

THE FLOER HOMOTOPY TYPE OF HEIGHT FUNCTIONS ON COMPLEX GRASSMANN MANIFOLDS

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ABSTRACT. A family of Floer functions on the infinite dimensional complex Grassmann manifold is defined by taking direct limits of height functions on adjoint orbits of unitary groups. The Floer cohomology of a generic function in the family is computed using the Schubert calculus. The Floer homotopy type of this function is computed and the Floer cohomology which was computed algebraically is recovered from the Floer homotopy type. Certain non-generic elements of this family of Floer functions were shown to be related to the symplectic action functional on the universal cover of the loop space of a finite dimensional complex Grassmann manifold in the author's preprint *The Floer homotopy type of complex Grassmann manifolds*.

1. INTRODUCTION

Recently Cohen, Jones, and Segal have been trying to develop an underlying homotopy theory for Floer homology [6], [8]. According to their theory a Floer function, i.e. a function that can be used to define Floer homology groups [9], [10], [11], should in many cases also give rise to an inverse system of spectra (a pro-spectrum). They call this inverse system of spectra the “Floer homotopy type”, and the Floer homology groups should be recoverable from the Floer homotopy type. The analogous situation in Morse theory is that a Morse function gives rise to a CW complex whose homology agrees with the Morse homology [14], [16].

In [8] Cohen, Jones, and Segal considered the example of a Floer function $f : \mathbb{C}P^\infty \rightarrow \mathbb{R}$. This function is a modification, due to Segal, of the Morse function on $\mathbb{C}P^k$ defined in [14]. Segal's idea was to use the function which appears in [14] extended to $\mathbb{C}P^\infty$ but with the basis elements of \mathbb{C}^∞ indexed over \mathbb{Z} instead of \mathbb{Z}^+ . The result is a function on $\mathbb{C}P^\infty$ whose critical points all have infinite index but whose relative indices are all finite. The Floer cohomology of $f : \mathbb{C}P^\infty \rightarrow \mathbb{R}$ has the structure of a module over $H^*(\mathbb{C}P^\infty)$. This Floer cohomology module is $H^*(\mathbb{C}P^\infty)[c_1^{-1}]$, where c_1 is the Chern class of the Hopf line bundle γ_1 over $\mathbb{C}P^\infty$. The Floer homotopy type of $\mathbb{C}P^\infty \rightarrow \mathbb{R}$ is

$$\mathbb{C}P^\infty \leftarrow (\mathbb{C}P^\infty)^{-\gamma_1} \leftarrow (\mathbb{C}P^\infty)^{-2\gamma_1} \leftarrow \dots$$

and the Floer cohomology module can be recovered from this pro-spectrum using the Thom Isomorphism Theorem as the direct limit of the following:

$$H^*(\mathbb{C}P^\infty) \xrightarrow{\cup c_1} H^{*+2}(\mathbb{C}P^\infty) \xrightarrow{\cup c_1} H^{*+4}(\mathbb{C}P^\infty) \xrightarrow{\cup c_1} \dots$$

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In this paper I define a family of Floer functions on $G_n(\mathbb{C}^\infty)$, the infinite dimensional complex Grassmann manifold. These Floer functions are direct limits of well known height functions defined on certain adjoint orbits of U_{n+k} diffeomorphic to $G_{n,n+k}(\mathbb{C})$ [4]. An explicit formula for the gradient flow lines of these height functions with respect to the Killing form is derived, and the functions are shown to satisfy the Morse-Bott-Smale transversality condition (see Theorems 3 and 5). For a generic height function on $G_{n,n+k}(\mathbb{C})$ I show that the unstable manifolds of the function are the Schubert cells of $G_{n,n+k}(\mathbb{C})$ (see Theorem 8).

One function $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ in the family is an extension of the function studied by Cohen, Jones, and Segal in [8]. This function has critical points whose Morse indices are all infinite; however, the relative index between any two points is finite. Using the relative index (see Definition 10), the Floer chain complex of $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ is defined. There is a natural filtration of the Floer chain complex such that the cohomology of each term in the filtration is isomorphic (after reindexing) to $H^*(G_n(\mathbb{C}^\infty))$. This isomorphism induces a direct system of chain complexes, and the cohomology of the direct limit is computed as a module over $H^*(G_n(\mathbb{C}^\infty))$ using the Schubert calculus. This cohomology module is the localization of $H^*(G_n(\mathbb{C}^\infty))$ at c_n , the n th Chern class of the tautological n -plane bundle γ_n over $G_n(\mathbb{C}^\infty)$, i.e.

$$HF^*(G_n(\mathbb{C}^\infty)) \approx H^*(G_n(\mathbb{C}^\infty))[c_n^{-1}].$$

Each term in the filtered Floer chain complex has the cohomology of $G_n(\mathbb{C}^\infty)^{-j\gamma_n}$ for some $j \in \mathbb{Z}_+$. Moreover, the maps in the direct system of chain complexes coming from the filtration are multiplication by the Euler class of γ_n . This algebraic information is reflected in the Floer homotopy type of f_A . The Floer homotopy type of $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ has

$$G_n(\mathbb{C}^\infty) \leftarrow G_n(\mathbb{C}^\infty)^{-\gamma_n} \leftarrow G_n(\mathbb{C}^\infty)^{-2\gamma_n} \leftarrow \dots$$

as a cofinal system, where the maps are induced by certain bundle inclusions. The Floer cohomology module can be recovered from this pro-spectrum using the Thom Isomorphism Theorem as the direct limit of the following:

$$H^*(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup c_n} H^{*+2n}(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup c_n} H^{*+4n}(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup c_n} \dots$$

The Floer homotopy type of $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ is constructed from its flow category [7], [13]. The flow category of f_A is a topological category whose objects are the critical points of f_A and whose morphisms are the unparameterized piecewise gradient flow lines of f_A . The flow category of $f_A : G_n(\mathbb{C}^\infty) \rightarrow \mathbb{R}$ is particularly easy to work with, in part because its morphism spaces are compact. The flow category associated to a certain non-generic element of the family of Floer functions defined in this paper is related to the flow category of the symplectic action functional on the universal cover of the loop space of $G_{n,n+k}(\mathbb{C})$ via a group action and an \mathbb{R} -equivariant fiber bundle. For the case $n = 1$ the group action is trivial and the flow category of this non-generic function compactifies the flow category of the action functional on the universal cover of the loop space of $\mathbb{C}P^\infty$. Details concerning the relationship between the flow categories defined in this paper and the flow category of the symplectic action functional can be found in [13].

