LIOUVILLE TYPE THEOREMS FOR FOURTH ORDER ELLIPTIC EQUATIONS IN A HALF PLANE

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ABSTRACT. Consider an elliptic equation $\omega\Delta\varphi - \Delta^2\varphi = 0$ in the half plane $\{(x, y),\ -\infty < x < \infty,\ y > 0\}$ with boundary conditions $\varphi = \varphi_y = 0$ if $y = 0$, $x > 0$ and $B_j\varphi = 0$ if $y = 0$, $x < 0$ where B_j $(j = 2, 3)$ are second and third order differential operators. It is proved that if $Re\,\omega \geq 0$, $\omega \neq 0$ and, for some $\varepsilon > 0$, $|\varphi| \leq Cr^\alpha$ if $r = \sqrt{x^2 + y^2} \to \infty$, $|\varphi| \leq Cr^\beta$ if $r \to 0$ where $\alpha = n + \frac{1}{2} - \varepsilon$, $\beta = n + \frac{1}{2} + \varepsilon$ for some nonnegative integer n, then $\varphi \equiv 0$. Results of this type are also established in case $\omega = 0$ under different conditions on α and β; furthermore, in one case $B_3\varphi$ has a lower order term which depends nonlocally on φ. Such Liouville type theorems arise in the study of coating flow; in fact, they play a crucial role in the analysis of the linearized version of this problem. The methods developed in this paper are entirely different for the two cases (i) $Re\,\omega \geq 0$, $\omega \neq 0$ and (ii) $\omega = 0$; both methods can be extended to other linear elliptic boundary value problems in a half plane.

0. INTRODUCTION

A Liouville type theorem for an elliptic operator $P(D)$ of order $2m$ with constant coefficients asserts that if

(0.1) $$P(D)\varphi = 0 \text{ in } \mathbb{R}^n$$

and

(0.2) $$\varphi(x) = O\left(r^N\right) \text{ as } r \to \infty$$

where $r = |x|$, then $\varphi(x)$ is a polynomial.

There are two simple and entirely different methods for proving such a result:

Method 1. Suppose $P(D)$ is a positive operator in the sense that

(0.3) $$\int_{\mathbb{R}^n} P(D)\psi \cdot \overline{\psi}dx \geq c_0 \int_{\mathbb{R}^n} |\psi|^2 dx\ (c_0 > 0)$$

for every ψ which decays at ∞, say like $O\left(r^{-M}\right)$. If $P(D)$ is homogenous then from (0.2) we can deduce, by applying elliptic estimates to a scaled solution, that

$$D^\alpha\varphi = O\left(r^{-M}\right) \text{ if } |\alpha| \geq M + N\ ,\quad \text{as } r \to \infty\ .$$

Using (0.3) with $\psi = D^\alpha \varphi$ we deduce that $\psi \equiv 0$, so that φ is a polynomial. If $P(D)$ is not homogenous but (0.2) holds with $N = -M$, then (0.3) immediately yields $\varphi \equiv 0$.

Method 2. The Fourier transform $\widehat{\varphi}(\xi)$ of $\varphi(x)$, taken in the distribution sense, satisfies

$$P(i\xi)\widehat{\varphi}(\xi) = 0 .$$

Hence, $\widehat{\varphi}$ is a distribution with support in the null set

$$N_P = \{\xi \in \mathbb{R}^n ; \quad P(i\xi) = 0\} .$$

If, for instance, N_P consists of just the origin, then

$$\widehat{\varphi}(\xi) = \sum_{|\gamma| \leq K} a_\gamma D^\gamma \delta(\xi)$$

for some K, where $\delta(\xi)$ is the Dirac measure and, consequently, $\varphi(x)$ is a polynomial.

Both methods can be extended to a solution of (0.1) in the half-space $\mathbb{R}_+^n = \{(x_1, \dots, x_n); \ x_n > 0\}$, satisfying boundary conditions

$$(0.4) \qquad\qquad Q_j(D)\varphi = 0 \text{ on } x_n = 0 \ (1 \leq j \leq m) .$$

To apply Method 1, we note that the tangential derivatives

$$D^\alpha \varphi = D_1^{\alpha_1}, \dots, D_{n-1}^{\alpha_{n-1}}$$

satisfy (0.4). Hence, if $P(D)$ is homogeneous and the positivity condition (0.3) holds in $\mathbb{R}_+^n$ for solutions of (0.1), (0.4) which decay like $O(r^{-M})$ at ∞, then we can deduce as before that $D^\alpha \varphi = 0$ and, consequently, $\varphi(x)$ is a polynomial in $x_1, \dots, x_{n-1}$. Using (0.1), (0.4) we can then easily deduce that $\varphi(x)$ is also a polynomial in x_n.

To apply Method 2, we take the Fourier transform $\widehat{\varphi}(\xi_1, \dots, \xi_{n-1}, x_n)$ in the variables $x_1, \dots, x_{n-1}$ and solve the resulting ODE problem for $\widehat{\varphi}$ as a function of x_n (using the boundary conditions resulting from (0.4)). Then we go back to the inverse transform of $\widehat{\varphi}$ and try to deduce, from the structure of $\widehat{\varphi}$, that φ is a polynomial.

In Part I of this paper we establish a Liouville type theorem for solutions φ of the elliptic equation

$$(0.5) \qquad \mathcal{L}\varphi \equiv \omega \Delta \varphi - \Delta^2 \varphi = 0 \text{ in } \mathbb{R}_+^2 \equiv \{(x, y); \ x \in \mathbb{R}, \ 0 < y < \infty\}$$

where $Re \ \omega \geq 0$, $\omega \neq 0$, satisfying the boundary conditions

$$(0.6) \qquad\qquad \varphi(x, 0) = \varphi_y(x, 0) = 0 \text{ if } x > 0 ,$$

$$(0.7) \qquad\qquad \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{if } y = 0, \ x < 0 ,$$

$$(0.8) \qquad\qquad 3\frac{\partial^3 \varphi}{\partial x^2 \partial y} + \frac{\partial^3 \varphi}{\partial y^3} - \omega \frac{\partial \varphi}{\partial y} = 0 \quad \text{if } y = 0, \ x < 0 .$$

We prove (Theorem 1.1) that if for some $\varepsilon > 0$,

$$(0.9) \qquad\qquad |\varphi(x, y)| \leq \begin{cases} Cr^\alpha & \text{as } r \to \infty , \\ Cr^\beta & \text{as } r \to 0 , \end{cases}$$

where $\alpha = n + \frac{1}{2} - \varepsilon$, $\beta = n + \frac{1}{2} + \varepsilon$, n nonnegative integer and $r = \sqrt{x^2 + y^2}$, then $\varphi \equiv 0$.

In Part II of this paper we consider the elliptic problem

$$(0.10) \qquad \Delta^2 \varphi = 0 \quad \text{in } \mathbb{R}^2_+$$

under the boundary conditions (0.6), (0.7) and either

$$(0.11) \qquad 3 \frac{\partial^3 \varphi}{\partial x^2 \partial y} + \frac{\partial^3 \varphi}{\partial y^3} = 0 \quad \text{if } y = 0, \ x < 0 \ ,$$

or

$$\mu f + \beta_0 \frac{df}{dx} - \frac{\partial \varphi}{\partial x} = 0 \qquad \qquad \text{if } y = 0, \ x < 0 \ ,$$

$$(0.12)$$

$$3 \frac{\partial^3 \varphi}{\partial x^2 \partial y} + \frac{\partial^3 \varphi}{\partial y^3} + \sigma \frac{d^3 f}{dx^3} = 0 \qquad \text{if } y = 0, \ x < 0 \ ,$$

where μ, β_0, σ are constants, $\beta_0 > 0$, $\sigma > 0$, and

$$Re\mu \geq 0 \ , \qquad \frac{\mu}{\sigma \pm 2\beta_0 i} \notin (-\infty, 0) \ .$$

For the case of (0.11) we prove (Theorem 10.1) that if, for some $\varepsilon > 0$, (0.9) holds where

$$\alpha = n + \frac{1}{2} - \varepsilon \ , \quad \beta = n - \frac{1}{2} + \varepsilon \ , \quad n \text{ a nonnegative integer} \ ,$$

then $\varphi \equiv 0$. For the case of (0.12) we prove (Theorem 10.2) that if, for some $\varepsilon > 0$, (0.9) holds and

$$|f(x)| \leq C(1 + |x|)^\alpha \qquad (-\infty < x < 0)$$

where

$$\alpha = n + \frac{1}{2} - \varepsilon, \ \beta = n - \rho_0 + \varepsilon, \ n \text{ a nonnegative integer,}$$

then $\varphi \equiv 0$, $f \equiv 0$; here

$$\rho_0 = \frac{1}{2\pi i} \log \frac{\sigma + 2\beta_0 i}{\sigma - 2\beta_0 i} \ , \quad 0 < \rho_0 < \frac{1}{2} \ .$$

Elliptic boundary value problems of the above type arise in the study of non-stationary coating flows in the plane, which are a small perturbation of the uniform flow. When the free boundary problem is linearized and the Laplace transform is taken with respect to the t variable, one obtains an elliptic equation of the form

$$\mathcal{L}\varphi = \mu \text{ in } D \equiv \{(x, y) \ ; \ x \in \mathbb{R}, \ 0 < y < 1\}$$

with boundary conditions (0.6) and

$$(0.13) \qquad B_1(\varphi, f) = a, \ B_2\varphi = b, \ B_3(\varphi, f) = c \text{ if } y = 0, \ x < 0 \ ,$$

and with Dirichlet data on $y = 1$. Here $B_2\varphi$ coincides with the left-hand side of (0.7), and $B_j(\varphi, f)$ is a differential operator of order j in φ and in the free boundary $f(x)$; the free boundary is such that $0 < f(x) < 1$ if $-\infty < x < 0$ and $f(0) = 0$.

