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L^2–HOMOLOGY OVER TRACED *-ALGEBRAS

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ABSTRACT. Given a unital complex *-algebra A, a tracial positive linear functional τ on A that factors through a *-representation of A on Hilbert space, and an A-module M possessing a resolution by finitely generated projective A-modules, we construct homology spaces $H_k(A,\tau,M)$ for $k = 0, 1, \dots$. Each is a Hilbert space equipped with a *-representation of A, independent (up to unitary equivalence) of the given resolution of M. A short exact sequence of A-modules gives rise to a long weakly exact sequence of homology spaces. There is a Künneth formula for tensor products. The von Neumann dimension which is defined for A-invariant subspaces of $L^2(A,\tau)^n$ gives well-behaved Betti numbers and an Euler characteristic for M with respect to A and τ.

1. INTRODUCTION

The ingredients for the theory developed in this paper are a complex unital *-algebra A, a tracial positive linear functional τ on A that factors through a *-representation of A on Hilbert space, and an A-module M possessing a resolution by finitely generated projective A-modules, that is, an exact sequence

$$\dots \to V_2 \to V_1 \to V_0 \to M \to 0$$

of A-modules, where each V_j is a direct summand of the direct sum of finitely many copies of A.

The prototypical example of this situation comes from from group cohomology. Suppose a group G acts by orientation-preserving cellular maps on a contractible oriented cell complex X with finite cell stabilizers and finitely many k-cell orbits for each k. Here, the algebra A is the complex group algebra $\mathbb{C}G$ with involution $g^* = g^{-1}$. Any positive-definite class function on G — the traditional choice is the indicator function of $\{1\}$ — furnishes a suitable trace τ. The module M is $\mathbb{C}$, with trivial G-action. To obtain a finite-rank projective resolution of M, let V_k be the space of finitely-supported complex functions on the set of k-cells, and map V_k to V_{k-1} by the usual boundary map of cellular homology. Contractibility of X makes the sequence exact. Each V_k is finite-rank projective because it is the direct sum of finitely many modules of the form $\mathbb{C}Gp_\sigma$, where p_σ is the idempotent in $\mathbb{C}G$ obtained by averaging the group elements in the stabilizer of the k-cell σ.

In the general setting, we use the trace τ to manufacture from the given resolution a chain complex of Hilbert spaces. The homology of this complex, taken in the weak sense of kernels modulo image closures, turns out to be quite well-behaved in several respects. More specifically, we choose for each k an embedding of V_k as a direct summand of some A^{n_k}. We have a map $J : A^{n_k} \to L^2(A,\tau)^{n_k}$, where the Hilbert

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space $L^2(A,\tau)$ comes from the GNS construction. The natural left action of A restricts to give a *-representation of A on the Hilbert space $\overline{J(V_k)}$. The maps Δ in the original resolution give rise to A-intertwining bounded operators $\overline{\Delta}$ between successive $\overline{J(V_k)}$'s such that $\overline{\Delta}J = J\Delta$. In the resulting sequence

$$\dots \xrightarrow{\overline{\Delta}_3} \overline{J(V_2)} \xrightarrow{\overline{\Delta}_2} \overline{J(V_1)} \xrightarrow{\overline{\Delta}_1} \overline{J(V_0)} \longrightarrow 0,$$

we have $\overline{\Delta}_k\ \overline{\Delta}_{k+1} = 0$.

The gist of section 2 below is that the homology spaces

$$H_k(A,\tau,M) = \ker \overline{\Delta}_k \ominus \overline{\operatorname{im} \overline{\Delta}_{k+1}}$$

depend neither on how the V_k's are realized as direct summands of finite-rank free modules, nor on which finitely generated projective resolution of M we use; different choices lead to Hilbert spaces with unitarily equivalent *-representations of A. In section 3, we show that a short exact sequence of A-modules with finitely generated projective resolutions gives rise to a long weakly exact sequence of homology spaces. Section 4 gives a Künneth formula for the homology of the tensor product of a pair of modules over a pair of algebras in terms of the homology of the two tensor factors. In the same section, we also consider the Betti numbers obtained as von Neumann dimensions of homology spaces, using the dimension function on left A-invariant subspaces of $L^2(A,\tau)^n$ that comes from taking traces of projections in the von Neumann algebra generated by the right action of $A \otimes M_n$. When there are only finitely many non-zero Betti numbers, the alternating sum of these gives an Euler characteristic which is well-behaved on tensor products and short exact sequences. The final section of the paper is an extended example involving finite-dimensional modules over the "quantum group" $U_q(\mathbf{sl}_2)$ recently introduced and studied in [9].

There is ample precedent for what we do here in the special case $A = \mathbb{C}G$, equipped with its natural trace τ_0 (the one that comes from the indicator function of $\{1\}$). The nearest predecessors are [11], for $H_1(\mathbb{C}G,\tau_0,\mathbb{C})$ in terms of G-actions on graphs, and [12], which treats $H_1(\mathbb{C}G,\tau_0,M)$ for finitely presented M in the spirit of the present paper. For other recent work in which $H_1(\mathbb{C}G,\tau_0,\mathbb{C})$ plays a major role, see [1] and [3]. Looking somewhat further back, the fundamental idea of forming weak homology or cohomology spaces from a complex of G-Hilbert spaces and then measuring their sizes by means of von Neumann dimension may be found for instance in [2], [8], and, considerably elaborated, in [5] and [6].

2. The basic apparatus

By a traced *-algebra, we mean a pair (A,τ), where A is a complex unital *-algebra, and τ is a linear functional on A satisfying: *(i)* $\tau(ab) = \tau(ba)\ \ \forall\ a,b \in A$; *(ii)* $\tau(a^*a) \geq 0\ \ \forall\ a \in A$; and *(iii)* $\sup_k \tau((a^*a)^k)^{1/k} < \infty\ \ \forall\ a \in A$. We will call such a τ a representable trace on A. Notice that τ is necessarily selfadjoint: $\tau(a^*) = \overline{\tau(a)}$ follows from $\tau((1+a^*)(1+a)) \geq 0$ and $\tau((1-ia^*)(1+ia)) \geq 0$. The GNS construction yields a Hilbert space $L^2(A,\tau)$ on which (thanks to condition *(iii)*) A acts boundedly by a *-representation and a commuting *-antirepresentation. In somewhat more detail, this works as follows. Let K denote the left (= right) kernel of τ, *i.e.* the *-ideal $\{a \in A : \tau(a^*a) = 0\}$. Then A/K has an inner product $\langle\cdot,\cdot\rangle$ defined by $\langle a+K, b+K\rangle = \tau(b^*a)$; the Hilbert space completion is $L^2(A,\tau)$. For a,b in A, we calculate

$$\| ab + K \|^2 = \tau(b^*a^*ab) \le \tau(b^*b)^{1/2}\tau(b^*(a^*a)^2b)^{1/2}$$
$$\le \tau(b^*b)^{1/2}\tau(b^*b)^{1/4}\tau(b^*(a^*a)^4b)^{1/4} \le \ldots \le \tau(b^*b)^{1-2^{-n}}\tau(b^*(a^*a)^{2^n}b)^{2^{-n}}$$
$$= \tau(b^*b)^{1-2^{-n}}\tau(bb^*(a^*a)^{2^n})^{2^{-n}} \le \tau(b^*b)^{1-2^{-n}}\tau((bb^*)^2)^{2^{-n-1}}\tau((a^*a)^{2^{n+1}})^{2^{-n-1}},$$

and in the limit we obtain $\| ab + K \|^2 \le \| b + K \|^2 \sup_k \tau((a^*a)^k)^{1/k}$. Because $\| c+K \| = \| c^*+K \|$, we also have $\| ba+K \|^2 \le \| b+K \|^2 \sup_k \tau((aa^*)^k)^{1/k}$. Thus for each a in A, the maps $b+K \mapsto ab+K$ and $b+K \mapsto ba+K$ extend to bounded operators $\lambda(a)$ and $\rho(a)$ on $L^2(A,\tau)$. The von Neumann algebras $\lambda(A)''$ and $\rho(A)''$ are commutants of one another, and $\xi_0 \equiv 1 + K$ is a cyclic and separating trace vector for each. With no great risk of ambiguity, we will denote the corresponding faithful trace on each of these algebras by τ.

We define an (A,τ)-space to be an A-invariant subspace of $L^2(A,\tau)^n = L^2(A,\tau) \otimes \mathbb{C}^n$ for some n, where A acts on the latter via the representation $\lambda \otimes \mathrm{id}_n$. Two (A,τ)-spaces will be called equivalent ($\approx$) if there is a unitary map from the closure of one onto the closure of the other intertwining the two A-actions. We obtain a well-behaved dimension function on the (A,τ)-spaces which respects this equivalence as follows. Let L be an (A,τ)-subspace of some $L^2(A,\tau)^n$. Invariance under $\lambda(A)\otimes\mathrm{id}_n$ forces the projection P of $L^2(A,\tau)^n$ on $\overline{L}$ to belong to $\rho(A)''\otimes M_n$. We have on the latter von Neumann algebra the faithful trace $\tau\otimes\mathrm{tr}_n$, where tr_n sums the diagonal entries of an $n\times n$ matrix. We write $\dim(L) = \dim_{A,\tau}(L)$ for the nonnegative real number $(\tau\otimes\mathrm{tr}_n)(P)$, *i.e.* if $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis for $\mathbb{C}^n$, then

$$\dim(L) = \sum_{j=1}^{n}\langle P(\xi_0\otimes\epsilon_j), \xi_0\otimes\epsilon_j\rangle.$$

This dimension function, called the von Neumann dimension, is familiar in the case of a complex group algebra and its natural trace (see [2], [5], [6], [7]). It works equally well in our more general setting. (For the underlying theory of finite von Neumann algebras that makes it work, see for instance Chapters 6 and 8 of [10].) Because it comes from a trace, $\dim(L)$ is independent of how L is realized as an (A,τ)-space, and equivalent (A,τ)-spaces have the same dimension. Because the (von Neumann algebra) trace in question is faithful, we have $\dim(L_0) = \dim(L)$ for an (A,τ)-subspace L_0 of L only if $\overline{L_0} = \overline{L}$. If L_1 and L_2 are closed (A,τ)-spaces and $t : L_2 \to L_1$ is a bounded operator intertwining the respective A-actions, then $\dim(\operatorname{im} t) = \dim(\operatorname{im} t^*)$ by polar decomposition, so we have the "rank-nullity" formula

$$\dim(\ker t) + \dim(\operatorname{im} t) = \dim(L_2).$$

As a consequence, if t is injective and has dense range, then L_1 and L_2 are equivalent; the requisite intertwining unitary operator comes from the polar decomposition of t. Further, the existence of bounded A-linear operators in both directions, with either both injective or both having dense range, implies $L_1 \approx L_2$. If K is a closed A-invariant subspace of the closed (A,τ)-space, then L/K becomes a closed (A,τ)-space when we identify it with $L \ominus K = L \cap K^\perp$, and of course $\dim(L/K) = \dim(L) - \dim(K)$.

