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DIFFERENTIAL OPERATORS ON STANLEY-REISNER RINGS

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Abstract. Let k be an algebraically closed field of characteristic zero, and let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Suppose that I is an ideal in R that may be generated by monomials.

We investigate the ring of differential operators $\mathcal{D}(R/I)$ on the ring R/I, and $\mathcal{I}_R(I)$, the idealiser of I in R. We show that $\mathcal{D}(R/I)$ and $\mathcal{I}_R(I)$ are always right Noetherian rings. If I is a square-free monomial ideal then we also identify all the two-sided ideals of $\mathcal{I}_R(I)$.

To each simplicial complex Δ on $V = \{v_1, \dots, v_n\}$ there is a corresponding square-free monomial ideal I_Δ, and the Stanley-Reisner ring associated to Δ is defined to be $k[\Delta] = R/I_\Delta$. We find necessary and sufficient conditions on Δ for $\mathcal{D}(k[\Delta])$ to be left Noetherian.

1. Introduction

Let k be an algebraically closed field of characteristic zero. If S is a commutative k-algebra, then $\mathcal{D}(S)$ denotes the ring of differential operators on S. Suppose that X is an affine variety over k with coordinate ring S. It is well known that if X is non-singular and irreducible then $\mathcal{D}(S)$ has a nice structure, for example, it is a Noetherian domain and it is simple. The problem now is to understand the properties of $\mathcal{D}(S)$ when X is singular. There are several interesting results, for example, in [2], [9] and [14, Section 7].

Now let $R = k[x_1, \dots, x_n]$ denote the polynomial ring in n variables over k. We investigate the case $S = R/I$ for I an ideal of R generated by monomials. This is inspired by Example 7.2 of [8] and [11, Theorem 2] and attempts to generalise them.

In the special case that I may be generated by square-free monomials R/I is called a Stanley-Reisner ring. There is a corresponding simplicial complex Δ, and R/I is usually written $k[\Delta]$. It is particularly satisfying to study $\mathcal{D}(k[\Delta])$ because it is possible to identify the elements of $\mathcal{D}(k[\Delta])$ computationally using the combinatorics of Δ.

We approach the ring of differential operators $\mathcal{D}(R/I)$ on R/I via the idealiser $\mathcal{I}_R(I)$:

$$\mathcal{I}_R(I) = \{\theta \in \mathcal{D}(R) : \theta \bullet I \subseteq I\} \qquad \text{and} \qquad \mathcal{D}(R/I) \cong \mathcal{I}_R(I)/I\mathcal{D}(R).$$

We show the following:

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Theorem A. *Suppose that I is an ideal in R generated by a finite set of monomials $\mathcal{M}$. Then:*

1. $\bigcap_{\mathbf{x}^{\mathbf{a}} \in \mathcal{M}} \mathcal{I}_R(\mathbf{x}^{\mathbf{a}} R)$ *is Noetherian;*
2. $\mathcal{I}_R(I)$ *and* $\mathcal{D}(R/I)$ *are right Noetherian;*
3. $\mathcal{I}_R(I)$ *is left Noetherian if and only if I is principal.*

When I is a square-free monomial ideal in R, then we can identify all the ideals in $\mathcal{I}_R(I)$ and characterise those square-free monomial ideals I for which $\mathcal{D}(R/I)$ is left Noetherian:

Theorem B. *Suppose that I is a square-free monomial ideal. Then:*

1. *each ideal of $\mathcal{I}_R(I)$ is a sum of ideals of $\mathcal{I}_R(I)$ of the form*
$$(I_{i_1} \cap \cdots \cap I_{i_r})\mathcal{D}(R) \cap \mathcal{I}_R(I),$$
where $I = I_1 \cap \cdots \cap I_s$ is a certain prime decomposition of I;
2. $\mathcal{D}(R/I)$ *is left Noetherian if and only if*
$$\mathcal{I}_R(I) = \mathcal{I}_R(\mathbf{x}^{\mathbf{d}} R) + I\mathcal{D}(R),$$
where $\mathbf{x}^{\mathbf{d}}$ is the lowest common multiple of $\mathcal{M}$, the finite set of monomials that generate I.

Now let $V := \{v_1, \ldots, v_n\}$ be a vertex set. There is a correspondence between (slack) simplicial complexes on V and square-free monomial ideals of R. If I_Δ is a square-free monomial ideal of R corresponding to the simplicial complex Δ, then $k[\Delta] := R/I_\Delta$ is the Stanley-Reisner ring associated to Δ. This correspondence provides a link between commutative algebra and the combinatorics of simplicial complexes. See, for example, the original paper of G. A. Reisner [10], where the condition that $k[\Delta]$ be Cohen-Macaulay is translated into a condition on Δ.

We characterise the simplicial complexes Δ on V such that $\mathcal{D}(k[\Delta])$ is Noetherian:

Theorem C. *Suppose that Δ is a simplicial complex on V with associated Stanley-Reisner ring $k[\Delta]$. Then $\mathcal{D}(k[\Delta])$ is left Noetherian if and only if* $\operatorname{Core}\Delta$ *is a T-space.*

The paper is organised as follows. Section 2 contains notation and some well-known results on idealisers and rings of differential operators. In Section 3 we prove the key result which is used in Theorem A:

Proposition. *If S is a right Noetherian k-algebra, and $\mathcal{M}$ is a finite set of positive integers, then*
$$S \otimes \bigcap_{m \in \mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$$
is right Noetherian.

This result is applied to idealisers of monomial ideals, and the corresponding rings of differential operators, where we show Theorem A. In Section 4 we extend these ideas to demonstrate Theorem B.

In Section 5 we first recall the definition of a simplicial complex, and the correspondence between simplicial complexes and square-free monomial ideals. We define a T-space to be a simplicial complex satisfying a certain separatedness property. If Δ is a simplicial complex on V with $k[\Delta]$ the associated Stanley-Reisner ring, then we relate the existence of certain monomial differential operators in $\mathcal{D}(k[\Delta])$ to combinatorial properties of Δ. Using this, we prove Theorem C.

The paper concludes with some examples in Section 6 to demonstrate that there is no apparent connection between $\mathcal{D}(k[\Delta])$ Noetherian and whether $k[\Delta]$ is Cohen-Macaulay.

2. Background on Differential Operators

All tensor products are taken over k, and in $R \otimes S$ we will always identify R with $R \otimes 1$, and S with $1 \otimes S$.

2.1 Idealisers and Rings of Differential Operators. Let S be a commutative k-algebra. The ring of differential operators on S is defined by

$$\mathcal{D}(S) = \bigcup_{i=0}^{\infty} \mathcal{D}^i(S),$$

where $\mathcal{D}^0(S) = S$ and $\mathcal{D}^i(S) = \{\theta \in \operatorname{End}_k(S) : \theta s - s\theta \in \mathcal{D}^{i-1}(S),\ \forall\, s \in S\}$. If $\theta \in \mathcal{D}(S)$ and $f \in S$, then write $\theta \bullet f \in S$ for the value of the operator θ evaluated at f.

Suppose I and J are two ideals of S; then we may define the *relative idealiser*:

$$\mathcal{I}_S(I; J) := \{\theta \in \mathcal{D}(S) : \theta \bullet I \subseteq J\}.$$

For another ideal K of S, composition then gives a map $\mathcal{I}_S(J; K) \times \mathcal{I}_S(I; J) \to \mathcal{I}_S(I; K)$. For I an ideal in S we define the *idealiser*:

$$\mathcal{I}_S(I) := \mathcal{I}_S(I; I) = \{\theta \in \mathcal{D}(S) : \theta \bullet I \subseteq I\}.$$

It follows then that $\mathcal{I}_S(I; J)$ is a $\mathcal{I}_S(J) - \mathcal{I}_S(I)$-bimodule, and the idealiser $\mathcal{I}_S(I)$ is a subalgebra of $\mathcal{D}(S)$. Note that $I\mathcal{D}(S)$ is a two-sided ideal of $\mathcal{I}_S(I)$ and there is a well-known isomorphism:

$$\mathcal{D}(S/I) \cong \mathcal{I}_S(I)/I\mathcal{D}(S);$$

for a proof see [14, Proposition 1.6], [8, Lemma 1.4], or [6, 15.5.6].

Throughout this paper R is a polynomial ring over k, with ring of differential operators $\mathcal{D}(R)$:

$$R = k[x_1, \ldots, x_n] \qquad \text{and} \qquad \mathcal{D}(R) = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n],$$

where, as usual, ∂_i denotes $\partial/\partial x_i$. We will use multi-index notation for the monomial elements of R and $\mathcal{D}(R)$: if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, then

$$\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \ldots x_n^{a_n} \qquad \text{and} \qquad \partial^{\mathbf{b}} := \partial_1^{b_1} \ldots \partial_n^{b_n}.$$

If I is a monomial ideal, then it may be generated irredundantly by a unique (finite) set of monomials. This set will be denoted by $\mathcal{M}$. The lowest common multiple of $\mathcal{M}$ will be denoted $\mathbf{x}^{\mathbf{d}}$. This is to be understood in the context of the polynomial ring R, that is, $d_i = \max\{a_i : \mathbf{x}^{\mathbf{a}} \in \mathcal{M}\}$ for each i.