The existence of a solution for the linearized problem with some "natural" a priori estimates depends upon deriving sharp estimates for the elliptic problem in D, with very precise dependence on the parameter ω. This, in turn, involves a blow-up analysis, i.e., working with a sequence of scaled solutions for which one must establish that the limit is uniquely determined. There are several possible limit problems, three of which were described above. Thus the Liouville theorems

stated above are a critical ingredient in the analysis for the linearized coating flow problem; for more details we refer to [5].

In Part I we try to follow the basic idea of Method 1: the main effort is in proving that the function φ *itself* already decays sufficiently fast at ∞. The proof depends on the derivation of very sharp a priori estimates for the solutions of the inhomogenous system corresponding to (0.5) when the boundary condition (0.6) holds on all of $\mathbb{R}^1$ (Sections 4–6), as well as for solutions of the inhomogenous system corresponding to (0.5) when the boundary conditions (0.7), (0.8) hold on all of $\mathbb{R}^1$ (Section 3); we shall refer to the latter boundary condition as "strange."

In Part II we try to follow the basic idea of Method 2; here the main effort is in the analysis of the Fourier transform $\widehat{\varphi}$ near the origin. This will be based on the Wiener-Hopf techniques. In the case of Theorem 10.2, we need to establish very sharp estimates on solutions of $\Delta^2\varphi = \mu$ in $\mathbb{R}^2_+$ satisfying "strange" inhomogenous boundary conditions (corresponding to (0.7), (0.12)) on all of $\mathbb{R}^1$.

Both methods (of Parts I and II) can be extended to other elliptic boundary value problems in the half-plane. One simple example is briefly described in the last section of this paper.

Part I. The Case $Re\,\omega \geq 0,\ \omega \neq 0$

1. The main result

For any $R > 0$ we define $B_R = \left\{(x,\,y);\ x^2 + y^2 < R^2\right\}$. In this Part I we prove the following theorem:

Theorem 1.1. *Let $Re\,\omega \geq 0$, $\omega \neq 0$, and let φ be a solution in $H^2(\mathbb{R}^2_+ \cap B_R)$, for any $R > 0$, of the elliptic system:*

$$(1.1) \qquad\qquad \omega\Delta\varphi = \Delta^2\varphi \ \textit{in}\ \mathbb{R}^2_+\,,$$

$$(1.2) \qquad\qquad \varphi(x,\,0) = \partial_y\varphi(x,\,0) = 0 \ \textit{if}\ x > 0\,,$$

$$(1.3) \qquad\qquad \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} = 0 \ \textit{if}\ y = 0,\ x < 0\,,$$

$$(1.4) \qquad\qquad 3\frac{\partial^3\varphi}{\partial x^2\partial y} + \frac{\partial^3\varphi}{\partial y^3} - \omega\frac{\partial\varphi}{\partial y} = 0 \ \textit{if}\ y = 0,\ x < 0\,.$$

If, for some $\varepsilon > 0$,

$$(1.5) \qquad\qquad |\varphi(x,\,y)| \leq Cr^\alpha \ \textit{as}\ r \to \infty\,,$$

$$(1.6) \qquad\qquad |\varphi(x,\,y)| \leq Cr^\beta \ \textit{as}\ r \to 0\,,$$

where

$$(1.7) \qquad \alpha = n + \frac{1}{2} - \varepsilon, \quad \beta = n + \frac{1}{2} + \varepsilon, \quad n\ \textit{a nonnegative integer}$$

and $r = \sqrt{x^2 + y^2}$, then $\varphi \equiv 0$.

A proof of Theorem 1.1 which is based on taking the Fourier transform and using the Wiener-Hopf method appears too complicated for the case $\omega \neq 0$. We shall instead use a simpler and entirely different approach motivated by Method 1 above. To explain our approach let us introduce two auxiliary problems P^+ and P^-:

Problem P^+ consists of (1.1) and of (1.2) for all $x \in \mathbb{R}$, and problem P^- consists of (1.1) and of (1.3), (1.4) for all $x \in \mathbb{R}$. We need to study these problems carefully and establish some a priori estimates. This is done in Section 3 for problem P^- and in Sections 4–6 for problem P^+.

Assume now that $\varphi \not\equiv 0$ and define the function

(1.8)
$$\rho(R) = \left[R^2 \int_{\left(B_{2R} \setminus B_{R/2}\right)^+ \setminus \Sigma_R} \left(|\nabla^2 \varphi|^2 + |\nabla^4 \varphi|^2 \right) \right]^{1/2} + \sup_{\Sigma_R} R \left[|\nabla \varphi| + |\nabla^4 \varphi| \right]$$

where G^+ means $G \cap \{y \geq 0\}$ and

(1.9)
$$\Sigma_R = \left\{ (x,\, y);\ \frac{R}{2} < \sqrt{x^2 + y^2} < 2R,\ x > 0 \text{ and } 0 < y < A \log R \right\}$$

for $A > 1$. It will be shown in Section 11 that under the growth condition (1.5), $\rho(R) \leq CR^\alpha$ for R large. If for some positive constants $\overline{C}$ and γ,

(1.10)
$$\rho(R) \leq \frac{\overline{C}}{R^\gamma} \text{ for all } R \geq 1\,,$$

then one can show (Section 8) that φ decays at infinity in such a way the if we integrate

$$\int_{\mathbb{R}^2_+ \cap B_R} \varphi \mathcal{L} \varphi = 0$$

by parts and let $R \to \infty$, then the boundary integrals on $(\partial B_R)^+$ go to zero. Consequently, we get an "energy equality" from which it follows that $\varphi \equiv 0$.

On the other hand, it is shown in Section 2 that if (1.10) is not true then there are sequences $R_j \to \infty$, $B_j \to \infty$ such that

(1.11)
$$\rho(R_j) \geq \frac{1}{A} \sup_{\eta \in [1,\, B_j]} \left\{ \frac{1}{\eta^\alpha} \sup_{\frac{R_j}{\eta} \leq \xi \leq \eta R_j} \rho(\xi) \right\}\,.$$

[This result is actually true for general positive functions $\rho(R)$ which are bounded by CR^α for R large.] We then work, in Sections 7, 8, with the scaled functions

(1.12)
$$\psi_j(x,\, y) = \frac{\varphi(R_j x,\, R_j y)}{\rho(R_j)}\,,$$

noting that

(1.13)
$$\int_{\left(B_2 \setminus B_{1/2}\right)^+ \setminus \widetilde{\Sigma}_{R_j}} \left[|\nabla^2 \psi_j|^2 + \frac{1}{R_j^4} |\nabla^4 \psi_j|^2 \right]^{1/2} + \sup_{\widetilde{\Sigma}_{R_j}} \left[|\nabla \psi_j| + \frac{1}{R_j^3} |\nabla^4 \psi_j| \right] = 1\,,$$

where

$$\widetilde{\Sigma}_{R_j} = \left\{ (x,\, y),\ \frac{1}{2} < \sqrt{x^2 + y^2} \leq 2,\ 0 < y < A \frac{\log R_j}{R_j} \right\}\,.$$

Assuming that $n = 0$ in (1.7), we show in Section 7 that

(1.14)
$$\psi_j \to 0 \text{ away from the positive } x\text{-axis}\,.$$

Let ζ be a cutoff function whose support does not intersect either the positive x-axis or the negative x-axis. In Section 8 we consider the elliptic problem for

$$\widetilde{\psi}_j = \zeta \psi_j$$

and apply the estimates for problems P^- and P^+ separately to deduce (from (1.14)) a stronger convergence to zero of the ψ_j. In fact, we show that the convergence is such that the left-hand side of (1.13) goes to zero as $j \to \infty$, which is a contradiction. This completes the proof of Theorem 1.1 in case $n = 0$. The proof for any positive integer n is obtained by induction, at the end of Section 9.