2. HEIGHT FUNCTIONS ON ADJOINT ORBITS

Let G be a compact connected semi-simple Lie group with Lie algebra \mathfrak{g} . Let $x_0 \in \mathfrak{g}$ and consider $G/G_{x_0} \approx G \cdot x_0 \subseteq \mathfrak{g}$, where \cdot denotes the adjoint action. For any $A \in \mathfrak{g}$ define $f_A : G \cdot x_0 \rightarrow \mathbb{R}$ by the formula $f_A(x) = \langle x, A \rangle$ where $\langle -, - \rangle$ denotes the Killing form. For the proof of the following theorem see [4], [12], and [3].

Theorem 1. $f_A : G \cdot x_0 \rightarrow \mathbb{R}$ is a Morse-Bott function for any $A \in \mathfrak{g}$. For a generic A , i.e. when A lies in an orbit of maximal dimension, f_A is a Morse function.

The following claim can be established by elementary means (see for instance [18] or [5]).

Claim 2.

1. For all $x \in G \cdot x_0$, the tangent space to the orbit at x is $T_x(G \cdot x_0) = [\mathfrak{g}, x]$.
2. For all $x \in G \cdot x_0$, the normal space to the orbit at x is $N_x(G \cdot x_0) = \{W \in \mathfrak{g} \mid [W, x] = 0\}$.
3. For any $x \in G \cdot x_0$ and for all $X \in T_x(G \cdot x_0)$ the differential of f_A is given by $D_X f_A = \langle X, A \rangle$.
4. $x_c \in G \cdot x_0$ is a critical point of f_A if and only if $[A, x_c] = 0$.

If G is a matrix group, e.g. U_n , O_n , or Sp_n , then the above Morse-Bott functions are compatible with an increase in dimension, e.g. $U_n \hookrightarrow U_{n+1}$, and so they can be used to define Morse-Bott functions on various orbits which arise as direct limits of matrix groups. Since this paper is mainly concerned with complex Grassmann manifolds, we will now restrict to the case where G is a unitary group. For any $n, k \in \mathbb{N}$ we choose

$$x_0 = i \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & 0_{k \times k} \end{pmatrix} \in u(n+k)$$

and we see that $G_{n,n+k}(\mathbb{C})$, the complex Grassmann manifold consisting of n -planes in \mathbb{C}^{n+k} , is diffeomorphic to the orbit $U_{n+k} \cdot x_0 \subseteq u(n+k)$ since the action of U_{n+k} on both spaces is transitive with stabilizer $U_n \times U_k$, i.e. both spaces are diffeomorphic to $U_{n+k}/U_n \times U_k$. From now on we will identify $G_{n,n+k}(\mathbb{C})$ with $U_{n+k} \cdot x_0 \subseteq u(n+k)$ by choosing the unique equivariant diffeomorphism which sends x_0 to $\mathbb{C}^n \subseteq \mathbb{C}^{n+k}$. This amounts to choosing a specific embedding of $G_{n,n+k}(\mathbb{C})$ into the $(n+k)^2$ dimensional Euclidean space $u(n+k)$ where the inner product is the Killing form.

Theorem 3. Let $p \in G_{n,n+k}(\mathbb{C})$ be any plane. For any matrix $A \in u(n+k)$ the gradient flow line of the Morse-Bott function f_A through p , with respect to the pullback of the Killing form, is $\gamma_p(t) = L_t(p)$, where $L_t : \mathbb{C}^{n+k} \rightarrow \mathbb{C}^{n+k}$ is the linear map determined by the matrix $\exp(-itA) \in GL_{n+k}(\mathbb{C})$ using column orientation for \mathbb{C}^{n+k} .

Note that if we use row orientation for \mathbb{C}^{n+k} , then the gradient flow is given by the linear map determined by the transpose of $\exp(-itA)$.

We will use the above theorem to prove that f_A satisfies the Morse-Bott-Smale transversality condition. This transversality condition can be stated as follows (see for instance [2]).

Definition 4. A function satisfies the Morse-Bott-Smale transversality condition if and only if for any two critical submanifolds M and N the stable manifold of N

intersects the unstable manifold of m transversely for all $m \in M$, i.e. $W^u(m) \pitchfork W^s(N)$ for all $m \in M$.

Theorem 5. For any $A \in \mathfrak{u}(n+k)$ the function $f_A : G_{n,n+k}(\mathbb{C}) \rightarrow \mathbb{R}$ satisfies the Morse-Bott-Smale transversality condition with respect to the pullback of the Killing form.

For the proof of Theorem 3 we will need the following elementary facts.

Claim 6.

1. For all $A \in \mathfrak{u}(n+k)$ and for all $x \in U_{n+k} \cdot x_0$ the projection of A onto $T_x(U_{n+k} \cdot x_0)$ is $-[A, x]$.
2. The gradient vector field of $f_A : U_{n+k} \cdot x_0 \rightarrow \mathbb{R}$ with respect to the Killing form is $(\nabla f_A)_x = -[A, x]$.

Proof. Let $A \in \mathfrak{u}(n+k)$ and $x \in U_{n+k} \cdot x_0$. The projection of A onto $T_x(U_{n+k} \cdot x_0)$ is the unique vector $X \in T_x(U_{n+k} \cdot x_0)$ such that $A - X \in N_x(U_{n+k} \cdot x_0)$. That is, $[A - X, x] = 0$ (see Claim 2) or equivalently, $[A, x] = [X, x]$. So the first part of the claim is simply the statement that $[A, x] = -[[A, x], x]$; i.e. $[-, x]$ is an almost complex structure on $U_{n+k} \cdot x_0$ (see for instance [1] p. 57).

To prove the second part of the claim recall that $(\nabla f_A)_x$ is the unique element of $T_x(U_{n+k} \cdot x_0)$ which satisfies $\langle (\nabla f_A)_x, X \rangle = D_X f_A$ for all $X \in T_x(U_{n+k} \cdot x_0)$. So we have by Claim 2 $\langle (\nabla f_A)_x, X \rangle = \langle A, X \rangle$ for all $X \in T_x(U_{n+k} \cdot x_0)$. That is, $(\nabla f_A)_x$ is the projection of A onto the tangent space $T_x(U_{n+k} \cdot x_0)$. \square

Lemma 7. Let $\eta : G \times M \rightarrow M$ be a smooth action of a Lie group G on a smooth manifold M . Let $m \in M$ and let $\eta_m : G \rightarrow M$ be the obvious map induced from η . Then $T_e(G_m) \subseteq \ker d\eta_m(e)$, where $e \in G$ is the identity and G_m is the stabilizer of m . Moreover, $T_e(G_{g \cdot m}) = \text{Ad}_g(T_e(G_m))$ for all $g \in G$.

