta) > \delta^+$ and $h(\delta)$ is regular.

We also assume without loss of generality that $h(\delta) = 0$ if δ isn't inaccessible and that if $\delta < \kappa$ is inaccessible and δ' is the least supercompact cardinal $> \delta$, then $h(\delta) < \delta'$. To see that the conditions on h imply this last restriction, assume that $h(\delta) \geq \delta'$ on the set of inaccessibles below κ. It must then be the case that $M \models$ "For some cardinal $\rho \leq \lambda$, ρ is supercompact". By the closure properties of M, $M \models$ "κ is ζ supercompact for all $\zeta < \rho$". It is a theorem of Magidor [Ma2] that if α is $< \beta$ supercompact and β is supercompact, then α is supercompact. It is thus the case that $M \models$ "κ is supercompact". Since $V \models$ "κ is the least supercompact limit of supercompact cardinals" and κ is the critical point of j, if $V \models$ "$\alpha < \kappa$ is supercompact", $M \models$ "$j(\alpha) = \alpha$ is supercompact". Putting these last two sentences together yields a contradiction to the fact that $M \models$ "$j(\kappa) > \kappa$ is the least supercompact limit of supercompact cardinals".

We next show the following fact about Laver indestructibility [L] which will play a critical role in the proof of Theorem 3.

Lemma 13. *If δ is a supercompact cardinal, then the definition of the partial ordering which makes δ Laver indestructible under δ-directed closed forcings can be reworked so that for any fixed $\gamma < \delta$, there are no strongly compact cardinals in the interval (γ, δ).*

Proof of Lemma 13. Let $f : \delta \to V_\delta$ be a Laver function; i.e., f is so that for every $x \in V$ and every $\sigma \geq |\mathrm{TC}(x)|$, there is a fine, δ-complete, normal ultrafilter $\mathcal{U}_\sigma$ over $P_\delta(\sigma)$ so that for j_σ the elementary embedding generated by $\mathcal{U}_\sigma$, $(j_\sigma f)(\delta) = x$.

The Laver partial ordering P^* which makes δ Laver indestructible under δ-directed closed forcings and destroys all strongly compact cardinals in the interval (γ, δ) is as usual defined as a δ stage Easton support iteration $\langle\langle P^*_\alpha, \dot{Q}^*_\alpha\rangle : \alpha < \delta\rangle$. As in the original definition, at each stage $\alpha < \delta$, an ordinal $\rho_\alpha < \delta$ is chosen, where at limit stages α, $\rho_\alpha = \bigcup_{\beta<\alpha} \rho_\beta$. We define $P^*_{\alpha+1} = P^*_\alpha * \dot{Q}^*_\alpha$, where $\dot{Q}^*_\alpha$ is a term for the trivial partial ordering $\{\emptyset\}$ and $\rho_{\alpha+1} = \rho_\alpha$, unless for all $\beta < \alpha$, $\rho_\beta < \alpha$ and $f(\alpha) = \langle \dot{R}, \sigma\rangle$, where $\sigma \neq \gamma$ is a regular cardinal $\geq \max(\gamma, \alpha)$ and $\dot{R}$ is a term so that $\Vdash_{P^*_\alpha}$ "$\dot{R}$ is $\max(\gamma, \alpha)$-directed closed". Under these circumstances, $\dot{Q}^*_\alpha$ is a term for the partial ordering $\dot{R} * \dot{P}^0_{\gamma,\sigma}$ and $\rho_\alpha = \sigma^+$, where $P^0_{\gamma,\sigma}$ has the same meaning as it did in Section 1.