2. A lemma for polynomially increasing functions

Lemma 2.1. *Fix $A > 1$ and let $\rho(R)$ be a continuous positive function such that*

$$(2.1) \qquad \rho(R) \leq CR^\alpha \text{ for all } R \geq 1,$$

where C, α are positive constants. Then, either

  (i) *there exist sequences $R_j \to \infty$, $B_j \to \infty$ such that (1.11) holds, or*

  (ii) *there exist positive constants $\overline{C}$, γ such that (1.10) holds.*

Proof. Suppose (i) is not true. Then there exist constants $L > 1$ and $B > 1$ such that

$$(2.2) \qquad \rho(R) \leq \frac{1}{A} \sup_{1 \leq \eta \leq B} \left[\frac{1}{\rho^\alpha} \sup_{\frac{R}{\eta} \leq \xi \leq \eta R} \rho(\xi) \right] \qquad \forall R \geq L.$$

Choose $\overline{\eta} = \overline{\eta}(R)$ such that $1 \leq \overline{\eta} \leq B$ and

$$\rho(R) \leq \frac{1}{A} \frac{1}{\overline{\eta}^\alpha} \sup_{\frac{R}{\overline{\eta}} \leq \xi \leq \overline{\eta} R} \rho(\xi),$$

and $\theta(R) \in [\frac{R}{\overline{\eta}}, \overline{\eta} R]$ such that

$$(2.3) \qquad \rho(R) \leq \frac{1}{A} \frac{1}{(\overline{\eta}(R))^\alpha} \rho(\theta(R)) \qquad \forall R \geq L.$$

If $\theta(R) \geq L$ then we can iterate the argument. After $N - 1$ iterations we get a function $\theta^N(R)$ such that

$$\theta^N(R) \in \left[\frac{\theta^{N-1}(R)}{\overline{\eta}^{N-1}(R)}, \overline{\eta}^{N-1}(R)\theta^{N-1}(R) \right]$$

and

$$\rho\left(\theta^{N-1}(R)\right) \leq \frac{1}{A} \frac{1}{\left(\overline{\eta}^N(R)\right)^\alpha} \rho\left(\theta^N(R)\right).$$

It follows that

$$(2.4) \qquad \rho(R) \leq \frac{1}{A^{N-1}} \frac{1}{\left[\overline{\eta}(R) \cdots \overline{\eta}^{N-1}(R)\right]^\alpha} \rho\left(\theta^N(R)\right)$$

and

$$\theta^N(R) \leq \overline{\eta}^{N-1}(R)\theta^{N-1}(R) \leq \cdots \leq \left[\overline{\eta}(R) \cdots \overline{\eta}^{N-1}(R)\right] R.$$

Using (2.1) we conclude that

$$\rho(R) \leq \frac{1}{A^{N-1}} \frac{CR^\alpha \left[\overline{\eta}(R) \cdots \overline{\eta}^{N-1}(R)\right]^\alpha}{\left[\overline{\eta}(R) \cdots \overline{\eta}^{N-1}(R)\right]^\alpha} = \frac{CR^\alpha}{A^{N-1}}.$$

Thus if the $\theta^N(R)$ remain in the interval $\{R \geq L\}$ for all N then $\rho(R) = 0$, a contradiction to the assumption that ρ is positive.

There must therefore exist a smallest positive number N such that

$$\theta^N(R) \leq L \,.$$

Then

$$L \geq \theta^N(R) \geq \frac{\theta^{N-1}(R)}{\overline{\eta}^{N-1}(R)} \geq \frac{\theta^{N-1}(R)}{B} \geq \cdots \geq \frac{\theta(R)}{B^{N-1}} \geq \frac{R}{B^N} \,,$$

so that $B^N \geq R/L$, or

$$(2.5) \qquad\qquad N \geq \frac{\log(R/L)}{\log B} \,.$$

We also have, by (2.4) and the inequalities $\overline{\eta}^j(R) \geq 1$ if $1 \leq j \leq N-1$ and $\theta^N(R) \leq L$,

$$(2.6) \qquad\qquad \rho(R) \leq \frac{C_0}{A^{N-1}} \,,$$

where $C_0 = \sup\limits_{1 \leq R \leq L} \rho(R)$.

Using (2.5) in (2.6), we get

$$\rho(R) \leq C \left(\frac{1}{A}\right)^N \leq \overline{C} R^{-\gamma} \qquad \left(\overline{C} = C_0 A\right) \,,$$

where

$$\gamma = \frac{\log A}{\log B} \,.$$

We have thus proved that if (i) is not true then (ii) holds, and this completes the proof of the lemma. □

3. A PRIORI ESTIMATE FOR "STRANGE" BOUNDARY CONDITIONS

In this section we consider solutions of the elliptic system

$$(3.1) \qquad\qquad R^2 \omega \, \Delta \varphi - \Delta^2 \varphi \;=\; \mu \text{ in } \mathbb{R}_+^2 \,,$$

$$(3.2) \qquad\qquad \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \;=\; a(x) \text{ if } y = 0, \; -\infty < x < \infty \,,$$

$$(3.3) \qquad 3\frac{\partial^3 \varphi}{\partial x^2 \partial y} + \frac{\partial^3 \varphi}{\partial y^3} - \omega R^2 \frac{\partial \varphi}{\partial y} \;=\; b(x) \text{ if } y = 0, \; -\infty < x < \infty \,,$$

where

$$(3.4) \qquad\qquad R > 0, \; Re\,\omega \geq 0, \; \omega \neq 0 \,;$$

for the sake of easy reference we call these boundary conditions "strange."

Lemma 3.1. *Let φ be a solution in $H^4(\mathbb{R}_+^2)$ of (3.1)–(3.4) and suppose that $\varphi(x,y) = 0$, $\mu(x,y) = 0$ if $x^2 + y^2 \geq R_0^2$ for some $R_0 > 0$. Then there exists a positive*

constant C depending only on R_0 such that

(3.5)
$$\int_{\mathbb{R}_+^2} \left[|\nabla^4 \varphi|^2 + R^4 |\nabla^2 \varphi|^2 \right]$$
$$\leq C \int_{-\infty}^{\infty} dx \left[|D_x^{3/2} a|^2 + |R^{3/2} a|^2 + |D_x^{1/2} b|^2 + |R^{1/2} b|^2 \right] + C \int_{\mathbb{R}_+^2} |\mu|^2 .$$

Here $\int dx |D_x^{j+1/2} c|^2$ is defined in terms of the Fourier transform.

Proof. It suffices to prove that

(3.6)
$$\int_{\mathbb{R}_+^2} \left[|\frac{\partial^4 \varphi}{\partial x^4}|^2 + |\frac{\partial^4 \varphi}{\partial y^4}|^2 + R^4 \left(|\frac{\partial^2 \varphi}{\partial x^2}|^2 + |\frac{\partial^2 \varphi}{\partial y^2}|^2 \right) \right]$$
$$\leq C \int_{-\infty}^{\infty} dx \left[|D_x^{3/2} a|^2 + |R^{3/2} a|^2 + |D_x^{1/2} b|^2 + |R^{1/2} b|^2 \right] + C \int_{\mathbb{R}_+^2} |\mu|^2 .$$

Indeed the additional terms that appear on the left-hand side of (3.5) can be estimated in the same way as the terms on the left-hand side of (3.6); alternatively, we can deduce (3.5) from (3.6) directly by interpolation.

Consider first the case $\mu \equiv 0$. We introduce the Fourier transform

$$\widehat{\varphi}(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \varphi(x, y) dx .$$

It satisfies

$$(-k^2 + D_y^2)(-k^2 - R^2 \omega + D_y^2) \, \widehat{\varphi}(k, y) = 0 .$$

Since $\widehat{\varphi}(k, y)$ remains bounded as $y \to \infty$, it must have the form

(3.7)
$$\widehat{\varphi}(k, y) = A(k) e^{-|k|y} + B(k) e^{-\sqrt{k^2 + R^2 \omega} \, y} .$$

Taking the Fourier transform in the boundary conditions (3.2), (3.3) and using (3.7), we get

$$2k^2 A(k) + (2k^2 + R^2 \omega) B(k) = -\widehat{a}(k) ,$$
$$|k|(2k^2 + R^2 \omega) A(k) + 2k^2 \sqrt{k^2 + R^2 \omega} \, B(k) = \widehat{b}(k) .$$

Solving for $A(k)$, $B(k)$ we find that

$$A(k) = \frac{(2k^2 + R^2 \omega) \widehat{b}(k) + 2k^2 \sqrt{k^2 + R^2 \omega} \, \widehat{a}(k)}{|k|(2k^2 + R^2 \omega)^2 - 4k^2 \sqrt{k^2 + R^2 \omega}} ,$$
$$B(k) = -\frac{|k|(2k^2 + R^2 \omega) \widehat{a}(k) + 2k^2 \widehat{b}(k)}{|k|(2k^2 + R^2 \omega) - 4k^2 \sqrt{k^2 + R^2 \omega}} .$$

Substituting this into (3.7) and setting

(3.8)
$$\theta = \sqrt{1 + \frac{R^2 \omega}{k^2}} , \quad \text{so that } 2k^2 + R^2 \omega = k^2 (1 + \theta^2) ,$$

we obtain, after some simplifications,

(3.9)
$$\widehat{\varphi}(k,\,y) = \frac{1}{|k|^3}\,\frac{1}{Q(\theta)}\,\left\{\left[(1+\theta^2)e^{-|k|y} - 2e^{-|k|\theta y}\right]\widehat{b}\,(k)\right.$$
$$\left. + |k|\left[2\theta e^{-|k|y} - (1+\theta^2)e^{-\theta|k|y}\right]\widehat{a}\,(k)\right\}\,,$$

where

(3.10)
$$Q(\theta) = (1+\theta^2)^2 - 4\theta\,.$$

Observe that

(3.11)
$$Q(\theta) = (\theta - 1)h(\theta)\,,\quad h(\theta) = \theta^3 + \theta^2 + 3\theta - 1.$$

To determine the location of the zeros of $h(\theta)$ we shall use the argument principle. One can check that h maps the boundary of any half circle $\{x^2 + y^2 < \rho^2,\ x > 0\}$ (with ρ large) in the complex θ-plane into a curve which surrounds the origin just once; see Figure 3.1.
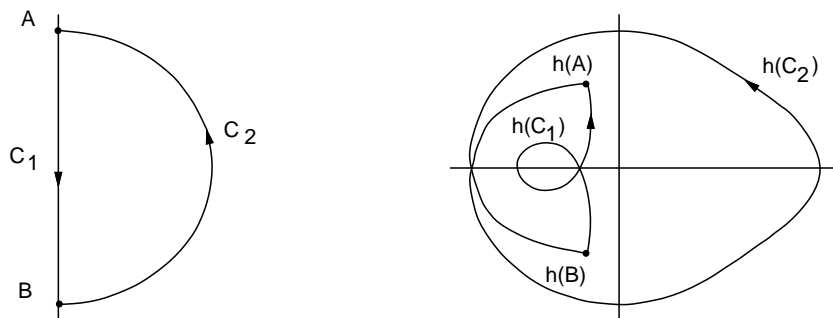

FIGURE 3.1

Hence h has only one zero $\widetilde{\theta}$ in the half-plane $Re\ \theta > 0$. Since $h(0) = -1$, $h(1) = 4$, we have $0 < \widetilde{\theta} < 1$. Recalling (3.8) and (3.11), we deduce that

(3.12)
$$|Q(\theta)| \geq c_0|\theta - 1|\,,\quad c_0 > 0\,,$$

where c_0 is a constant independent of $R^2\omega$.