Proof. Let $\gamma(t)$ be a path in G_m with $\gamma(0) = e$. Then $\eta_m(\gamma(t))$ is constant and so $\frac{d}{dt}|_{t=0} \eta_m(\gamma(t)) = d\eta_m(e)(\gamma'(0)) = 0$. This shows that $T_e(G_m) \subseteq \ker d\eta_m(e)$. To prove the second statement one need merely recall that $G_{g \cdot m} = gG_m g^{-1}$ and then consider paths of the form $g\sigma(t)g^{-1}$, where $\sigma(t) \in G_m$ and $\sigma(0) = e$. \square

Proof of Theorem 3. Let $\mathbb{C}^n \subseteq \mathbb{C}^{n+k}$ be the vector subspace spanned by the first n standard basis vectors of \mathbb{C}^{n+k} . Pick any unitary matrix $U \in U_{n+k}$ such that $U(\mathbb{C}^n) = p$ and let $x = Ux_0 U^{-1} \in \mathfrak{u}(n+k)$. The path $\sigma_x(t) = \exp(-t[A, x]) \cdot x \in \mathfrak{u}(n+k)$ satisfies $\sigma'_x(0) = -[A, x]$, the gradient vector of f_A at x (see p. 115 of [18]). Notice that under the diffeomorphism $U_{n+k} \cdot x_0 \approx G_{n,n+k}(\mathbb{C})$ this path goes to the path $\tilde{\sigma}_x(t) = \exp(-t[A, x])(p)$. Because of this and the fact that $\gamma_p(t+t_0) = \gamma_{\tilde{p}}(t)$ for some $\tilde{p} \in G_{n,n+k}(\mathbb{C})$, it suffices to show that $\gamma'_p(0) = \tilde{\sigma}'_x(0)$.

Let $\eta : GL_{n+k}(\mathbb{C}) \times G_{n,n+k}(\mathbb{C}) \rightarrow G_{n,n+k}(\mathbb{C})$ be the smooth action of $GL_{n+k}(\mathbb{C})$ on $G_{n,n+k}(\mathbb{C})$ given by treating an element of $GL_{n+k}(\mathbb{C})$ as a linear map and evaluating it on a plane, and let $\eta_p : GL_{n+k}(\mathbb{C}) \rightarrow G_{n,n+k}(\mathbb{C})$ be induced from η . In this notation $\gamma_p(t) = \eta_p(\exp(-itA))$ and $\tilde{\sigma}_x(t) = \eta_p(\exp(-t[A, x]))$. By the chain rule we have $\gamma'_p(0) = d\eta_p(I_{n+k})(-iA)$ and $\tilde{\sigma}'_x(0) = d\eta_p(I_{n+k})(-[A, x])$, where I_{n+k} is the identity matrix of size $(n+k) \times (n+k)$. Thus we will be done once we show that $d\eta_p(I_{n+k})(-iA + [A, x]) = 0$.

To show that $-iA + [A, x] \in \ker d\eta_p(I_{n+k})$ we apply the previous lemma and show that

$$-iA + [A, x] \in T_{I_{n+k}}(GL_{n+k}(\mathbb{C})_p) = U(T_{I_{n+k}}(GL_{n+k}(\mathbb{C})_{\mathbb{C}^n}))U^{-1}.$$

If we use column orientation for \mathbb{C}^{n+k} , then the stabilizer of $\mathbb{C}^n \in G_{n,n+k}(\mathbb{C})$ consists of those elements of $GL_{n+k}(\mathbb{C})$ whose lower left $k \times n$ block is zero, and since $GL_{n+k}(\mathbb{C})$ is an open subset of $\mathbb{C}^{(n+k)^2}$ the tangent space at I_{n+k} of the stabilizer of \mathbb{C}^n consists of those matrices in the tangent space whose lower left $k \times n$ block is zero. Therefore all we need to do is to write $-iA + [A, x]$ as such a matrix conjugated with U :

$$\begin{aligned} -iA + [A, x] &= -iA + [A, Ux_0U^{-1}] \\ &= -iA + U[U^{-1}AU, x_0]U^{-1} \\ &= U(-iU^{-1}AU + [U^{-1}AU, x_0])U^{-1} \end{aligned}$$

Letting $Y = U^{-1}AU$, one easily sees that $-iY + [Y, x_0]$ does have its lower left $k \times n$ block identically zero. \square

Proof of Theorem 5. Let $A \in u(n+k)$. Since the Killing form is invariant under the adjoint action and every skew-Hermitian matrix can be conjugated by a unitary matrix to a diagonal matrix, we may assume that A is diagonal. Moreover, by reordering the basis elements if necessary we may assume that the eigenvalues of A (divided by i) are increasing. The matrix $\exp(-itA)$ is then a diagonal matrix whose diagonal entries are all of the form $e^{a_j t}$, where $a_j \in \mathbb{R}$ and $a_j \leq a_{j+1}$ for all $1 \leq j \leq n+k$. Let $V_{n,n+k}(\mathbb{C})$ be the complex Stiefel manifold. Then $\pi : V_{n,n+k}(\mathbb{C}) \rightarrow G_{n,n+k}(\mathbb{C})$ is a smooth fiber bundle. If we think of an element of $V_{n,n+k}(\mathbb{C})$ as an $(n+k) \times n$ matrix of complex numbers, then $\exp(-itA)$ acts on $V_{n,n+k}(\mathbb{C})$ by multiplication on the left. This gives a flow on $V_{n,n+k}(\mathbb{C})$, which commutes with the projection π since $GL_n(\mathbb{C})$ acts on $V_{n,n+k}(\mathbb{C})$ by multiplication on the right (using column orientation for \mathbb{C}^{n+k}) to give the quotient space $V_{n,n+k}(\mathbb{C})/GL_n(\mathbb{C}) \approx G_{n,n+k}(\mathbb{C})$.

Let M and N be critical submanifolds of $G_{n,n+k}(\mathbb{C})$ such that there exists a flow from M to N . Let $m \in M$ and choose $n \in N$ such that there is a flow from m to n . Let $\tilde{m}, \tilde{n} \in V_{n,n+k}$ be the matrices in reduced column echelon form such that $\pi(\tilde{m}) = m$, $\pi(\tilde{n}) = n$ and the first non-zero entry in each column of m and n is 1. Since $\exp(-itA)$ is diagonal with its eigenvalues increasing, it is easy to see that $\pi^{-1}(W^u(m))$ contains all matrices in $V_{n,n+k}(\mathbb{C})$ whose topmost non-zero entries in column j agree with all the non-zero entries in column j of \tilde{m} for $j = 1, \dots, n$. Similarly, $\pi^{-1}(W^s(n))$ contains all matrices in $V_{n,n+k}(\mathbb{C})$ whose bottommost non-zero entries in column j agree with all the non-zero entries in column j of \tilde{n} for $j = 1, \dots, n$. Hence, $\pi^{-1}(W^u(m)) \cap \pi^{-1}(W^s(n))$, and since $\pi : V_{n,n+k}(\mathbb{C}) \rightarrow G_{n,n+k}(\mathbb{C})$ is locally trivial we have $W^u(m) \cap W^s(n)$. \square

3. THE FLOER COHOMOLOGY OF $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$

Consider $\mathbb{C}[z, z^{-1}]$, the ring of Laurent polynomials. As a vector space $\mathbb{C}[z, z^{-1}]$ is isomorphic to \mathbb{C}^∞ and so the Grassmann manifold $G_n(\mathbb{C}[z, z^{-1}])$, which consists of all n -dimensional planes in $\mathbb{C}[z, z^{-1}]$, is homeomorphic to $G_n(\mathbb{C}^\infty)$. In order to put a smooth structure on $G_n(\mathbb{C}[z, z^{-1}])$ we write it as a direct limit as follows.