To show P^* is as desired, let $\dot{Q}$ be a term in the forcing language with respect to P^* so that $\Vdash_{P^*}$ "$\dot{Q}$ is δ-directed closed", and let $\eta \geq \delta$. Let $\sigma > |\mathrm{TC}(\dot{Q})|$ be a regular cardinal so that $\Vdash_{P^* * \dot{Q}}$ "$\sigma > \zeta$ for $\zeta = \max(|\mathrm{TC}(\dot{Q})|, 2^{|[\eta]^{<\delta}|})$". Take $\mathcal{U}_\sigma$ and the associated elementary embedding $j_\sigma : V \to M$ so that $(j_\sigma f)(\delta) = \langle \dot{Q}, \sigma\rangle$, and call j_σ k. By the definition of P^*, in M we have $P^*_{\delta+1} = P^*_\delta * \dot{Q} * \dot{P}^0_{\gamma,\sigma} = (P^*)^V * \dot{Q} * P^0_{\gamma,\sigma}$. Thus, for $G * H$ V-generic over $P^* * \dot{Q}$, since P^*_δ is δ-c.c., the usual arguments (see Lemma 6.4 of [Ba]) show that $M[G*H]$ is closed under ζ sequences (in the sense of Lemma 9) with respect to $V[G*H]$. Further, since $\Vdash_{P^* * \dot{Q}}$ "$\sigma > \zeta$", by Lemma 8, the closure properties of $M[G*H]$ with respect to $V[G*H]$, and the definitions of P^* and $k(P^*)$ (including the fact both P^* and $k(P^*)$ are Easton support iterations of partial orderings satisfying a certain degree of strategic closure), the partial ordering $T \in M[G*H]$ so that $P^*_\delta * \dot{Q} * \dot{P}^0_{\gamma,\sigma} * \dot{T}^* = P^* * \dot{Q} * \dot{T} = P^*_{k(\delta)}$ (where $\dot{T}^*$ is a term for the appropriate partial ordering) is ζ-strategically closed in both $M[G*H]$ and $V[G*H]$. As in Lemma 9, this means that if H' is $V[G*H]$-generic over T, then $M[G*H*H']$ is closed under ζ sequences with respect to $V[G*H*H']$, and the embedding k extends in $V[G*H*H']$ to $k^* : V[G] \to M[G*H*H']$. The definition of ζ and the fact $k(Q)$ is ζ-directed closed in both $M[G*H*H']$ and $V[G*H*H']$ allow us to find in $V[G*H*H']$ a master condition q extending each $p \in k^{*\prime\prime}H$. If H'' is now a $V[G*H*H']$-generic object over $k^*(Q)$ containing q, then k^* extends further in $V[G*H*H'*H'']$ to $k^{**} : V[G*H] \to M[G*H*H'*H'']$. By the fact that $T * k(\dot{Q})$ is ζ-strategically closed in either $V[G*H]$ or $M[G*H]$ and the definition of ζ, the ultrafilter over $(P_\delta(\eta))^{V[G*H]}$ definable via k^{**} in $V[G*H*H'*H'']$ is present in $V[G*H]$. Since η was arbitrary, $V[G*H] \models$ "δ is supercompact".

It remains to show that $V^{P^*} \models$ "No cardinal in the interval (γ, δ) is strongly compact". To see this, choose an embedding $k' : V \to M$ so that $(k'f)(\delta) = \langle \dot{Q}, \delta\rangle$, where $\dot{Q}$ is a term with respect to P^* for the trivial partial ordering $\{\emptyset\}$. By the definition of P^*, it will then be the case that $M \models$ "$P^*_{\delta+1} = P^*_\delta * \dot{Q} * \dot{P}^0_{\gamma,\delta} = (P^*)^V * \dot{Q} * \dot{P}^0_{\gamma,\delta}$ and the $\dot{T}$ so that $P^*_{\delta+1} * \dot{T} = k'(P^*)$ is such that $\Vdash_{k'(P^*)}$ "δ contains a non-reflecting stationary set of ordinals of cofinality γ". By reflection, $\{\beta < \delta : P^*_{\beta+1} = P^*_\beta * \dot{Q} * \dot{P}^0_{\gamma,\beta}$ and the $\dot{T}$ so that $P^*_{\beta+1} * \dot{T} = P^*$ is such that $\Vdash_{P^*}$ "β contains a non-reflecting stationary set of ordinals of cofinality γ"$\}$ is unbounded in δ, where $\dot{Q}$ is a term with respect to P^*_β for the trivial partial ordering $\{\emptyset\}$. By a theorem of [SRK], if β contains a non-reflecting stationary set of ordinals of cofinality γ, then there are no strongly compact cardinals in the interval (γ, β). Thus, the last two sentences immediately imply $V^{P^*} \models$ "No cardinal in the interval (γ, δ) is strongly compact". This proves Lemma 13. $\square$

We note that in the proof of Lemma 13, GCH is not assumed. If GCH were assumed, then as in Lemma 9, we could have taken the generic object H' as an element of $V[G * H]$.