2.2 Definition: Supports. Let $I \subseteq J$ be two ideals of R. The *preimage in R of the submodule of elements of R/I with support in J/I* is

$$\Gamma_{J/I} := \{a \in R \,:\, aJ^n \subseteq I \text{ for some } n > 0\}.$$

The following proposition is the idealiser version of the similar well-known result for differential operators. The proof is almost exactly the same.

Proposition "Idealisers preserve supports".

$$\mathcal{I}_R(I) \bullet \Gamma_{J/I} \subseteq \Gamma_{J/I},$$

for ideals I and J of R, $I \subseteq J$.

2.3 Morita Equivalences. We give now a Morita equivalence between idealisers of monomial ideals. This will not be used in this generality in the paper, and is only included for completeness.

Definition. We define an equivalence relation on the set of monomial ideals as follows: if I and I' are two monomial ideals generated irredundantly by finite sets of monomials $\mathcal{M}$ resp. $\mathcal{M}'$, then $I \sim I'$ if there exists a bijection denoted $\mathbf{x}^{\mathbf{a}} \mapsto \mathbf{x}^{\mathbf{a}'}$, and integers $d_1, \ldots, d_n$, such that for each $\mathbf{x}^{\mathbf{a}} \in \mathcal{M}$ we have $a_i > 0$ if and only if $a'_i > 0$, and further if this happens then $a'_i = a_i + d_i$.

2.4 Proposition. *Suppose that I and I' are two monomial ideals in R, and that they are generated by $\mathcal{M}$ and $\mathcal{M}'$, two finite sets of monomials, such that $I \sim I'$. Then*

$$\begin{aligned} &\mathcal{I}_R(I) \quad \textit{is Morita equivalent to} \quad \mathcal{I}_R(I') \quad \textit{and} \\ &\mathcal{D}(R/I) \quad \textit{is Morita equivalent to} \quad \mathcal{D}(R/I'). \end{aligned}$$

Proof. The result follows from the following lemma by induction. □

Lemma. *Let I be a monomial ideal in R with finite set of monomial generators $\mathcal{M}$. Let I' be the monomial ideal generated by $\mathcal{M}' = \{\mathbf{x}^{\mathbf{a}'} : \mathbf{x}^{\mathbf{a}} \in \mathcal{M}\}$, where*

$$\mathbf{x}^{\mathbf{a}'} = \begin{cases} \mathbf{x}^{\mathbf{a}}, & \textit{if } a_i = 0, \\ \mathbf{x}^{\mathbf{a}} x_i, & \textit{if } a_i > 0. \end{cases}$$

Then $\mathcal{I}_R(I)$ is Morita equivalent to $\mathcal{I}_R(I')$ and $\mathcal{D}(R/I)$ is Morita equivalent to $\mathcal{D}(R/I')$.

Proof. The result follows if we can show:

$$\mathcal{I}_R(I';I)\mathcal{I}_R(I;I') = \mathcal{I}_R(I) \qquad \text{and} \qquad \mathcal{I}_R(I;I')\mathcal{I}_R(I';I) = \mathcal{I}_R(I').$$

If $a_i = 0$ for all $\mathbf{x}^{\mathbf{a}} \in \mathcal{M}$, then $\mathcal{M} = \mathcal{M}'$ and there is nothing further to show. So assume that $a_i \neq 0$ for some $\mathbf{x}^{\mathbf{a}} \in \mathcal{M}$. Let $a = \min(a_i \neq 0 : \mathbf{x}^{\mathbf{a}} \in \mathcal{M})$, $b = \max(a_i : \mathbf{x}^{\mathbf{a}} \in \mathcal{M})$. By assumption $a > 0$. For convenience we drop the subscript i, and write $x = x_i$, $\partial = \partial_i$.

We will need the following elements of these relative idealisers:

$$\begin{aligned} x, \quad \text{and} \quad (x\partial - a)(x\partial - (a+1)) \ldots (x\partial - b) &\in \mathcal{I}_R(I;I') \\ 1, \partial, \partial(x\partial), \ldots, \partial(x\partial)^{b-a} &\in \mathcal{I}_R(I';I) \end{aligned}$$

So we see that $x\partial$, $(x\partial)^2$, …, $(x\partial)^{b-a+1}$ and $(x\partial - a)(x\partial - (a+1)) \ldots (x\partial - b) \in \mathcal{I}_R(I;I')\mathcal{I}_R(I';I)$. Expanding out this last expression as a polynomial in $x\partial$ and cancelling, we see that $1 \in \mathcal{I}_R(I;I')\mathcal{I}_R(I';I)$.

Similarly, by using $\partial x = x\partial + 1$ we see that

$$\begin{aligned} x, \quad \text{and} \quad (\partial x - (a+1))(\partial x - (a+2)) \ldots (\partial x - (b+1)) &\in \mathcal{I}_R(I;I'), \\ 1, \partial, (\partial x)\partial, \ldots, (\partial x)^{b-a}\partial &\in \mathcal{I}_R(I';I). \end{aligned}$$

So ∂x, $(\partial x)^2$, $(\partial x)^3$, ..., $(\partial x)^{b-a+1}$ and $(\partial x-(a+1))(\partial x-(a+2))\dots(\partial x-(b+1)) \in \mathcal{I}_R(I';I)\mathcal{I}_R(I;I')$. Similarly $1 \in \mathcal{I}_R(I';I)\mathcal{I}_R(I;I')$.

Now $\mathcal{D}(R)$ is known to be simple, and hence if K is any non-empty subset of $\mathcal{D}(R)$, then $\mathcal{D}(R)K\mathcal{D}(R) = \mathcal{D}(R)$, for otherwise it would be a proper non-zero ideal of $\mathcal{D}(R)$. Thus it follows that $I\mathcal{D}(R)$ is the unique minimal ideal contained in $\mathcal{I}_R(I)$, for if J is any non-zero ideal of $\mathcal{I}_R(I)$, then $I\mathcal{D}(R) = I\mathcal{D}(R)JI\mathcal{D}(R) \subseteq J$.

Since $\mathcal{D}(R/I)$ and $\mathcal{D}(R/I')$ are the factors of $\mathcal{I}_R(I)$ and $\mathcal{I}_R(I')$ by their unique minimal ideals, they are Morita equivalent by [6, 3.5.9 (ii)]. □

Example. This demonstrates, for example, that $\mathcal{I}_R((x^4zw, yz^2w, x^2y^3)R)$ is Morita equivalent to $\mathcal{I}_R((x^3z^4w, yz^5w, xy^3)R)$, where $R = k[x, y, z, w]$.

For reference we single out the special case that is required:

2.5 Corollary. *Suppose that S is a k-algebra and that m is a non-zero positive integer. Then $S \otimes \mathcal{I}_{k[x]}(xk[x])$ is Morita equivalent to $S \otimes \mathcal{I}_{k[x]}(x^m k[x])$.*

Proof. It follows from Proposition 2.4 that $\mathcal{I}_{k[x]}(xk[x])$ is Morita equivalent to $\mathcal{I}_{k[x]}(x^m k[x])$. Thus the result follows, since Morita equivalence extends over tensor products. □

3. Idealisers of Monomials

First we look at idealisers in $\mathcal{D}(k[x])$, where the following is well known:

3.1 Lemma. *Let m be a positive integer, then*

$$\begin{aligned}\mathcal{I}_{k[x]}(x^m k[x]) &= \mathcal{D}(k[x]) \cap x^m \mathcal{D}(k[x]) x^{-m} \\ &= \sum_{0\le i,j\le m-1} f_{ij} x^i \partial^j + x^m \mathcal{D}(k[x])\end{aligned}$$

where

$$f_{ij} = \begin{cases} k(x\partial - (m-j+i))\dots(x\partial-(m-1)), & \text{if } i<j, \\ k, & \text{otherwise.}\end{cases}$$

If $\mathcal{M}$ is a finite set of positive integers, then

$$\bigcap_{m\in\mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x]) = \mathcal{D}(k[x]) \cap \bigcap_{m\in\mathcal{M}} x^m \mathcal{D}(k[x]) x^{-m}.$$

There is a similar expression for $\bigcap_{m\in\mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$, as a sum of $f_{ij}x^i\partial^j$, but it is involved and we don't need it. It is enough to note that if $n = \max(m \in \mathcal{M})$, then $\bigcap_{m\in\mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$ has finite vector space codimension in $\mathcal{I}_{k[x]}(x^n k[x])$.

Now we show that we can build up arbitrary intersections of monomials from idealisers of the form $\bigcap_{m\in\mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$.

