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HERZ-SCHUR MULTIPLIERS AND WEAKLY ALMOST PERIODIC FUNCTIONS ON LOCALLY COMPACT GROUPS

GUANGWU XU

Abstract. For a locally compact group G and $1 < p < \infty$, let $A_p(G)$ be the Herz–Figà-Talamanca algebra and $B_p(G)$ the Herz-Schur multipliers of G, and $MA_p(G)$ the multipliers of $A_p(G)$. Let $W(G)$ be the algebra of continuous weakly almost periodic functions on G. In this paper, we show that (1), if G is a noncompact nilpotent group or a noncompact [IN]-group, then $W(G)/B_p(G)^-$ contains a linear isometric copy of $l^\infty(\mathbb{N})$; (2), for a noncommutative free group $F, B_p(F)$ is a proper subset of $MA_p(F) \cap W(F)$.

1. Introduction

Let G be a locally compact group, $C(G)$ the space of bounded continuous functions on G with the sup norm. For a subset S of $C(G)$, S^- denotes the uniform closure of S in $C(G)$. Let $A_p(G)$ be the Herz–Figà-Talamanca algebra of G and $B_p(G)$ the algebra of Herz-Schur multipliers, with $1 < p < \infty$. Note that $A_2(G) = A(G)$ is the Fourier algebra of G, introduced by Eymard [12], and $B_2(G)$ is the completely bounded multipliers $M_0A(G)$ of $A(G)$, as was shown by Bożejko and Fendler [5]. The Fourier-Stieltjes algebra $B(G)$ of G is the space of coefficients of strongly continuous unitary representations of G. It is known that $B(G) \subseteq M_0A(G)$, and they are equal if G is amenable. Also, $M_0A(G) \subseteq B_p(G)$ for every $1 < p < \infty$ (see [1], [15]). Let $W(G)$ be the algebra of continuous weakly almost periodic functions on G. Then it can be shown that $B_p(G) \subseteq W(G)$ for every $1 < p < \infty$. In answering a question raised by Eberlein, i.e., whether for an abelian group G, $B(G)^- = W(G)$, Rudin [32] showed that $B(G)^- \subsetneq W(G)$ if G is abelian and contains a discrete subgroup which is not of bounded order, and Ramirez [31] later showed that Rudin's conclusion holds for all noncompact abelian groups. More general results on this topic were obtained by Chou [7]. He extended the Rudin-Ramirez result to include many nonabelian groups: if G is either a noncompact nilpotent group or a noncompact [IN]-group, then $W(G)/B(G)^-$ contains a linear isometric copy of $l^\infty(\mathbb{N})$, in particular $B(G)^- \subsetneq W(G)$. In the first part of this paper, we are able to replace $B(G)$ by some larger spaces. More precisely, we have the following result: for every $1 < p < \infty$, $W(G)/B_p(G)^-$ contains a linear isometric copy of $l^\infty(\mathbb{N})$, if G is a noncompact nilpotent group or a noncompact [IN]-group. This generalizes Chou's result mentioned above. This will be the contents of sections 3 and 4.

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If G is a locally compact group, we denote by $MA_p(G)$ the space of multipliers of $A_p(G)$, $1 < p < \infty$. When G is amenable, we have the equality $B_p(G) = MA_p(G)$. It was shown by Bożejko [2], [4] that for a noncommutative free group F, $B_p(F)$ is a proper subset of $MA_p(F)$. In fact he constructed a function ϕ in [4] such that $\phi \in MA_p(F)$ but $\phi \notin W(F)$, hence $\phi \notin B_p(F)$. It is therefore interesting to decide whether $B_p(F) = MA_p(F) \cap W(F)$. In section 5, by constructing a Leinert set and using the discussions in section 3, we are able to show that $B_p(F)$ is a proper subset of $MA_p(F) \cap W(F)$ for a free group F on at least two generators.

2. Preliminaries

Let G be a locally compact group with a fixed left Haar measure and $L^p(G), 1 \leq p \leq \infty$, the usual Lebesgue spaces on G with the norm $\|\cdot\|_p$.

Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The Herz–Figà-Talamanca algebra $A_p(G)$ is the space of continuous functions u which can be represented as

$$u = \sum_{i=1}^{\infty} f_i * \check{g}_i,$$

where $f_i \in L^q(G)$, $g_i \in L^p(G)$ ($\check{g}_i(x) = g_i(x^{-1})$) and $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$, with norm the infimum of the last expression over all such representations of u. $A_p(G)$ is a Banach algebra with pointwise multiplication. Note that $A_p(G)$ is contained in $C_0(G)$, the subspace of $C(G)$ consisting of functions vanishing at infinity, and for every $u \in A_p(G)$, $\|u\|_\infty \leq \|u\|_{A_p}$.

Denote $MA_p(G) = \{u \in C(G) : uv \in A_p(G) \text{ for all } v \in A_p(G)\}$ with the norm $\|u\|_M = \sup\{\|uv\|_{A_p} : v \in A_p(G), \|v\|_{A_p} \leq 1\}$. It is called the space of multipliers of $A_p(G)$.

Let $V_p(G) = \{\psi : G \times G \to \mathbb{C} : \psi F \in L^p(G) \otimes_\gamma L^q(G) \text{ for all } F \in L^p(G) \otimes_\gamma L^q(G)\}$. It is the space of pointwise multipliers of the projective tensor product $L^p(G) \otimes_\gamma L^q(G)$. The norm on $V_p(G)$ is the operator norm on $L^p(G) \otimes_\gamma L^q(G)$.

Let $\phi : G \to \mathbb{C}$ be a function. Define $M\phi : G \times G \to \mathbb{C}$ by

$$M\phi(x, y) = \phi(xy^{-1})$$

for all $x, y \in G$. The space of Herz-Schur mulitipliers is defined to be

$$B_p(G) = \{\phi : G \to \mathbb{C} : M\phi \in V_p(G)\}.$$

The norm $\|\phi\|_{B_p}$ is given by $\|\phi\|_{B_p} = \|M\phi\|_{V_p}$. Elements of $B_p(G)$ are continuous, and $\|u\|_\infty \leq \|u\|_{B_p}$ for every $u \in B_p(G)$.

For each $1 < p < \infty$, let $\mathcal{B}_p$ denote the category of p-spaces (see [21]). It is a subcategory of the category of Banach spaces. The following characterisation of the space $B_p(G)$ is due to Fendler [13, Theorem 4.4] (see also Pisier [29, Theorem 2.1] for a general treatment): a function ϕ on G is in $B_p(G)$ if and only if there exist $B \in \mathcal{B}_p$ and (continuous) bounded maps $a : G \to B$ and $b : G \to B^*$ such that

$$\phi(yx^{-1}) = \langle a(x), b(y) \rangle$$

for all $x, y \in G$.

If $f \in C(G)$ and $x \in G$, then $\lambda(x)f$, the left translate of f by x, is defined by $\lambda(x)f(y) = f(x^{-1}y)$. $f \in C(G)$ is said to be a weakly almost periodic function (w.a.p. for short) if the set $\{\lambda(x)f; x \in G\}$ is relatively compact with respect to the weak topology of $C(G)$. We denote by $W(G)$ the space of w.a.p. functions. It is known that $W(G)$ has a unique translation-invariant mean m_G.

Finally, we point out that the inclusion $B_p(G) \subseteq W(G)$ $(1 < p < \infty)$ follows from the description of $B_p(G)$ and the *Grothendieck criterion*, which says that $f \in C(G)$ is w.a.p. if and only if whenever $\{x_n\}$ and $\{y_m\}$ are two sequences in G and $\lim_n \lim_m f(x_n y_m)$ and $\lim_m \lim_n f(x_n y_m)$ exist, then they are equal.

3. Discrete Groups

Throughout this section, we will assume that G is a discrete group.

First, let us give an alternative description of $B_p(G)$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let $END_p(G)$ be the Banach algebra of bounded linear operators on $l^p(G)$. Every element $k \in END_p(G)$ can be identified with a function $k : G \times G \to \mathbb{C}$ such that

$$\|k\|_{END_p} = sup\left\{ |\sum_{x,y\in G} k(x,y)u(y)v(x)| : \|u\|_p \leq 1, \|v\|_q \leq 1 \right\}$$

is a finite number, where $u \in l^p(G)$ and $v \in l^q(G)$. The algebra of Herz-Schur multipliers $B_p(G)$ is the space of functions ϕ such that

$$M\phi \cdot END_p(G) \subseteq END_p(G),$$

where $M\phi \cdot k$ is the pointwise multiplication for $k \in END_p(G)$. The norm $\|\phi\|_{B_p}$ is given by

$$\|\phi\|_{B_p} = sup\{\|M\phi \cdot k\|_{END_p} : \|k\|_{END_p} \leq 1\}.$$

Let X_p be the completion of $l^1(G)$ with respect to the norm

$$\|f\|_{X_p} = sup\left\{ |\sum_{x\in G} f(x)\phi(x)| : \phi \in B_p(G), \|\phi\|_{B_p} \leq 1 \right\}.$$

Then ${X_p}^* = B_p(G)$, as was shown in [4], [13].