We can now define in the manner of Theorem 1 the partial ordering P used in the proof of Theorem 3 by defining a κ stage Easton support iteration $P_\kappa = \langle\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha < \kappa\rangle$ and then defining $P = P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$ for a certain partial ordering Q_κ definable in V^{P_κ}. We first let $\langle\delta_\alpha : \alpha < \kappa\rangle$ be the continuous, increasing enumeration of the set $\{\delta < \kappa : \delta$ is a supercompact cardinal$\} \cup \{\delta < \kappa : \delta$ is a limit of supercompact cardinals$\}$. We take as an inductive hypothesis that the field of P_α is $\{\delta_\beta : \beta < \alpha\}$ and that if δ_α is supercompact, then $|P_\alpha| < \delta_\alpha$. The definition is then as follows:

1. P_0 is trivial.
2. Assuming P_α has been defined for $\alpha < \kappa$, we consider the following three cases.
   (1) δ_α is a supercompact cardinal. By the inductive hypothesis, since $|P_\alpha| < \delta_\alpha$, the Lévy-Solovay results [LS] show that $V^{P_\alpha} \models$ "δ_α is supercompact". It is thus possible in V^{P_α} to make δ_α Laver indestructible under δ_α-directed closed forcings. We therefore let $\dot{Q}_\alpha$ be a term for the partial ordering of Lemma 13 making δ_α Laver indestructible so that $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is defined using partial orderings that are at least $(2^{\gamma_\alpha})^+$-directed closed and add non-reflecting stationary sets of ordinals of cofinality $(2^{\gamma_\alpha})^+$ for $\gamma_\alpha = \max(\sup(\{\delta_\beta : \beta < \alpha\}), h(\sup(\{\delta_\beta : \beta < \alpha\})))$", and we define $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$. (If $\alpha = 0$, then $\dot{Q}_\alpha$ is a term for the Laver partial ordering of Lemma 13 where $\gamma = \omega$, and $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$.) Since $\dot{Q}_\alpha$ can be chosen so that $|P_{\alpha+1}| = \delta_\alpha < \delta_{\alpha+1}$, the inductive hypothesis is easily preserved.
   (2) δ_α is a regular limit of supercompact cardinals. Then $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, with $\dot{Q}_\alpha$ a term for $P^0_{\delta_\alpha, h(\delta_\alpha)} * P^1_{\delta_\alpha, h(\delta_\alpha)}[\dot{S}_{h(\delta_\alpha)}]$, where $\dot{S}_{h(\delta_\alpha)}$ is a term for the non-reflecting stationary subset of $h(\delta_\alpha)$ introduced by $P^0_{\delta_\alpha, h(\delta_\alpha)}$. Since $\delta_{\alpha+1}$ must be supercompact, by the conditions on h, $|P_{\alpha+1}| < \delta_{\alpha+1}$, so the inductive hypothesis is once again preserved.
   (3) δ_α is a singular limit of supercompact cardinals. Then $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where $\dot{Q}_\alpha$ is a term for the trivial partial ordering $\{\emptyset\}$.
3. $\dot{Q}_\kappa$ is a term for $P^0_{\kappa,\lambda} * (P^1_{\kappa,\lambda}[\dot{S}_\lambda] \times P^2_{\kappa,\lambda}[\dot{S}_\lambda])$, where again, $\dot{S}_\lambda$ is a term for the non-reflecting stationary subset of λ introduced by $P^0_{\kappa,\lambda}$.

Note that if $\alpha < \kappa$ is a limit ordinal, then since κ is the least supercompact limit of supercompact cardinals, $\sup(\{\delta_\beta : \beta < \alpha\}) = \delta < \delta'$, where δ' is the least supercompact cardinal in the interval $[\delta, \kappa)$. It is this fact that preserves the inductive hypothesis at limit ordinals $\alpha < \kappa$ and at successor stages $\alpha + 1$ when δ_α is a singular limit of supercompact cardinals.