Module maps between finitely generated projective A-modules give rise to operators between (A,τ)-spaces as follows. For a finite-rank free A-module A^n, factoring out K in each entry gives a map $J = J_n : A^n \to L^2(A,\tau)^n$. To avoid needlessly burdensome subscripting, we will use J in all-purpose fashion to denote the map from any submodule of a finite-rank free module to the corresponding (A,τ)-space. Let

$\Delta : V \to W$ be a module map between two finitely generated projective A-modules. By realizing V and W as direct summands of A^m and A^n respectively, we obtain a module map $D : A^m \to A^n$ whose restriction to V is Δ. The map D is given by the right action of an $m \times n$ matrix with entries in A on rows of length m with entries in A. Applying the *-antirepresentation ρ to each entry of this matrix gives a bounded operator $\overline{D} : \overline{J(A^m)} \to \overline{J(A^n)}$. We obtain $\overline{\Delta} : \overline{J(V)} \to \overline{J(W)}$ by restricting $\overline{D}$ to $\overline{J(V)}$; more precisely, $\overline{\Delta} = \overline{ED}|_{\overline{J(V)}}$, where $E : A^n \to A^n$ is an idempotent map with range W. This makes $\overline{\Delta}$ a bounded A-linear operator between (A, τ)-spaces satisfying $J\Delta = \overline{\Delta}J$. Although our notation does not reflect this, the operator $\overline{\Delta}$ depends on how V and W fit inside the two free modules. It follows easily from our remarks in the previous paragraph, however, that the domain, codomain, kernel, and image of $\overline{\Delta}$ change to equivalent (A, τ)-spaces when the complemented embeddings of V and W are changed. Furthermore, if $\Gamma : W \to X$ is a module map to a third finitely generated projective module, and the same free module embedding of W is used to define both $\overline{\Delta}$ and $\overline{\Gamma}$, then $\overline{\Gamma\Delta} = \overline{\Gamma}\,\overline{\Delta}$ (because these two bounded operators agree on $J(V)$.)

Let M be an A-module with a finitely generated projective resolution, that is, an exact sequence $\mathcal{V}$

$$\cdots \to V_2 \xrightarrow{\Delta_2} V_1 \xrightarrow{\Delta_1} V_0 \xrightarrow{\delta} M \to 0$$

of A-modules and module maps, where each V_j is finitely generated and projective. Our abbreviated notation for this is

$$\mathcal{V} : \quad V \xrightarrow{\Delta} V \xrightarrow{\delta} M.$$

(Of course, not every A-module is resoluble in this way — M as above must be finitely generated, finitely presented in the sense of [12], and so forth — and we offer no standard method for resolving modules even under favorable circumstances. See section 5 below for some examples of "bare hands" construction of resolutions.) Upon realizing the V_j's as direct summands of finite-rank free modules, we obtain a chain complex $\overline{\mathcal{V}}$ of (A, τ)-spaces, namely

$$\cdots \to \overline{J(V_2)} \xrightarrow{\overline{\Delta}_2} \overline{J(V_1)} \xrightarrow{\overline{\Delta}_1} \overline{J(V_0)} \to 0.$$

Let $H^w_*(\overline{\mathcal{V}})$ denote the homology of this complex, taken in the weak sense of kernels modulo image closures. Thus

$$H^w_j(\overline{\mathcal{V}}) = \ker \overline{\Delta}_j \ominus \overline{\operatorname{im} \overline{\Delta}_{j+1}} = \ker \overline{\Delta}_j \ominus \overline{J(\operatorname{im} \Delta_{j+1})} = \ker \overline{\Delta}_j \cap J(\ker \Delta_j)^\perp$$

for $j = 0, 1, \dots$, where $\Delta_0 = \delta$ and $\overline{\Delta}_0$ is the zero map.

The H^w_j's somewhat resemble (left) derived functors in homological algebra. The analogy would be much closer, and our labors correspondingly lighter, if the ranges of the operators $\overline{\Delta}_j$ were always closed. The following simple example shows that this is too much to hope for. Let A be the algebra of complex polynomials in two commuting variables x, y, with the involution that makes x and y selfadjoint. We have the resolution

$$0 \longrightarrow A \xrightarrow{\;(-y \quad x)\;} A^2 \xrightarrow{\;\binom{x}{y}\;} A \xrightarrow{\;\delta\;} \mathbb{C} \longrightarrow 0,$$

where δ is evaluation at $(0, 0)$. Let τ be given by integration with respect to, say, area measure on the square $[0, 1] \times [0, 1]$. Then $\overline{\Delta}_2 : L^2(A, \tau) \to L^2(A, \tau)^2$ is injective but not bounded below, so cannot have closed range. For essentially the same reason, the range of $\overline{\Delta}_1 : L^2(A,\tau)^2 \to L^2(A,\tau)$ is dense but not closed. (It is easy to check that $H_*^w = (0)$ here, incidentally. To get a nonzero result, integrate instead with respect to a compactly supported measure with an atom at $(0,0)$.)

Adapting standard arguments from homological algebra, we now proceed to show that the (A,τ)-spaces $H_j^w(\overline{\mathcal{V}})$ are the same no matter which resolution $\mathcal{V}$ of M is used. Given a chain map $t = (t_n)$, by bounded operators $t_n : L_n \to L'_n$, between two Hilbert space chain complexes

$$\mathcal{L} : L_n \xrightarrow{d_n} L_{n-1} \quad \text{and} \quad \mathcal{L}' : L'_n \xrightarrow{d'_n} L'_{n-1},$$

with each d_n and d'_n bounded, the relations $t_{j-1}d_j = d'_j t_j$ imply that t_n takes $\ker d_n$ to $\ker d'_n$, $\operatorname{im} d_{n+1}$ to $\operatorname{im} d'_{n+1}$, and therefore $\overline{\operatorname{im} d_{n+1}}$ to $\overline{\operatorname{im} d'_{n+1}}$. Write t_* for the resulting map from $H_*^w(\mathcal{L})$ to $H_*^w(\mathcal{L}')$. When we identify $H_n^w(\mathcal{L}) = \ker d_n / \overline{\operatorname{im} d_{n+1}}$ with the subspace $\ker d_n \ominus \overline{\operatorname{im} d_{n+1}}$ of L_n (and likewise for $\mathcal{L}'$), the induced operator from $H_n^w(\mathcal{L})$ to $H_n^w(\mathcal{L}')$ is the restriction of t_n to $H_n^w(\mathcal{L})$, followed by the orthogonal projection on $H_n^w(\mathcal{L}')$. As in the familiar algebraic setting, it is plain that if $t' : \mathcal{L}' \to \mathcal{L}''$ is a chain map to a third complex, then $(t't)_* = t'_* t_*$. The Hilbert space chain maps with which we will work come from algebraic chain maps — morphisms, for short — between resolutions. Given a morphism $T = (\ldots, T_1, T_0, \theta)$ between two finitely generated projective resolutions $\mathcal{V}$ and $\mathcal{V}'$, that is, a ladder of module maps

$$\begin{array}{ccccccc}
\cdots \to & V'_1 & \longrightarrow & V'_0 & \longrightarrow & M' & \to 0 \\
& \Big\uparrow T_1 & & \Big\uparrow T_0 & & \Big\uparrow \theta & \\
\cdots \to & V_1 & \longrightarrow & V_0 & \longrightarrow & M & \to 0
\end{array}$$

in which each square commutes, we define $\operatorname{op}_*(T) : H_*^w(\overline{\mathcal{V}}) \to H_*^w(\overline{\mathcal{V}'})$ by $\operatorname{op}_j(T) = P'_j \overline{T_j}|_{H_j^w(\overline{\mathcal{V}})}$, where P'_j is the (orthogonal) projection of $\overline{J(V'_j)}$ on $H_j^w(\overline{\mathcal{V}'})$. That is, $\operatorname{op}_*(T) = \overline{T}_*$. If $T' : \mathcal{V}' \to \mathcal{V}''$ is a morphism to a third projective resolution, we have $\operatorname{op}_*(T'T) = \overline{T'T}_* = \operatorname{op}_*(T')\operatorname{op}_*(T)$.

The following lemma is the analog of the comparison theorem in homological algebra; see 2.2.6 of [14].

Lemma 2.1. *Let $\mathcal{V}$ and $\mathcal{V}'$ be resolutions of M and M' respectively, and let $\theta : M \to M'$ be a module map. Then:*

(a) there is a morphism $T = (\ldots, T_1, T_0, \theta)$ from $\mathcal{V}$ to $\mathcal{V}'$ with -1 entry θ;

(b) if $S = (\ldots, S_1, S_0, \theta)$ is another such morphism, then $\operatorname{op}_*(S) = \operatorname{op}_*(T)$.

Proof. Part (a), which is entirely algebraic, is proved in [14]. (Use the projectivity of V_0 and the surjectivity of δ' to get $T_0 : V_0 \to V'_0$ such that $\delta' T_0 = \theta\delta$. Since T_0 maps $\ker\delta = \operatorname{im}\Delta_1$ to $\ker\delta' = \operatorname{im}\Delta'_1$, the projectivity of V_1 yields $T_1 : V_1 \to V'_1$ such that $T_0\Delta_1 = \Delta'_1 T_1$, and so on.) For part (b), we first construct a chain contraction of $T - S$ as in [14], that is, a sequence of maps $r_n : V_n \to V'_{n+1}$ such that $T_0 - S_0 = \Delta'_1 r_0$ and $T_n - S_n = r_{n-1}\Delta_n + \Delta'_{n+1} r_n$ for $n > 0$. The existence of r_0 follows from $\operatorname{im}(T_0 - S_0) \subseteq \ker\delta' = \operatorname{im}\Delta'_1$ and the projectivity of V_0. Using $\Delta'_1(T_1 - S_1) = (T_0 - S_0)\Delta_1 = \Delta'_1 r_0 \Delta_1$ — so $\operatorname{im}(T_1 - S_1 - r_0\Delta_1) \subseteq \ker\Delta'_1 = \operatorname{im}\Delta'_2$ — we likewise obtain the desired r_1. Use

$$\Delta'_2(T_2 - S_2) = (T_1 - S_1)\Delta_2 = (T_1 - S_1 - r_0\Delta_1)\Delta_2 = \Delta'_2 r_1 \Delta_2$$

to get r_2, and continue inductively. To finish the proof, let P'_j be the projection of $\overline{J(V'_j)}$ on $H_j^w(\overline{\mathcal{V}'})$, and observe that $P'_0(\overline{T_0} - \overline{S_0}) = 0$ because $\operatorname{im}(\overline{T_0} - \overline{S_0})$ lies in

$\overline{J(\ker \delta')}$, which is orthogonal to $H_0^w(\overline{\mathcal{V}'})$. Hence $\mathrm{op}_0(T) = \mathrm{op}_0(S)$. For ξ in $\ker \overline{\Delta}_1$, we have

$$(\overline{T_1} - \overline{S_1})\xi = (\overline{T_1} - \overline{S_1} - \overline{r_0\Delta_1})\xi = \overline{\Delta_2' r_1}\xi \in \mathrm{im}\,\overline{\Delta_2'} \subseteq \overline{J(\ker \Delta_1')},$$

so $P_1'(\overline{T_1} - \overline{S_1})\xi = 0$. Thus $\mathrm{op}_1(T) = \mathrm{op}_1(S)$, and so on for higher indices. □

That $H_*^w(\overline{\mathcal{V}})$ depends only on M (and of course on τ) follows easily from what we have shown so far (*cf.* 2.4.1 in [14]).

Theorem 2.2. *If $\mathcal{V}$ and $\mathcal{V}'$ are resolutions of the same A-module, then $H_j^w(\overline{\mathcal{V}}) \approx H_j^w(\overline{\mathcal{V}'})$ for all j.*

Proof. Apply part (a) of Lemma 2.1 with $M = M'$ and $\theta = \mathrm{id}_M$ to obtain morphisms T and T' from $\mathcal{V}$ to $\mathcal{V}'$ and *vice versa* with -1 entry id_M. Then $\mathrm{op}_j(T'T) = \mathrm{id}_{H_j^w(\overline{\mathcal{V}})}$ and $\mathrm{op}_j(TT') = \mathrm{id}_{H_j^w(\overline{\mathcal{V}'})}$ by part (b) of the same lemma. Thus, $\mathrm{op}_j(T)$ and $\mathrm{op}_j(T')$ are inverses of one another. The desired A-linear unitary operator is $\mathrm{op}_j(T)|\,\mathrm{op}_j(T)|^{-1}$. □

What we have at this point is a sequence of functors $\mathcal{V} \mapsto H_j^w(\overline{\mathcal{V}})$ — from the category of finitely generated projective resolutions over A, with morphisms as described above, to the category of closed $(A,\tau)-$spaces and bounded $A-$intertwining operators — that ignore, up to invertible intertwining operators, all but the lowest term of the resolution. Blurring the distinction between different but equivalent closed $(A,\tau)-$spaces, we will henceforth write $H_*(M) = H_*(A,\tau,M)$ for $H_*^w(\overline{\mathcal{V}})$, where $\mathcal{V}$ is any resolution of M by finitely generated projective A-modules. This assumes, of course, that M has such a resolution to begin with. In case M has only a partial resolution, terminating on the left with

$$V_n \xrightarrow{\Delta_n} V_{n-1} \to \cdots,$$

where $\ker \Delta_n$ is not (known to be) finitely generated, our arguments above still show that the (A,τ)-spaces $H_j(M)$ are defined unambiguously for $0 \le j \le n$ by $H_j(M) = \ker \overline{\Delta}_j \cap J(\ker \Delta_j)^{\perp}$.