In [28], Picardello introduced the concept of weak Sidon sets, which was later made use of in [7]. In our situation, we need the following

Definition 3.1. A subset $S \subseteq G$ is said to be a B_p*-Sidon set*, if given any $f \in l^\infty(G)$ there exists $u \in B_p(G)$ such that $f|_S = u|_S$.

If g is a function defined on a subset S of G, we can regard g as a function on G by setting its values to be zero outside of S. Thus, it is natural to identify $l^p(S)$ as a closed subspace of $l^p(G)$, for $1 \leq p \leq \infty$.

Proposition 3.2. *Let S be a subset of the discrete group G. Then the following conditions are equivalent:*

(1) S is a B_p-Sidon set;

(2) $l^1(S)$ is closed in X_p;

(3) $\|\cdot\|_1$ and $\|\cdot\|_{X_p}$ are equivalent on $l^1(S)$.

Proof. (1) $\Rightarrow$ (2). Suppose that S is a B_p-Sidon set; then

$$B_p(G) \xrightarrow{T} l^\infty(S), \qquad u \longmapsto u|_S,$$

is continuous and surjective; hence T is an open mapping. Therefore, there is a $\delta > 0$ such that

$$Ball_{l^\infty(G)}(0,\delta) \subseteq T(Ball_{B_p}(0,1)).$$

So for any $f \in l^\infty(G)$, there exists $u_f \in B_p(G)$ with $f|_S = u_f|_S$ and $\|f|_S\|_\infty \geq \frac{\delta}{2}\|u_f\|_{B_p}$.

Suppose that $\{g_n\}$ is a sequence in $l^1(S)$ that converges in the norm $\|\cdot\|_{X_p}$. For every $f \in L^\infty(S)$ with $\|f\|_\infty = 1$, we have

$$\begin{aligned}|\langle g_n - g_m, f\rangle| &= |\langle g_n - g_m, u_f|_S\rangle| \\ &\leq \|g_n - g_m\|_{X_p}\|u_f\|_{B_p} \\ &\leq \frac{2}{\delta}\|g_n - g_m\|_{X_p}.\end{aligned}$$

So, $\{g_n\}$ is a Cauchy sequence in the norm $\|\cdot\|_1$, and hence $l^1(S)$ is closed in X_p.

(2) $\Rightarrow$ (1). Let $f \in l^\infty(S) = l^1(S)^*$. Since $l^1(S)$ is closed in X_p, we can get an extension $T \in X_p^*$ of f. Note that we can identify T with a function $u \in B_p(G)$ by setting

$$u(x) = T(\delta_x),$$

where δ_x is the function on G which is 1 at x and 0 elsewhere. It is easy to see that $u(x) = f(x)$ holds for $x \in S$.

(2) $\Rightarrow$ (3). Note that $\|\cdot\|_{X_p} \leq \|\cdot\|_1$. So (3) is a consequence of the open mapping theorem.

(3) $\Rightarrow$ (2). Trivial. □

We give another useful criterion of B_p-Sidon sets, similar to the Lemma 3.11 of [7].

Corollary 3.3. *A subset S of G is a B_p-Sidon set if and only if there is a positive constant $c < 1$ such that for every $f \in l^\infty(S)$ with $\|f\|_\infty = 1$ there exists a $u \in B_p(G)$ with $\|f - u|_S\|_\infty \leq c$.*

Proof. One direction is trivial.