The intuitive motivation behind the above definition is much the same as in Theorem 1. Specifically, below κ at any inaccessible limit δ of supercompact cardinals, we must force to ensure that δ becomes non-measurable and is so that $2^\delta = h(\delta)$. At κ, however, we must force so as simultaneously to make $2^\kappa = \lambda$ while first destroying and then resurrecting the $< \lambda$ supercompactness of κ. The forcing will preserve the supercompactness of every V-supercompact cardinal below κ and will ensure there are no measurable limits of supercompacts below κ. In addition, the forcing will guarantee that the only strongly compact cardinals below κ are those that were supercompact in V. Thus, κ will have become the least measurable limit of supercompact and strongly compact cardinals in the generic extension.

Lemma 14. $V^P \models$ "*If* $\delta < \kappa$ *is supercompact in* V*, then* δ *is supercompact*".

Proof of Lemma 14. Let $\delta < \kappa$ be a V-supercompact cardinal. Write $P = R_\delta * \dot{R}^\delta$, where R_δ is the portion of P whose field is all cardinals $\le \delta$ and $\dot{R}^\delta$ is a term for the rest of P. By case 1 in clause 2 of the inductive definition of P, $V_1 = V^{R_\delta} \models$ "δ is supercompact and is indestructible under δ-directed closed forcings".

Assume now that $V^P = V_1^{R^\delta} \models$ "δ isn't supercompact", and let $p = \langle \dot{p}_\alpha : \alpha \le \kappa\rangle \in R^\delta$ be so that over V_1, $p \Vdash_{R_\delta}$ "δ isn't supercompact". By the remark after the proof of Lemma 3, case 1 in clause 2 of the inductive definition of P, and the fact each $P^0_{\delta_\alpha, h(\delta_\alpha)}$ is δ_α-directed closed for $\alpha < \kappa$, it inductively follows that if H is a V_1-generic object over R^δ so that $p \in H$, then H must be V_1-generic over a partial ordering $T^\delta \in V_1$ so that $p \in T^\delta$ and so that $V_1 \models$ "T^δ is δ-directed closed". This means $V_1[H] \models$ "δ is supercompact". This contradicts that over V_1, $p \Vdash_{R_\delta}$ "δ isn't supercompact". This proves Lemma 14. $\square$

Lemma 15. $V^P \models$ "*No inaccessible* $\delta < \kappa$ *which is a limit of* V*-supercompact cardinals is measurable*".

Proof of Lemma 15. If $\delta < \kappa$ is in V^P an inaccessible limit of V-supercompact cardinals, then since $V^{P_\delta} \subseteq V^P$, this fact must be true in V^{P_δ} as well. Hence, since δ is so that P_δ is the direct limit of the system $\langle\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha < \delta\rangle$ and $V \models$ GCH, $V^{P_\delta} \models$ "All cardinals $\gamma \ge \delta$ are the same as in V and GCH holds for all cardinals $\gamma \ge \delta$". Therefore, the same arguments as in Lemmas 3 and 6 show that $V^{P_{\delta+1}} \models$ "δ isn't measurable and $2^\gamma = h(\delta)$ if $\gamma \in [\delta, h(\delta))$ is a cardinal". (The same argument as in Lemma 8 also tells us that $V^P \models$ "$2^\gamma = \lambda$ if $\gamma \in [\kappa, \lambda)$ is a cardinal".) It then follows by case 1 in clause 2 of the inductive definition of P that $V^P \models$ "δ isn't measurable and $2^\gamma = h(\delta)$ if $\gamma \in [\delta, h(\delta))$ is a cardinal". This proves Lemma 15. $\square$

Lemma 16. $V^P \models$ "*For any* $\delta < \kappa$*,* δ *is supercompact in* V *iff* δ *is supercompact iff* δ *is strongly compact*".