Returning to (3.9), we first bound

(3.13)
$$\int_0^\infty |k|^4\,|\widehat{\varphi}(k,\,y)|^2 dy \leq \frac{C}{|Q(\theta)|^2}\,\left\{\left(\int_0^\infty |(1+\theta^2)e^{-\xi} - 2e^{-\theta\xi}|^2 d\xi\right)|k|\,|\widehat{b}(k)|\right.$$
$$\left. + \left(\int_0^\infty |2\theta e^{-\xi} - (1+\theta^2)e^{-\theta\xi}|^2 d\xi\right)|k|^3\,|\widehat{a}(k)|^2\right\}\,,$$

where on the right-hand side we substitute ξ for $|k|y$. The integrand of

$$\int_0^\infty |(1+\theta^2)e^{-\xi} - 2e^{-\theta\xi}|^2 d\xi$$

vanishes quadrically at $\theta = 1$, whereas, as $|\theta| \to \infty$, it remains bounded by $C|\theta|^4 e^{-2\xi}$. Recalling (3.12) and noting that

$$|Q(\theta)| < \frac{1}{2}|\theta|^4 \text{ if } |\theta| \to \infty\,,$$

we obtain from (3.13) the estimate

$$(3.14) \qquad \int_0^\infty |k|^4 |\widehat{\varphi}(k,y)|^2 dy \le C \left\{ |k| |\widehat{b}(k)|^2 + |k|^3 |\widehat{a}(k)|^2 \right\} .$$

Next we apply D_y^2 to (3.9) and then deduce, analogously to (3.13), that

$$(3.15)$$
$$\int_0^\infty |D_y^2 \widehat{\varphi}(k,y)|^2 dy \le \frac{C}{|k|^3 |Q(\theta)|^2} \left\{ \left(\int_0^\infty |(1+\theta^2)e^{-\xi} - 2\theta^2 e^{-\theta\xi}|^2 d\xi \right) |\widehat{b}(k)|^2 \right.$$
$$\left. + |k|^2 \left(\int_0^\infty |2\theta e^{-\xi} - (1+\theta^2)\theta^2 e^{-\theta\xi}|^2 d\xi \right) |\widehat{a}(k)|^2 \right\} .$$

By the same analysis as before we can deduce that

$$(3.16) \qquad \int_0^\infty \left(|k|^2 |D_y^2 \widehat{\varphi}(k,y)| \right)^2 dy \le C \left\{ |k| |\widehat{b}(k)|^2 + |k|^3 |\widehat{a}(k)|^2 \right\} .$$

Similarly, using (3.8) we get

$$(3.17)$$
$$R^4 \int_0^\infty |D_y^2 \widehat{\varphi}(k,y)|^2 dy \le C|\theta|^4 |k|^4 \int_0^\infty |D_y^2 \widehat{\varphi}(k,y)|^2 dy$$
$$\le C \left\{ |k| |\widehat{b}(k)|^2 + |k|^3 |\theta|^3 |\widehat{a}(k)|^2 \right\} ;$$

note that if we actually compute the second integral on the right-hand side of (3.15), we get an expression which is $O(|\theta|^7)$ as $|\theta| \to \infty$, and this explains the factor $|\theta|^3$ in (3.17).

Finally, we apply D_y^4 to (3.9) and obtain

$$(3.18)$$
$$\int_0^\infty |D_y^4 \widehat{\varphi}(k,y)|^2 dy \le \frac{C|k|^2}{|Q(\theta)|^2} \frac{1}{|k|} \left\{ \left(\int_0^\infty |(1+\theta^2)e^{-\xi} - 2\theta^4 e^{-\theta\xi}|^2 d\xi \right) |\widehat{b}(k)|^2 \right.$$
$$\left. + |k|^2 \left(\int_0^\infty |2\theta e^{-\xi} - (1+\theta^2)\theta^4 e^{-\theta\xi}|^2 d\xi \right) |\widehat{a}(k)|^2 \right\}$$
$$\le C \left\{ |k| |\widehat{b}(k)|^2 + |\theta|^3 |k|^3 |\widehat{a}(k)|^2 \right\} .$$

Observe that in the right-hand sides of both (3.17) and (3.18) we may replace $|\theta|^3 |k|^3$ by CR^3. Hence in conjunction with (3.14), (3.16), we obtain (after applying the Parseval equality) the assertion (3.6).

So far we have assumed that $\mu \equiv 0$. Consider now the case $\mu \not\equiv 0$ and extend μ by $\mu(x,-y) = \mu(x,y)$ to $y < 0$. We shall denote by $\widehat{f}(k_1, k_2)$ the Fourier transform of a function $f(x,y)$ in $\mathbb{R}^2$, and introduce a special solution $\psi(x,y)$ of (3.1) by

$$(3.19) \qquad \widehat{\psi}(k_1, k_2) = \frac{\widehat{\mu}(k_1, k_2)}{(R^2\omega + k_1^2 + k_2^2)(k_1^2 + k_2^2)} .$$

Since clearly $\psi(x,y) = \psi(x,-y)$ and $\psi \in W^{4,2}(\mathbb{R}^2)$,

$$(3.20) \qquad \frac{\partial \psi(x,0)}{\partial y} = \frac{\partial^3 \psi(x,0)}{\partial y^3} = 0$$

in the trace sense.

Set $\varphi = \psi + \Phi$. Then

$$(3.21) \qquad R^2\omega\Delta\Phi - \Delta^2\Phi = 0 \text{ in } \mathbb{R}_+^2 ,$$

(3.22) $$\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} = a_1 \text{ if } y = 0, \quad -\infty < x < \infty ,$$

(3.23) $$3\frac{\partial^3 \Phi}{\partial x^2 \partial y} + 3\frac{\partial^3 \Phi}{\partial y^3} - \omega R^2 \frac{\partial \Phi}{\partial y} = b \text{ if } y = 0, \quad -\infty < x < \infty ,$$

where (3.23) is a consequence of (3.20).

We easily compute that $a_1 = a + A$, where

$$|k_1|^3 |\widehat{A}(k_1)|^2 \leq C \int_{-\infty}^{\infty} |\widehat{\mu}(k_1, k_2)|^2 \frac{|k_1^2 - k_2^2|^2 |k_1|^3 dk_2}{(R^2\omega + k_1^2 + k_2^2)^2 (k_2^2 + k_2^2)^2}$$

$$\leq C \int_{-\infty}^{\infty} |\widehat{\mu}(k_1, k_2)|^2 dk_2 ;$$

the boundedness of the second integral on the right-hand side of the first inequality is obtained by substituting $k_2 = k_1 \lambda$.

Similarly,

$$R^3 |\widehat{B}(k_1)|^2 \leq C \int_{-\infty}^{\infty} |\widehat{\mu}(k_1, k_2)|^2 dk_2 .$$

Hence, if we apply the lemma in the special case $\mu \equiv 0$ to the function Φ, we find that (3.5) holds with φ replaced by Φ. From (3.19) and Parseval's equality we also readily deduce that (3.5) holds with φ replaced by ψ, and $a = b = c = 0$. Combining this with the estimates for Φ, the proof of (3.5) is complete. $\qquad \square$

Given a function $\psi(x, y)$ defined in $\mathbb{R}_+^2$, we define

$$\partial_y^{-1} \psi(x, y) = \int_{\infty}^{y} \psi(x, y) dy , \qquad \partial_y^{-n} \psi = \partial_y^{-1} \left(\partial_y^{-(n-1)} \psi \right) ,$$

and

$$\|\partial_x^{-l} \partial_y^{-n} \psi\|_{L^2(\mathbb{R}_+^2)} = \left\{ \int_0^{\infty} dy \int_{\mathbb{R}} \frac{|\partial_y^{-n} \widehat{\psi}(k, y)|^2}{(1 + |k|^2)^l} \, dk \right\}^{1/2}$$

for any nonnegative integers n, l. Similarly we define

$$\|\partial_x^{-l} a\|_{L^2(\mathbb{R})} = \left\{ \int_{\mathbb{R}} \frac{|\widehat{a}(k)|^2}{(1 + |k|^2)^l} \, dk \right\}^{1/2}$$

for any function $a(x)$.

We shall need the following extension of Lemma 3.1.