Now suppose that S is not a B_p-Sidon set. Then $\|\cdot\|_1$ and $\|\cdot\|_{X_p}$ are not equivalent on $l^1(S)$ by the above proposition. Let us choose $g_1 \in l^1(S)$ with finite support F_1, and $\|g_1\|_1 = 1, \|g_1\|_{X_p} < 1$. Note that $B_p(G)$ contains all functions with finite support; hence $S \setminus F_1$ is again not a B_p-Sidon set. Therefore, we can choose $g_2 \in l^1(S \setminus F_1)$ with finite support F_2, and $\|g_2\|_1 = 1, \|g_2\|_{X_p} < \frac{1}{2}$. Continuing this procedure, we can get a sequence of functions $\{g_n\}$ in $l^1(S)$ with disjoint supports F_n, and $\|g_n\|_1 = 1, \|g_n\|_{X_p} < \frac{1}{n}$, for $n = 1, 2, \ldots$.

Define

$$f(x) = \begin{cases} \frac{g_n(x)}{|g_n(x)|}, & x \in F_n \text{ for some } n, \\ 0, & x \notin \bigcup_{n=1}^\infty F_n. \end{cases}$$

Then $f \in l^\infty(S)$ and $\|f\|_\infty = 1$. Note that for every $u \in B_p(G)$,

$$\begin{aligned}\|f - u|_S\|_\infty &\geq |\langle f - u|_S, g_n\rangle| \\ &\geq |\langle f, g_n\rangle| - |\langle u, g_n\rangle| \\ &\geq 1 - \|u\|_{B_p}\|g_n\|_{X_p} \\ &\geq 1 - \frac{\|u\|_{B_p}}{n}.\end{aligned}$$

As n can be arbitrarily large, $\|f - u|_S\|_\infty = 1$, and the condition in the statement is not satisfied. □

A subset C of G is called an *n-square* if $C = AB$ where $A, B \subseteq G$ and $|A| = |B| = n$ and $|C| = n^2$ ($|X|$ denotes the cardinality of the set X). A subset S of G is said to contain large squares if for each positive integer k, S contains a k-square.

Proposition 3.4. *Suppose $S \subseteq G$ contains large squares. Then $\|\cdot\|_1$ and $\|\cdot\|_{X_p}$ are not equivalent on $l^1(S)$, i.e., S is not a B_p-Sidon set.*

Proof. For each integer $n > 0$, choose an n-square $C = \{a_1, \dots, a_n\}\{b_1, \dots, b_n\}$.

It was shown by Bennett [1, Proposition 3.2] that there exist an $n \times n$ matrix $A = (a_{ij})$ all of whose entries are ± 1, and a constant D, which is independent of n, such that the norm of the linear operator

$$A : l^p(Z_n) \to l^p(Z_n),$$

where $Z_n = \{1, \dots, n\}$, satisfies

$$\|A\|_{p,p} \leq D max\{n^{\frac{1}{p}}, n^{\frac{1}{q}}\}.$$

Let

$$g = \sum_{i,j=1}^{n} a_{ij}\delta_{a_i b_j};$$

then $g \in l^1(S)$ and $\|g\|_1 = n^2$.

Now let us estimate $\|g\|_{X_p}$. By the definition,

$$\|g\|_{X_p} = sup\left\{ |\sum_{x\in G} g(x)\phi(x)| : \phi \in B_p(G), \|\phi\|_{B_p} \leq 1 \right\}.$$

For $\phi \in B_p(G)$ with $\|\phi\|_{B_p} \leq 1$, we have

$$|\sum_{x\in G} g(x)\phi(x)| = |\sum_{i,j=1}^{n} a_{ij}\phi(a_i b_j)|.$$

Let $k \in END_p(G)$ be defined as

$$k(x,y) = \begin{cases} a_{ij}, & \text{if } (x,y) = (a_i, {b_j}^{-1}), \\ 0, & \text{otherwise;} \end{cases}$$

then

$$\|k\|_{END_p} = \|A\|_{p,p} \leq D max\{n^{\frac{1}{p}}, n^{\frac{1}{q}}\}.$$

Let $k_1 = M\phi \cdot k$ and $u = \sum_{j=1}^{n} \delta_{b_j^{-1}}$; then for every $v \in l^q(G)$,

$$|\sum_{x,y\in G} k_1(x,y)u(y)v(x)| \leq \|k_1\|_{END_p}\|u\|_p\|v\|_q,$$

i.e.,

$$|\sum_{i,j=1}^{n} a_{ij}\phi(a_i b_j)v(a_i)| \leq n^{\frac{1}{p}}\|k_1\|_{END_p}\|v\|_q.$$