3.2 Proposition. *For $\mathbf{x^a} \in R$*

$$\mathcal{I}_R(\mathbf{x^a}R) = \big(\mathcal{I}_{k[x_1]}(x_1^{a_1}k[x_1])\big), \otimes\dots\otimes \big(\mathcal{I}_{k[x_n]}(x_n^{a_n}k[x_n])\big),$$

and if $\mathcal{M} \subset R$ is a finite set of monomials, then

$$\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R) = \big(\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_{k[x_1]}(x_1^{a_1}k[x_1])\big) \otimes\dots\otimes \big(\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_{k[x_n]}(x_n^{a_n}k[x_n])\big).$$

Proof. By inspection, $\mathcal{I}_R(x_1^{a_1}R) \cap \cdots \cap \mathcal{I}_R(x_n^{a_n}R) \subseteq \mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R)$. Now $\partial_i^{a_i} \bullet \mathbf{x}^{\mathbf{a}}R = (\mathbf{x}^{\mathbf{a}}/x_i^{a_i})R$ is an ideal of R containing $\mathbf{x}^{\mathbf{a}}R$, and notice that

$$\Gamma_{(\partial_i^{a_i} \bullet \mathbf{x}^{\mathbf{a}}R)/\mathbf{x}^{\mathbf{a}}R} = x_i^{a_i}R,$$

so, since idealisers preserve supports, $\mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R) \subseteq \mathcal{I}_R(x_i^{a_i}R)$, for each i, $1 \leq i \leq n$. Hence

$$\mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R) = \mathcal{I}_R(x_1^{a_1}R) \cap \cdots \cap \mathcal{I}_R(x_n^{a_n}R).$$

Recall the following identity for two vector spaces C and T and subspaces A and B of C and R and S of T:

$$(A \otimes R) \cap (B \otimes S) = (A \cap B) \otimes (R \cap S). \tag{3.2.1}$$

Again by inspection $\mathcal{I}_R(x_i^{a_i}R) = \mathcal{I}_{k[x_i]}(x_i^{a_i}k[x_i]) \otimes \mathcal{D}(k[x_j : j \neq i])$, for each i, $1 \leq i \leq n$. So by 3.2.1,

$$\mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R) = \mathcal{I}_{k[x_1]}(x_1^{a_1}k[x_1]) \otimes \cdots \otimes \mathcal{I}_{k[x_n]}(x_n^{a_n}k[x_n]).$$

The second part follows from the first by applying 3.2.1 again. □

Now we need some results on extending right Noetherian rings over tensor products. For similiar results see also [12, 2.3(i)] and [13, Proposition 1.a].

3.3 Proposition. *Let S be a right Noetherian k-algebra. Then $S \otimes \mathcal{I}_{k[x]}(xk[x])$ is right Noetherian.*

Proof. For short let $\mathcal{I} := \mathcal{I}_{k[x]}(xk[x])$. Now

$$S \otimes \mathcal{D}(k[x]) = S \otimes (\mathcal{I} + \sum_{j \in \mathbb{N}^+} k\partial^j) = S \otimes \mathcal{I} + \sum_{j \in \mathbb{N}^+} S\partial^j$$

is an iterated Ore extension of S, so is also right Noetherian. Let I be a right ideal of $S \otimes \mathcal{I}$. Since $S \otimes \mathcal{D}(k[x])$ is right Noetherian there exists a finite set of generators Ψ in I for $I(S \otimes \mathcal{D}(k[x]))$. If $I = \Psi(S \otimes \mathcal{I})$ then the ideal is finitely generated so there is nothing further to prove; so assume that $\Psi(S \otimes \mathcal{I}) \subsetneqq I$. If we can show that $I = \Psi(S \otimes \mathcal{I}) + (\Psi S\partial \cap I)\mathcal{I}$, then since $\Psi S\partial \cap I$ is a right Noetherian S-module, this shows I is finitely generated. Thus the result follows from the next lemma. □

Lemma. *Let β be an element of $I \setminus \Psi(S \otimes \mathcal{I})$, say $\beta = \beta_0 + \sum_{i=1}^r \Gamma_i \partial^i$, where $\beta_0 \in \Psi(S \otimes \mathcal{I})$ and $\Gamma_i \in \Psi S$. Then*

1. *each $\Gamma_i\partial \in \Psi S\partial \cap I$, $i = 1, \ldots, r$, and furthermore*
2. *$\beta \in \Psi(S \otimes \mathcal{I}) + \sum_{i=1}^r \Gamma_i \partial \mathcal{I}$.*

Proof. Note first of all that for any non-negative integer N,

$$x\partial^N \in \mathcal{I}, \text{ and } \Gamma_i x \partial^N \in \Psi(S \otimes \mathcal{I}).$$

1. First we show $\Gamma_r \partial \in \Psi S\partial \cap I$: Now

$$\sum_{i=1}^r i\Gamma_i\partial^{i-1} = \sum_{i=1}^r \Gamma_i(\partial^i x - x\partial^i) = \sum_{i=1}^r -\Gamma_i x\partial^i + (\sum_{i=1}^r \Gamma_i\partial^i)x \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}.$$

But the first term of this summand is $\Gamma_1 \in \Psi(S \otimes \mathcal{I})$, so

$$\sum_{i=2}^{r} i\Gamma_i \partial^{i-1} \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}.$$

Apply the same procedure so this to show that:

$$\sum_{i=3}^{r} i(i+1)\Gamma_i \partial^{i-2} \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}$$

$$\dots$$

$$r!\Gamma_r \partial \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}$$

Using this we see that

$$\Gamma_r \partial^r = -\Gamma_r x \partial^{r+1} + \Gamma_r \partial x \partial^r \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}.$$

Subtract this from β to get:

$$\beta' := \sum_{i=1}^{r-1} \Gamma_i \partial^i \in \Psi(S \otimes \mathcal{I}) + \beta\mathcal{I}.$$

Now apply the above procedure to β' a further $r-1$ times to complete the proof.

2. It remains to show that $\beta \in \Psi(S \otimes \mathcal{I}) + \sum_{i=1}^{r} \Gamma_i \partial \mathcal{I}$. This follows since we have already shown above that $\Gamma_i \partial^i \in \Psi(S \otimes \mathcal{I}) + \Gamma_i \partial \mathcal{I}$.

□

3.4 Lemma. *Let $A \subseteq B$ be k-algebras with B/A a finite dimensional k-vector space. Let S be any right Noetherian k-algebra such that $S \otimes B$ is right Noetherian. Then $S \otimes A$ is right Noetherian.*

Proof. We will identify $S \otimes A$ with a subring of $S \otimes B$. Let $\mathcal{V}$ be a finite set of generators for B/A.

Let I be any right ideal of $S \otimes A$. Take a finite set of generators in I, say Ψ, for $I(S \otimes B)$ as a right ideal of $S \otimes B$. Now $S \otimes B = S \otimes A + \mathcal{V}S$, so $I(S \otimes B) = \Psi(S \otimes B) = \Psi(S \otimes A) + \Psi\mathcal{V}S$ and hence $I = \Psi(S \otimes A) + I \cap \Psi\mathcal{V}S$. But $I \cap \Psi\mathcal{V}S$ is an S-submodule of $\Psi\mathcal{V}S$, which is a Noetherian S-module. So I is finitely generated, as required. □

3.5 Proposition. *Let S be a right Noetherian k-algebra, and $\mathcal{M}$ a finite set of positive integers. Then $S \otimes \bigcap_{m \in \mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$ is a right Noetherian ring.*

Proof. Suppose that $\mathrm{Card}(\mathcal{M}) = 1$, and let m be the unique element of $\mathcal{M}$. By Proposition 3.3, $S \otimes \mathcal{I}_{k[x]}(xk[x])$ is right Noetherian, and by Corollary 2.5, this is Morita equivalent to $S \otimes \mathcal{I}_{k[x]}(x^m k[x])$; hence this is also right Noetherian.

Now suppose that $\mathrm{Card}(\mathcal{M}) > 1$, and let $n = \max\{m \in \mathcal{M}\}$. By the above $S \otimes \mathcal{I}_{k[x]}(x^n k[x])$ is right Noetherian, and by Lemma 3.1, $\mathcal{I}_{k[x]}(x^n k[x])$ is an extension of $\bigcap_{m \in \mathcal{M}} \mathcal{I}_{k[x]}(x^m k[x])$ by a finite dimensional k-vector space hence the result follows from Lemma 3.4. □

3.6 Theorem. *Suppose that I is a monomial ideal, generated by a finite set of monomials, $\mathcal{M}$. Then*

1. $\bigcap_{\mathbf{x}^{\mathbf{a}} \in \mathcal{M}} \mathcal{I}_R(\mathbf{x}^{\mathbf{a}} R)$,

2. *$\mathcal{I}_R(I)$ and*
3. *$\mathcal{D}(R/I)$*

are all right Noetherian.

Proof. For short let $\mathcal{C} := \bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R)$ and note that $\mathcal{C} \subset \mathcal{I}_R(I)$.

1. That $\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R)$ is right Noetherian follows from Proposition 3.2 and Proposition 3.5 by induction.
2. Let $\mathbf{x^d}$ denote the lowest common multiple of $\mathcal{M}$. We know by 1 that $\mathcal{C}$ is right Noetherian. Now $\mathbf{x^d}\mathcal{D}(R)$ is a right ideal of $\mathcal{C}$ which is isomorphic to $\mathcal{D}(R)$ as a right $\mathcal{C}$ module. So $\mathcal{I}_R(I)$ is a subring of a Noetherian $\mathcal{C}$-module, and since $\mathcal{C}$ is contained in $\mathcal{I}_R(I)$, it follows that $\mathcal{I}_R(I)$ is right Noetherian.
3. Since $\mathcal{D}(R/I)$ is a quotient of $\mathcal{I}_R(I)$, this follows from 2.