Lemma 3.2.
(3.24)
$$\sum_{j_1+j_2=4} \|\partial_x^{j_1-m_1} \partial_y^{j_2-m_2} \varphi\|_{L^2(\mathbb{R}_+^2)} + R^2 \sum_{j_1+j=2} \|\partial_x^{j_1-m_1} \partial_y^{j_2-m_2} \varphi\|_{L^2(\mathbb{R}_+^2)}$$

$$\leq C \Big[\|\partial_x^{-m} D_x^{3/2} a\|_{L^2(\mathbb{R})} + \|R^{3/2} \partial_x^{-m} a\|_{L^2(\mathbb{R})}$$

$$+ \|\partial_x^{-m} D_x^{1/2} b\|_{L^2(\mathbb{R})} + \|R^{1/2} \partial_x^{-m} b\|_{L^2(\mathbb{R})} \Big]$$

$$+ C \|\partial_x^{-m_1} \partial_y^{-m_2} \mu\|_{L^2(\mathbb{R}_+^2)}$$

for any integers $m_1 \geq 0$, $m_2 \geq 0$, $m = m_1 + m_2 \leq 3$, where C is a constant depending only on R_0.

Proof. We proceed similarly to the proof of Lemma 3.1. Consider first the case $\mu \equiv 0$. Then in (3.9) we just multiply by a factor $(1 + |k|^2)^{-m_1/2}$, and the square of this factor then appears in (3.13), (3.17), (3.18). To control $\partial_y^{-m_2}$ applied to the various derivatives of φ, we observe that in (3.9) each y-integration (from ∞ to y) yields a factor $1/|k|$. Therefore, as long as $|k|$ stays away from 0, say $|k| \geq \delta > 0$, all the desired estimates can be obtained as before. Similarly we can handle the case $\mu \not\equiv 0$, using (3.19) as before. We conclude that

$$\int_{|k|>\delta} \left[\sum_{j_1+j_2=4} |k|^{2(j_1-m_1)} |\partial_y^{j_2-m_2} \widehat{\varphi}(k, y)|^2 \right.$$

$$\left. + R^2 \sum_{j_1+j_2=2} |k|^{2(j_1-m_1)} |\partial_y^{j_2-m_2} \widehat{\varphi}(k, y)|^2 \right] dk$$

is bounded by the right-hand side of (3.24).

Finally, the same estimates for the integral with respect to k, over $|k| \leq \delta$, follow from the following version of the "uncertainty principle." □

Lemma 3.3. *Let* $f \in L^2(\mathbb{R})$ *with support in some integral* $(-R_0, R_0)$. *Then*

$$(3.25) \qquad \int_{-\delta}^{\delta} |\widehat{f}(k)|^2 dk \leq \int_{|k|\geq\delta} |\widehat{f}(k)|^2 dk \quad \text{if } \delta < \frac{\pi}{4R_0} .$$

Note that the lemma gives a quantitative expression to the assertion that if f has compact support then $\widehat{f}$ contains a large component of high frequencies.

Proof. Clearly

$$|\widehat{f}(k)|^2 = \frac{1}{\pi} \int_{-R_0}^{R_0} dx \int_{-R_0}^{R_0} dy \, f(x) \, \overline{f(y)} \, e^{ik(y-x)} .$$

Hence

$$\int_{-\delta}^{\delta} |\widehat{f}(k)|^2 dk \; = \frac{1}{2\pi} \int_{-R_0}^{R_0} dx \int_{-R_0}^{R_0} dy \, f(x) \, f(y) \, \frac{e^{i\delta(y-x)} - e^{-i\delta(y-x)}}{y - x}$$

$$\leq \frac{2\delta}{2\pi} \left[\int_{-R_0}^{R_0} dx \, f(x) \right]^2 \leq \frac{2R_0\delta}{\pi} \int_{-R_0}^{R_0} |f(x)|^2 dx$$

$$= \frac{2R_0\delta}{\pi} \int_{-\infty}^{\infty} |\widehat{f}(k)|^2 dk \leq \frac{1}{2} \left[\int_{-\delta}^{\delta} |\widehat{f}(k)|^2 dk + \int_{|k|\geq\delta} |\widehat{f}(k)|^2 dk \right]$$

and (3.25) follows.   □

4. Special solution for a Dirichlet problem

Consider the Dirichlet problem:

$$(4.1) \qquad\qquad \omega \Delta \psi - \Delta^2 \psi = 0 \text{ in } \mathbb{R}_+^2 \qquad (Re \, \omega \geq 0, \, \omega \neq 0),$$

$$(4.2) \qquad\qquad \psi(x, 0) = \psi_0(x), \; \psi_y(x, 0) = 0 \text{ if } x \in \mathbb{R} ,$$

where ψ_0 is, say, with compact support. If ψ is polynomially bounded then by taking the Fourier transform in x, viewed as a tempered distribution, we find that

$$\widehat{\psi}(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} \psi(x, y) dx$$

is given by

$$(4.3) \qquad \widehat{\psi}(k, y) = \frac{\sqrt{k^2 + \omega}\, e^{-|k|y} - |k| e^{-\sqrt{k^2 + \omega}\, y}}{\sqrt{k^2 + \omega^2} - |k|} \, \widehat{\psi}_0(k) \,,$$

so that

$$(4.4) \qquad \psi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{k^2 + \omega}\, e^{-|k|y} - |k| e^{-\sqrt{k^2 + \omega}\, y}}{\sqrt{k^2 + \omega} - |k|} \, \widehat{\psi}_0(k) e^{ikx} dk \,.$$

In this section we consider the special case of (4.1), (4.2) where

$$\psi_0(x) = \delta(x) \qquad \text{(the Dirac function)}$$

and denote the solution by $K(x, y)$. We shall prove:

Lemma 4.1. *There exist small positive constants ε_0, γ and a positive constant C such that*

$$(4.5) \qquad |K(x, y)| \quad \leq \quad \frac{Cy}{|x|^2} + C e^{-\gamma |x|} \ \text{if } 0 \leq y \leq \varepsilon_0 |x|, \ |x| \geq 1 \,,$$

$$(4.6) \qquad |\nabla^j K(x, y)| \quad \leq \quad \frac{C}{|x|^{j+1}} \ \text{if } 0 \leq y \leq \varepsilon_0 |x|, \ |x| \geq 1 \quad (1 \leq j \leq 4) \,.$$

Proof. Clearly $\widehat{K}(k, y)$ is given by (4.3) with $\widehat{\psi}_0(k) \equiv 1$, so that, by (4.4),

(4.7)

$$\begin{aligned} \sqrt{2\pi}\, K(x, y) \quad &= \quad \int_0^{\infty} e^{ikx} \frac{\sqrt{k^2 + \omega}\, e^{-ky} - k\, e^{-\sqrt{k^2 + \omega}\, y}}{\sqrt{k^2 + \omega}\, - k} \, dk \\ &+ \int_{-\infty}^{0} e^{ikx} \frac{\sqrt{k^2 + \omega}\, e^{ky} + k\, e^{-\sqrt{k^2 + \omega}\, y}}{\sqrt{k^2 + \omega} + k} \, dk \equiv I_1 + I_2 \,. \end{aligned}$$

The function $\sqrt{k^2 + \omega}$ is analytic in the complex plane except for two branch points $k = \pm \sqrt{-\omega}$. We distinguish two cases:

(i) $Im\,\omega \geq 0$, so that $\sqrt{-\omega}$ lies in the second quadrant, $Im\,\sqrt{-\omega} > 0$, and then $-\sqrt{-\omega}$ lies in the fourth quadrant;

(ii) $Im\,\omega < 0$, so that $\sqrt{-\omega}$ lies in the first quadrant $Im\,\sqrt{-\omega} > 0$, and then $-\sqrt{-\omega}$ lies in the third quadrant.

Here the fact that $\omega \neq 0$ was used to assert that $Im\,\sqrt{-\omega} > 0$ in case (i).

For definiteness we consider only case (i); case (ii) can be treated similarly. It will also be enough to establish (4.5), (4.6) in case $x > 0$; the case $x < 0$ can be treated similarly.

We cut the complex plane by the rays

$$\Lambda = \left\{ k = \sqrt{-\omega} + \lambda i, \ \lambda > 0 \right\}, \ \Lambda_1 = \left\{ k = -\sqrt{-\omega} + \lambda i, \ \lambda < 0 \right\} \,.$$

Then $\sqrt{k^2 + \omega}$ is analytic in the cut plane. We deform the line integral of I_1 from $(0, +\infty)$ to $i(0, \infty)$ (the part of the integral at ∞ is negligible). Substituting $k = iu$ we get

$$I_1 = i \int_0^{+\infty} e^{-ux} \frac{\sqrt{\omega - u^2}\, e^{-iu\,y} - iu\, e^{-\sqrt{\omega - u^2}\, y}}{\sqrt{\omega - u^2} - iu} \, du \,.$$

Similarly we deform the line integral of I_2 to obtain

$$I_2 = \int_\Lambda + \int_\ell \equiv J_\Lambda + I_3 \ ,$$

where the integral on Λ is traced on the vertical half lines

$$\left(\sqrt{-\omega} + i\infty, \ \sqrt{-\omega} \right) \ \text{and} \ \left(\sqrt{-\omega}, \ \sqrt{-\omega} + i\infty \right) \ ,$$

and ℓ is the vertical half line $i(\infty, 0)$ (from $i\infty$ to 0). In I_3 we substitute $k = iu$ to get

$$I_3 = -i \int_0^\infty e^{-ux} \frac{\sqrt{\omega - u^2} \, e^{iu\,y} + iu \, e^{-\sqrt{\omega - u^2}\,y}}{\sqrt{\omega - u^2} + iu} \, du \ .$$

We also have

$$(4.8) \qquad J_\Lambda = \int_\Lambda e^{ikx} \frac{\sqrt{k^2 + \omega} \, e^{ky} + k \, e^{-\sqrt{k^2 + \omega}\,y}}{\sqrt{k^2 + \omega} + k} \, dk$$

with Λ traced twice as explained above.