We conclude this section with an intrinsic description of $H_0(A,\tau,M)$, valid for any representable trace τ on A and any finitely generated A-module M. Denote by $M^{\#}$ the vector space dual of M, and let $M_\tau^{\#}$ be the subspace of $M^{\#}$ consisting of those linear functionals $f : M \to \mathbb{C}$ such that for each m in M there is a nonnegative constant c_m satisfying $|f(am)| \le c_m \tau(a^*a)^{1/2}$ for all a in A. Notice that $M^{\#}$ and $M_\tau^{\#}$ are right A-modules via $(fa)(m) = f(am)$. We construct an anti-A-linear bijection between $M_\tau^{\#}$ and $H_0(A,\tau,M)$ as follows. Let $\delta : A^n \to M$ be a surjective A-module map, so $H_0(A,\tau,M)$ is realized in $L^2(A,\tau)^n$ as $J(\ker\delta)^{\perp}$, where $J : A^n \to L^2(A,\tau)^n$ is the usual GNS map. Given ξ in $J(\ker\delta)^{\perp}$, define $\xi^{\#}$ in $M^{\#}$ by $\xi^{\#}(\delta(\vec{a})) = \langle J(\vec{a}), \xi\rangle$ for $\vec{a}$ in A^n. Notice that $(\lambda(a)\xi)^{\#} = \xi^{\#}a^*$ for all a in A.

Proposition 2.3. *The map $\xi \mapsto \xi^{\#}$ is a bijection from $H_0(A,\tau,M)$ onto $M_\tau^{\#}$.*

Proof. The map in question is injective because $J(A^n)$ is dense in $L^2(A,\tau)^n$. For ξ in $J(\ker\delta)^{\perp}, a$ in A, and $\vec{a} = (a_1,\dots,a_n)$ in A^n, we have

$$|\xi^{\#}(a\delta(\vec{a}))| \;\le \|\, \xi \,\|\, \|\, J(a\vec{a}) \,\| \le \|\, \xi \,\|\, \tau(a^*a)^{1/2} \sum_1^n \|\, \rho(a_j) \,\|,$$

so $\xi^\# \in M_\tau^\#$. Let $m_1, \ldots, m_n$ in M be the δ-images of $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ in A^n. Given f in $M_\tau^\#$, let $c_1, \ldots, c_n \geq 0$ be such that $|f(am_j)| \leq c_j \tau(a^* a)^{1/2}$ for a in A. We then have

$$|f(\delta(\vec{a}))| \;\leq \sum_1^n c_j \tau(a_j^* a_j)^{1/2} \;\leq\; \left(\sum_1^n c_j^2\right)^{1/2} \| \, J(\vec{a}) \, \|$$

for all $\vec{a} = (a_1, \ldots, a_n)$ in A^n. Thus, there exists ξ in $J(\ker \delta)^\perp$ such that $\xi^\# = f$. □

3. Weakly exact sequence of homology spaces

In this section we show that a short exact sequence of resoluble A-modules gives rise to a long weakly exact sequence of homology spaces. (Weakly exact here means that the closure of the range of each map coincides with the kernel of the next map.) The argument is the familiar diagram chase from homological algebra, modified in certain respects so as to work in our analytic setting. The properties of of (A, τ)-spaces and bounded A-linear maps that make this possible are highlighted in the following lemma.

Lemma 3.1. *Let $t : K \to L$ be a bounded A-linear operator between two closed (A, τ)-spaces. Then:*

(a) for any A-invariant subspace L_0 of $\operatorname{im} t$, we have $t^{-1}(\overline{L_0}) = \overline{t^{-1}(L_0)}$;

(b) for any closed A-invariant subspace L_1 of L, we have $\overline{\operatorname{im} t \cap L_1} = \overline{\operatorname{im} t} \cap L_1$.

Proof. For (a), let t_1 and t_2 be the restrictions of t to $\overline{t^{-1}(L_0)}$ and $t^{-1}(\overline{L_0})$, respectively. Then $\ker t_1 = \ker t_2 = \ker t$. Further, $\overline{\operatorname{im} t_1} = \overline{\operatorname{im} t_2} = \overline{L_0}$ because of our assumption that $L_0 \subseteq \operatorname{im} t$. It follows from the rank-nullity formula that the domains of t_1 and t_2 have the same $\dim_{A,\tau}$. Since $\overline{t^{-1}(L_0)} \subseteq t^{-1}(\overline{L_0})$, these two closed subspaces must coincide.

We deduce part (b) from part (a) as follows. Let $Q = \overline{\operatorname{im} t} / \overline{\operatorname{im} t \cap L_1}$ and let $\pi : \overline{\operatorname{im} t} \to Q$ be the quotient map. Define $\theta : K \oplus (\overline{\operatorname{im} t} \cap L_1) \to Q$ by $\theta(\xi, \eta) = \pi(t\xi + \eta)$. We have $\theta^{-1}(\operatorname{im} \pi t) \subseteq K \oplus \overline{\operatorname{im} t \cap L_1}$ because $\theta(\xi, \eta) \in \operatorname{im} \pi t$ implies $\eta - t\xi' \in \overline{\operatorname{im} t \cap L_1}$ for some ξ' in K; since $\eta \in L_1$, this puts $t\xi'$ in $\operatorname{im} t \cap L_1$, and hence $\eta \in \overline{\operatorname{im} t \cap L_1}$. By part (a), then,

$$K \oplus (\overline{\operatorname{im} t} \cap L_1) = \theta^{-1}(\overline{\operatorname{im} \pi t}) = \overline{\theta^{-1}(\operatorname{im} \pi t)} \subseteq K \oplus \overline{\operatorname{im} t \cap L_1},$$

so $\overline{\operatorname{im} t \cap L_1} = \overline{\operatorname{im} t} \cap L_1$. □

Suppose now that

$$0 \to \mathcal{L} \xrightarrow{r} \mathcal{L}' \xrightarrow{s} \mathcal{L}'' \to 0$$

is an (algebraically) exact sequence of complexes of (A, τ)-spaces in which all operators that appear are bounded and A-linear. The diagram we will chase is

$$\begin{array}{ccccc}
L''_{n+1} & \xrightarrow{d''_{n+1}} & L''_n & \xrightarrow{d''_n} & L''_{n-1} \\
\uparrow{\scriptstyle s_{n+1}} & & \uparrow{\scriptstyle s_n} & & \uparrow{\scriptstyle s_{n-1}} \\
L'_{n+1} & \xrightarrow{d'_{n+1}} & L'_n & \xrightarrow{d'_n} & L'_{n-1} \\
\uparrow{\scriptstyle r_{n+1}} & & \uparrow{\scriptstyle r_n} & & \uparrow{\scriptstyle r_{n-1}} \\
L_{n+1} & \xrightarrow{d_{n+1}} & L_n & \xrightarrow{d_n} & L_{n-1}
\end{array}$$

where $d_n d_{n+1} = 0$ (and likewise for d', d''), each r_j is injective, each s_j is surjective with $\ker s_j = \operatorname{im} r_j$, and the squares commute. We will use $[\ \cdot\ ]$ to denote the quotient map from $\ker d_j$ to the weak homology space $H_j^w(\mathcal{L}) = \ker d_j / \overline{\operatorname{im} d_{j+1}}$ (and likewise for d', d''). The boundary map $\partial_n : H_n^w(\mathcal{L}'') \to H_{n-1}^w(\mathcal{L})$ is constructed by the usual algebraic procedure: starting with ξ'' in $\ker d_n''$, we obtain ξ' in L_n' such that $s_n \xi' = \xi''$, so $s_{n-1} d_n' \xi' = d_n'' \xi'' = 0$, so $d_n' \xi' = r_{n-1}\eta$ for some η in L_{n-1} which must lie in $\ker d_{n-1}$ because $r_{n-1}\eta \in \ker d_{n-1}'$ and r_{n-2} is injective. Any such η must belong to $\overline{\operatorname{im} d_n}$ if the ξ'' whence it comes belongs to $\overline{\operatorname{im} d_{n+1}''}$. [To see this, write

$$\xi'' = s_n \xi' = \lim_k d_{n+1}'' s_{n+1} \rho_k' = \lim_k s_n d_{n+1}' \rho_k'$$

for an appropriate sequence $\{\rho_k'\}$ in L_{n+1}'. Because s_n is a bounded surjective map of Banach spaces, we obtain, using $\ker s_n = \operatorname{im} r_n$, a sequence $\{\xi_k\}$ in L_n such that

$$\xi' = \lim_k (d_{n+1}' \rho_k' + r_n \xi_k),$$

and hence

$$r_{n-1}\eta = d_n' \xi' = \lim_k d_n' r_n \xi_k = \lim_k r_{n-1} d_n \xi_k.$$

Because r_{n-1} is a Banach space isomorphism of L_{n-1} with $\operatorname{im} r_{n-1}$, it follows that $\eta = \lim_k d_n \xi_k$.] We thus have a well-defined A-linear map $\partial_n : H_n^w(\mathcal{L}'') \to H_{n-1}^w(\mathcal{L})$ such that $\partial_n[\xi''] = [\eta]$, where ξ'' and η are related as above. If $\| \xi'' \|$ is small, then we can choose ξ' above with small norm by the open mapping theorem, making $\| d_n' \xi' \|$ small, and hence $\| \eta \|$ small. It follows that ∂_n is bounded.

The next proposition is Theorem 2.1 of [5], whose proof via spectral projections extends without change from the setting of a group algebra with its natural trace to our more general context. We provide a different proof based on Lemma 3.1 above.

Proposition 3.2. *With*

$$0 \to \mathcal{L} \xrightarrow{r} \mathcal{L}' \xrightarrow{s} \mathcal{L}'' \to 0$$

as above, the sequence

$$\cdots \longrightarrow H_n^w(\mathcal{L}) \xrightarrow{r_{n*}} H_n^2(\mathcal{L}') \xrightarrow{s_{n*}} H_n^w(\mathcal{L}'')$$
$$\xrightarrow{\partial_n} H_{n-1}^w(\mathcal{L}) \xrightarrow{r_{(n-1)*}} H_{n-1}^w(\mathcal{L}') \longrightarrow \cdots$$

is weakly exact.

Proof. We have $s_{n*} r_{n*} = 0$ because $s_n r_n = 0$. It is apparent from the definition of ∂_n that $\partial_n[s_n \xi'] = 0$ for ξ' in $\ker d_n'$, and that $r_{n-1}\eta \in \operatorname{im} d_n'$ for $[\eta]$ in $\operatorname{im} \partial_n$; that is, $\partial_n s_{n*} = 0 = r_{(n-1)*}\partial_n$.