□

3.7 Proposition. *Let $\mathcal{M}$ be a finite set of monomials in R. Then $\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R)$ is Noetherian.*

Proof. Let $\mathbf{x^d}$ be the lowest common multiple of $\mathcal{M}$. For each i, $1 \le i \le n$, let

$$\mathcal{I}_i := \bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_{k[x_i]}(x_i^{a_i}k[x_i]) = \mathcal{D}(k[x_i]) \cap \bigcap_{\mathbf{x^a}\in\mathcal{M}} x_i^{a_i}\mathcal{D}(k[x_i])x_i^{-a_i}.$$

Let θ_i be the anti-automorphism of $\mathcal{I}_{k[x_i]}(x_i^{d_i}k[x_i])$ given by $\alpha_i \circ \beta_i$, where β_i is the anti-automorphism of $\mathcal{D}(k[x_i])$ given on the generators by $x_i \mapsto x_i$, $\partial_i \mapsto -\partial_i$, and α_i is given by conjugation with $x_i^{d_i}$: $b \mapsto x_i^{d_i}bx_i^{-d_i}$. This is an anti-automorphism of $\mathcal{I}_{k[x_i]}(x_i^{d_i}k[x_i])$, since

$$\begin{aligned}\theta(\mathcal{I}_{k[x_i]}(x_i^{d_i}k[x_i])) &= \theta(\mathcal{D}(k[x_i]) \cap x_i^{d_i}\mathcal{D}(k[x_i])x_i^{-d_i}) \\ &= x_i^{d_i}(\mathcal{D}(k[x_i]) \cap x_i^{-d_i}\mathcal{D}(k[x_i])x_i^{d_i})x_i^{-d_i} \\ &= x_i^{d_i}\mathcal{D}(k[x_i])x_i^{-d_i} \cap \mathcal{D}(k[x_i])\end{aligned}$$

as required. So

$$\begin{aligned}\theta_i(\mathcal{I}_i) &= x_i^{d_i}\big(\mathcal{D}(k[x_i]) \cap \bigcap_{\mathbf{x^a}\in\mathcal{M}} x_i^{-a_i}\mathcal{D}(k[x_i])x_i^{a_i}\big)x_i^{-d_i} \\ &= x_i^{d_i}\mathcal{D}(k[x_i])x_i^{-d_i} \cap \bigcap_{\mathbf{x^a}\in\mathcal{M}} x_i^{d_i-a_i}\mathcal{D}(k[x_i])x_i^{a_i-d_i} \\ &= \mathcal{D}(k[x_i]) \cap \bigcap_{n_j\in\mathcal{N}_i} x_i^{n_j}\mathcal{D}(k[x_i])x_i^{-n_j}, \\ &= \bigcap_{n_j\in\mathcal{N}_i} \mathcal{I}_{k[x_i]}(x_i^{n_j}k[x_i]),\end{aligned}$$

where $\mathcal{N}_i = \{d_i - a_i : \mathbf{x^a} \in \mathcal{M} : a_i \ne d_i\} \cup \{d_i\}$.

We know by Proposition 3.2 that $\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R) = \mathcal{I}_1 \otimes \cdots \otimes \mathcal{I}_n$, so $\theta_1 \otimes \cdots \otimes \theta_n$ is an anti-isomorphism between $\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a})$ and $\bigcap_{\mathbf{x^b}\in\mathcal{N}} \mathcal{I}_R(\mathbf{x^b}R)$, where $\mathcal{N}$ is any set of monomials such that $\mathcal{N}_i = \{b_i : \mathbf{x^b} \in \mathcal{N}\}$.

By Theorem 3.6, $\bigcap_{\mathbf{x^b}\in\mathcal{N}} \mathcal{I}_R(\mathbf{x^b}R)$ is right Noetherian; hence $\bigcap_{\mathbf{x^a}\in\mathcal{M}} \mathcal{I}_R(\mathbf{x^a}R)$ is left Noetherian. □

3.8 Theorem. *Suppose that I is a monomial ideal. Then $\mathcal{I}_R(I)$ is left Noetherian if and only if I is principal.*

Proof. If I is principal, then this follows directly from Proposition 3.7, so suppose that I is not principal. Let $\mathcal{M}$ denote the irredundant set of monomials which generate I, and let $\mathbf{x}^{\mathbf{d}}$ denote the lowest common multiple of $\mathcal{M}$. Let $\mathbf{x}^{\mathbf{a}} \in I$ be an element and i an index such that $a_i < d_i$. Then $\mathbf{x}^{\mathbf{a}}\partial_i^{d_i} \in I\mathcal{D}(R) \subset \mathcal{I}_R(I)$. The result follows from the following two lemmas, taking $J = 0$. □

3.9 Lemma. *Suppose that I is a monomial ideal, with finite set of monomial generators $\mathcal{M}$, and $\mathbf{x}^{\mathbf{d}}$ the lowest common multiple of $\mathcal{M}$. Suppose also that $\mathbf{x}^{\mathbf{a}}\partial_i^{d_i} \in \mathcal{I}_R(I)$. Then $\mathbf{x}^{\mathbf{a}}\partial_i^{N} \in \mathcal{I}_R(I)$ for all positive integers N.*

Proof. Firstly, if $N \leq d_i$ then this follows by repeatedly commutating $\mathbf{x}^{\mathbf{a}}\partial_i^{d_i}$ by x_i. So assume that $N > d_i$. Since I is a monomial ideal, to show that $\mathbf{x}^{\mathbf{a}}\partial_i^{N} \in \mathcal{I}_R(I)$ it is enough to show that $\mathbf{x}^{\mathbf{a}}\partial_i^{N} \bullet \mathbf{x}^{\mathbf{b}} \in I$ for all $\mathbf{x}^{\mathbf{b}} \in I$. Consider an $\mathbf{x}^{\mathbf{b}} \in I$. For $N > b_i$ we have $\mathbf{x}^{\mathbf{a}}\partial_i^{N} \bullet \mathbf{x}^{\mathbf{b}} = 0 \in I$, so there is nothing further to prove; so assume that $b_i \geq N > d_i$. Write $\mathbf{x}^{\mathbf{b}} = x_i^{b_i}\mathbf{x}^{\mathbf{b}'}$; then by definition of d_i, we have $x_i^{d_i}\mathbf{x}^{\mathbf{b}'} \in I$. By assumption $\mathbf{x}^{\mathbf{a}}\partial_i^{d_i} \bullet x_i^{d_i}\mathbf{x}^{\mathbf{b}'} \in I$; but this is $d_i!\mathbf{x}^{\mathbf{a}+\mathbf{b}'}$. So $\mathbf{x}^{\mathbf{a}}\partial_i^{N} \bullet \mathbf{x}^{\mathbf{b}} = \frac{b_i!}{N!}x_i^{b_i-N}\mathbf{x}^{\mathbf{a}+\mathbf{b}'} \in I$, as required. □

The next lemma is a generalisation of [11, Theorem 2].

3.10 Lemma. *Let I be a monomial ideal, with finite set of monomial generators $\mathcal{M}$, and let $\mathbf{x}^{\mathbf{d}}$ denote the lowest common multiple of $\mathcal{M}$. Let J be a completely prime ideal in $\mathcal{I}_R(I)$, and suppose that there exists a monomial $\mathbf{x}^{\mathbf{a}}$ such that $a_i < d_i$ and $\mathbf{x}^{\mathbf{a}}\partial_i^{d_i} \in \mathcal{I}_R(I)\backslash J$. Then $\mathcal{I}_R(I)/J$ is not left Noetherian.*

Proof. Suppose that $\mathcal{I}_R(I)/J$ is left Noetherian. By Lemma 3.9, each $\mathbf{x}^{\mathbf{a}}\partial_i^{k} \in \mathcal{I}_R(I)$, so

$$\sum_{k=1}^{\infty} \mathcal{I}_R(I)\mathbf{x}^{\mathbf{a}}\partial_i^{k} + J$$

is a left ideal of $\mathcal{I}_R(I)/J$ and is finitely generated. Now $\mathbf{x}^{\mathbf{a}}/x_i^{a_i}$ commutes with ∂_i, and $\mathcal{I}_R(I)/J$ is a domain, so

$$\sum_{k=1}^{\infty} \mathcal{I}_R(I)x_i^{a_i}\partial_i^{k} + J$$

is a finitely generated left $\mathcal{I}_R(I)/J$-module. Therefore for some positive integer M, $\exists r_k \in \mathcal{I}_R(I)$, $1 \leq k \leq M$, such that

$$x_i^{a_i}\partial_i^{M+1} - \sum_{k=1}^{M} r_k x_i^{a_i}\partial_i^{k} \in J.$$

(Without loss of generality assume that $a_i \leq M$.)