Therefore, we get

$$\left(\sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}\phi(a_i b_j)|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}}\|k_1\|_{END_p}.$$

Hence

$$\begin{aligned}|\sum_{i,j=1}^{n} a_{ij}\phi(a_ib_j)| &\leq \sum_{i=1}^{n}|\sum_{j=1}^{n} a_{ij}\phi(a_ib_j)| \\ &\leq n^{\frac{1}{q}}\left(\sum_{i=1}^{n}|\sum_{j=1}^{n} a_{ij}\phi(a_ib_j)|^p\right)^{\frac{1}{p}} \\ &\leq n\|k_1\|_{END_p} \\ &\leq n\|k\|_{END_p}\|\phi\|_{B_p} \\ &\leq Dmax\{n^{1+\frac{1}{p}}, n^{\frac{1}{q}}\}.\end{aligned}$$

So,

$$\|g\|_{X_p} \leq Dmax\{n^{1+\frac{1}{p}}, n^{1+\frac{1}{q}}\}.$$

Since n can be arbitrarily large, we conclude that $\|\cdot\|_1$ and $\|\cdot\|_{X_p}$ are not equivalent. □

A subset $T \subseteq G$ is said to be a *t-set* if $(T \cap Tx) \cup (T \cap xT)$ is finite for every $x \in G \setminus \{e\}$. It is known that if T is a t-set, then any $f \in l^\infty(G)$ with supp$f \subseteq T$ is in $W(G)$; see [6], [7].

A consequence of Proposition 3.4 related to the concept of t-set is the following:

Corollary 3.5. *If S is a countable B_p-Sidon set, then S is a finite union of t-sets.*

Proof. Since S does not contain large squares, by Theorem 4.1 of [8] S is a finite union of t-sets. □

The following result of Chou [7] will play a very important role in our proof of the main result of this section.

Theorem (Chou). *If G is an infinite group, then there is a t-set T of G such that $T = \bigcup_{n=1}^{\infty} S_n$ is a disjoint union and each S_n contains large squares.*

By applying Proposition 3.4 and a device in Chou [7], we are able to show the following.

Theorem 3.6. *Let G be an infinite discrete group. Then $W(G)/B_p(G)^-$ contains a linear isometric copy of $l^\infty(\mathbb{N})$. In particular, $B_p(G)^-$ is a proper subset of $W(G)$.*

Proof. Let T be the t-set as in Chou's construction. So $T = \bigcup_{n=1}^{\infty} S_n$ is a disjoint union of S_n's with each S_n containing large squares. Thus each S_n is not a B_p-Sidon set, and hence by Corollary 3.3, there exists a function $f_n \in l^\infty(G)$ with the following properties: $\|f_n\|_\infty = 1, suppf_n \subseteq S_n$ and $\|(f_n - u)|_{S_n}\|_{l^\infty(S_n)} \geq 1$ for every $u \in B_p(G)$.

Since S_n is a t-set, $f_n \in W(G)$, $n = 1, 2, ...$

Define

$$\xi : l^\infty(\mathbb{N}) \to W(G)/B_p(G)^-,$$

$$(c_n) \mapsto \sum_{i=1}^{\infty} c_n f_n + B_p(G)^-.$$

It is not hard to see that ξ is an isometry. □

4. Nilpotent Groups and [IN]-Groups

First, let us recall that a locally compact group G is called an [IN]-group if it has a compact neighborhood of the identity which is invariant under all inner automorphisms of G.

Let H be a closed normal subgroup of G and

$$\pi : G \to G/H$$

be the canonical homomorphism. For $1 < p < \infty$ and $f \in B_p(G/H)$, there exist $B \in \mathcal{B}_p$ and continuous bounded maps $a_0 : G/H \to B, b_0 : G/H \to B^*$ such that

$$f(\pi(y)\pi(x)^{-1}) = \langle a_0(\pi(x)), b_0(\pi(y))\rangle$$

for all $x, y \in G$. Let $a = a_0 \circ \pi, b = b_0 \circ \pi$; then the continuous function $f \circ \pi$ on G satisfies

$$(f \circ \pi)(yx^{-1}) = \langle a(x), b(y)\rangle$$

for all $x, y \in G$. So, $f \circ \pi \in B_p(G)$ and the map

$$\Phi : B_p(G/H) \to B_p(G),$$

$$f \mapsto f \circ \pi,$$

is an isometry from $B_p(G/H)$ onto the subspace of $B_p(G)$ consisting of functions that are constant on the left cosets of H .