To see that $\ker s_{n*} \subseteq \overline{\operatorname{im} r_{n*}}$, take ξ' in $\ker d_n'$ with $s_n \xi'$ in $\overline{\operatorname{im} d_{n+1}''}$. We have $\operatorname{im} d_{n+1}'' = \operatorname{im} d_{n+1}'' s_{n+1} = \operatorname{im} s_n d_{n+1}'$. Let t be the restriction of s_n to $\ker d_n'$, so $\operatorname{im} s_n d_{n+1}' = \operatorname{im} t d_{n+1}'$. Part (a) of Lemma 3.1, with $L_0 = \operatorname{im} t d_{n+1}'$, yields $t^{-1}(\overline{\operatorname{im} t d_{n+1}'}) = \overline{t^{-1}(\operatorname{im} t d_{n+1}')}$. We thus have

$$\xi' \in \overline{t^{-1}(\operatorname{im} t d_{n+1}')} = \overline{\ker t + \operatorname{im} d_{n+1}'} = \overline{(\operatorname{im} r_n \cap \ker d_n') + \operatorname{im} d_{n+1}'}.$$

Let $\{\xi_k\}$ and $\{\rho_k'\}$ be sequences in L_n and L_{n+1}' respectively such that $d_n' r_n \xi_k = 0$ for all k and

$$\xi' = \lim_k (r_n \xi_k + d_{n+1}' \rho_k').$$

Then each ξ_k lies in $\ker d_n$ — because $r_{n-1}d_n\xi_k = 0$ and r_{n-1} is injective — and factoring out $\overline{\operatorname{im} d'_{n+1}}$ gives $[\xi'] = \lim_k[r_n\xi_k]$.

For the inclusion $\ker \partial_n \subseteq \overline{\operatorname{im} s_{n*}}$, let ξ'', ξ', and η be as in the recipe above for ∂_n, and suppose furthermore that $\eta \in \overline{\operatorname{im} d_n}$. Then

$$d'_n\xi' = r_{n-1}\eta \in \overline{\operatorname{im} r_{n-1}d_n} = \overline{\operatorname{im} d'_n r_n}.$$

Applying part (a) of Lemma 3.1 with $t = d'_n$ and $L_0 = \operatorname{im} d'_n r_n$, we obtain

$$\xi' \in \overline{(d'_n)^{-1}(\operatorname{im} d'_n r_n)} = \overline{\ker d'_n + \operatorname{im} r_n},$$

and hence $\xi'' = s_n\xi' \in \overline{s_n(\ker d'_n)}$.

We conclude the proof by showing that $\ker r_{(n-1)*} \subseteq \overline{\operatorname{im} \partial_n}$. Suppose η in $\ker d_{n-1}$ is such that $r_{n-1}\eta \in \overline{\operatorname{im} d'_n}$. By part (b) of Lemma 3.1, with $t = r_{n-1}$ and $L_1 = \ker s_{n-1} = \operatorname{im} r_{n-1}$, we have

$$r_{n-1}\eta \in \overline{\operatorname{im} d'_n} \cap \operatorname{im} r_{n-1} = \overline{\operatorname{im} d'_n \cap \operatorname{im} r_{n-1}}.$$

This means there are sequences $\{\eta_k\}$ and $\{\xi'_k\}$ in L_{n-1} and L'_n respectively such that $r_{n-1}\eta_k = d'_n\xi'_k$ for all k and $r_{n-1}\eta = \lim_k r_{n-1}\eta_k$. Let $\xi''_k = s_n\xi'_k$. Then $\xi''_k \in \ker d''_n$ because $d''_n s_n\xi'_k = s_{n-1}d'_n\xi_k = s_{n-1}r_{n-1}\eta_k = 0$, and referring to the definition of ∂_n we see that $\partial_n[\xi''_k] = [\eta_k]$. We have $\eta_k \to \eta$ by the open mapping theorem, so $[\eta] = \lim_k \partial_n[\xi''_k]$. □

We digress briefly to comment on the relation between Lemma 3.1 and Proposition 3.2 in the larger context of Banach space categories. The gist of our remarks is that the two results mentioned are essentially equivalent. Suppose $\mathcal{C}$ is a category of Banach spaces and bounded linear maps — say, $\mathcal{C}$-spaces and $\mathcal{C}$-maps between them — with the closure properties required to make the discussion in this section intelligible: roughly, $\mathcal{C}$ is closed under quotients and direct sums, and if t and v are $\mathcal{C}$-maps with the same codomain, then $\overline{\operatorname{im} v}, t^{-1}(\overline{\operatorname{im} v})$, and $\overline{t^{-1}(\operatorname{im} v)}$ are $\mathcal{C}$-spaces. An additional property that $\mathcal{C}$ might have is (*): $t^{-1}(\overline{\operatorname{im} v}) = \overline{t^{-1}(\operatorname{im} v)}$ for any two $\mathcal{C}$-maps t and v into the same $\mathcal{C}$-space such that $\operatorname{im} v \subseteq \operatorname{im} t$. (For instance, the category of (A,τ)-spaces and bounded A-linear maps has (*), but (*) fails in the full Hilbert space category.) If $\mathcal{C}$ has (*), then the proof of Lemma 3.1 shows that $\mathcal{C}$ has (**): $\overline{\operatorname{im} t \cap L_1} = \overline{\operatorname{im} t} \cap L_1$ for every $\mathcal{C}$-map t and every $\mathcal{C}$-subspace L_1 of the codomain of t. The proof of Proposition 3.2, unchanged except for using (*) and (**) in place of Lemma 3.1, establishes the weak exactness of the weak homology sequence arising from a short exact sequence of complexes of $\mathcal{C}$-spaces and $\mathcal{C}$-maps. Assume, conversely, that $\mathcal{C}$ has this exactness property. Let t and v be $\mathcal{C}$-maps into the same $\mathcal{C}$-space with $\operatorname{im} v \subseteq \operatorname{im} t$, and consider the three complexes

$$\cdots \to 0 \to \ker t \hookrightarrow \overline{t^{-1}(\operatorname{im} v)} \xrightarrow{t} \overline{\operatorname{im} v} \to 0 \to \cdots$$

$$\cdots \to 0 \to \ker t \hookrightarrow t^{-1}(\overline{\operatorname{im} v}) \xrightarrow{t} \overline{\operatorname{im} v} \to 0 \to \cdots$$

$$\cdots \to 0 \to 0 \to \frac{t^{-1}(\overline{\operatorname{im} v})}{\overline{t^{-1}(\operatorname{im} v)}} \to 0 \to 0 \to \cdots .$$

Reading down gives a short exact sequence of complexes, and H^w_* vanishes for the first two, so it must vanish for the third, forcing $t^{-1}(\overline{\operatorname{im} v}) = \overline{t^{-1}(\operatorname{im} v)}$.

Returning now to the (A,τ) setting, we obtain the desired long weakly exact sequence for $H_*(A,\tau,\cdot)$ from Proposition 3.2 by standard arguments.

Theorem 3.3. *Let*

$$0 \to M \xrightarrow{\iota} M' \xrightarrow{\pi} M'' \to 0$$

be a short exact sequence of A-modules. If M and M'' are resoluble by finitely generated projective modules, then so is M', and there is a long exact sequence

$$\cdots \to H_n(M) \to H_n(M') \to H_n(M'') \to H_{n-1}(M) \to \cdots,$$

where the maps from $H_n(M)$ and $H_n(M')$ are induced by ι and π, respectively.

Proof. Let $\mathcal{V} : V \xrightarrow{\Delta} V \xrightarrow{\delta} M$ and $\mathcal{V}'' : V'' \xrightarrow{\Delta''} V'' \xrightarrow{\delta''} M''$ be finitely generated projective resolutions. By 2.2.8 in [14], we obtain a resolution $\mathcal{V}'$

$$\cdots V_1 \oplus V_1'' \xrightarrow{\Delta_1'} V_0 \oplus V_0'' \xrightarrow{\delta'} M' \to 0$$

of M' making

$$0 \to \mathcal{V} \to \mathcal{V}' \to \mathcal{V}'' \to 0$$

an exact sequence of resolutions, where the maps $V_j \to V_j \oplus V_j'' \to V_j''$ are the natural ones associated with the direct sum. (Here, δ' is given by $\delta'(v_0, v_0'') = \iota\delta v_0 + S_0 v_0''$, where $S_0 : V_0'' \to M'$ lifts δ''. Since $\operatorname{im} S_0\Delta_1'' \subseteq \ker \pi = \operatorname{im} \iota = \operatorname{im} \iota\delta$, we obtain $S_1 : V_1'' \to V_0$ such that $\iota\delta S_1 = S_0\Delta_1''$. We then define Δ_1' by

$$\Delta_1'(v_1, v_1'') = (\Delta_1 v_1 - S_1 v_1'', \Delta_1'' v_1''),$$

and so on.) The theorem now follows by applying Proposition 3.2 to the short exact sequence

$$0 \to \overline{\mathcal{V}} \to \overline{\mathcal{V}'} = \overline{\mathcal{V}} \oplus \overline{\mathcal{V}''} \to \overline{\mathcal{V}''} \to 0$$

of (A, τ)-space complexes. □

We remark that the theorem applies when any two of the three modules M, M', and M'' have finitely generated projective resolutions; in the two cases not treated — M and M' resoluble, respectively M' and M'' resoluble — it is more or less straightforward to construct a resolution of the remaining module by finitely generated projectives.

4. Tensor products, Betti numbers, Euler characteristic

We begin our consideration of tensor products with some operator bookkeeping in doubly indexed Hilbert space complexes. Suppose we have Hilbert spaces $L_{i,j}$, with $L_{i,j} = (0)$ if either i or j is negative, and bounded operators

$$r_{i,j} : L_{i,j} \to L_{i,j-1} \quad \text{and} \quad d_{i,j} : L_{i,j} \to L_{i-1,j}$$

such that $r^2 = 0$ along each row and $d^2 = 0$ down each column in the diagram

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \to & & L_{i,j} & \xrightarrow{r_{i,j}} & L_{i,j-1} & & \to \\ & & \downarrow{\scriptstyle d_{i,j}} & & \downarrow{\scriptstyle d_{i,j-1}} & & \\ \to & & L_{i-1,j} & \xrightarrow{r_{i-1,j}} & L_{i-1,j-1} & & \to \\ & & \downarrow & & \downarrow & & \end{array}$$

and each square not only commutes ($d_{i,j-1}r_{i,j} = r_{i-1,j}d_{i,j}$) but also *-commutes ($d_{i,j-1}^* r_{i-1,j} = r_{i,j} d_{i,j}^*$). Write $\mathcal{L}_{i,\bullet}$ for the i^{th} row and $\mathcal{L}_{\bullet,j}$ for the j^{th} column of the double complex. Notice that

$$H_j^w(\mathcal{L}_{i,\bullet}) = \ker r_{i,j} \ominus \overline{\operatorname{im} r_{i,j+1}} = \ker r_{i,j} \cap \ker r_{i,j+1}^*$$

and

$$H_i^w(\mathcal{L}_{\bullet,j}) = \ker d_{i,j} \cap \ker d^*_{i+1,j}.$$

Both of these weak homology spaces are subspaces of $L_{i,j}$ — so it is meaningful to intersect them. Form the total complex $\mathcal{K}$

$$\dots K_2 \xrightarrow{g_2} K_1 \xrightarrow{g_1} K_0 \to 0,$$

where

$$K_k = \bigoplus_{i+j=k} L_{i,j},$$

and $g_k : K_k \to K_{k-1}$ is given by

$$g_k(\xi_{k,0},\ \xi_{k-1,1}, \dots ,\ \xi_{0,k})$$
$$= (d_{k,0}\xi_{k,0} + (-1)^k r_{k-1,1}\xi_{k-1,1}\ , \dots ,\ d_{1,k-1}\xi_{1,k-1} + (-1)^k r_{0,k}\xi_{0,k}).$$

(This construction is standard in homological algebra. A direct calculation, not requiring *-commutativity, shows that $g^2 = 0$.) The lemma below relates the weak homology of $\mathcal{K}$ to that of the row and column complexes of the original double complex.