Note that by the definition of lowest common multiple, there exists an element $\mathbf{x}^{\mathbf{c}} \notin I$ with $c_i = d_i - 1$, and $\mathbf{x}^{\mathbf{c}}x_i \in I$.

Now consider

$$(x_i^{a_i}\partial_i^{M+1} - \sum_{k=1}^{M} r_k x_i^{a_i}\partial_i^{k}) \bullet (\mathbf{x}^{\mathbf{c}}x_i^{M+1-a_i}).$$

By definition this is an element of $J \bullet I$, and since J is an ideal of $\mathcal{I}_R(I)$, this is an element of I. But $x_i^{a_i}\partial_i^{M+1} \bullet (\mathbf{x}^{\mathbf{c}}x^{M+1-a_i}) = \mu\mathbf{x}^{\mathbf{c}} \notin I$, where $\mu \in k^*$, since $a_i \leq c_i$. On the other hand, for $1 \leq k \leq M$, we have $x_i^{a_i}\partial_i^{k} \bullet (\mathbf{x}^{\mathbf{c}}x_i^{M+1-a_i}) =$

$\lambda_k \mathbf{x}^{\mathbf{c}} x_i^{M+1-k} \in I$, where $\lambda_k \in k^*$. Hence each $r_i x_i^{a_i} \partial_i^k \bullet (\mathbf{x}^{\mathbf{c}} x_i^{M+1-a_i}) \in I$, giving a contradiction. □

It appears to be a difficult problem to characterise the monomial ideals I with ring of differential operators $\mathcal{D}(R/I)$ left Noetherian. However it follows from the paper of I.M. Musson [9] that if R/I has an embedded prime, then $\mathcal{D}(R/I)$ is not left Noetherian. For the rest of this paper we shall restrict our attention to square-free monomial ideals.

4. Idealisers of Square-Free Monomial Ideals

For this section fix a square-free monomial ideal I. We may write I uniquely as an intersection of minimal primes: $I = I_1 \cap \cdots \cap I_s$, and in this case we may take these to be of the form $(x_{i_1}, \ldots, x_{i_r})$. See [4, Theorem 5.1.4].

4.1 Lemma. *Let I be as above. Then*

$$\mathcal{I}_R(I) = \mathcal{I}_R(I_1) \cap \cdots \cap \mathcal{I}_R(I_s).$$

Proof. First, by inspection, we see that any element of $\mathcal{I}_R(I_1) \cap \cdots \cap \mathcal{I}_R(I_s)$ must idealize $I_1 \cap \cdots \cap I_s$. Thus, since this is I, we see $\mathcal{I}_R(I_1) \cap \cdots \cap \mathcal{I}_R(I_s) \subseteq \mathcal{I}_R(I)$.

On the other hand it is well known that for each minimal prime I_i we have $\mathcal{I}_R(I) \subseteq \mathcal{I}_R(I_i)$ (see [1]), or since I is reduced then it is easy to show that $I_i = \Gamma_{J/I}$, where $J = \bigcap_{j \neq i} I_j$, and use the fact that idealisers preserve supports. □

Using the commutation relations in $\mathcal{D}(R)$, each element $f \in \mathcal{D}(R)$ will always be written in the form

$$f = \sum \lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}},$$

for some scalars $\lambda_{\mathbf{a},\mathbf{b}} \in k$. We call the $\lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}}$ with $\lambda_{\mathbf{a},\mathbf{b}} \neq 0$ the summands of f.

4.2 Remark. Suppose that I is a monomial ideal in R and that R' is a polynomial ring in R. Then $f \in I\mathcal{D}(R')$ if and only if each summand of f is in $I\mathcal{D}(R')$. (This is because $\alpha = \sum \lambda_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ is an element of I if and only if each $\mathbf{x}^{\mathbf{a}}$ is an element of I.)

Now define polynomial rings $S_i := k[x_j : x_j \notin I_i]$ for each i, $1 \leq i \leq s$.

4.3 Proposition. *Suppose that $f = \sum \lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}}$ is an element of $\mathcal{I}_R(I_i)$, for some minimal prime I_i. Then each $\lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}} \in \mathcal{I}_R(I_i)$, and*

$$\mathcal{I}_R(I_i) = \mathcal{D}(S_i) + I_i \mathcal{D}(R).$$

Proof. By inspection, $\mathcal{D}(S_i) + I_i \mathcal{D}(R) \subset \mathcal{I}_R(I_i)$. Let $f = \sum \lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}}$ be an element of $\mathcal{I}_R(I_i)$ written in the standard form.

Suppose that $I_i = (x_{i_1}, \ldots, x_{i_r})$. Then $\partial^{\mathbf{b}} \in \mathcal{D}(S_i)$ if and only if $b_{i_1} = \cdots = b_{i_r} = 0$ if and only if $\mathbf{x}^{\mathbf{b}} \notin I_i$.

Consider $\lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}}$ a summand of f. If $\mathbf{x}^{\mathbf{a}} \in I_i$, then $\lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}} \in I_i \mathcal{D}(R)$, whereas if $\mathbf{x}^{\mathbf{a}} \notin I_i$ and $\mathbf{x}^{\mathbf{b}} \notin I_i$, then $\mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}} \in \mathcal{D}(S_i)$. So in either case $f - \lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}} \in \mathcal{I}_R(I_i)$. Therefore by subtracting from f all summands of the above form we may assume without loss of generality that each summand of f is of the following form: $\lambda_{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{a}} \partial^{\mathbf{b}}$, where $\mathbf{x}^{\mathbf{a}} \notin I_i$ and $\mathbf{x}^{\mathbf{b}} \in I_i$.

To finish the proof it is enough to show that $f = 0$. Suppose that $f \neq 0$. Define the usual lexicographic ordering on $k[\partial_1, \ldots, \partial_n]$: $\partial^{\mathbf{a}} < \partial^{\mathbf{b}}$ if $a_1 < b_1$ or $a_1 = b_1$ and if i is the first index for which $a_i \neq b_i$ then $a_i < b_i$. Let $\partial^{\mathbf{c}}$ be the unique minimal monomial amongst all $\partial^{\mathbf{b}}$ such that $\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$ is a summand of f. Now $\mathbf{x}^{\mathbf{c}} \in I_i$, so consider $f \bullet \mathbf{x}^{\mathbf{c}}$. Since $f \in \mathcal{I}_R(I_i)$, this is an element of I_i.

If $\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$ is a summand of f such that $\mathbf{x}^{\mathbf{c}} < \mathbf{x}^{\mathbf{b}}$, then by definition of the ordering, $\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \bullet \mathbf{x}^{\mathbf{c}} = 0$. Hence $f \bullet \mathbf{x}^{\mathbf{c}} = c_1! \dots c_n! \sum \lambda_{\mathbf{a},\mathbf{c}}\mathbf{x}^{\mathbf{a}}$. However each $\mathbf{x}^{\mathbf{a}} \notin I_i$, so $f \bullet \mathbf{x}^{\mathbf{c}} \notin I_i$, giving a contradiction. □

Let $\mathcal{V} := \{1, \dots, s\}$. If $\mathcal{J}$ is any subset of $\mathcal{V}$, define

$$I_{\mathcal{J}} := \bigcap_{i \in \mathcal{J}} I_i \quad \text{and} \quad R_{\mathcal{J}} := \bigcap_{i \notin \mathcal{J}} S_i.$$

4.4 Corollary. *Let I be as above; then $f \in \mathcal{I}_R(I)$ if and only if each summand of f is in $\mathcal{I}_R(I)$.*

In particular, by Remark 4.2, if $\mathcal{J} \subseteq \mathcal{V}$, then $f \in I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$ if and only if each summand of f is in $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$.

Proof. By Lemma 4.1, if $f \in \mathcal{I}_R(I)$, then $f \in \mathcal{I}_R(I_i)$ for each minimal prime I_i, so by Proposition 4.3, each summand of f is in $\mathcal{I}_R(I_i)$. Since this is true for each minimal prime, then again by Lemma 4.1, each summand is in $\mathcal{I}_R(I)$. So the result follows. □

We can now characterise the ideals in $\mathcal{I}_R(I)$, for I a square-free monomial ideal. We keep the notation as above.

4.5 Lemma. *For each $\mathcal{J} \subseteq \mathcal{V}$, then $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}}) \subseteq \mathcal{I}_R(I)$.*

Proof. By Lemma 4.1, it is enough to show that $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}}) \subseteq \mathcal{I}_R(I_j)$ for each mimimal prime I_j. If $j \in \mathcal{J}$, then $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}}) \subseteq I_j\mathcal{D}(R) \subseteq \mathcal{I}_R(I_j)$ as required, so assume that $j \notin \mathcal{J}$. Consider some $\mathbf{x}^{\mathbf{a}} \in I_j$. So for some i, we have $x_i^{a_i} \in I_j$. But then $x_i^{a_i} \notin S_j$ and hence $x_i^{a_i} \notin R_{\mathcal{J}}$. Thus $\mathcal{D}(R_{\mathcal{J}})$ commutes with x_i. So $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}}) \bullet \mathbf{x}^{\mathbf{a}} \in x_i^{a_i}R \subseteq I_j$ as required. □

Now we need another way of looking at $\mathcal{I}_R(I)$:

4.6 Proposition. *For each $\mathcal{J} \subseteq \mathcal{V}$. Then*

$$I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I) = \sum_{\mathcal{K}\,:\,\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{V}} I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}).$$

In particular, when $\mathcal{J} = \emptyset$,

$$\mathcal{I}_R(I) = \sum_{\mathcal{K}\,:\,\mathcal{K} \subseteq \mathcal{V}} I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}).$$

Proof. If $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{V}$ then $I_{\mathcal{K}} \subseteq I_{\mathcal{J}}$; hence $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq I_{\mathcal{J}}\mathcal{D}(R)$. By Lemma 4.5, $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq \mathcal{I}_R(I)$, so $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$.