Fix $x \in G$. For any function f on G, define a function

$$f_x : H \to \mathbb{C}$$

by $f_x(t) = f(xt)$, $t \in H$. If $f \in B_p(G)$ and

$$f(yx^{-1}) = \langle a(x), b(y)\rangle$$

for some space $B \in \mathcal{B}_p$ and bounded maps $a : G \to B, b : G \to B^*$, then

$$f_x(ts^{-1}) = f(xts^{-1}) = \langle a(s), b(xt)\rangle$$

for $s, t \in H$; so $f_x \in B_p(H)$.

Let m_H be the unique invariant mean of $W(H)$. For $f \in B_p(G)$ and $x \in G$; since $f_x \in B_p(H) \subseteq W(H)$, we can define

$$\phi(x) = m_H(f_x)$$

for $x \in G$.

Proposition 4.1. *Let ϕ be defined as above. Then $\phi \in B_p(G)$ and ϕ is constant on the left cosets of H.*

Proof. Since m_H is H-invariant, ϕ is constant on left cosets of H. The function ϕ is continuous, since $x \mapsto f_x$ is continuous.

By a result of Davis [10], there exists a net of open and relatively compact subsets $\{U_\alpha\}$ of H such that

$$m_H(k) = \lim_{\alpha} \lambda(U_\alpha)^{-1} \int_{U_\alpha} k(t) d\lambda(t),$$

where $k \in W(H)$ and λ is a fixed left Haar measure of H.

Let $B \in \mathcal{B}_p$. If $p : H \to B$ or B^* is a continuous bounded map, $U \subseteq H$ is a relatively compact open set, the vector-valued integral

$$\int_U p(t) d\lambda(t)$$

exists, and $\| \int_U p(t) d\lambda(t) \| \leq \int_U \|p(t)\| d\lambda(t)$.

For fixed $x, y \in G$,

$$\begin{aligned}\phi(yx^{-1}) &= m_H(f_{yx^{-1}}) \\ &= \lim_\alpha \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} f(yx^{-1}t) d\lambda(t) \\ &= \lim_\alpha \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} \langle a(t^{-1}x), b(y) \rangle d\lambda(t) \\ &= \lim_\alpha \langle c_\alpha(x), b(y) \rangle\end{aligned}$$

where

$$c_\alpha(x) = \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} a(t^{-1}x) d\lambda(t).$$

Note that

$$\begin{aligned}\|c_\alpha(x)\| &\leq \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} \|a(t^{-1}x)\| d\lambda(t) \\ &\leq sup_{x \in G} \|a(x)\|,\end{aligned}$$

and that any space in $\mathcal{B}_p$ is reflexive [21, Proposition 7], the net $\{c_\alpha(x)\}$ has a weak limit, say $c(x)$, in B. Clearly, $\|c(x)\| \leq sup_{x\in G} \|a(x)\|$ and

$$\phi(yx^{-1}) = \langle c(x), b(y) \rangle$$

for all $x, y \in G$. So $\phi \in B_p(G)$. □

Let $A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$ be two matrices. The *Schur product* of A and B is the matrix

$$A * B = (a_{ij} b_{ij})_{n \times n}.$$

Let $\|A\|_{(p)} = \sup\{\|A * B\|_{p,p} : \|B\|_{p,p} \leq 1\}$. Recall that $\| \cdot \|_{p,p}$ is the norm of a linear operator on $l^p(Z_n)$.

We will use the following characterisation of $B_p(G)$ due to Fendler [13]:

Lemma. *A function ϕ on G is in $B_p(G)$ if and only if ϕ is continuous and there is a constant C such that for any finite set $\{x_1, \ldots, x_n\} \subseteq G$, $\|(\phi(x_i x_j^{-1}))_{n \times n}\|_{(p)} \leq C$.*

In order to prove the main result of this section, we need the next lemma.

Lemma 4.2. *Let H be an open subgroup of G. Extend $f \in C(H)$ to $f^\circ \in C(G)$ by setting $f^\circ(x) = 0$ if $x \in G \setminus H$. If $f \in B_p(H)$, then $f^\circ \in B_p(G)$.*

Proof. f° is clearly a continuous function on G. Since $f \in B_p(H)$, there exists a constant C such that for any finite set $\{t_1, \ldots, t_k\} \subseteq H$, $\|(f(t_j t_j^{-1}))_{k \times k}\|_{(p)} \leq C$.