Lemma 4.1. *For a doubly commuting complex $\mathcal{L}$ of Hilbert spaces as above with total complex $\mathcal{K}$, one has*

$$H_k^w(\mathcal{K}) = \bigoplus_{i+j=k} \left(H_i^w(\mathcal{L}_{\bullet,j}) \cap H_j^w(\mathcal{L}_{i,\bullet})\right)$$

for $k = 0, 1, \dots$.

Proof. Take $k \geq 1$; for notational reasons, the argument for the case $k = 0$ is special (and easier). Suppose $\xi = (\xi_{k,0},\ \xi_{k-1,1}, \dots ,\ \xi_{0,k}) \in H_k^w(\mathcal{K})$, so $g_k\xi = 0 = g^*_{k+1}\xi$. For $i = 1, \dots , k$, we have

$$(i) \quad d_{i,k-i}\, \xi_{i,k-i} + (-1)^k r_{i-1,k-i+1}\, \xi_{i-1,k-i+1} = 0$$

because $g_k\xi = 0$ and

$$(ii) \quad (-1)^{k+1} r^*_{i,k-i+1}\, \xi_{i,k-i} + d^*_{i,k-i+1}\, \xi_{i-1,k-i+1} = 0$$

because $g^*_{k+1}\xi = 0$. At the two ends of ξ, this condition also forces

$$(iii) \quad d^*_{k+1,0}\, \xi_{k,0} \ = \ 0 \ = r^*_{0,k+1}\, \xi_{0,k}.$$

Apply $d^*_{i,k-i}$ to *(i)*, then use *-commutativity and *(ii)* to see that

$$d^*_{i,k-i} d_{i,k-i}\, \xi_{i,k-i} + r_{i,k-i+1} r^*_{i,k-i+1}\, \xi_{i,k-i} = 0$$

for $i = 1, \dots , k$. Thus, each summand in *(i)* and *(ii)* vanishes, meaning that

$$(iv) \quad \xi_{i,k-i} \in \ker d_{i,k-i} \cap \ker d^*_{i+1,k-i} \cap \ker r_{i,k-i} \cap \ker r^*_{i,k-i+1}$$

for $i = 1, \dots k-1$, and also (with *(iii)*)

$$(v) \quad \xi_{k,0} \in \ker d_{k,0} \cap \ker d^*_{k+1,0} \cap \ker r^*_{k,1}$$
$$\text{and} \quad \xi_{0,k} \in \ker d^*_{1,k} \cap \ker r_{0,k} \cap \ker r^*_{0,k+1}.$$

(Notice that $r_{k,0}$ and $d_{0,k}$ are both the zero map.) We have shown that each component of ξ is where the lemma asserts it should be. If, conversely, ξ belongs to K_k and its components satisfy *(iv)* and *(v)*, the calculation above reverses to show that $g_k\xi = 0 = g^*_{k+1}\xi$. □

In what follows, we will use $\otimes$ for the tensor product of complex vector spaces (and linear maps), and for the Hilbert space tensor product of Hilbert spaces (and bounded operators). The good behavior of $\otimes$ in both of these settings plays an understated but essential role in the proof of the theorem below.

Theorem 4.2. *Let (A, τ) and (B, σ) be traced *-algebras, and let M and N be modules over A and B respectively having resolutions by finitely generated projective modules. Then $M \otimes N$ has such a resolution over $A \otimes B$, and*

$$H_k(A \otimes B, \tau \otimes \sigma, M \otimes N) \approx \bigoplus_{i+j=k} H_i(A, \tau, M) \otimes H_j(B, \sigma, N)$$

for $k = 0, 1, \ldots$.

Proof. Let the given resolutions of M and N be

$$\mathcal{V} : V \xrightarrow{\Delta} V \xrightarrow{\delta} M \quad \text{and} \quad \mathcal{W} : W \xrightarrow{\Phi} W \xrightarrow{\phi} N.$$

Form the double complex $\mathcal{V} \otimes \mathcal{W}$ over $A \otimes B$, with maps

$$\Delta_{i,j} \equiv \Delta_i \otimes \mathrm{id}_{W_j} : V_i \otimes W_j \to V_{i-1} \otimes W_j$$

and

$$\Phi_{i,j} \equiv \mathrm{id}_{V_i} \otimes \Phi_j : V_i \otimes W_j \to V_i \otimes W_{j-1},$$

where $V_{-1} = M, W_{-1} = N, \Delta_0 = \delta$, and $\Phi_0 = \phi$. (Rows and columns here are exact because we are tensoring complex vector spaces.) The corresponding total complex gives a resolution of $M \otimes N$ by finitely generated projective modules. Specifically, we set

$$X_k = \bigoplus_{\substack{i+j=k \\ i,j \geq 0}} V_i \otimes W_j \quad (k = 0, 1, \ldots),$$

and define $\Gamma_k : X_k \to X_{k-1}$ by

$$\Gamma_k(e_{k,0},\ e_{k-1,1}, \ldots ,\ e_{0,k})$$

$$= (\Delta_{k,0} e_{k,0} + (-1)^k \Phi_{k-1,1} e_{k-1,1}\ , \ldots ,\ \Delta_{1,k-1} e_{1,k-1} + (-1)^k \Phi_{0,k} e_{0,k}).$$

Let $\gamma = \delta \otimes \phi$. Exactness of the sequence $\mathcal{X}$

$$\cdots \to X_2 \xrightarrow{\Gamma_2} X_1 \xrightarrow{\Gamma_1} X_0 \xrightarrow{\gamma} M \otimes N \to 0$$

is proved by diagram chasing (see 2.7.3 in [14]). In the prologue to Lemma 4.1, let $L_{i,j}$ be the $(A \otimes B, \tau \otimes \sigma)$-space $\overline{J(V_i \otimes W_j)}$, with $d_{i,j} = \overline{\Delta}_{i,j}$ and $r_{i,j} = \overline{\Phi}_{i,j}$. This double complex of Hilbert spaces *-commutes; indeed

$$d^*_{i,j-1} r_{i-1,j} = \overline{\Delta}^*_i \otimes \overline{\Phi}_j = r_{i,j} d^*_{i,j}.$$

The Hilbert space total complex is simply $\overline{\mathcal{X}}$, that is, $K_k = \overline{J(X_k)}$ and $g_k = \overline{\Gamma}_k$. Notice that $\ker d_{i,j} = (\ker \overline{\Delta}_i) \otimes \overline{J(W_j)}$, and similarly for $d^*_{i,j}, r_{i,j}$, and $r^*_{i,j}$. Since

$$(L_1 \otimes L') \cap (L \otimes L'_1) = L_1 \otimes L'_1$$

for closed subspaces L_1 and L'_1 of Hilbert spaces L and L', the theorem now follows from Lemma 4.1. $\square$

We define the n^{th} Betti number b_n of an A-module M having a finitely generated projective resolution by $b_n(A,\tau,M) = \dim_{A,\tau}(H_n(A,\tau,M))$. We will say that M has finite homological dimension relative to τ if $H_n(A,\tau,M)$ vanishes for sufficiently large n. For such M, we define the Euler characteristic χ by

$$\chi(A,\tau,M) = \sum_{n=0}^{\infty}(-1)^n b_n(A,\tau,M).$$

Our first result concerning these numerical invariants is that they depend additively on the trace τ. This is an easy consequence of the following lemma.

Lemma 4.3. *Let $\tau = \tau_1 + \tau_2$ be the sum of representable traces on A and let E be an A-submodule of A^n for some n. Then*

$$\dim_{A,\tau}(J(E)) = \dim_{A,\tau_1}(J^1(E)) + \dim_{A,\tau_2}(J^2(E)),$$

where J is the usual map of A^n into $L^2(A,\tau)^n$ and J^1 and J^2 are its counterparts for the traces τ_1 and τ_2.

Proof. Let P, P_1, and P_2 be the projections on $\overline{J(E)}$, $\overline{J^1(E)}$, and $\overline{J^2(E)}$ respectively. We must show that

$$(\tau\otimes \mathrm{tr}_n)(P) = (\tau_1\otimes \mathrm{tr}_n)(P_1) + (\tau_2\otimes \mathrm{tr}_n)(P_2).$$

For $i=1,2$, the map $J(a)\mapsto J^i(a)$ extends to a contraction $W_i : L^2(A,\tau)^n \to L^2(A,\tau_i)^n$. One checks easily that $W_1^*W_1 + W_2^*W_2$ is the identity operator on $L^2(A,\tau)^n$ and that

$$(\tau\otimes \mathrm{tr}_n)(W_i^*P_iW_i) = (\tau_i\otimes \mathrm{tr}_n)(P_i).$$

It will suffice to show that $P = W_1^*P_1W_1 + W_2^*P_2W_2$. For a in E, we have

$$\begin{aligned}(W_1^*P_1W_1 + W_2^*P_2W_2)J(a) &= W_1^*J^1(a) + W_2^*J^2(a)\\ &= (W_1^*W_1 + W_2^*W_2)J(a) = J(a),\end{aligned}$$

so

$$(W_1^*P_1W_1 + W_2^*P_2W_2)P = P = P(W_1^*P_1W_1 + W_2^*P_2W_2).$$

Notice now that W_i intertwines the action of $\rho(A)\otimes M_n$ on $L^2(A,\tau)^n$ with the corresponding action of $\rho_i(A)\otimes M_n$ on $L^2(A,\tau_i)^n$. It follows that

$$W_i^*W_i \in (\rho(A)\otimes M_n)' = \lambda(A)''\otimes \mathrm{id}_n\,.$$

Because E is an A-submodule, we thus have $W_i^*W_iJ(E) \subseteq \overline{J(E)}$, and hence $PW_i^*P_iW_i = W_i^*P_iW_i$. We conclude that $P = W_1^*P_1W_1 + W_2^*P_2W_2$, as promised. □

Theorem 4.4. *For representable traces τ_1 and τ_2 on A, and an A-module M having a finitely generated projective resolution, we have*

$$b_n(A,\tau_1+\tau_2,M) = b_n(A,\tau_1,M) + b_n(A,\tau_2,M).$$

Proof. Let $\mathcal{V} : V \xrightarrow{\Delta} V \xrightarrow{\delta} M$ be a finitely generated projective resolution of M. We have

$$\begin{aligned} b_n(A,\tau,M) &= \dim_{A,\tau}(\ker\overline{\Delta}_n) - \dim_{A,\tau}(J(\mathrm{im}\,\Delta_{n+1}))\\ &= \dim_{A,\tau}(J(V_n)) - \dim_{A,\tau}(J(\mathrm{im}\,\Delta_{n+1})) - \dim_{A,\tau}(J(\mathrm{im}\,\Delta_{n+1}))\end{aligned}$$

by the rank-nullity formula. Now apply the previous lemma, with $E = V_n$, $\mathrm{im}\,\Delta_n$, and $\mathrm{im}\,\Delta_{n+1}$. □

Standard arguments show that the Euler characteristic treats quotients and tensor products in standard fashion.