On the other hand, if f is any element of $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$, then by Proposition 4.3, we may assume that $f = \lambda\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$. Take $\mathcal{K}$ a subset such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{V}$ and maximal (with respect to containment) under the condition that $\mathbf{x}^{\mathbf{a}} \in I_{\mathcal{K}}$. So for each $i \notin \mathcal{K}$, we have $\mathbf{x}^{\mathbf{a}} \notin I_{\mathcal{K}\cup\{i\}}$, hence $\mathbf{x}^{\mathbf{a}} \notin I_i$. Since $f \in \mathcal{I}_R(I)$, then Proposition 4.3 implies that $\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \in \mathcal{D}(S_i)$. This is true for each $i \notin \mathcal{K}$, and so $\partial^{\mathbf{b}} \in \mathcal{D}(R_{\mathcal{K}})$. Thus $f \in I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})$. □

4.7 Lemma. *Suppose that $\mathcal{J}$ and $\mathcal{K}$ are subsets of $\mathcal{V}$. Then*

$$I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}})I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq I_{\mathcal{J}\cup\mathcal{K}}\mathcal{D}(R) \cap \mathcal{I}_R(I).$$

Proof. It follows from Lemma 4.5 that this product is in $\mathcal{I}_R(I)$. Consider some $f \in I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})$. By Remark 4.2, we may suppose that $f = \mu \mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$. Consider some $j \in \mathcal{K}\backslash\mathcal{J}$. Then $\mathbf{x}^{\mathbf{a}} \in I_j$, so $x_i^{a_i} \in I_j$ for some index i. Hence, $x_i^{a_i} \notin S_j$, and therefore $x_i^{a_i} \notin R_{\mathcal{J}}$. So x_i commutes with $\mathcal{D}(R_{\mathcal{J}})$. Thus $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}})\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \subset I_{(\mathcal{J}\cup\{j\})}\mathcal{D}(R)$. This is true for each $j \in \mathcal{K}\backslash\mathcal{J}$, and so $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}})I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq I_{\mathcal{J}\cup\mathcal{K}}\mathcal{D}(R)$. □

Now we can identify at least some of the ideals in $\mathcal{I}_R(I)$.

4.8 Proposition. *For each $\mathcal{J}$, then $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$ is an ideal of $\mathcal{I}_R(I)$ and $I_j\mathcal{D}(R) \cap \mathcal{I}_R(I)$ is a completely prime ideal of $\mathcal{I}_R(I)$ for each minimal prime I_j.*

Proof. By Proposition 4.6, we only need to show that

$$I_{\mathcal{H}}\mathcal{D}(R_{\mathcal{H}})I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})I_{\mathcal{L}}\mathcal{D}(R_{\mathcal{L}}) \subseteq I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I),$$

for all subsets $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{V}$ such that $\mathcal{J} \subseteq \mathcal{K}$. This follows from Lemma 4.7.

By Lemma 4.1, $\mathcal{I}_R(I) \subseteq \mathcal{I}_R(I_j)$, so consider the following subring of $\mathcal{D}(R/I_j)$:

$$\frac{\mathcal{I}_R(I) + I_j\mathcal{D}(R)}{I_j\mathcal{D}(R)} \cong \frac{\mathcal{I}_R(I)}{I_j\mathcal{D}(R) \cap \mathcal{I}_R(I)}.$$

Since I_j is prime, it follows that $\mathcal{D}(R/I_j)$ is a domain, and hence $I_j\mathcal{D}(R) \cap \mathcal{I}_R(I)$ is completely prime. □

In fact, these are all the ideals of $\mathcal{I}_R(I)$:

4.9 Theorem. *Let I be as above. Then each ideal of $\mathcal{I}_R(I)$ is a sum of ideals of the form*

$$I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I) \qquad \text{for } \mathcal{J} \subseteq \mathcal{V}.$$

Proof. Consider some ideal M of $\mathcal{I}_R(I)$, and let M' be the ideal in $\mathcal{I}_R(I)$ which is the sum of all $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$ such that there exists an $f \in M$ with a summand in $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}})$. This M' is of the required form, and by construction and Proposition 4.6, $M \subset M'$. It remains to prove that $M = M'$. The proof follows from the next lemma: □

4.10 Lemma. *Suppose that $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I)$ is a summand of M'. Then $I_{\mathcal{J}}\mathcal{D}(R) \cap \mathcal{I}_R(I) \subseteq M$.*

Proof. By Proposition 4.6, we only need prove that $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq M$ for each subset $\mathcal{K}$ such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{V}$.

If $\mathcal{K}$ has cardinality s, then $\mathcal{K} = \mathcal{V}$ and $I_{\mathcal{K}}\mathcal{D}(R) \subseteq M$, since this is the unique minimal ideal.

For a proof by descending induction, suppose that $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq M$ for each subset $\mathcal{K}$ of cardinality $\mathrm{Card}(\mathcal{K}) > t$. Consider a subset $\mathcal{K}$ of cardinality t such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{V}$. By definition of $\mathcal{J}$, there is an element $f \in M$ with a summand in $I_{\mathcal{J}}\mathcal{D}(R_{\mathcal{J}})$. Since $\mathcal{J} \subseteq \mathcal{K}$ then $\mathcal{D}(R_{\mathcal{J}}) \subseteq \mathcal{D}(R_{\mathcal{K}})$, and so this summand is in $\mathcal{D}(R_{\mathcal{K}})$ (although not necessarily $\mathcal{I}_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})$).

By Lemma 4.5, $I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq \mathcal{I}_R(I)$, so consider

$$I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})fI_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq M.$$

Consider $\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$ a summand of f with $\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \notin \mathcal{D}(R_{\mathcal{K}})$. So $\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \notin \mathcal{D}(S_i)$ for some $i \notin \mathcal{K}$. Now $f \in M \subset \mathcal{I}_R(I_i)$, and so by Lemma 4.3, and Corollary 4.4, this implies that $\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \in I_i\mathcal{D}(R) \cap \mathcal{I}_R(I)$. By Lemma 4.7,

$$I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}}) \subseteq I_{\mathcal{K}\cup\{i\}}\mathcal{D}(R) \cap \mathcal{I}_R(I).$$

By induction and Proposition 4.6, $I_{\mathcal{K}\cup\{i\}}\mathcal{D}(R)\cap\mathcal{I}_R(I)\subseteq M$. So

$$I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})(f-\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}})I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})\subseteq M.$$

So without loss of generality assume that each summand $\lambda_{\mathbf{a},\mathbf{b}}\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$ of f is in $\mathcal{D}(R_{\mathcal{K}})$. By assumption f has such a non-zero summand, and since $\mathcal{D}(R_{\mathcal{K}})$ is simple then

$$I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})=I_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})fI_{\mathcal{K}}\mathcal{D}(R_{\mathcal{K}})\subseteq M,$$

as required. □

In order to get some feel for the above notation, consider the following examples:

4.11 Examples. 1. $R=k[x,y,z]$, and $I=xyzR=xR\cap yR\cap zR$. Let $I_1=xR$, $I_2=yR$ and $I_3=zR$. Then $S_1=k[y,z]$, $S_2=k[x,z]$ and $S_3=k[x,y]$, so $\mathcal{I}_R(I_1)=\mathcal{D}(k[y,z])+x\mathcal{D}(R)$.

Therefore also:

$$\begin{aligned}I_{\{1,2,3\}}\mathcal{D}(R_{\{1,2,3\}})&=I\mathcal{D}(R),\\ I_{\{1,2\}}\mathcal{D}(R_{\{1,2\}})&=xy\mathcal{D}(k[x,y]),\\ I_{\{1\}}\mathcal{D}(R_{\{1\}})&=x\mathcal{D}(k[x]),\\ \mathcal{I}_R(I)&=k+x\mathcal{D}(k[x])+y\mathcal{D}(k[y])+z\mathcal{D}(k[z])\\ &\quad+xy\mathcal{D}(k[x,y])+xz\mathcal{D}(k[x,z])+yz\mathcal{D}(k[y,z])\\ &\quad+xyz\mathcal{D}(R).\end{aligned}$$

2. $R=k[x,y,z]$, and $I=(xy,xz)R=xR\cap(y,z)R$. Here

$$\begin{aligned}\mathcal{I}_R(I)&=k+x\mathcal{D}(k[x])+(y,z)\mathcal{D}(k[y,z])+I\mathcal{D}(R)\\ &=\mathcal{I}_R(x)\cap\mathcal{I}_R(y,z)\\ &=(\mathcal{D}(k[y,z])+x\mathcal{D}(R))\cap(\mathcal{D}(k[x])+(y,z)\mathcal{D}(R).\end{aligned}$$

The ideals of $\mathcal{I}_R(I)$ are $I\mathcal{D}(R)$, $x\mathcal{D}(k[x])+I\mathcal{D}(R)$, $(y,z)\mathcal{D}(k[y,z])+I\mathcal{D}(R)$ and their sum.