Consider now a finite set $\{x_1, \ldots, x_n\} \subseteq G$ of cardinality n. Since the norm of a matrix remains the same after interchanging any two rows or any two columns, we may assume that $x_1, \ldots, x_l$ belong to a right coset of H and $x_{l+1}, \ldots, x_n$ belong to another right coset of H (the proof for the case of more cosets is similar).

Let $A = (a_{ij})$ be an $n \times n$ matrix with $\|A\|_{p,p} \leq 1$. Then

$$(f^\circ(x_i x_j^{-1}))_{n\times n} * A = \begin{pmatrix} T_1 * A_1 & 0 \\ 0 & T_2 * A_2 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdot & \cdots & \cdot \\ a_{l1} & \cdots & a_{ll} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{l+1,l+1} & \cdots & a_{l+1,n} \\ \cdot & \cdots & \cdot \\ a_{n,l+1} & \cdots & a_{nn} \end{pmatrix},$$

and

$$T_1 = \begin{pmatrix} f(x_1 x_1^{-1}) & \cdots & f(x_1 x_l^{-1}) \\ \cdot & \cdots & \cdot \\ f(x_l x_1^{-1}) & \cdots & f(x_l x_l^{-1}) \end{pmatrix}, \quad T_2 = \begin{pmatrix} f(x_{l+1} x_{l+1}^{-1}) & \cdots & f(x_{l+1} x_n^{-1}) \\ \cdot & \cdots & \cdot \\ f(x_n x_{l+1}^{-1}) & \cdots & f(x_n x_n^{-1}) \end{pmatrix}.$$

Note that

$$\|A_1\|_{p,p} \leq 1, \quad \|A_2\|_{p,p} \leq 1,$$

so

$$\|T_1 * A_1\|_{p,p} \leq C, \quad \|T_2 * A_2\|_{p,p} \leq C.$$

Since $(f^\circ(x_i x_j^{-1}))_{n\times n} * A$ is in diagonal form, we have the following equality:

$$\|(f^\circ(x_i x_j^{-1}))_{n\times n} * A\|_{p,p} = \max\{\|T_1 * A_1\|_{p,p}, \|T_2 * A_2\|_{p,p}\}.$$

Since A can be arbitrary, we conclude that

$$\|(f^\circ(x_i x_j^{-1}))_{n\times n}\|_{(p)} \leq C.$$

Hence, by the above lemma, $f^\circ \in B_p(G)$. □

Applying Proposition 4.1, Lemma 4.2 and Theorem 3.6, a proof similar to that of Theorem 4.5 of Chou [7] gives us

Theorem 4.3. *Let G be a noncompact nilpotent group or a noncompact [IN]-group. Then $W(G)/B_p(G)^-$ contains a linear copy of $l^\infty(\mathbb{N})$. In particular, $B_p(G)^-$ is a proper subset of $W(G)$.*

4. Free Groups

It is well known that for an amenable locally compact group G, and $1 < p < \infty$, $MA_p(G) = B_p(G)$, and in particular, $MA(G) = M_0A(G) = B(G)$. Losert [26] showed that $MA(G) = B(G)$ implies the amenability of G (the discrete case was due to Nebbia [27]), and for a discrete group G, Bożejko [3] showed that $M_0A(G) = B(G)$ implies the amenability of G. He also obtained in [2], [4] the following result: for a noncommutative free group F, $B_p(F) \subsetneq MA_p(F)$. The proof in [4] gives a function ϕ with $\phi \in MA(F)$ but $\phi \notin W(F)$, hence $\phi \notin B_p(F)$. Thus it is natural to ask whether $B_p(F) = MA_p(F) \cap W(F)$. In this section, we show that this is not the case.

Recall that a subset E of a discrete group G is a *Leinert set* if there is $C > 0$ such that for every $f \in l^2(E)$

$$\|f\|_{VN} = sup\{\|f * g\|_2 : \|g\|_2 = 1\} \leq C\|f\|_2,$$

or, equivalently,

$$\chi_E A(G) = l^2(E).$$

It was shown by Bożejko that E is a Leinert set if and only if $l^\infty(E) \subseteq MA(G)$, and if E is a Leinert set then $l^\infty(E) \subseteq MA_p(G)$, for $1 < p < \infty$; see [2]. Now we are ready for the main result of this section.