Combining the expressions for I_1 and I_3 we easily compute that

$$(4.9)$$

$$\begin{aligned} I_1 + I_3 \ = \ & \frac{i}{\omega} \int_0^{+\infty} e^{-ux} \Big\{ (\omega - u^2) \left(e^{-iu\,y} - e^{iu\,y} \right) \\ & + iu \sqrt{\omega - u^2} \left(e^{iu\,y} - e^{-iu\,y} \right) - 2iu \sqrt{\omega - u^2} \, e^{-\sqrt{\omega - u^2}\,y} \Big\} du \ . \end{aligned}$$

Writing

$$e^{-ux} = -\frac{1}{x} \frac{d}{du} \left(e^{-ux} \right)$$

in (4.9) and integrating by parts, we get

$$(4.10) \qquad I_1 + I_3 = \frac{i}{\omega} \frac{1}{x} \int_0^\infty e^{-ux} \frac{d}{du} \{ \cdots \} \, du$$

and

$$\left| \frac{d}{du} \{ \cdots \} \right| \le Cy \text{ if } 0 \le u \le 1 \ .$$

If $u > 1$ then

$$\left| e^{-(u/2)x} \frac{d}{du} \{ \cdots \} \right| \le Cy \text{ if } |x| \ge 1 \ .$$

It follows that

$$(4.11) \qquad |I_1 + I_3| \le \frac{Cy}{|x|^2} \text{ if } |x| \ge 1 \ .$$

To estimate J_Λ (defined in (4.8)) we split the integrand:

$$J_\Lambda = \int_\Lambda \frac{e^{ikx} \sqrt{k^2 + \omega} \, e^{ky}}{\sqrt{k^2 + \omega} + k} \, dk + \int_\Lambda \frac{k \, e^{ikx} e^{-\sqrt{k^2 + \omega}\,y}}{\sqrt{k^2 + \omega} + k} \, dk \equiv J_1 + J_2$$

Writing

$$k = Re \sqrt{-\omega} + i \, Im \sqrt{-\omega} + i\lambda, \ \lambda \in (0, \infty)$$

for $k \in \Lambda$, we estimate

$$|J_1| \leq C \int_0^\infty e^{-Im\sqrt{-\omega}\,x} e^{-\frac{1}{2}\lambda x} \, d\lambda \leq C e^{-\gamma x} \qquad (\gamma > 0)$$

if $y \leq \varepsilon_0 |x|$, $x > 0$; recall that $Im\sqrt{-\omega} > 0$. Similarly

$$\begin{aligned}
|J_2| &\leq C \int_0^\infty e^{-Im\sqrt{-\omega}\,x} e^{-\lambda x} e^{-\sqrt{k^2+\omega}\,y} \, d\lambda \\
&\leq C \int_0^\infty e^{-\frac{1}{2}Im\sqrt{-\omega}\,x} e^{-\frac{1}{2}\lambda x} \, d\lambda \leq C e^{-\gamma x}
\end{aligned}$$

if $y \leq \varepsilon_0 |x|$, $x > 0$. Combining the estimates of J_1, J_2 with (4.11), the inequality (4.5) follows.

To estimate $\partial_x K$ we proceed as before to deform the contours of integration. The only difference is in the factor ik which now appears in the integrals I_1, I_2. Consequently

$$\frac{\partial}{\partial x}(I_1 + I_3)$$

is equal to the integral in (4.9) with a factor u inserted in the integrand. Proceeding as before, we get

$$\left| \frac{\partial}{\partial x}(I_1 + I_3) \right| \leq \frac{C}{|x|} \int_0^\infty \left[yu e^{-(u/2)x} + e^{-(u/2)x} \right] du \leq \frac{C}{|x|^2} .$$

dJ_Λ/dx can be estimated as J_Λ, that is,

$$\left| \frac{\partial J_\Lambda}{\partial x} \right| = \left| \frac{\partial J_1}{\partial x} \right| + \left| \frac{\partial J_2}{\partial x} \right| \leq C e^{-\gamma x} ,$$

and thus $|\partial K/\partial x|$ is bounded by $C/|x|^2$. In the same way we can estimate $|\partial K/\partial y|$, as well as any higher order derivatives of K. $\qquad \square$

5. Another special solution for a Dirichlet problem

We shall need the following formula:

$$(5.1) \qquad \log(1 + \xi^2) = -\int_{-\infty}^\infty du \, \frac{e^{-|u|}}{|u|} \left(e^{iu\xi} - 1 \right) .$$

The proof follows by noting that the right-hand side vanishes at $\xi = 0$ and that its derivative is equal to

$$-i \int_{-\infty}^\infty du(sgn\,u)e^{-|u|+iu\xi} = \frac{2\xi}{1 + \xi^2} = \frac{d}{d\xi} \log(1 + \xi^2) .$$

Using (5.1) we can write, for any $\theta > 0$,

$$\log(x^2 + \theta^2) = \log\theta^2 + \log\left(1 + \frac{x^2}{\theta^2}\right) = \log\theta^2 - \int_{-\infty}^\infty \frac{e^{-|u|}}{|u|} \left(e^{iu\,x/\theta} - 1 \right) du ,$$

or

$$(5.2) \qquad \log(x^2 + \theta^2) = \log\theta^2 - \int_{-\infty}^\infty \frac{e^{-|k|\theta}}{|k|} \left(e^{ikx} - 1 \right) dk .$$

Consider the function

$$(5.3) \ \Phi(x, y) = -\int_{-\infty}^\infty dk \, \frac{e^{-|k|\theta}}{|k|} \left[\frac{\sqrt{k^2+\omega}\, e^{-|k|y} - |k| e^{-\sqrt{k^2+\omega}\,y}}{\sqrt{k^2+\omega} - |k|} e^{ikx} - 1 \right] .$$

One can easily check that

$$(5.4) \qquad\qquad \omega \Delta \Phi - \Delta^2 \Phi = 0 \text{ in } \mathbb{R}^2_+$$

and that, by (5.2),

$$(5.5) \qquad\qquad \Phi(x, 0) = \log\left(1 + \frac{x^2}{\theta^2}\right) ;$$

also

$$(5.6) \qquad\qquad \Phi_y(x, 0) = 0 .$$

Thus Φ is a solution of a special Dirichlet problem.

In this section we shall estimate the first derivatives of Φ.

Lemma 5.1. *For any positive constant C_0 there exists a positive constant C such that*

$$(5.7) \qquad \left| \frac{\partial}{\partial x} \Phi(\theta\xi, \theta\eta) - \frac{2}{\theta} \frac{\xi}{\xi^2 + (1+\eta)^2} \right| \;\leq\; \frac{C}{\theta^2} + \frac{C}{\theta} e^{-\left(Re\,\sqrt{\omega}\right)\theta\eta} ,$$

$$(5.8) \qquad \left| \frac{\partial}{\partial y} \Phi(\theta\xi, \theta\eta) - \frac{2}{\theta} \frac{1+\eta}{\xi^2 + (1+\eta)^2} \right| \;\leq\; \frac{C}{\theta^2} + \frac{C}{\theta} e^{-\left(Re\,\sqrt{\omega}\right)\theta\eta}$$

for all $|\xi| \leq C_0$, $0 \leq \eta \leq C_0$ and $\theta \geq 1$.

Lemma 5.2. *For any positive constant C_0 there exists a positive constant C such that*

$$(5.9) \qquad \left| \frac{\partial}{\partial x} \Phi(\theta\xi, \theta\eta) - \frac{2}{\theta} \frac{\xi}{1+\xi^2} \right| \;\leq\; C \frac{\log\theta}{\theta^2} ,$$

$$(5.10) \qquad \left| \frac{\partial}{\partial y} \Phi(\theta\xi, \theta\eta) - \frac{2}{\theta} \left(1 - e^{-\sqrt{\omega}\,\theta\eta}\right) \frac{1}{1+\xi^2} \right| \;\leq\; C \frac{\log\theta}{\theta^2}$$

for all $\xi \in \mathbb{R}$, $0 < \eta < C_0(\log\theta)/\theta$ and $\theta \geq 2$.

We shall need these lemmas only for θ large. Note that the estimates of Lemma 5.2 are sharper than the estimates in Lemma 5.1 when $\eta \to 0$; (5.10) indicates a boundary layer at $\eta = 0$.

Proof of Lemma 5.1. We can write

$$(5.11)$$

$$\begin{aligned}
\frac{\partial}{\partial y} \Phi(\theta\xi, \theta\eta) \;=\;& \int_{-\infty}^{\infty} \sqrt{k^2 + \omega}\, e^{-|k|\theta}\, e^{ik\theta\xi}\, \frac{e^{-|k|\theta\eta}}{\sqrt{k^2 + \omega} - |k|}\, dk \\
& - \int_{-\infty}^{\infty} \sqrt{k^2 + \omega}\, e^{-|k|\theta}\, e^{ik\theta\xi}\, \frac{e^{-\sqrt{k^2+\omega}\,\theta\eta}}{\sqrt{k^2 + \omega} - |k|}\, dk \equiv J_1 + J_2 .
\end{aligned}$$

As can easily be seen,

$$|J_2| \leq \frac{C}{\theta} e^{-\left(Re\,\sqrt{\omega}\right)\theta\eta} .$$

To estimate J_1 we write

$$\begin{aligned}
J_1 \;=\;& \int_{-\infty}^{\infty} e^{-|k|\theta}\, e^{ik\theta\xi}\, e^{-|k|\theta\eta}\, dk \\
& + \int_{-\infty}^{\infty} |k|\, \frac{e^{-|k|\theta}\, e^{ik\theta\xi}\, e^{-|k|\theta\eta}}{\sqrt{k^2 + \omega} - |k|}\, dk = J_{11} + J_{12}
\end{aligned}$$

and note that

$$|J_{12}| \leq C \int_{-\infty}^{\infty} |k| \, e^{-|k|\theta} \leq \frac{C}{\theta^2} \, .$$

J_{11} can be computed explicitly:

$$J_{11} = \frac{2}{\theta} \, \frac{1+\eta}{(1+\eta)^2 + \xi^2} \, .$$

Combining this with the previous estimates of J_{12} and J_2, the assertion (5.8) follows.