Theorem 4.5. *For a short exact sequence*

$$0 \to N \to M \to M/N \to 0$$

of A-modules of finite homological dimension relative to a representable trace τ on A, we have

$$\chi(A, \tau, M) = \chi(A, \tau, N) + \chi(A, \tau, M/N).$$

Proof. For appropriately large n, we obtain from Theorem 3.2 a weakly exact sequence

$$0 \to H_n(N) \to H_n(M) \to H_n(M/N) \to H_{n-1}(N) \to \dots$$
$$\to H_1(M/N) \to H_0(N) \to H_0(M) \to H_0(M/N) \to 0.$$

It follows easily from the rank-nullity formula that the alternating sum of the $\dim_{A,\tau}$'s of the terms in the sequence is zero, so $\chi(N) - \chi(M) + \chi(M/N) = 0$. □

Theorem 4.6. *Let (A, τ) and (B, σ) be traced *-algebras, and let M (resp. N) be an A- (resp. B-) module of finite homological dimension relative to τ (resp. σ). Then $M \otimes N$ has finite homological dimension rel. $\tau \otimes \sigma$, and*

$$\chi(A \otimes B, \tau \otimes \sigma, M \otimes N) = \chi(A, \tau, M)\chi(B, \sigma, N).$$

Proof. If $H_k(A, \tau, M) = 0 = H_k(B, \sigma, N)$ for $k > n$, then

$$H_k(A \otimes B, \tau \otimes \sigma, M \otimes N) = 0 \text{ for } k > 2n$$

by Theorem 4.2. The calculation for the product of the Euler characteristics is

$$\begin{aligned}(\sum_{i=0}^{\infty}(-1)^i b_i(M))(\sum_{j=0}^{\infty}(-1)^j b_j(N)) &= \sum_{k=0}^{\infty}(-1)^k \sum_{i+j=k} b_i(M) b_j(N) \\ &= \sum_{k=0}^{\infty}(-1)^k b_k(M \otimes N),\end{aligned}$$

where the last equality comes from Theorem 4.2 and the observation that

$$\dim_{A\otimes B, \tau\otimes\sigma}(L \otimes K) = \dim_{A,\tau}(L) \dim_{B,\sigma}(K)$$

for any (A, τ)-space L and any (B, σ)-space K. □

When M is 'fully' resolved by a projective resolution of finite length, the Euler characteristic is especially easy to calculate.

Proposition 4.7. *Let*

$$0 \longrightarrow V_n \xrightarrow{\Delta_n} V_{n-1} \xrightarrow{\Delta_{n-1}} \cdots \xrightarrow{\Delta_1} V_0 \xrightarrow{\delta} M \longrightarrow 0$$

be an exact sequence of A-modules, where $V_0, \dots, V_n$ are finitely generated projective modules. Then

$$\chi(M) = \sum_{j=0}^{n}(-1)^j \dim(J(V_j)).$$

Proof. As in the proof of Theorem 4.4, we have

$$b_j(M) = \dim(J(V_j)) - \dim(J(\operatorname{im}\Delta_j)) - \dim(J(\operatorname{im}\Delta_{j+1})).$$

Hence

$$\chi(M) = \sum_{j=0}^{n} (-1)^j [\dim(J(V_j)) - \dim(J(\operatorname{im}\Delta_j)) - \dim(J(\operatorname{im}\Delta_{j+1}))],$$

where Δ_0 and Δ_{n+1} are zero maps. The contribution from the $\operatorname{im}\Delta$'s to this sum telescopes to 0. □

We remark that in the above situation, the number $\chi(M)$ can be arrived at in a purely algebraic fashion. It follows easily from Schanuel's lemma in homological algebra (see [4]) that the alternating sum

$$\chi_0(M) = \sum_{j=0}^{n} (-1)^j [V_j]$$

in $K_0(A)$ depends only on M. (See section 1.7 of [13] for a discussion of this K_0-valued Euler characteristic.) When each K_0-class $[V_k]$ is paired with the trace τ, one obtains $\dim(J(V_k))$. (Recall that for a finite-rank projective module V, the pairing of $[V]$ with τ works by writing $V = A^n E$ for some idempotent matrix E in $A \otimes M_n$ and then taking τ of the sum of the diagonal entries of E. Because τ is a trace, this gives the same result as working with the orthogonal projection of $L^2(A,\tau)^n$ on $\overline{J(V)}$.) Thus, $\chi(A,\tau,M)$ is the number obtained by pairing the K_0-element $\chi_0(M)$ with the trace τ. For example, let G be a group acting on a contractible cell complex X as in the introduction, and assume furthermore that X is finite-dimensional (so that the associated resolution $\mathcal{V}$ of $\mathbb{C}$ over $\mathbb{C}G$ has finite length). In $K_0(\mathbb{C}G)$, we have

$$\sum_k (-1)^k [V_k] = \sum_\sigma (-1)^{\dim\sigma} [p_\sigma],$$

where σ ranges over a complete set of orbit representatives for the action of G on the cells of X, and p_σ is the average of the group elements in the stabilizer G_σ of σ. Since $\tau_0(p_\sigma) = |G_\sigma|^{-1}$ for the natural trace τ_0 on $\mathbb{C}G$, this makes

$$\chi(\mathbb{C}G, \tau_0, \mathbb{C}) = \sum_\sigma \frac{(-1)^{\dim\sigma}}{|G_\sigma|},$$

the rational Euler characteristic of G (see [4]).

5. A quantum group example

For a positive real number q different from 1, we consider the complex unital *-algebra $A = A_q$ generated by a selfadjoint element h and another element x modulo the relations

$$qhx - xh = 2x, \tag{1}$$

$$xx^* - qx^*x = h + \frac{1-q}{4}h^2. \tag{2}$$

This is the algebra $U_q(\mathbf{sl}_2)$ studied in [9], with a parameter restriction permitting an involutory structure with an ample supply of representable traces. The coefficient of h^2 in (2) is far from whimsical; it is chosen so that $U_q(\mathbf{sl}_2)$ can be endowed in a natural way with a comultiplication that makes it a quantum group (*i.e.* a noncommutative and noncocommutative Hopf algebra). As we will see shortly, the given relations also impose particularly good behavior on the *-representations of A.

Our purpose here is to calculate $H_*(A,\tau,M)$ (at least in principle) for all finite-dimensional A-modules M and all representable traces τ on A. We begin by describing certain features of A on which the calculations will hinge. (The reference for these preliminary facts is [9].) First of all, the set of monomials

$$\{x^{*i}h^jx^k : i,j,k \in \mathbb{N}\}$$

is a basis for A, and the same is true with the x^*'s written on the right and the x's on the left. There is a natural $\mathbb{Z}$-grading of A given by the *-automorphic action of the circle group that fixes h and multiplies x by scalars of modulus 1. This decomposes A as the direct sum of subspaces $A^{(n)}$ $(n \in \mathbb{Z})$, where

$$A^{(n)} = \operatorname{span}\{x^{*i}h^jx^k : i,j,k \in \mathbb{N},\ k-i=n\},$$

so that $A^{(m)}A^{(n)} \subseteq A^{(m+n)}$ and $A^{(n)*} = A^{(-n)}$. For $n > 0$, we furthermore have $A^{(n)} = x^nA^{(0)} = A^{(0)}x^n$ and $A^{(-n)} = x^{*n}A^{(0)} = A^{(0)}x^{*n}$. The following two identities, which follow by induction from (1) and (2), are worth noting:

$$hx^{*n} = q^nx^{*n}h - 2[n]x^{*n} \tag{3}$$

$$xx^{*n} - q^nx^{*n}x = [n]x^{*n-1}\left(\frac{q^{n-1}-q^n}{4}h^2 + q^{n-1}h - [n-1]\right), \tag{4}$$

for $n = 0,1,\ldots$, where $[n]$ is the q-integer $(q^n-1)/(q-1)$.

The finite-dimensional representation theory of A is easily summarized. The only finite-dimensional irreducible A-modules in which x is invertible are the one-dimensional modules M_β $(\beta \in \mathbb{C}\setminus\{0\})$ given by the complex homomorphisms

$$h \mapsto \frac{2}{q-1},\ x \mapsto \beta,\ x^* \mapsto -\frac{1}{\beta(q-1)^2}.$$

Additionally, for each positive integer n there are two irreducible n-dimensional A-modules $M^+(n)$ and $M^-(n)$, which may be characterized as follows: $\ker x = \mathbb{C}\xi$; $x^{*n-1}\xi \neq 0$; $x^{*n}\xi = 0$. It follows from (1) that ξ must be an eigenvector for h. What distinguishes $M^+(n)$ from $M^-(n)$ is the eigenvalue, namely $h\xi = \lambda^\pm(n)\xi$ in $M^\pm(n)$, where

$$\lambda^\pm(n) = 2\frac{1 \pm q^{\frac{1-n}{2}}}{q-1}.$$

The M_β's and the $M^\pm(n)$'s exhaust the finite-dimensional irreducible A-modules.

In our setting, the modules $M^\pm(n)$ can be realized by *-representations of A on $\mathbb{C}^n$. For $n=1$, we have the *-homomorphisms π_1^+ (resp.π_0^+) : $A \to \mathbb{C}$ annihilating x and sending h to $4/(q-1)$ (resp. 0). For $n \geq 2$, let $a_k = \lambda^\pm(n-2k+2)$ $(1 \leq k \leq n)$ and $b_j = [j][n-j]/q^{n-j}$ $(1 \leq j \leq n-1)$. It is easily checked that $a_{k+1} = qa_k - 2$ and that

$$\frac{1-q}{4}a_k^2 + a_k = \frac{[n-2k+1]}{q^{n-2k+1}} = \begin{cases} b_1 & \text{if } k=1, \\ b_k - qb_{k-1} & \text{if } 2 \leq k \leq n-1, \\ -qb_{n-1} & \text{if } k=n. \end{cases}$$

If H is the $n \times n$ diagonal matrix with diagonal $(a_1,\ldots,a_n)$ and X is the $n \times n$ matrix with superdiagonal $(\sqrt{b_1},\ldots,\sqrt{b_{n-1}})$ and zeroes elsewhere, we thus obtain *-homomorphisms $\pi_n^\pm : A \to M_n(\mathbb{C})$ sending h to H and x to X.

The existence of these representations facilitates the proof of the following lemma.

Lemma 5.1. *The zero eigenspace $A^{(0)}$ in the $\mathbb{Z}$-grading of A is isomorphic to the algebra of complex polynomials in two commuting variables via $P(\cdot,\cdot) \mapsto P(x^*x, h)$, and the same is true if x^*x is replaced by xx^*.*

Proof. It is immediate from relation (1) and its adjoint that x^*x and h commute. Each monomial $x^{*i}h^kx^i$ can be written as a polynomial in x^*x and h using (1) and (2), so the indicated map is surjective. If P is a polynomial such that $P(x^*x, h) = 0$, then the diagonal matrices $\pi_n^{\pm}(x^*x)$ and $\pi_n^{\pm}(h)$ satisfy $P(\pi_n^{\pm}(x^*x), \pi_n^{\pm}(h)) = 0$, and in particular $P(0, \lambda^{\pm}(n)) = 0$ for all positive integers n. This makes $P(0,\cdot) = 0$, so $P(x^*x, h) = x^*xQ(x^*x, h)$ for some polynomial $Q(\cdot,\cdot)$. We have $Q(x^*x, h) = 0$ because x is not a divisor of 0 in A. By induction on the degree of P in its first variable, we conclude that P must be the zero polynomial. Replacing x^*x by xx^* is harmless because of (2). □

For $n = 1, 2, \ldots$, let $\tau_n^{\pm}$ be the pair of representable traces on A defined by $\tau_n^{\pm} = \mathrm{tr}_n \circ \pi_n^{\pm}$. We write $\mathbb{C}_{\pm}^n$ for $\mathbb{C}^n$ as an $(A, \tau_n^{\pm})$-space (that is, acted on by the the *-representation $\pi_n^{\pm}$). It is an immediate consequence of the following proposition — plus what we have said about the irreducible representations of A — that every nonzero representable trace on A is the sum of finitely many positive multiples of these special traces.