Proof of Lemma 16. Let $\delta < \kappa$ be strongly compact and not V-supercompact, and let $\delta' \in (\delta, \kappa)$ be the least V-supercompact cardinal $> \delta$. Since Lemma 15 shows that no inaccessible limit of V-supercompact cardinals is measurable, $\sup(\{\beta < \delta : \beta$ is a V-supercompact cardinal$\}) = \delta_\alpha < \delta$, where $\alpha < \kappa$ and δ_α is as in the definition of P. (If $\delta < \delta_0$, then let $\alpha = -1$ and $\delta_\alpha = 0$.) Thus, $\delta \in (\delta_\alpha, \delta')$ and $\delta' = \delta_{\alpha+1}$. By the definition of P, $P_{\alpha+2} = P_{\alpha+1} * \dot{Q}_{\alpha+1}$, where $\dot{Q}_{\alpha+1}$ is so that $\Vdash_{P_{\alpha+1}}$ "$\dot{Q}_{\alpha+1}$ destroys all strongly compact cardinals in the interval $(\delta_\alpha, \delta_{\alpha+1})$ by adding non-reflecting stationary sets of ordinals of cofinality $(2^{\gamma_{\alpha+1}})^+$ to unboundedly many in $\delta_{\alpha+1}$ cardinals, where as before, $\gamma_{\alpha+1} = \max(\sup(\{\delta_\beta : \beta < \alpha+1\}), h(\sup(\{\delta_\beta : \beta < \alpha+1\}))) = \max(\delta_\alpha, h(\delta_\alpha))$". (If $\alpha = -1$, then $\dot{Q}_{\alpha+1}$ is so that $\Vdash_{P_{\alpha+1}}$ "$\dot{Q}_{\alpha+1}$ destroys all strongly compact cardinals in the interval $(\delta_\alpha, \delta_{\alpha+1})$ by adding non-reflecting stationary sets of ordinals of cofinality ω to unboundedly many in $\delta_{\alpha+1}$ cardinals".) Again by the definition of P, for the $\dot{T}$ so that $P_{\alpha+2} * \dot{T} = P$, $\Vdash_{P_{\alpha+2}}$ "$\dot{T}$ is $(2^{\gamma_{\alpha+2}})^+ = (2^{h(\delta_{\alpha+1})})^+$-strategically closed", so $\Vdash_P$ "There are unboundedly many in $\delta_{\alpha+1}$ cardinals in the interval $(\delta_\alpha, \delta_{\alpha+1})$

containing non-reflecting stationary sets of ordinals of either cofinality $(2^{\gamma_{\alpha+1}})^+$ or ω". This means $V^P \models$ "No cardinal in the interval $(\delta_\alpha, \delta_{\alpha+1})$ is strongly compact", a contradiction. This, combined with Lemma 14, proves Lemma 16. □

Lemma 17. $V^P \models$ "*κ is the least measurable limit of either strongly compact or supercompact cardinals*".

Proof of Lemma 17. By Lemma 16, if $\delta < \kappa$ is strongly compact, then δ must be supercompact in both V and V^P. By Lemma 15, there are no measurable limits of V-supercompact cardinals in V^P below κ. This proves Lemma 17. □

Lemmas 13–17, together with the observation that the same arguments as in Lemma 9 yield $V^P \models$ "κ is $< \lambda$ supercompact", complete the proof of Theorem 3.

4. Concluding remarks

In conclusion to this paper, we outline an alternate notion of forcing that can be used to construct models witnessing Theorems 1 and 2 in which cardinals and cofinalities are the same as in the ground model. The forcing we use is a slight variation of the forcing used in [AS]. Specifically, as in Section 1, we let $\delta < \lambda$ be cardinals with δ inaccessible, $\lambda > \delta^+$ regular, and λ either inaccessible or the successor of a cardinal of cofinality $> \delta$. We also assume as in Section 1 that our ground model V is so that GCH holds in V for all cardinals $\kappa \geq \delta$, and we fix $\gamma < \delta$ a regular cardinal. As before, we define three notions of forcing. $P^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals of cofinality γ to λ; i.e., $P^0_{\delta,\lambda}$ is defined as in Section 1, only replacing δ in the definition with γ. If $\dot{S}$ is a term for the non-reflecting stationary set of ordinals of cofinality γ introduced by $P^0_{\delta,\lambda}$, then $P^2_{\delta,\lambda}[S] \in V_1 = V^{P^0_{\delta,\lambda}}$ is the standard notion of forcing for introducing a club set C which is disjoint to S; i.e., $P^2_{\delta,\lambda}[S]$ essentially has the same definition as in Section 1.