Let Λ denote the set of all finite subsets of \mathbb{Z} . That is, $\Lambda = \{\lambda \in \mathcal{P}(\mathbb{Z}) \mid |\lambda| < \infty\}$. Λ is a directed set whose partial ordering is given by inclusion \subseteq . For any $\alpha \in \Lambda$ let $\mathbb{C}[z, z^{-1}]^\alpha \subseteq \mathbb{C}[z, z^{-1}]$ be the vector space spanned by $\{z^j\}_{j \in \alpha}$. Let $\alpha, \beta \in \Lambda$ with $\alpha \subseteq \beta$. Then $\mathbb{C}[z, z^{-1}]^\alpha \subseteq \mathbb{C}[z, z^{-1}]^\beta$ induces an inclusion $j_\alpha^\beta :$

$G_n(\mathbb{C}[z, z^{-1}]^\alpha) \hookrightarrow G_n(\mathbb{C}[z, z^{-1}]^\beta)$ which sends an n -plane in $\mathbb{C}[z, z^{-1}]^\alpha$ to itself viewed as a plane in $\mathbb{C}[z, z^{-1}]^\beta$. Thus $\{G_n(\mathbb{C}[z, z^{-1}]^\alpha), j_\alpha^\beta\}_{\alpha, \beta \in \Lambda}$ is a direct system of smooth manifolds whose direct limit is $G_n(\mathbb{C}[z, z^{-1}])$. This describes the smooth structure on $G_n(\mathbb{C}[z, z^{-1}])$. Note that $G_n(\mathbb{C}[z, z^{-1}])$ is a smooth manifold modeled on the space $\bigoplus_{j=-\infty}^\infty \mathbb{C}$. In particular, $G_n(\mathbb{C}[z, z^{-1}])$ is **not** a Banach manifold.

We can also use the indexing set Λ to define direct systems of unitary groups and their Lie algebras. Fix any $\lambda \in \Lambda$ and let $U(\lambda)$ denote the unitary group of $\mathbb{C}[z, z^{-1}]^\lambda$. Notice that the indexing set for the matrix entries is λ . This gives for all $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ an inclusion $U(\alpha) \subseteq U(\beta)$ given by $(a_{rs}) \mapsto (b_{rs})$, where

$$b_{rs} = \begin{cases} a_{rs} & \text{if } r, s \in \alpha, \\ 0 & \text{if } r \text{ or } s \notin \alpha \text{ and } r \neq s, \\ 1 & \text{if } r \text{ or } s \notin \alpha \text{ and } r = s. \end{cases}$$

In terms of maps this inclusion sends a unitary map $U_\alpha \in U(\alpha)$ to the unitary map $U_\beta \in U(\beta)$ which acts on $\mathbb{C}[z, z^{-1}]^\alpha \subseteq \mathbb{C}[z, z^{-1}]^\beta$ the same way U_α acts on $\mathbb{C}[z, z^{-1}]^\alpha$ and which fixes $\mathbb{C}[z, z^{-1}]^\beta \setminus \mathbb{C}[z, z^{-1}]^\alpha$. Similarly for any $\lambda \in \Lambda$ we define the Lie algebra of skew-Hermitian matrices $u(\lambda)$ to be the associated Lie algebra where the matrix entries are indexed by λ .

If for every $\lambda \in \Lambda$ we choose a skew-Hermitian matrix $A_\lambda \in u(\lambda)$ such that $j_\alpha^\beta(A_\alpha) = A_\beta$ for every $\alpha, \beta \in \Lambda$, then we can define functions $f_{A_\lambda}(x) = \langle x, A_\lambda \rangle$ such that $f_{A_\beta} \circ j_\alpha^\beta = f_{A_\alpha}$. The functions f_{A_λ} are all smooth and hence in the limit define a smooth function $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$. Since the Killing form is compatible with the inclusions j_α^β the gradient flow lines of f_{A_α} get mapped to gradient flow lines of f_{A_β} under j_α^β , and so it makes sense to talk about the gradient flow lines of f_A .

We now choose a set of generic skew-Hermitian matrices $A_\lambda \in u(\lambda)$ and compute the Floer cohomology of the function $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$. A matrix $A_\lambda \in u(\lambda)$ is generic if and only if it has distinct eigenvalues. By Theorem 1, f_{A_λ} will have discrete critical points in this case. For the sake of exposition we will from now on limit ourselves to indexing sets $\lambda \in \Lambda$ of the form

$$\lambda = \{m_1, m_1 + 1, m_1 + 2, \dots, m_2 - 1, m_2\}$$

for some $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$. For every such $\lambda \in \Lambda$ we let $A_\lambda \in u(\lambda)$ be the skew-Hermitian matrix whose m th diagonal entry is $-im$ for every $m \in \lambda$ and all its other entries are zero. We define $f_A = \lim f_{A_\lambda}$. Part 4 of Claim 2 implies that the critical points of f_{A_λ} are the diagonal matrices in the orbit. Since conjugation by a unitary matrix does not change the eigenvalues, the critical points of f_{A_λ} all have exactly n square roots of minus one along the diagonal and all other entries zero. We denote these critical points by x_σ , where $\sigma = (r_1, \dots, r_n)$ is an n -tuple of integers with $r_j \in \lambda$ for all $j = 1, \dots, n$ and $r_1 < r_2 < \dots < r_n$. x_σ is the matrix with i in its (r_j, r_j) entry for all $j = 1, \dots, n$ and 0 in all its other entries. (Recall that we are indexing the entries of the matrices in $u(\lambda)$ by λ .) The corresponding point of $G_n(\mathbb{C}[z, z^{-1}])$ will be denoted by p_σ .

We now describe the unstable manifolds $W^u(p_\sigma)$ of f_{A_λ} . Given any Schubert symbol $\sigma = (r_1, \dots, r_n)$, we recall (see [15], pp. 74-75) that the associated Schubert cell $e(\sigma) \subseteq G_n(\mathbb{C}[z, z^{-1}]^\lambda)$ is defined to be the set of all planes $p \in G_n(\mathbb{C}[z, z^{-1}]^\lambda)$ such that

$$\dim(p \cap \mathbb{C}[z, z^{-1}]^{\{r_1, r_1+1, \dots, r_j\}}) = j$$

and

$$\dim(p \cap \mathbb{C}[z, z^{-1}]^{\{r_1, r_1+1, \dots, r_j-1\}}) = j - 1$$

for all $j = 1, \dots, n$. The following theorem relates the unstable manifolds of f_{A_λ} to the Schubert cells of $G_n(\mathbb{C}[z, z^{-1}]^\lambda)$.