Proposition 5.2. *Let π be a *-representation of A on Hilbert space such that the von Neumann algebra $\pi(A)''$ is finite. Then $\pi(A)$ is finite-dimensional.*

Proof. We may of course assume that π is unital. Let $\mathcal{H}$ denote the Hilbert space in question. The operators $\pi(x)$ and $\pi(x^*)$ must both have nonzero kernel. [Otherwise, because these operators lie in a finite von Neumann algebra, we would have $\pi(x) = U|\pi(x)|$ with U unitary and $\mathrm{im}\,|\pi(x)|$ dense. The relation (1) makes $\pi(h)$ commute with $|\pi(x)|$, so we would have $U\pi(h)U^* = q\pi(h) - 2$, forcing $\pi(h) = 2(q-1)^{-1}$ and hence, by (2), $\pi(xx^* - qx^*x) = (q-1)^{-1}$. This last equality is incompatible with the existence of a nonzero positive tracial linear functional on a finite von Neumann algebra.] Let $\mathcal{H}_0 = \ker \pi(x^*)$ and $\mathcal{H}_n = \overline{\pi(x^n)\mathcal{H}_0}$ for $n = 1, 2, \ldots$. From the adjoint of formula (4) above, we obtain quadratic polynomials $r_1, r_2, \ldots$ such that $\pi(x^*x^n)\xi = \pi(r_n(h)x^{n-1})\xi$ for ξ in $\mathcal{H}_0$. It now follows readily from (1) that the subspaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ are mutually orthogonal and $\pi(h)$-invariant, and that their (Hilbert space) direct sum is $\pi(A)$-invariant. Restricting $\pi(A)$ to the orthogonal complement of this direct sum gives a *-representation $\tilde{\pi}$ of A with $\tilde{\pi}(A)''$ finite and $\ker \tilde{\pi}(x^*) = (0)$. We conclude that $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. Because $X\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$, we have

$$(0) \neq \ker \pi(x) = \bigoplus_{n=0}^{\infty} (\ker \pi(x) \cap \mathcal{H}_n).$$

The polynomials r_n satisfy $\pi(x^*x)\eta = \pi(r_{n+1}(h))\eta$ for η in $\mathcal{H}_n$, so $\pi(r_{n+1}(h))$ must annihilate $\ker \pi(x) \cap \mathcal{H}_n$. It follows that each subspace $\ker \pi(x) \cap \mathcal{H}_n$ is either (0), an eigenspace for $\pi(h)$, or the direct sum of two such, and thus that $\ker \pi(x)$ is the direct sum of eigenspaces for $\pi(h)$.

Because $0 < q \neq 1$, there is a positive integer k such that

$$\max\{|\lambda^+(n)|, |\lambda^+(2-n)|\},\ \max\{|\lambda^-(n)|, |\lambda^-(2-n)|\} \ > \| \pi(h) \|$$

for all $n > k$. We claim that $\pi(x^{*k}A)$ must annihilate $\ker \pi(x)$. [Let ξ be an eigenvector for $\pi(h)$ in $\ker \pi(x)$. Calculating as above, it is easily seen that the vectors $\xi, \pi(x^*)\xi, \pi(x^{*2})\xi, \dots$ are othogonal and span $\pi(A)\xi$. Because the restriction of $\pi(A)''$ to the closure of this subspace is a finite von Neumann algebra, we must have $\ker \pi(x^*) \cap \overline{\pi(A)\xi} \neq (0)$, forcing $\pi(x^{*n})\xi = 0$ for some n (and hence $\pi(x^{*n})\pi(A)\xi = (0)$). Suppose n is the smallest positive power of $\pi(x^*)$ annihilating ξ. Then $\pi(A)\xi$ must be one of the irreducible A-modules $M^+(n)$ or $M^-(n)$, forcing $n \leq k$ by our choice of k.] But $\pi(A) \ker \pi(x)$ spans a dense subspace of $\mathcal{H}$ (for the same reason that $\pi(A) \ker \pi(x^*)$ does), so $\pi(x^{*k}) = 0$. Further, the spectrum of $\pi(h)$ must be contained in the union of the spectra of $\pi_j^{\pm}(h)$ for $j = 1, \dots, k$, so the algebra F generated by $\pi(h)$ is finite-dimensional. Thus $\pi(A)$, which is the vector space direct sum of $\{\pi(x^{*i})F\pi(x^j) : 0 \leq i, j \leq k-1\}$, must be finite-dimensional. □

We turn now to the calculation of homology spaces. The essential result is the following: $H_k(A, \tau_n^+, M^+(n))$ is $\mathbb{C}_+^n$ for $k = 0$ and $k = 3$ and vanishes for all other k; $H_*(A, \tau, M^+(n))$ vanishes for all other $\tau \in \{\tau_m^{\pm} : m = 1, 2, \dots\}$; and likewise with $+$ replaced by $-$.

It is feasible, but rather unpleasant, to construct a finite-rank free resolution for $M^{\pm}(n)$ over A and then apply the definitions. Instead, we will express $M^{\pm}(n)$ as a quotient of modules whose homology is more readily calculated. For γ in $\mathbb{R}$, let $L(\gamma)$ be the A-module $A/(Ax + A(h-\gamma))$, and let $\delta : A \to L(\gamma)$ be the quotient map.

Lemma 5.3. *The subset $\{\delta(1), \delta(x^*), \delta(x^{*2}), \dots\}$ is a basis for $L(\gamma)$.*

Proof. The indicated vectors span $L(\gamma)$ because $\delta(x^{*i}h^jx^k)$ is $\gamma^j\delta(x^{*i})$ if $k = 0$ and 0 if $k > 0$. If a nontrivial linear combination of $\delta(x^{*k})$'s were zero, the $\mathbb{Z}$-grading of A would give

$$x^{*k} \in A^{(-k)} \cap (Ax + A(h-\gamma)) = x^{*k}(A^{(0)}x^*x + A^{(0)}(h-\gamma))$$

for some $k \geq 0$, and hence $1 \in A^{(0)}x^*x + A^{(0)}(h-\gamma)$, contradicting Lemma 5.1 above. □

It is apparent from (3) and (4) that h acts diagonally on this basis, and that x acts by a weighted backward shift.

Lemma 5.4. *The sequence*

$$0 \longrightarrow A \xrightarrow{(\gamma+2-qh\ x)} A^2 \xrightarrow{\binom{x}{h-\gamma}} A \xrightarrow{\delta} L(\gamma) \longrightarrow 0$$

is exact.

Proof. The map from A to A^2 is injective because x is not a divisor of 0 in A. For exactness at A^2, suppose that a, b in A satisfy $ax + b(h-\gamma) = 0$. Use the $\mathbb{Z}$-grading to write

$$a = \sum_{j<0} x^{*(-j)}a_j + a_0 + \sum_{j>0} x^j a_j,$$
$$bx^* = \sum_{j<0} x^{*(-j)}c_j + c_0 + \sum_{j>0} x^j c_j,$$

where the a_j's and c_j's belong to $A^{(0)}$. Since $(h-\gamma)x^* = x^*(qh-2-\gamma)$, we have

$$a_j xx^* + c_j(qh - 2 - \gamma) = 0$$

for every j. By Lemma 5.1 above, then, we obtain d_j in $A^{(0)}$ such that

$$a_j = d_j(\gamma + 2 - qh) \text{ and } c_j = d_j xx^*.$$

Setting

$$d = \sum_{j<0} x^{*(-j)} d_j + d_0 + \sum_{j>0} x^j d_j,$$

we have $a = d(\gamma+2-qh)$ and $bx^* = dxx^*$, and hence $(a \;\; b) = d(\gamma+2-qh \quad x)$. □

With this resolution in hand, the homology spaces of $L(\gamma)$ are easily determined.

Proposition 5.5. *For any representable trace τ on A and any real γ, we have*

$$H_0(A,\tau,L(\gamma)) \approx \ker \rho(\gamma - h) \cap \ker \rho(x^*),$$
$$H_2(A,\tau,L(\gamma)) \approx \ker \rho(\gamma + 2 - qh) \cap \ker \rho(x),$$
$$H_1(A,\tau,L(\gamma)) \approx H_0(A,\tau,L(\gamma)) \oplus H_2(A,\tau,L(\gamma)),$$

*where ρ is the *-antirepresentation of A on $L^2(A,\tau)$ defined in section 1 above.*

Proof. The identifications of H_0 ($=\ker \overline{\Delta}_1^*$) and of H_2 ($=\ker \overline{\Delta}_2$ here) are clear. Writing ξa for $\rho(a)\xi$ $(\xi \in L^2(A,\tau), a \in A)$, we have

$$(\xi,\eta) \in \ker \overline{\Delta}_2^* \cap \ker \overline{\Delta}_1$$
$$\Longrightarrow \left\{ \begin{array}{ccc} \xi(\gamma+2-qh) + \eta x^* & = & 0 \\ \xi x + \eta(h-\gamma) & = & 0 \end{array} \right\}$$
$$\Longrightarrow \left\{ \begin{array}{ccc} \xi(\gamma+2-qh)^2 + \eta x^*(\gamma+2-qh) & = & 0 \\ \xi xx^* - \eta x^*(\gamma+2-qh) & = & 0 \end{array} \right\}$$
$$\Longrightarrow \xi(\gamma+2-qh)^2 + \xi xx^* = 0$$
$$\Longrightarrow \xi \in \ker \rho(\gamma+2-qh) \cap \ker \rho(x),$$

and furthermore

$$\eta \in \ker \rho(\gamma - h) \cap \ker \rho(x^*).$$

This takes care of H_1. □

In case $\tau = \tau_n^{\pm}$, it easily seen that $L^2(A,\tau)$ is $\mathbb{C}_{\pm}^n \otimes \mathbb{C}_{\pm}^n$, with $\lambda(a)$ and $\rho(a)$ given respectively by

$$\lambda(a)(\xi \otimes \eta) = (\pi_n^{\pm}(a)\xi) \otimes \eta, \quad \rho(a)(\xi \otimes \eta) = \xi \otimes (\eta \pi_n^{\pm}(a)).$$

The proposition just proved and the recipe for $\pi_n^{\pm}$ yield the following:

$$H_0(A,\tau_n^{\pm},L(\gamma)) \approx \left\{ \begin{array}{ll} \mathbb{C}_{\pm}^n & \text{if } \gamma = \lambda^{\pm}(n), \\ (0) & \text{else,} \end{array} \right. \tag{5}$$

$$H_2(A,\tau_n^{\pm},L(\gamma)) \approx \left\{ \begin{array}{ll} \mathbb{C}_{\pm}^n & \text{if } \gamma = \lambda^{\pm}(-n), \\ (0) & \text{else.} \end{array} \right. \tag{6}$$

(In the formula for H_2, we are using $q\lambda^{\pm}(2-n) - 2 = \lambda^{\pm}(-n)$.)

We next exhibit an injective map $\theta : L(\lambda^{\pm}(-n)) \to L(\lambda^{\pm}(n))$ whose cokernel is $M^{\pm}(n)$. Write the identity (4) in the form

$$xx^{*n} - q^n x^{*n} x = x^{*n-1} p_n(h). \tag{7}$$

The zeroes of the quadratic polynomial p_n are easily seen to be $\lambda^+(n)$ and $\lambda^-(n)$, so there are two linear polynomials P_n^+ and P_n^- such that

$$p_n(h) = P_n^+(h)(h - \lambda^+(n)) = P_n^-(h)(h - \lambda^-(n)).$$

Let $S : A^2 \to A^2$ be the module map with matrix

$$\begin{pmatrix} q^n x^{*n} & x^{*n-1} P_n^{\pm}(h) \\ 0 & q^n x^{*n}(h - \lambda^{\pm}(n)) \end{pmatrix},$$

acting (as usual) on the right. By (3) and (7), plus $q^n \lambda^{\pm}(n) = 2[n] + \lambda^{\pm}(n)$, we have the commutative diagram

$$\begin{array}{ccc} A^2 & \xrightarrow{\left(\begin{smallmatrix} x \\ h-\lambda^{\pm}(n) \end{smallmatrix}\right)} & A \\ {\scriptstyle S}\Big| & & \Big|{\scriptstyle x^{*n}} \\ A^2 & \xrightarrow[\left(\begin{smallmatrix} x \\ h-\lambda^{\pm}(-n) \end{smallmatrix}\right)]{} & A \end{array}$$

There is thus a map $\theta : L(\lambda^{\pm}(-n)) \to L(\lambda^{\pm}(n))$ such that $\theta(\delta'(a)) = \delta(ax^{*n})$ for a in A, where δ and δ' are the quotient maps from A to $L(\lambda^{\pm}(n))$ and $L(\lambda^{\pm}(-n))$. It follows from Lemma 5.3 that θ is injective, and that $\{\delta(x^{*k}) : k \geq n\}$ is a basis for its image. (Notice that $x\delta(x^{*n}) = x^{*n-1}\delta(p_n(h)) = 0$ because $p_n(\lambda^{\pm}(n)) = 0$ — otherwise this set would not span a submodule of $L(\lambda^{\pm}(n))$.) Let $\langle\cdot\rangle$ be the quotient map from $L(\lambda^{\pm}(n))$ to its quotient by the image of θ. The action of x on the quotient has kernel $\mathbb{C}\langle\delta(1)\rangle$, and

$$x^{*n-1}\langle\delta(1)\rangle \;=\; \langle\delta(x^{*n-1})\rangle \;\neq 0 = x^{*n}\langle\delta(1)\rangle.$$

Since furthermore $h\langle\delta(1)\rangle \;=\; \lambda^{\pm}(n)\langle\delta(1)\rangle$, the quotient must be isomorphic to $M^{\pm}(n)$.