To define $P^1_{\delta,\lambda}[S]$ in V_1, as in [AS] or Section 1, we first fix in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\beta : \beta \in S\rangle$. (Since each element of S has cofinality γ, either Lemma 1 of [AS] or our Lemma 1 shows each $x \in X$ can be assumed to be so that order type$(x) = \gamma$.) Then, in analogy to the definition given in Section 1 of [AS], $P^1_{\delta,\lambda}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z\rangle$ satisfying the following properties.

1. $w \in [\lambda]^{<\delta}$.
2. $\alpha < \delta$.
3. $\bar{r} = \langle r_i : i \in w\rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a sequence of subsets of α.
4. $Z \subseteq \{x_\beta : \beta \in S\}$ is a set so that if $z \in Z$, then for some $y \in [w]^\gamma$, $y \subseteq z$ and $z - y$ is bounded in the β so that $z = x_\beta$.

As in [AS], the definition of Z implies $|Z| < \delta$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1\rangle \leq \langle w^2, \alpha^2, \bar{r}^2, Z^2\rangle$ iff the following hold.

1. $w^1 \subseteq w^2$.
2. $\alpha^1 \leq \alpha^2$.
3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.
4. $Z^1 \subseteq Z^2$.

5. If $z \in Z^1 \cap [w^1]^\gamma$ and $\alpha^1 \leq \alpha < \alpha^2$, then
$$|\{i \in z : r_i^2(\alpha) = 0\}| = |\{i \in z : r_i^2(\alpha) = 1\}| = \gamma.$$

The intuition behind the definition of $P^1_{\delta,\lambda}[S]$ just given is essentially the same as in [AS] or in the remarks immediately following the definition of $P^1_{\delta,\lambda}[S]$ in Section 1 of this paper. Specifically, we wish to be able simultaneously to make $2^\delta = \lambda$, destroy the measurability of δ, and be able to resurrect the $< \lambda$ supercompactness of δ if necessary. $P^1_{\delta,\lambda}[S]$ has been designed so as to allow us to do all of these things.

The proof that $V_1^{P^1_{\delta,\lambda}[S]} \models$ "δ is non-measurable" is as in Lemma 3 of [AS]. In particular, the argument of Lemma 3 of [AS] will show that δ can't carry a γ-additive uniform ultrafilter. We can then carry through the proof of Lemma 4 of [AS] to show $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is equivalent to $\mathcal{C}(\lambda) * \dot{C}(\delta, \lambda)$. The proofs of Lemma 5 of [AS] and Lemma 6 of this paper will then show $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ preserves cardinals and cofinalities, is λ^+-c.c., and is so that $V^{P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\kappa = \lambda$ for every cardinal $\kappa \in [\delta, \lambda)$". Then, if we assume κ is $< \lambda$ supercompact with λ and $h : \kappa \to \kappa$ as in Theorem 1 and define in V an iteration P as in Section 2 of this paper, we can combine the arguments of Lemma 8 of [AS] and Lemma 8 of this paper to show V and V^P have the same cardinals and cofinalities and $V^P \models$ "For all inaccessible $\delta < \kappa$ and all cardinals $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$, for all cardinals $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$, and no cardinal $\delta < \kappa$ is measurable". We can then prove as in Lemma 9 of this paper that $V^P \models$ "κ is $< \lambda$ supercompact". The proof now of Theorem 2 is as before, this time using the just described iteration as the building blocks of the forcing. This will allow us to conclude that the model witnessing the conclusions of Theorem 2 thereby constructed is so that it and the ground model contain the same cardinals and cofinalities.

We finish by explaining our earlier remarks that it is impossible to use the just described definitions of $P^0_{\delta,\lambda}$, $P^1_{\delta,\lambda}[S]$, and $P^2_{\delta,\lambda}[S]$ to give a proof of Theorem 3 of this paper. This is since if $V \models$ "κ is a supercompact limit of supercompact cardinals, $\mu < \kappa$ is supercompact, δ and λ are both regular cardinals, and $\gamma < \mu < \delta < \lambda$", then forcing with either $P^0_{\delta,\lambda}$ or $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ will kill the λ strong compactness of μ, as S will be a non-reflecting stationary set of ordinals of cofinality γ through λ in either $V^{P^0_{\delta,\lambda}}$ or $V^{P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]}$. (See [SRK] or [KiM] for further details.) This type of forcing must of necessity occur if we use the iteration described in the proof of Theorem 3.

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