The proof of (5.7) is similar. We write

(5.12)

$$
\begin{aligned}
\frac{\partial}{\partial x} \, \Phi(\theta\xi, \, \theta\eta) \quad &= \quad -i \int_{-\infty}^{\infty} (sgn \, k) \, e^{-|k|\theta} \, \frac{\sqrt{k^2 + \omega}}{\sqrt{k^2 + \omega} - |k|} \, e^{-|k|\theta\eta} \, e^{ik\theta\xi} \, dk \\
&\quad + i \int_{-\infty}^{\infty} (sgn \, k) \, e^{-|k|\theta} \, \frac{|k|}{\sqrt{k^2 + \omega} - |k|} \, e^{-\sqrt{k^2 + \omega} \, \theta\eta} \, e^{ik\theta\xi} \, dk \\
&\equiv \quad K_1 + K_2 \, .
\end{aligned}
$$

As can easily be seen,

$$|K_2| \leq C \int_{-\infty}^{\infty} |k| \, e^{-|k|\theta} \leq \frac{C}{\theta} \, e^{-\left(Re \, \sqrt{\omega} \right)\theta\eta} \, .$$

Next we write

$$
\begin{aligned}
K_1 \quad &= \quad -i \int_{-\infty}^{\infty} (sgn \, k) \, e^{-|k|\theta} \, e^{-|k|\theta\eta} \, e^{ik\theta\xi} \, dk \\
&\quad -i \int_{-\infty}^{\infty} (sgn \, k) \, e^{-|k|\theta} \, \frac{|k|}{\sqrt{k^2 + \omega} - |k|} \, e^{-|k|\theta\eta} \, e^{ik\theta\xi} \, dk \equiv K_{11} + K_{12}
\end{aligned}
$$

and note that

$$|K_{12}| \leq C \int_{-\infty}^{\infty} |k| \, e^{-|k|\theta} \leq \frac{C}{\theta^2} \, .$$

Finally, K_{11} can be computed explicitly:

$$K_{11} = \frac{2}{\theta} \, \frac{\xi}{(1+\eta)^2 + \xi^2} \, .$$

Combining this and the preceding estimates, the inequality (5.7) follows. $\square$

Proof of Lemma 5.2. We break the integral of $\partial\Phi/\partial x$ differently than in (5.12):

(5.13)

$$
\begin{aligned}
&\frac{\partial}{\partial x} \, \Phi(\theta\xi, \, \theta\eta) \\
&= -i \int_{|k| \leq 1/(\log \theta)} dk \, (sgn \, k) \, e^{-|k|\theta} \, \frac{\sqrt{k^2 + \omega} \, e^{-|k|y} - |k| \, e^{-\sqrt{k^2 + \omega} \, y}}{\sqrt{k^2 + \omega} - |k|} \, e^{ik\theta\xi} \\
&\quad + i \int_{|k| > 1/(\log \theta)} dk \, (sgn \, k) \, e^{-|k|\theta} \, \frac{\sqrt{k^2 + \omega} \, e^{-|k|y} - |k| \, e^{\sqrt{k^2 + \omega} \, y}}{\sqrt{k^2 + \omega} - |k|} \, e^{ik\theta\xi} \\
&\equiv M_1 + M_2 \, .
\end{aligned}
$$

where $y = \theta\eta$. Clearly

(5.14) $$|M_2| \leq C \int_{|k| > 1/(\log \theta)} e^{-|k|\theta} \, dk \leq \frac{C}{\theta} \, e^{-\theta/(\log \theta)} \, .$$

In the integrand of M_1

$$e^{-|k|y} = 1 + O(|k|y), \ |k| \, e^{-\sqrt{k^2+\omega}\, y} = |k| + O(|k|y)$$

since $|k|y \leq C$. Hence the quotient in the integrand is equal to $1 + O(|k|y)$. It follows that

$$
\begin{aligned}
M_1 &= -i \int_{-\infty}^{\infty} dk \, (sgn\ k) \, e^{-|k|\theta} \, e^{ik\theta\xi} + i \int_{|k|\geq 1/(\log \theta)} dk \, (sgn\ k) \, e^{-|k|\theta} \, e^{ik\theta\xi} \\
&\quad + \int_{|k|\leq 1/\log \theta} dk \, O(1) y |k| \, e^{-|k|\theta} \, .
\end{aligned}
$$

The last two integrals are bounded respectively by

$$(5.15) \qquad\qquad \frac{C}{\theta} e^{-\theta/(\log \theta)} \text{ and } \frac{C}{\theta^2} \, ,$$

whereas the first integral is equal to

$$\frac{2}{\theta} \frac{\xi}{1 + \xi^2} \, .$$

Combining these facts with (5.14) and (5.13), the estimate (5.9) follows.

To prove (5.10) we break the integral of $\partial \Phi / \partial y$ differently than in (5.11):

$$(5.16)$$

$$
\begin{aligned}
\frac{\partial}{\partial y} \Phi(\theta\xi,\, y) &= \int_{|k|\leq 1/(\log \theta)} dk \, \sqrt{k^2+\omega} \, e^{-|k|\theta} \, e^{ik\theta} \, \frac{e^{-|k|y} - e^{-\sqrt{k^2+\omega}\, y}}{\sqrt{k^2+\omega} - |k|} \\
&\quad + \int_{|k|\geq 1/(\log \theta)} dk \, \sqrt{k^2+\omega} \, e^{-|k|\theta} \, e^{ik\theta} \, \frac{e^{-|k|y} - e^{-\sqrt{k^2+\omega}\, y}}{\sqrt{k^2+\omega} - |k|} \\
&\equiv N_1 + N_2 \, .
\end{aligned}
$$

As before,

$$|N_2| \leq \frac{C}{\theta} e^{-\theta/(\log \theta)} \, .$$

The quotient in the integrand of N_1 is equal to

$$\left[1 + O(|k|y) - e^{-\sqrt{\omega}\, y} \left(1 + O(k^2 y) \right) \right] \left[\frac{1}{\sqrt{\omega}} + O|k| \right] = \frac{1 - e^{-\sqrt{\omega}\, y}}{\sqrt{\omega}} + O \left(|k| + |k|y \right) \, .$$

Hence

$$
\begin{aligned}
N_1 &= \int_{-\infty}^{\infty} dk \, e^{-|k|\theta} \, e^{ik\theta\xi} \left(1 - e^{-\sqrt{\omega}\, y} \right) \\
&\quad - \int_{|k|\geq 1/(\log \theta)} dk \, e^{-|k|\theta} \, e^{ik\theta\xi} \left(1 - e^{-\sqrt{\omega}\, y} \right) \\
&\quad + \int_{|k|\leq 1/(\log \theta)} e^{-|k|\theta} \left[O(|k|) + O(|k|y) \right] \, .
\end{aligned}
$$

The last two integrals are bounded, respectively, by the expressions in (5.15), whereas the first integral is equal to

$$\frac{2}{\theta} \left(1 - e^{-\sqrt{\omega}\, y} \right) \frac{1}{1 + \xi^2} \, .$$

Combining this with the estimate on N_2 and with (5.16), the proof of (5.10) follows.

$\square$

The proof of Lemma 5.1 can be extended to higher order derivatives. For our purposes we just need the following crude version:

Lemma 5.3. *For any positive constant C_0 and nonnegative integer j there exists a positive constant C such that*

$$(5.17) \qquad |\nabla_{\xi\eta}^{j} \nabla_{xy} \Phi(\theta\xi, \theta\eta)| \leq \frac{C}{\theta}$$

for all $|\xi| \leq C_0$, $0 \leq \eta \leq C_0$ and $\theta \geq 1$.

6. GREEN'S FUNCTION IN A HALF PLANE

In this section we analyze the Green function for the Dirichlet problem in $\mathbb{R}^2_+$ for the operator $\omega\Delta - \Delta^2$. Since the problem is invariant under translation in the x-direction, it suffices to consider the Green function $V(x, y, \theta)$ with pole at $(0, \theta)$, $\theta > 0$. The function V should satisfy:

$$(6.1) \qquad \omega\,\Delta V - \Delta^2 V \;=\; \delta(x)\,\delta(y - \theta) \text{ in } \mathbb{R}^2_+\,,$$

$$(6.2) \qquad V(x, 0) \;=\; V_y(x, 0) = 0 \text{ for } x \in \mathbb{R}\,.$$

We begin by considering the equation

$$-\Delta\psi + \omega\psi = \delta(x)\,\delta(y) \text{ in } \mathbb{R}^2\,.$$

It has a radially symmetric solution

$$\psi(r) = \frac{K_0\left(\sqrt{\omega}\,r\right)}{2\pi}\,,$$

where K_0 is the zero-th order modified Bessel function. Since $Re\,\sqrt{\omega} > 0$, $\psi(r)$ decays exponentially as $r \to \infty$.