Before we can apply Theorem 3.3 to the exact sequence

$$0 \to L(\lambda^{\pm}(-n)) \to L(\lambda^{\pm}(n)) \to M^{\pm}(n) \to 0,$$

we need to understand the map on H_1 induced (as in Section 2) by θ. In what follows, we consider only the traces $\tau_n^{\pm}$ and abbreviate $H_*(A, \tau_n^{\pm}, \cdot)$ as $H_*(\cdot)$.

Lemma 5.6. *The map from $H_1(L(\lambda^{\pm}(-n)))$ to $H_1(L(\lambda^{\pm}(n)))$ induced by θ is an isomorphism.*

Proof. Consider the subspaces E_0 and E_2 of $L^2(A, \tau_n^{\pm})$ defined by

$$E_0 = \ker\rho(\lambda^{\pm}(n) - h) \cap \ker\rho(x^*),$$
$$E_2 = \ker\rho(\lambda^{\pm}(2-n) - h) \cap \ker\rho(x).$$

By (the proof of) Proposition 5.5 and its specialization in (5) and (6), the homology spaces $H_1(L(\lambda^{\pm}(n \text{ resp. } -n)))$ are realized inside $L^2(A, \tau_n^{\pm})^2$ as

$$H_1(L(\lambda^{\pm}(n))) = 0 \oplus E_0 \text{ and } H_1(L(\lambda^{\pm}(-n))) = E_2 \oplus 0.$$

For ξ in E_2 and $S : A^2 \to A^2$ as above, we have

$$\overline{S}(\xi, 0) = (q^n \xi x^{*n}, \xi x^{*n-1} P_n^{\pm}(h)) = (0, \xi x^{*n-1} P_n^{\pm}(h))$$

because $\rho(x^{*n}) = 0$. Recall from the construction of $\pi_n^{\pm}$ that $\pi_n^{\pm}(x)$ is superdiagonal with nonzero superdiagonal entries, so $\rho(x^{*n-1})$ maps $\ker\rho(x)$ bijectively to $\ker\rho(x^*)$, which is the $\lambda^{\pm}(n)$-eigenspace for $\rho(h)$. Since $\lambda^+(n) \neq \lambda^-(n)$, we have $P_n^+(\lambda^+(n)) \neq 0 \neq P_n^-(\lambda^-(n))$. It follows that $\overline{S}$ maps $E_2 \oplus 0$ bijectively to $0 \oplus E_0$. ∎

The outcome of our calculations is summarized as follows.

Proposition 5.7. *If M is one of the modules $M^{\pm}(n)$ and τ is one of the traces $\tau_m^{\pm}$, then*

$$H_k(A,\tau,M) \approx \begin{cases} \mathbb{C}_+^n & \text{if } k \in \{0,3\}, M = M^+(n), \tau = \tau_n^+, \\ \mathbb{C}_-^n & \text{if } k \in \{0,3\}, M = M^-(n), \tau = \tau_n^-, \\ 0 & \text{else.} \end{cases}$$

Proof. In the third case, $H_*(L(\lambda^{\pm}(-n)), A, \tau)$ and $H_*(A, \tau, L(\lambda^{\pm}(n)))$ both vanish by (5) and (6), so $H_*(A, \tau, M\pm(n))$ vanishes by Theorem 3.3. In the first two cases, the weakly (here, in fact, truly) exact sequence in Theorem 3.3 is

$$\ldots \to 0 \to H_3(M^{\pm}(n)) \to \mathbb{C}_{\pm}^n \to 0 \to H_2(M^{\pm}(n)) \to \mathbb{C}_{\pm}^n \overset{\approx}{\to} \mathbb{C}_{\pm}^n \to H_1(M^{\pm}(n))$$
$$\to 0 \to \mathbb{C}_{\pm}^n \to H_0(M^{\pm}(n)) \to 0,$$

by (5), (6), and Lemma 5.6. □

It remains to deal with the special one-dimensional modules M_β described at the beginning of this section. The calculations are like those leading to Proposition 5.7, but easier.

Proposition 5.8. *For any representable trace τ on A and any nonzero complex number β, the homology spaces $H_*(A, \tau, M_\beta)$ all vanish.*

Proof. (*sketch*) Write $k = h - 2(q-1)^{-1}$; the relations (1) and (2) then become

$$qkx = xk, \tag{8}$$

$$xx^* - qx^*x = \frac{1-q}{4}k^2 + \frac{1}{q-1}. \tag{9}$$

Let $N_\beta = A/(Ak + A(x-\beta))$. Arguing as for Lemma 5.4, it is not difficult to show that the sequence

$$0 \longrightarrow A \xrightarrow{(\beta - x\ qk)} A^2 \xrightarrow{\binom{k}{x-\beta}} A \xrightarrow{\delta} N_\beta \longrightarrow 0$$

is exact. Since k has no kernel in any of the representations $\pi_n^{\pm}$, it follows from Proposition 5.2 that $\ker \rho(k) = (0)$ for any representable trace τ. Calculating as in Proposition 5.5, we then obtain $H_*(A, \tau, N_\beta) = (0)$.

We conclude the proof by exhibiting an exact sequence

$$0 \to N_{q\beta} \to N_\beta \to M_\beta \to 0$$

and appealing to Theorem 3.3. Let $\gamma = -(q-1)^{-2}\beta^{-1}$. It follows from (8) and (9) that

$$\begin{pmatrix} qx^* - \gamma & 0 \\ \frac{1-q}{4}k & qx^* - \gamma \end{pmatrix} \begin{pmatrix} k \\ x - \beta \end{pmatrix} = \begin{pmatrix} k \\ x - q\beta \end{pmatrix} (x^* - \gamma),$$

so right multiplication on A by $x^* - \gamma$ induces a map $\phi : N_{q\beta} \to N_\beta$. It is straightforward to show that ϕ is injective with cokernel isomorphic to M_β. □

Notice that $H_*(A, \tau_n^{\pm}, \cdot)$ "sees" (among the irreducible A-modules) only the module $M^{\pm}(n)$ that gives rise to $\tau_n^{\pm}$. Proposition 2.3 accounts for this in dimension 0, but in higher dimensions the phenomenon is somewhat special, owing here to the circumstance that the spectra of h in any two different irreducible *-representations are disjoint. By way of contrast, we consider an easy example, instructive in its own right, that does not exhibit this type of orthogonality.

Let A now be the complex group algebra $\mathbb{C}G$, where $G = PSL(2,\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$. Denote by u (resp.v) the generator of the free product factor $\mathbb{Z}_2$ (resp. $\mathbb{Z}_3$), and let p, q, r in A be the spectral idempotents of v, that is

$$\begin{aligned} p &= \frac{1}{3}(1 + v + v^2), \\ q &= \frac{1}{3}(1 + \lambda^2 v + \lambda v^2), \\ r &= \frac{1}{3}(1 + \lambda v + \lambda^2 v^2), \end{aligned}$$

where $\lambda = \exp(2\pi i/3)$. Consider the A-module M with projective resolution

$$0 \to A(1-p) \oplus A(1-uqu) \xrightarrow{\Delta} A \xrightarrow{\delta} M \to 0,$$

where Δ adds the two direct summands. [That Δ is injective follows from the observation that if $\phi : G \to \mathbb{C}$ satisfies

$$\phi(g) + \phi(gv) + \phi(gv^2) = 0 = \phi(g) + \lambda\phi(guvu) + \lambda^2\phi(guv^2u)$$

for all g in G, then either $\phi \equiv 0$ or the support of ϕ must be infinite. This is because $\phi(g) \neq 0$ implies that at least one of gv and gv^2 and at least one of $guvu$ and guv^2u belongs to the support of ϕ.] The module M is easily seen to be two-dimensional, with basis $\{\delta(1), \delta(u)\}$ on which v acts by $v\delta(1)$ $(= v\delta(p)) = \delta(1)$ and $v\delta(u)$ $(= vu\delta(uqu)) = \lambda\delta(u)$. Thus, M comes from the irreducible *-representation $\pi_2 : A \to M_2(\mathbb{C})$ sending

$$u \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad v \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

For any trace τ on A, the spaces H_0 and H_1 are simply the kernels of $\overline{\Delta}^*$ and $\overline{\Delta}$ respectively, so we have

$$\begin{aligned} H_0(A,\tau,M) &\approx \operatorname{im}\rho(p) \cap \operatorname{im}\rho(uqu), \\ H_1(A,\tau,M) &\approx \ker\rho(p) \cap \ker\rho(uqu). \end{aligned}$$

Let τ_2 be the trace $\operatorname{tr}_2 \circ \pi_2$. As an A-module, $L^2(A,\tau_2)$ is $M \otimes \mathbb{C}^2$. One checks that $\{J(p), J(q), J(pu), J(qu)\}$ is an orthonormal basis, and that $H_0(A,\tau_2,M)$ and $H_1(A,\tau_2,M)$ are respectively spanned by $\{J(p), J(qu)\}$ and $\{J(q), J(pu)\}$ — that is, both are the Hilbert space $\mathbb{C}^2$ with A acting via π_2. Here, the two Betti numbers b_0 and b_1 are both 1. Consider also the irreducible *-representation $\pi_3 : A \to M_3(\mathbb{C})$ sending

$$u \mapsto \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}, \qquad v \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

and let τ_3 be the trace $\operatorname{tr}_3 \circ \pi_3$. Observe that $\operatorname{im}\pi_3(p) \cap \operatorname{im}\pi_3(uqu) = (0)$ and $\ker\pi_3(p) \cap \ker\pi_3(uqu)$ is one-dimensional, whence it follows that $H_0(A,\tau_3,M) = (0)$ and that $H_1(A,\tau_3,M) \approx \mathbb{C}^3$ (with A acting via π_3). The Betti numbers in this case are $b_0 = 0$ and $b_1 = 1$. For an arbitrary representable trace τ on A, one has $b_1(A,\tau,M) - b_0(A,\tau,M) = \tau(r)$ because the $K_0(A)$-valued Euler characteristic (see section 5) $\chi_0(M)$ is $-[r]$. In particular, $H_*(A,\tau,M)$ is nonvanishing for any τ (for example τ_3) such that $\tau(r) > 0$. If no positive multiple of τ majorizes τ_2 (that is, there exists no $c > 0$ such that $\tau_2(a^*a) \le c\tau(a^*a)$ for all a in A), it follows from

Proposition 2.3 or from a direct argument that $H_0(A, \tau, M) = (0)$, and thus that $b_1(A, \tau, M) = \tau(r)$.

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