Now we have enough to characterise those square-free monomial ideals I for which $\mathcal{D}(R/I)$ is left Noetherian. First we note:

4.12 Proposition. *Suppose that I is a square-free monomial ideal in R. Then $\mathcal{I}_R(\mathbf{x}^{\mathbf{d}}R)+I\mathcal{D}(R)\subseteq\mathcal{I}_R(I)$, where I has finite set of generators $\mathcal{M}$, and $\mathbf{x}^{\mathbf{d}}$ is the lowest common multiple of $\mathcal{M}$.*

Proof. By inspection we see that $\bigcap_{\mathbf{x}^{\mathbf{a}}\in\mathcal{M}}\mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R)\subseteq\mathcal{I}_R(I)$. By Proposition 3.2,

$$\bigcap_{\mathbf{x}^{\mathbf{a}}\in\mathcal{M}}\mathcal{I}_R(\mathbf{x}^{\mathbf{a}}R)=\bigcap_{\mathbf{x}^{\mathbf{a}}\in\mathcal{M}}\mathcal{I}_{k[x_1]}(x_1^{a_1}k[x_1])\otimes\cdots\otimes\bigcap_{\mathbf{x}^{\mathbf{a}}\in\mathcal{M}}\mathcal{I}_{k[x_n]}(x_n^{a_n}k[x_n]).$$

Since I is square-free, then $d_i=1$ or 0 for each i. If $a_i=0$, then $\mathcal{I}_{k[x_i]}(x_i^{a_i}k[x_i])=\mathcal{D}(k[x_i])$. So this is $\mathcal{I}_{k[x_1]}(x_1^{d_1}k[x_1])\otimes\cdots\otimes\mathcal{I}_{k[x_n]}(x_n^{d_n}k[x_n])$, which is $\mathcal{I}_R(\mathbf{x}^{\mathbf{d}}R)$, again by Proposition 3.2.

We have also already noted that $I\mathcal{D}(R)\subseteq\mathcal{I}_R(I)$. □

4.13 Theorem. *Suppose that I is a square-free monomial ideal in R, with $\mathbf{x}^{\mathbf{d}}$ as above. Then $\mathcal{D}(R/I)$ is left Noetherian if and only if $\mathcal{I}_R(I)=\mathcal{I}_R(\mathbf{x}^{\mathbf{d}})+I\mathcal{D}(R)$.*

Proof. 1. First suppose that

$$\mathcal{I}_R(I) = \mathcal{I}_R(\mathbf{x}^{\mathbf{d}}) + I\mathcal{D}(R).$$

Then $\mathcal{D}(R/I) \cong \mathcal{I}_R(I)/I\mathcal{D}(R)$ is a quotient of $\mathcal{I}_R(\mathbf{x}^{\mathbf{d}})$, which is Noetherian by Theorem 3.6.

2. Now suppose that there exists some $f \in \mathcal{I}_R(I)$ such that $f \notin \mathcal{I}_R(\mathbf{x}^{\mathbf{d}}) + I\mathcal{D}(R)$. By Corollary 4.4, we may assume without loss of generality that $f = \mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}}$. By definition $\mathbf{x}^{\mathbf{a}} \notin I$, and $\mathbf{x}^{\mathbf{a}}\partial^{\mathbf{b}} \notin \mathcal{I}_R(\mathbf{x}^{\mathbf{d}})$. So for some i such that $d_i = 1$, we have $a_i = 0$ and $b_i > 0$. Fix this i. By repeatedly commuting with x_j, for all j we see that $\mathbf{x}^{\mathbf{a}}\partial_i \in \mathcal{I}_R(I)$.

   Write I as an intersection of minimal primes $I = I_1 \cap \cdots \cap I_s$. Since $\mathbf{x}^{\mathbf{a}} \notin I$, then $\mathbf{x}^{\mathbf{a}} \notin I_j$ for some j. Fix this j. For short let $J := I_j\mathcal{D}(R) \cap \mathcal{I}_R(I)$; by Proposition 4.8, this is a completely prime ideal of $\mathcal{I}_R(I)$.

   By Lemma 3.10, $\mathcal{I}_R(I)/J$ is not left Noetherian. Now $I\mathcal{D}(R) \subseteq I_j\mathcal{D}(R) \cap \mathcal{I}_R(I)$, so there exists a surjection $\mathcal{D}(R/I) \twoheadrightarrow \mathcal{I}_R(I)/J$. Hence $\mathcal{D}(R/I)$ is not left Noetherian either.

□

4.14 Remark. 1. Note that to check whether $\mathcal{D}(R/I)$ is not left Noetherian it is sufficient to find a derivation $\mathbf{x}^{\mathbf{a}}\partial_i \in \mathcal{I}_R(I)$ such that $\mathbf{x}^{\mathbf{a}} \notin I$, $a_i = 0$ and $d_i = 1$.

2. Suppose now that I is a monomial ideal, with set of monomial generators $\mathcal{M}$, and $\mathbf{x}^{\mathbf{d}}$ the lowest common multiple of $\mathcal{M}$. If $\{a_i : \mathbf{x}^{\mathbf{a}} \in \mathcal{M}\} = \{0, d_i\}$ for each i, then by Morita equivalence we may assume that I is square-free, and so Theorem 4.13 may be used.

5. Stanley-Reisner Rings

In this section we define a simplicial complex; then, following [10], we define the Stanley-Reisner ring $k[\Delta]$ associated to a simplicial complex Δ. We define T-spaces, and then show that $\mathcal{D}(k[\Delta])$ is left Noetherian if and only if $\operatorname{Core}\Delta$ is a T-space.

5.1 Simplicial Complexes. Let V be a vertex set. We say that a set Δ of subsets of V is a *simplicial complex* if the following axioms hold:

1. $F \in \Delta$ and $G \subseteq F$ imply that $G \in \Delta$, and
2. $\{v\} \in \Delta$ for all $v \in V$.

If only 1 holds for some set Δ of subsets of V, then we say that Δ is a slack simplicial complex.

The elements of Δ are called *faces*, and a *facet* is a maximal face under inclusion.

Now let Δ be any simplicial complex on a vertex set V. Let $W \subseteq V$ be any subset. We define the localised complex at W to be the following simplicial complex on W:

$$\Delta_W = \{F \in \Delta : F \subset W\}.$$

For any $F \in \Delta$, the *star of F* is

$$\operatorname{st}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}.$$

Define $\operatorname{Core} V = \{v \in V : \operatorname{st}_\Delta v \neq \Delta\}$, and then $\operatorname{Core}\Delta = \Delta_{\operatorname{Core} V}$.

For further background on this notation see [3, 5.5]. Note that in this paper we shall always refer to a slack simplicial complex simply as a simplicial complex.

5.2 T-spaces. Suppose that Δ is a simplicial complex and let F, G be two faces of Δ. Then we say that F *may be separated from* G *in* Δ if there exists a facet H such that $F \subseteq H$ and $G \nsubseteq H$.

We say that Δ is a T*-space* if F may be separated from G for all faces F and G of Δ such that $G \nsubseteq F$.

5.3 Lemma. *Let Δ be a simplicial complex on V. Then:*

1. *Δ is a T-space if and only if F may be separated from $\{v\}$ for all faces $F \in \Delta$ and $v \in V$ such that $v \notin F$.*
2. *Let F be a face of Δ, and $v \in \operatorname{Core} V$ a vertex with $v \notin F$. Then F may be separated from $\{v\}$ in Δ if and only if $F \cap \operatorname{Core} V$ (a face of $\operatorname{Core}\Delta$) may be separated from $\{v\}$ in $\operatorname{Core}\Delta$.*

Proof. 1. Assume that F may be separated from $\{v\}$ for all vertices $v \in V$ and all faces F such that $v \notin F$. Then if F and G are any two faces of Δ such that $G \nsubseteq F$, then there exists some vertex $v \in G \backslash F$. By assumption, then there is a facet H such that H contains F and not v. Therefore H cannot contain G, so is the required facet.

2. Suppose that $v \in \operatorname{Core} V$ and that F is a face of Δ, with $v \notin F$. If F may be separated from $\{v\}$, then there exists a facet H containing F and not v, so in particular H contains $F \cap \operatorname{Core} V$ and not v, so $F \cap \operatorname{Core} V$ may be separated from $\{v\}$.