Theorem 8. *For any Schubert symbol $\sigma = (r_1, \dots, r_n)$,*

$$W^u(p_\sigma) = e(\sigma).$$

Proof. Let $\sigma = (r_1, \dots, r_n)$ be any Schubert symbol and let $p \in e(\sigma)$. We must show that for any open neighborhood $U \in G_n(\mathbb{C}[z, z^{-1}]^\lambda)$ containing p_σ , there exists $T < 0$ such that for all $t < T$ we have $\gamma_p(t) = \exp(-itA)(p) \in U$. Since $p \in e(\sigma)$ we can pick a basis v_1, \dots, v_n of p such that for $j = 1, \dots, n$ the vector v_j has 1 in the r_j th entry and 0 in all entries below r_j . The vectors

$$e^{r_1 t} \exp(-itA)(v_1), \dots, e^{r_n t} \exp(-itA)(v_n)$$

span the plane $\exp(-itA)(p)$. Note that for $j = 1, \dots, n$ the entries of the vector $e^{r_j t} \exp(-itA)(v_j)$ are as follows:

$$m\text{th entry of } e^{r_j t} \exp(-itA)(v_j) = \begin{cases} ze^{(r_j-m)t} & \text{some } z \in \mathbb{C} \text{ if } m < r_j, \\ 1 & \text{if } m = r_j, \\ 0 & \text{if } m > r_j. \end{cases}$$

If we consider $\tilde{\gamma}_p(t) = (e^{r_1 t} \exp(-itA)(v_1), \dots, e^{r_n t} \exp(-itA)(v_n))$ as a path in $V_n(\mathbb{C}[z, z^{-1}]^\lambda)$, then we have $\pi(\tilde{\gamma}_p(t)) = \gamma_p(t)$ for all $t \in \mathbb{R}$. Since $(e_{r_1}, \dots, e_{r_n}) \in \pi^{-1}(U)$ where $\pi^{-1}(U)$ is open in $V_n(\mathbb{C}[z, z^{-1}]^\lambda)$ and $\lim_{t \rightarrow -\infty} \tilde{\gamma}_p(t) = (e_{r_1}, \dots, e_{r_n})$, we can choose a $T < 0$ such that for all $t < T$ we have $\tilde{\gamma}_p(t) \in \pi^{-1}(U)$. But this implies that $\gamma_p(t) \in U$ for all $t < T$.

We have shown that $e(\sigma) \subseteq W^u(p_\sigma)$. Now let $p \in W^u(p_\sigma)$. Since the Schubert cells partition $G_n(\mathbb{C}[z, z^{-1}]^\lambda)$, there exists a unique a Schubert symbol $\tilde{\sigma}$ such that $p \in e(\tilde{\sigma}) \subseteq W^u(p_{\tilde{\sigma}})$. But then $p \in W^u(p_{\tilde{\sigma}})$. Since the unstable manifolds are disjoint, this says that $\tilde{\sigma} = \sigma$, and so $p \in e(\sigma)$. \square

Recalling that for any critical point p_σ the unstable manifold theorem says that $W^u(p_\sigma)$ is an embedded disk whose dimension is equal to the index of p_σ , we have the following.

Corollary 9. *The index of p_σ in $G_n(\mathbb{C}[z, z^{-1}]^\lambda)$ can be computed by adding up the number of rows of $x_\sigma \in u(\sigma)$ above each i which consist entirely of zeros and then multiplying by 2.*

The preceding corollary shows that the index of a critical point is not preserved under the inclusions $j_\alpha^\beta : G_n(\mathbb{C}[z, z^{-1}]^\alpha) \rightarrow G_n(\mathbb{C}[z, z^{-1}]^\beta)$. However, the relative index of critical points is preserved under the inclusion $j_\alpha^\beta : G_n(\mathbb{C}[z, z^{-1}]^\alpha) \rightarrow G_n(\mathbb{C}[z, z^{-1}]^\beta)$, where α is in the cofinal system consisting of those sets which include all integers between the smallest and the largest in the set. That is, if p_σ and $p_{\tilde{\sigma}}$ are any two critical points in $G_n(\mathbb{C}[z, z^{-1}]^\alpha)$ where $\alpha = \{m_1, m_1+1, \dots, m_2\}$ for some $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$, then for all $\beta \in \Lambda$

$$\text{index}(p_\sigma) - \text{index}(p_{\tilde{\sigma}}) = \text{index}(j_\alpha^\beta(p_\sigma)) - \text{index}(j_\alpha^\beta(p_{\tilde{\sigma}})).$$

This allows us to define a relative index between critical points of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ analogous to Floer's relative index for critical points of the action functional (see for instance [11], p. 584).

Definition 10. Let $p_\sigma, p_{\bar{\sigma}} \in G_n(\mathbb{C}[z, z^{-1}])$ be critical points of the function $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$. Let $p_{\sigma'}$ be any critical point of f_A such that $p_{\sigma'} \leq p_\sigma$ and $p_{\sigma'} \leq p_{\bar{\sigma}}$. The relative index of p_σ and $p_{\bar{\sigma}}$ is defined to be

$$\text{ind}(p_\sigma, p_{\bar{\sigma}}) = \dim W(p_\sigma, p_{\sigma'}) - \dim W(p_{\bar{\sigma}}, p_{\sigma'}),$$

where $W(a, b) = W^u(a) \cap W^s(b)$ for any two critical points a and b .

We define the Floer chain complex \mathcal{C}_* as follows. Fix any critical point p_0 and for each $j \in \mathbb{Z}$ let C_j be the free abelian group on the critical points p_σ such that $\text{ind}(p_\sigma, p_0) = j$. The Floer chain complex is then defined to be:

$$\cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow C_{-1} \rightarrow C_{-2} \rightarrow \cdots,$$

where the boundary homomorphisms are given on the generators by counting (with sign) the number of gradient flow lines between two critical points of relative index 1. In this case all the boundary homomorphisms are zero, since $C_j = 0$ for j odd. Note that the Floer chain complex is independent of the choice of base point up to degree shift. The homology of this complex is $\bigoplus_{j=1}^{\infty} \mathbb{Z}$ in all even dimensions (positive and negative) and 0 in all odd dimensions. For the sake of exposition we will now fix $p_0 = p_{(0,1,\dots,n-1)}$.