Theorem 5.1. *Let F be the free group on k generators with $k > 1$. Then $B_p(F)$ is a proper subset of $MA_p(F) \cap W(F)$.*

Proof. First, let us consider the case that $k = \infty$. Let $E = \{x_1, x_2, \cdots\}$ be the set of free generators of F. By the Haagerup convolution theorem [18] (see also [14]), we conclude that $E^2 = \{x_i x_j : i, j = 1, 2, \cdots\}$ is a Leinert set.

For an integer $k > 0$, define

$$T_k = \{x_i x_j \in E^2 : 2^{k-1} \leq i, j < 2^k\}$$

and set

$$T = \bigcup_{k=1}^{\infty} T_k.$$

Since T contains large squares, by Proposition 3.4 it is not a B_p-Sidon set. Therefore, by Corollary 3.3, we can find a function $\phi \in l^\infty(F)$ such that $supp\phi \subseteq T$, $\|\phi\|_\infty = 1$ and $\|\phi - u|_T\|_{l^\infty(T)} \geq 1$ for every $u \in B_p(F)$. In particular, $\phi \notin B_p(F)$. We claim that $\phi \in MA_p(F) \cap W(F)$.

$\phi \in MA_p(F)$, since $supp\phi \subseteq T$ and T is a Leinert set, being a subset of the Leinert set E^2.

To show that $\phi \in W(F)$, it suffices to show that T is a t-set. Indeed, let $x \in F \setminus \{e\}$ and $x = x_{i_1}^{u_1} \cdots x_{i_n}^{u_n}$ be the reduced form, where $u_i = \pm 1, i = 1, \cdots, n$. If $y \in T \cap xT$, then

$$y = x_i x_j \in T_k$$

for some positive integer k, and

$$y = x_{i_1}^{u_1} \cdots x_{i_n}^{u_n} x_u x_v$$

for some $x_u x_v \in T$.

Comparing the two forms of y, and noticing that $x \neq e$, we get $i = i_1$. Moreover, x can take the forms $x_{i_1}^{u_1} x_{i_2}^{u_2} x_{i_3}^{u_3} x_{i_4}^{u_4}$ and $x_{i_1}^{u_1} x_{i_2}^{u_2}$. In the first case, we have at most one choice of y, namely $y = x_{i_1} x_{i_2}$, provided $u_1 = u_2 = 1$ and $x_{i_3}^{u_3} x_{i_4}^{u_4} x_u x_v = e$. In the second case, let k be $[log_2 i_1] + 1$; then we have at most 2^{k-1} choices of y, namely $y = x_{i_1} x_j$ with $j = 2^{k-1}, 2^{k-1} + 1, ..., 2^k - 1$, provided $u_2 = -1, i_2 = u$. So

$$|T \cap xT| \leq i_1 + 1 < \infty.$$

Similarly , $|T \cap Tx| < \infty$.

Now let F be the free group on k generators with $k > 1$. We can find a subgroup H of F with $H \cong F_\infty$. Therefore, there exists a function $\phi \in W(H) \cap MA_p(H)$, but $\phi \notin B_p(H)$, as in the proof above.

Let us extend ϕ to a function ϕ^0 on F by setting $\phi^0(x) = 0$ for $x \notin H$. Using the definition of Herz-Schur multipliers, we can check that $\phi^0 \notin B_p(F)$. Also, by Lemma 4.1 of [7], $\phi^0 \in W(F)$. Notice that our ϕ has support in a Leinert set E of H. To show $\phi \in MA_p(F)$, it suffices to show that E is a Leinert set of F. By a theorem of Herz [22], $u|_H \in A(H)$ whenever $u \in A(F)$. Let $u \in A(F)$ since E is a

Leinert set of H, $(\chi_E u)|_H \in l^2(H)$. Hence $\chi_E u \in l^2(F)$, which shows that E is a Leinert set of F.

So, $\phi^0 \in MA_p(F) \cap W(F) \setminus B_p(F)$, and the proof is complete. □

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Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214-3093

Current address: Department of Mathematical Sciences, University of Alberts, Edmonton, AB, T6G 2G1, Canada

E-mail address: `gxu@vega.math.ualberta.ca`