   On the other hand, suppose that $F \cap \operatorname{Core} V$ may be separated from $\{v\}$ in $\operatorname{Core}\Delta$. So there exists a facet H in $\operatorname{Core}\Delta$ as above. Consider $H' := H \cup \{V \backslash \operatorname{Core} V\}$. By definition of $\operatorname{Core} V$, for each $w \in V \backslash \operatorname{Core} V$, then $\{w\} \cup H$ is a face of Δ, since H is already a facet of Δ. So H' is a face of Δ and so F may be separated from $\{v\}$ in Δ.

□

Remarks. 1. In [7], G. Müller defined a T_1*-space* to be a simplicial complex Δ such that v_i may be separated from v_j for all vertices v_i and v_j of V such that $v_i \neq v_j$. So the concept of T-space defined here is a slight generalisation of a T_1-space.

2. In [3, 3.2.4], P. Brumatti and A. Simis make the following definitions: let Δ be a simplicial complex. The *star corners* of a vertex $v \in \Delta$ are defined to be the faces $F \in \operatorname{st}_\Delta v$ such that $\operatorname{st}_\Delta F \subseteq \operatorname{st}_\Delta v$ and $v \notin F$. A vertex is *cornerless* if $v \notin \operatorname{Core}\Delta$ or it has no star corners, and Δ is *cornerless* if every vertex is cornerless.

   It is easy to see that if $F \in \Delta$ and v is a vertex such that $v \notin F$, then F may be separated from $\{v\}$ if and only if F is not a star corner of v. So $\operatorname{Core}\Delta$ is a T-space if and only if Δ is cornerless.

5.4. Recall that $R = k[x_1, \dots, x_n]$, and let $V := \{v_1, \dots, v_n\}$ be a vertex set. In [10] we see how to set up a correspondence between square-free monomial ideals of R and simplicial complexes on V.

For each monomial $\mathbf{x}^{\mathbf{a}} \in R$ we define the *support* of $\mathbf{x}^{\mathbf{a}}$ to be

$$|\mathbf{x}^{\mathbf{a}}| := \{v_i \in V : a_i \geq 1\}.$$

(Notice that there is a unique square-free monomial corresponding to each subset of V. For example, if $F = \{v_{i_1}, \dots, v_{i_s}\}$, then this monomial is $x_{i_1} \dots x_{i_s}$.)

First fix a square-free monomial ideal I in R. We may define a simplicial complex Δ on V as follows:

$$|\mathbf{x^a}| \text{ is a face of } \Delta \text{ if and only if } \mathbf{x^a} \notin I.$$

Conversely, given a simplicial complex Δ on V, using the above rule, we may associate to it a square-free monomial ideal I_Δ. We define the *Stanley-Reisner ring* associated to the complex Δ to be $k[\Delta] := R/I_\Delta$.

This gives the required correspondence.

5.5 Lemma. *Suppose that Δ is a simplicial complex on V with I the associated square-free monomial ideal in R. Then $v_i \in \operatorname{Core} V$ if and only if $d_i = 1$, where $\mathbf{x^d}$ is the lowest common multiple of the minimal set of monomial generators of I.*

Proof. Now $v_i \in \operatorname{Core} V$ if and only if $\operatorname{st}_\Delta v_i \neq \Delta$. This happens if and only if there exists a face $G \in \Delta$ such that $\{v_i\} \cup G \notin \Delta$, that is, if and only if there exists a monomial $\mathbf{x^a} \notin I$ such that $x_i\mathbf{x^a} \in I$, that is, if and only if $d_i = 1$. □

5.6 Proposition. *Let Δ be a simplicial complex on V with I the associated square-free monomial ideal in R. Suppose that $\mathbf{x^a}$, $\mathbf{x^b} \in R$ are such that $|\mathbf{x^a}|$ and $|\mathbf{x^b}|$ are faces of Δ. Then $|\mathbf{x^a}|$ may be separated from $|\mathbf{x^b}|$ in Δ if and only if $\mathbf{x^a}\partial^\mathbf{b} \notin \mathcal{I}_R(I)$.*

Proof.

1. Suppose that $\mathbf{x^a}\partial^\mathbf{b} \notin \mathcal{I}_R(I)$, so there exists some $\mathbf{x^c} \in R$ such that $\mathbf{x^{b+c}} \in I$ and $\mathbf{x^{a+c}} \notin I$. Without loss of generality assume that $|\mathbf{x^{a+c}}|$ is a facet of Δ. Now suppose that $|\mathbf{x^b}| \subseteq |\mathbf{x^{a+c}}|$. Since $|\mathbf{x^c}| \subseteq |\mathbf{x^{a+c}}|$ anyway, then $|\mathbf{x^{b+c}}| \subseteq |\mathbf{x^{a+c}}|$. Axiom 2 for simplicial complexes implies that $|\mathbf{x^{b+c}}| \in \Delta$, contradicting the fact that $\mathbf{x^{b+c}} \in I$. So $|\mathbf{x^b}| \nsubseteq |\mathbf{x^{a+c}}|$, and $|\mathbf{x^{a+c}}|$ is the required facet.
2. Now suppose that $|\mathbf{x^a}|$ may be separated from $|\mathbf{x^b}|$. So there exists some facet H such that $|\mathbf{x^a}| \subseteq H$ and $|\mathbf{x^b}| \nsubseteq H$. Let $\mathbf{x^e} \in R$ be any monomial with $|\mathbf{x^e}| = H$. Now H is a facet and $|\mathbf{x^b}| \nsubseteq H$, so by definition of facet, $|\mathbf{x^b}| \cup H = |\mathbf{x^{b+e}}|$ is not a face of Δ. So $\mathbf{x^{b+e}} \in I$, and $\mathbf{x^a}\partial^\mathbf{b} \bullet \mathbf{x^{b+e}} = \lambda\mathbf{x^{a+e}}$, where $\lambda \in k^*$ is some scalar. But $|\mathbf{x^{a+e}}| \subseteq H$, so by axiom 2 for simplicial complexes, $\mathbf{x^{a+e}} \notin I$. Hence $\mathbf{x^a}\partial^\mathbf{b} \notin \mathcal{I}$.

□

5.7 Theorem. *Suppose that Δ is a simplicial complex. Then $\mathcal{D}(k[\Delta])$ is left Noetherian if and only if $\operatorname{Core}\Delta$ is a T-space.*

Proof. Let I be the square-free monomial ideal associated to Δ. By Theorem 4.13, $\mathcal{D}(k[\Delta])$ is left Noetherian if and only if $\mathcal{I}_R(I) = \mathcal{I}_R(\mathbf{x^d}R) + I\mathcal{D}(R)$. This is equivalent to $\mathbf{x^a}\partial_i \notin \mathcal{I}_R(I)$ for all $\mathbf{x^a} \notin I$ such that $d_i = 1$ and $a_i = 0$. By Proposition 5.6, this is equivalent to the condition that $|\mathbf{x^a}|$ may be separated from $\{v_i\}$ for all faces $|\mathbf{x^a}| \in \Delta$ and $v_i \in \operatorname{Core} V$ such that $v_i \notin |\mathbf{x^a}|$.

By Lemma 5.3, $|\mathbf{x^a}|$ may be separated from $\{v_i\}$ in Δ if and only if $|\mathbf{x^a}| \cap \operatorname{Core} V$ may be separated from $\{v_i\}$ in $\operatorname{Core}\Delta$.

So $\mathcal{D}(k[\Delta])$ is left Noetherian if and only if $\operatorname{Core}\Delta$ is a T-space. □

6. Examples

For this section, we will consider only Stanley-Reisner rings. If $k[\Delta]$ is Gorenstein then $\mathcal{D}(k[\Delta])$ is isomorphic to its opposite ring (see [5, Proposition 1.4(a)]), and in particular is Noetherian. However, there seems to be no connection between $k[\Delta]$ Cohen-Macaulay and $\mathcal{D}(k[\Delta])$ left Noetherian.

Now suppose that Core Δ is a *graph*. That is, all faces of Δ are of cardinality 1 or 2. Then it is easy to see that Core Δ is a T-space if and only if each vertex is contained in either no edges or at least two edges. Thus it is easy to check whether $\mathcal{D}(k[\Delta])$ is left Noetherian. For example:

Let $R = k[x, y, z, w]$, $V = \{v_x, v_y, v_z, v_w\}$ and let Δ be the simplicial complex on V where Δ is the union of the point v_w and the boundary of the 2-simplex spanned by v_x, v_y, v_z. Then the associated square-free monomial ideal is $I = (xyz, xw, yw, zw) = (x, w) \cap (y, w) \cap (z, w) \cap (x, y, z)$. Then $k[\Delta]$ is not Cohen-Macaulay since Δ is not *pure*, i.e. not all facets have the same dimension. However, it is easy to see that Δ is a T-space, so $\mathcal{D}(k[\Delta])$ is left Notherian.

Conversely, if Δ is a line joining v_x to v_y and a line joining v_z to v_w, then $I = (xz, xw, yz, yw)$. Then $k[\Delta]$ is Cohen-Macaulay but $\mathcal{D}(k[\Delta])$ is not left Noetherian, since every facet containing v_x also contains v_y.

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