We describe the module structure of $HF^*(G_n(\mathbb{C}[z, z^{-1}]))$ over $H^*(G_n(\mathbb{C}^\infty))$. The filtration

$$\mathbb{C}[z, z^{-1}]^{\{0,1,\dots\}} \subseteq \mathbb{C}[z, z^{-1}]^{\{-1,0,\dots\}} \subseteq \mathbb{C}[z, z^{-1}]^{\{-2,-1,\dots\}} \cdots \subseteq \mathbb{C}[z, z^{-1}]$$

induces a filtration of the Floer chain complex \mathcal{C}_* as follows. For each $N \in \mathbb{Z}_+$ we have

$$G_n(\mathbb{C}[z, z^{-1}]^{\{-N,-N+1,\dots\}}) \subseteq G_n(\mathbb{C}[z, z^{-1}]),$$

and we let $\mathcal{C}_*|_{-N} \subseteq \mathcal{C}_*$ be the chain complex generated by the critical points which lie in $G_n(\mathbb{C}[z, z^{-1}]^{\{-N,-N+1,\dots\}})$. We define the dual of this filtered complex to be $\mathcal{C}^*|_{-N} = \text{Hom}(\mathcal{C}_*|_{-N}, \mathbb{Z})$. Note that $\mathcal{C}^*|_{-N}$ is the dual of the Morse complex for f_A restricted to $G_n(\mathbb{C}[z, z^{-1}]^{\{-N,-N+1,\dots\}})$ but with the groups shifted down $2Nn$ dimensions. In particular, $H^*(\mathcal{C}^*|_{-N})$ is \mathbb{Z} in dimension $-2Nn$, and all the groups below dimension $-2Nn$ are zero.

Definition 11. The Floer cohomology of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ as a module over $H^*(G_n(\mathbb{C}^\infty))$ is defined to be the following direct limit:

$$HF^*(G_n(\mathbb{C}[z, z^{-1}])) = \varinjlim_N H^*(\mathcal{C}^*|_{-N}),$$

where the maps in the direct system are given by identifying each term in the limit with the dual of the Morse complex of f_A restricted to $G_n(\mathbb{C}[z, z^{-1}]^{\{-N,-N+1,\dots\}})$.

In order to make the isomorphisms in the preceding definition explicit we define a degree $2n$ chain homomorphism $S_1 : \mathcal{C}_* \rightarrow \mathcal{C}_{*+2n}$ by $S_1(p_{(r_1,\dots,r_n)}) = p_{(r_1+1,\dots,r_n+1)}$. S_1 sends a critical point of relative index j to a critical point of relative index $j+2n$, and for all $j \in \mathbb{Z}$ it induces an isomorphism

$$H_j(\mathcal{C}_*|_{-1}) \xrightarrow{S_1} H_{j+2n}(\mathcal{C}_*|_0) \approx H_{j+2n}(G_n(\mathbb{C}[z, z^{-1}]^{\{-1,0,\dots\}})).$$

For all $j \in \mathbb{Z}$ we define $p_{(r_1,\dots,r_n)}^* \in \text{Hom}(C_j, \mathbb{Z})$ to be the homomorphism which takes the value 1 on $p_{(r_1,\dots,r_n)} \in C_j$ and 0 on all other critical points. $\mathcal{C}^j|_{-N}$ is the free abelian group on all $p_{(r_1,\dots,r_n)}^*$ such that $p_{(r_1,\dots,r_n)} \in G_n(\mathbb{C}[z, z^{-1}]^{\{-N,-N+1,\dots\}})$

and $\text{ind}(p_{(r_1, \dots, r_n)}, p_0) = j$. $S_1 : \mathcal{C}_* \rightarrow \mathcal{C}_{*+2n}$ induces a degree $2n$ cochain homomorphism $S_1^* : \mathcal{C}^* \rightarrow \mathcal{C}^{*+2n}$ defined by $S_1^*(p_{(r_1, \dots, r_n)}^*) = p_{(r_1+1, \dots, r_n+1)}^*$. For all $j \in \mathbb{Z}$, S_1^* induces an isomorphism

$$H^j(\mathcal{C}^*|_{-1}) \xrightarrow{S_1^*} H^{j+2n}(\mathcal{C}^*|_0) \approx H^{j+2n}(G_n(\mathbb{C}[z, z^{-1}]\{-1, 0, \dots\})).$$

Similarly, for all $N \in \mathbb{Z}$ we define a degree $2Nn$ chain homomorphism $S_N : \mathcal{C}_* \rightarrow \mathcal{C}_{*+2Nn}$ by $S_N(p_{(r_1, \dots, r_n)}) = p_{(r_1+N, \dots, r_n+N)}$ and a degree $2Nn$ cochain homomorphism $S_N^* : \mathcal{C}^* \rightarrow \mathcal{C}^{*+2Nn}$ by $S_N^*(p_{(r_1, \dots, r_n)}^*) = p_{(r_1+N, \dots, r_n+N)}^*$. For all $j \in \mathbb{Z}$, S_N and S_N^* induce isomorphisms

$$\begin{aligned} H_j(\mathcal{C}_*|_{-N}) &\xrightarrow{S_N} H_{j+2Nn}(\mathcal{C}_*|_0) \approx H_{j+2Nn}(G_n(\mathbb{C}[z, z^{-1}]\{-N, -N+1, \dots\})), \\ H^j(\mathcal{C}^*|_{-N}) &\xrightarrow{S_N^*} H^{j+2Nn}(\mathcal{C}^*|_0) \approx H^{j+2Nn}(G_n(\mathbb{C}[z, z^{-1}]\{-N, -N+1, \dots\})). \end{aligned}$$

Theorem 12. *For all $N \in \mathbb{Z}_+$ the following diagram commutes:*

$$\begin{array}{ccc} H^*(\mathcal{C}^*|_{-N}) & \xrightarrow{\quad \quad \quad} & H^*(\mathcal{C}^*|_{-(N+1)}) \\ S_N^* \Big\downarrow & & \Big\downarrow S_{N+1}^* \\ H^{*+2Nn}(G_n(\mathbb{C}[z, z^{-1}]\{-N, \dots\})) & \xrightarrow{\cup c_n} & H^{*+2(N+1)n}(G_n(\mathbb{C}[z, z^{-1}]\{-(N+1), \dots\})) \end{array}$$

where c_n is the Euler class of γ_n , the tautological n -plane bundle over $G_n(\mathbb{C}^\infty)$.

Corollary 13. *The Floer cohomology of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ as a module over $H^*(G_n(\mathbb{C}^\infty))$ is*

$$HF^*(G_n(\mathbb{C}[z, z^{-1}])) = \varinjlim_N H^*(\mathcal{C}^*|_{-N}) \approx H^*(G_n(\mathbb{C}^\infty))[c_n^{-1}],$$

where c_n denotes the Euler class of γ_n , the tautological n -plane bundle over $G_n(\mathbb{C}^\infty)$.

Proof of Theorem 12. Recall from Theorem 8 that for all Schubert symbols σ we have $W^u(p_\sigma) = e(\sigma)$. This makes the isomorphism $H^*(\mathcal{C}^*|_0) \approx H^*(G_n(\mathbb{C}^\infty))$ very explicit. The isomorphism identifies $[p_{(r_1, \dots, r_n)}^*] \in H^*(\mathcal{C}^*|_0)$ with the cohomology class in $H^*(G_n(\mathbb{C}^\infty)) \approx \text{Hom}(H_*(G_n(\mathbb{C}^\infty)), \mathbb{Z})$ which has value 1 on $[e(r_1, \dots, r_n)] \in H_*(G_n(\mathbb{C}^\infty))$ and is zero on all other homology classes. From this description we see that the map in question sends the cohomology class represented by the dual of $[e(r_1, \dots, r_n)]$ to the cohomology class represented by the dual of $[e(r_1+1, \dots, r_n+1)]$. Using the Schubert calculus one sees that this map is cupping with c_n (see for instance [17], Th. 2.5, p. 229, and Th. 3.0, p. 230). \square

4. THE PRO-SPECTRUM ASSOCIATED TO \mathcal{C}_{f_A}

Let X be any compact space and let $\xi \in K(X)$. There exist a vector bundle $E \rightarrow X$ and a number $m \in \mathbb{Z}_+$ such that $\xi = E - \varepsilon_m$, where ε_m is the trivial bundle of dimension m . We define

$$\mathbf{X}^\xi = S^{-m} \mathbf{X}^E,$$

where \mathbf{X}^E is the suspension spectrum of the Thom space X^E . If X is a CW complex of finite type, then (following [8], p. 322) for all $m \in \mathbb{Z}_+$ we choose vector bundles E_m over the m -skeleton $X^{(m)}$ such that $\xi|_{X^{(m)}} = E_m - \varepsilon_{k_m} \in K(X^{(m)})$

for some $k_m \in \mathbb{Z}_+$ and such that for any $m_1, m_2 \in \mathbb{Z}_+$ there is a bundle map $E_{m_1} \oplus \varepsilon_{(k_{m_2}-k_{m_1})} \rightarrow E_{m_2}$ covering the inclusion $X^{(m_1)} \rightarrow X^{(m_2)}$ which is an isomorphism on fibers. This gives for all $m \in \mathbb{Z}_+$ maps

$$S^{k_{m+1}-k_m}(X^{(m)})^{E_m} \rightarrow (X^{(m+1)})^{E_{m+1}},$$

which together with the spaces $\{S^{m+|\xi|-|E_m|}(X^{(m)})^{E_m}\}$ define the Thom spectrum \mathbf{X}^ξ . This Thom spectrum is independent of the choices made, up to homotopy.

Now consider the following situation for X a compact topological space. Let η, ζ and ξ be complex vector bundles over X with $\eta \oplus \zeta = \xi$. Then the inclusion $E(\eta) \subseteq E(\xi)$ induces a map $X^\eta \rightarrow X^\xi$. Now assume $\eta, \xi \in K(X)$. Then we get a similar map as follows. Let $\eta = F - \varepsilon_m$ and $\xi = F \oplus \zeta - \varepsilon_m$. (We are still assuming that ζ is a vector bundle.) By definition we have

$$\mathbf{X}^\xi = S^{-m}(\mathbf{X}^{F+\zeta}) \text{ and } \mathbf{X}^\eta = S^{-m}(\mathbf{X}^F),$$

and so we have an induced map $\mathbf{X}^\eta \rightarrow \mathbf{X}^\xi$. For any vector bundle $E \rightarrow X$ and for any $N \in \mathbb{Z}_+$ we have $-(N+1)E + E = -NE$. This defines a map

$$\mathbf{X}^{-NE} \leftarrow \mathbf{X}^{-(N+1)E}.$$

A similar situation holds when X is a CW complex of finite type. (See [8], pp. 320-323, for more details.)

The following theorem shows that the algebraic computations done in the previous section can be realized geometrically.

Theorem 14. *For all $N \in \mathbb{Z}_+$ the maps*

$$G_n(\mathbb{C}^\infty)^{-(N+1)\gamma_n} \rightarrow G_n(\mathbb{C}^\infty)^{-N\gamma_n}$$

induced by $-(N+1)\gamma_n + \gamma_n = -N\gamma_n$ make the following diagram commute:

$$\begin{array}{ccc} H^*(G_n(\mathbb{C}^\infty)^{-N\gamma_n}) & \longrightarrow & H^*(G_n(\mathbb{C}^\infty)^{-(N+1)\gamma_n}) \\ \downarrow & & \downarrow \\ H^*(\mathcal{C}^*|_{-N}) & \longrightarrow & H^*(\mathcal{C}^*|_{-(N+1)}) \end{array}$$

where the vertical maps are given by the Thom Isomorphism Theorem.

Proof. By Theorem 12 the following diagram commutes:

$$\begin{array}{ccc} H^*(\mathcal{C}^*|_{-N}) & \longrightarrow & H^*(\mathcal{C}^*|_{-(N+1)}) \\ S_N^* \downarrow & & S_{N+1}^* \downarrow \\ H^{*+2Nn}(G_n(\mathbb{C}^\infty)) & \xrightarrow{\cup c_n} & H^{*+2(N+1)n}(G_n(\mathbb{C}^\infty)) \end{array}$$

and it follows from the Thom Isomorphism Theorem that the following diagram commutes:

$$\begin{array}{ccc} H^*(G_n(\mathbb{C}^\infty)^{-N\gamma_n}) & \longrightarrow & H^*(G_n(\mathbb{C}^\infty)^{-(N+1)\gamma_n}) \\ \downarrow & & \downarrow \\ H^{*+2Nn}(G_n(\mathbb{C}^\infty)) & \xrightarrow{\cup c_n} & H^{*+2(N+1)n}(G_n(\mathbb{C}^\infty)) \end{array}$$

□

In [6] and [8] Cohen, Jones, and Segal defined the Floer homotopy type of a function such as $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ which takes into account the above geometry. The Floer homotopy type is an inverse system of spectra constructed from the flow category of f_A (see [7] and [13]).

Definition 15. *The flow category of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$, denoted \mathcal{C}_{f_A} , is the topological category whose objects are the critical points of f_A topologized as a subspace of $G_n(\mathbb{C}[z, z^{-1}])$ and whose morphisms are the unparameterized piecewise gradient flow lines of f_A . That is, for any two critical points a and b , $\text{Mor}(a, b)$ is defined to be the space of all continuous curves $\omega : [f_A(b), f_A(a)] \rightarrow G_n(\mathbb{C}[z, z^{-1}])$ satisfying*

1. $\omega(f_A(b)) = b$,
2. $\omega(f_A(a)) = a$,
3. *away from the critical points of f_A the map ω is smooth and satisfies the following differential equation:*

$$\frac{d\omega}{dt} = \frac{\nabla(f_A)}{\|\nabla(f_A)\|^2}.$$

$\text{Mor}(a, b)$ is topologized as a subset of the space of all continuous maps from the closed interval $[f_A(b), f_A(a)]$ to $G_n(\mathbb{C}[z, z^{-1}])$. This space of continuous maps is given the compact open topology.

Let $a, b \in \text{Ob}(\mathcal{C}_{f_A})$ with $b \leq a$, i.e. there exists a gradient flow line of f_A beginning at a and ending at b . We will use the notation $\mathcal{C}_{f_A}|_b^a$ to denote the full subcategory whose objects are the critical points between a and b . $BC_{f_A}|_b^a$ denotes the geometric realization of this restricted category. In [6] Cohen, Jones, and Segal prove the following theorem.

Theorem 16. *Let $f : M \rightarrow \mathbb{R}$ be a Morse function defined on a closed Riemannian manifold M . Then associated to f is a topological category \mathcal{C}_f whose objects are the critical points of f and whose space of morphisms between critical points a and b is the space of piecewise flow lines, of the gradient flow of f , joining a to b .*

(1) If f is a generic Morse function (one whose gradient flow satisfies the Morse-Smale transversality condition), then there is a homeomorphism

$$BC_f \cong M.$$

(2) For any Morse function there is a homotopy equivalence

$$BC_f \simeq M.$$

To define the Floer homotopy type of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ we will use the following notation. Let

$$I = \{b \in \text{Ob}(\mathcal{C}_{f_A}) \mid \overline{W(p_0, b)} \text{ is a manifold}\}.$$

For each $b \in I$ let

$$I_b = \{a \in \text{Ob}(\mathcal{C}_{f_A}) \mid p_0 \leq a \text{ and } \overline{W(a, b)} \text{ is a manifold}\}.$$

Note that for every $a \in I_b$ the preceding theorem says that $BC_{f_A}|_b^{p_0} \cong \overline{W(p_0, b)}$ and $BC_{f_A}|_b^a \cong \overline{W(a, b)}$, and so we have smooth structures defined on $BC_{f_A}|_b^{p_0}$ and $BC_{f_A}|_b^a$. For each $a \in I_b$ let ν_a be the normal bundle

$$BC_{f_A}|_b^{p_0} \xrightarrow{\nu_a} BC_{f_A}|_b^a.$$

Note that if $a_1, a_2 \in I_b$ with $a_1 \leq a_2$, then $\nu_{a_1} \oplus \eta = \nu_{a_2}$, where η is the normal bundle of $BC_{f_A}|_b^{a_1} \hookrightarrow BC_{f_A}|_b^{a_2}$ restricted to $BC_{f_A}|_b^{p_0}$, and so we have a map

$$(BC_{f_A}|_b^{p_0})^{-\nu_{a_1}} \leftarrow (BC_{f_A}|_b^{p_0})^{-\nu_{a_2}}$$

defined. These maps define an inverse system of Thom spaces. The inverse systems $\{(BC_{f_A}|_b^{p_0})^{-\nu_a}\}_{a \in I_b}$ form a direct system as follows. For any $b_1, b_2 \in I$ we say $b_1 \triangleleft b_2$ if and only if $BC_{f_A}|_{b_1}^{p_0} \subseteq BC_{f_A}|_{b_2}^{p_0}$ and $I_{b_1} \subseteq I_{b_2}$. If $b_1 \triangleleft b_2$ then we have an inclusion map

$$\{(BC_{f_A}|_{b_1}^{p_0})^{-\nu_a}\}_{a \in I_{b_1}} \rightarrow \{(BC_{f_A}|_{b_2}^{p_0})^{-\nu_a}\}_{a \in I_{b_2}}.$$

Definition 17. *The Floer homotopy type of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ is defined to be*

$$\varinjlim_{b \in I} \{(BC_{f_A}|_b^{p_0})^{-\nu_a}\}_{a \in I_b}.$$

The following theorem shows how the above algebraic computation of the Floer cohomology of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ is reflected in its Floer homotopy type.

Theorem 18. *The Floer homotopy type of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ has the following inverse system of spectra:*

$$G_n(\mathbb{C}^\infty) \leftarrow G_n(\mathbb{C}^\infty)^{-\gamma_n} \leftarrow G_n(\mathbb{C}^\infty)^{-2\gamma_n} \leftarrow \dots$$

as a cofinal system, where the maps are induced from $-(N+1)\gamma_n + \gamma_n = -N\gamma_n \in K(G_n(\mathbb{C}^\infty))$ for all $N \in \mathbb{Z}_+$.

Proof. Let $p_j = p_{(j, j+1, \dots, j+n-1)}$ for all $j \in \mathbb{Z}$. Then for $j < -n$ we have $BC_{f_A}|_{p_j}^{p_0} \cong G_n(\mathbb{C}[z, z^{-1}]^{\{j, j+1, \dots, n-1\}})$. Moreover, the normal bundle of $BC_{f_A}|_{p_j}^{p_0} \hookrightarrow BC_{f_A}|_{p_j}^{p_m}$ is $m\gamma_n$ for all $m \in \mathbb{N}$. Hence $\{(BC_{f_A}|_{p_j}^{p_0})^{-\nu_a}\}_{a \in I_{p_j}}$ has

$$G_n(\mathbb{C}[z, z^{-1}]^{\{j, j+1, \dots, n-1\}}) \leftarrow G_n(\mathbb{C}[z, z^{-1}]^{\{j, j+1, \dots, n-1\}})^{-\gamma_n} \leftarrow \dots$$

as a cofinal system. Taking a direct limit as $j \rightarrow -\infty$, we see that

$$G_n(\mathbb{C}[z, z^{-1}]^{\{\dots, n-2, n-1\}}) \leftarrow G_n(\mathbb{C}[z, z^{-1}]^{\{\dots, n-2, n-1\}})^{-\gamma_n} \leftarrow \dots$$

is a cofinal system of the Floer homotopy type of $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$. \square

Note that the Floer cohomology module computed algebraically in the previous section can be recovered from the above cofinal system using the Thom Isomorphism Theorem as the direct limit of the following:

$$H^*(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup_{c_n}} H^{*+2n}(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup_{c_n}} H^{*+4n}(G_n(\mathbb{C}^\infty)) \xrightarrow{\cup_{c_n}} \dots$$

Hence the Floer cohomology $f_A : G_n(\mathbb{C}[z, z^{-1}]) \rightarrow \mathbb{R}$ is recoverable from its Floer homotopy type.

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