Consider next the equation

$$-\omega\Delta\varphi + \Delta^2\varphi = \delta(x)\,\delta(y) \text{ in } \mathbb{R}^2\,.$$

We again seek a radially symmetric solution $\varphi(r)$. We find it by solving

$$\Delta\varphi = \frac{1}{2\pi}\,K_0\left(\sqrt{\omega}\,r\right)\,,$$

or

$$\frac{1}{r}\,(r\varphi_r)_r = \frac{1}{2\pi}\,K_0\left(\sqrt{\omega}\,r\right)\,,$$

i.e.,

$$(6.3) \qquad \varphi(r) = \frac{1}{2\pi}\int_0^r \frac{du}{u}\int_0^u \xi K_0\left(\sqrt{\omega}\,\xi\right)d\xi\,.$$

Set

$$(6.4) \qquad \Gamma = \frac{1}{2\pi}\int_0^\infty \xi K_0\left(\sqrt{\omega}\,\xi\right)d\xi$$

and introduce the function $\varphi(x, y) = \varphi(r)$ where $r = \sqrt{x^2 + y^2}$, that is,

$$(6.5) \qquad \varphi(x, y) = \frac{1}{2\pi}\int_0^{\sqrt{x^2+y^2}} \frac{du}{u}\int_0^u \xi K_0\left(\sqrt{\omega}\,\xi\right)d\xi\,.$$

We want to construct V in the form

$$(6.6) \qquad V(x, y, \theta) = \varphi(x, y - \theta) + \varphi(x, y + \theta) - G(x, y; \theta)\,.$$

Then

(6.7) $$\omega \Delta G - \Delta^2 G = 0 \text{ in } \mathbb{R}^2_+ ,$$

(6.8) $$G(x,\, 0;\, \theta) = 2\varphi(x,\, \theta), \qquad G_y(x,\, 0;\, \theta) = 0 \text{ for } x \in \mathbb{R} .$$

To simplify notation we shall usually drop the parameter θ, writing $G = G(x,\, y)$.

Introducing the Fourier transform

$$\widehat{G}(k,\, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, G(x,\, y) dx ,$$

we can write (see (4.4))

(6.9) $$\widehat{G}(k,\, y) = 2\, \frac{\sqrt{k^2 + \omega}\, e^{-|k|y} - |k|\, e^{-\sqrt{k^2+\omega}\, y}}{\sqrt{k^2 + \omega} - |k|}\, \widehat{\varphi}(k,\, \theta) .$$

We have the crude estimate (see (6.5), (6.4))

$$\varphi(x,\, y) \sim \Gamma \log \sqrt{x^2 + y^2} \text{ as } x^2 + y^2 \to \infty ,$$

but we shall need the sharper estimate

(6.10)
$$\varphi(x,\, \theta) = \Gamma \log \sqrt{x^2 + \theta^2} + \frac{B}{2\pi} + W(x,\, \theta), \text{ where}$$
$$|\nabla^j W(x,\, y)| \le C\, e^{-\delta r} \quad \left(\delta > 0,\, r = \sqrt{x^2 + y^2} \right) \text{ for } 0 \le j \le 6 ;$$

$W(x,\, \theta)$ is of course a function $W\left(\sqrt{x^2 + \theta^2}\right)$.

To prove (6.10) we write

$$
\begin{aligned}
2\pi\varphi(r) &= \int_0^1 \frac{du}{u} \int_0^u \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi + \int_1^r \frac{du}{u} \int_0^u \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi \\
&= A + \int_1^r \frac{du}{u} 2\pi\Gamma - \int_1^r \frac{du}{u} \left[2\pi\Gamma - \int_0^u \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi \right] .
\end{aligned}
$$

The last integral is equal to

$$\int_1^r \frac{du}{u} \int_u^{\infty} \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi ,$$

and it converges exponentially fast (as $r \to \infty$) with all its derivatives. Consequently, (6.10) holds with

$$B = A - \int_1^{\infty} \frac{du}{u} \int_u^{\infty} \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi , \quad \text{where } A = \int_0^1 \frac{du}{u} \int_0^u \xi K_0\left(\sqrt{\omega}\, \xi\right) d\xi .$$

If we substitute φ from (6.10) into (6.9) we get

$$\widehat{G}(k,\, y) = \widehat{G}_1(k,\, y) + \frac{B}{\pi}\, \delta(k) + \widehat{G}_2(k,\, y)$$

or

(6.11) $$G(x,\, y) = G_1(x,\, y) + \frac{B}{\pi} + G_2(x,\, y) ,$$

where

(6.12) $$G_2(x,\, y) = 2 \int_{-\infty}^{\infty} K(x - \xi,\, y) W(\xi,\, \theta) d\xi ,$$

$K(x,\, y)$ is as in Lemma 13.1, and

(6.13) $$G_1(x,\, y) = 2\Gamma \log \theta + \Gamma \Phi(x,\, y) ,$$

where $\Phi(x,\,y)$ is the function defined in (5.3). From (6.12) we also have

$$(6.14) \qquad \widehat{G}_2(k,\,y) = 2\,\frac{\sqrt{k^2+\omega}\,e^{-|k|y} - |k|\,e^{-\sqrt{k^2+\omega^2}\,y}}{\sqrt{k^2+\omega} - |k|}\,\widehat{W}(k,\,\theta)\,.$$

We are interested in estimating G_2 and its derivatives in the region

$$(6.15) \qquad 2c_0 R \le \sqrt{x^2+\theta^2},\quad y \le \eta_0 R\,,\quad \text{where } R \ge 1 \text{ and } \eta_0 \ll c_0$$

where c_0 is some positive constant.

Observe that $|k|^j D_y^\ell \widehat{G}_2(k,\,y)$ is bounded by $C|k|^{j+\ell}|\widehat{W}(k,\,\theta)|$ for $0 \le j+\ell \le 6$. If $\theta \ge c_0 R$ then, using the bounds on the derivatives of W (in (6.10)) and Parseval's equation, we get

$$\int_{-\infty}^{\infty} |D_y^\ell D_x^j G_2(x,\,y)|^2 dx \le C e^{-\gamma R}\quad (0 \le j+\ell \le 6)$$

and, by Sobolev's inequality,

$$(6.16) \qquad |D_y^\ell D_x^j G_2(x,\,y)| \le C e^{-\gamma R}\quad (0 \le j+\ell \le 4)\,.$$

To analyze the case $\theta < c_0 R$ we introduce a C^∞ function $\lambda_1(\xi)$ such that

$$\lambda_1(\xi) = \begin{cases} 0 & \text{if } -\infty < \xi < \frac{1}{4}c_0 R\,, \\[2mm] 1 & \text{if } \frac{1}{2}c_0 R < \xi < \infty\,, \end{cases}$$

and $0 \le \lambda_1 \le 1$ elsewhere, and $\lambda_2 = 1 - \lambda_1$. We break up G_2 into a sum of two functions,

$$G_{2j}(x,\,y) = 2\int_{-\infty}^{\infty} K(x-\xi,\,y)W(\xi,\,\theta)\lambda_j(\xi)d\xi\,.$$

By the same proof as for (6.16) we get

$$(6.17) \qquad |D_y^\ell D_x^j G_{21}(x,\,y)| \le C e^{-\gamma R}\,,$$

and it only remains to estimate the derivatives of $G_{22}(x,\,y)$. Since $\theta < c_0 R$, we have $x > \theta_0 R$, so that $|x-\xi| \ge \frac{1}{2}c_0 R$ at the points ξ where $\lambda_2(\xi) \ne 0$. Also, $y \le \eta R \le \varepsilon_0|x-\xi|$ if $\eta < \frac{1}{2}c_0\varepsilon_0$ (ε_0 as in Lemma 13.1) provided $\eta \ll c_0$ (as in (6.15)). We can therefore apply (4.5) of Lemma 13.1:

$$|G_{22}(x,\,y)| \le C\int_{-\infty<\xi<\frac{1}{2}c_0 R} \left(\frac{y}{|x-\xi|^2} + e^{-\gamma|x-\xi|}\right)e^{-\delta|\xi|}d\xi\,,$$

and therefore

$$|G_{22}(x,\,y)| \le \frac{Cy}{R^2} + C e^{-\gamma R}\,,$$

with another $\gamma > 0$. Finally, by (4.6) of Lemma 13.1,

$$|\nabla^j G_{22}(x,\,y)| \le \frac{C}{R^{j+1}}\quad (0 \le j \le 4)\,.$$

Combining these estimates with (6.16), (6.17) we conclude:

Lemma 6.1. *The inequalities*

$$(6.18) \qquad |G_2(x, y)| \leq \frac{Cy}{R^2} + Ce^{-\gamma R},$$

$$(6.19) \qquad |\nabla^j G_2(x, y)| \leq \frac{C}{R^{j+1}} \quad (1 \leq j \leq 4)$$

hold for all (x, y, θ) *in the region* (6.15).

7. Proof of Theorem 1.1 for $n = 0$

Define $\rho(R)$ and Σ_R as in (1.8), (1.9).

We shall need the following lemma.

Lemma 7.1. *Let* φ *be a solution of* (1.1)–(1.4) *in* $\mathbb{R}^2_+$, *satisfying* (1.5) *for some* $\alpha > 0$. *Then*

$$(7.1) \qquad \rho(R) \leq CR^\alpha \quad \text{if } R > 1 .$$

The lemma does not follow from standard elliptic estimates [1], [2] since, upon scaling to

$$\psi_R(x, y) = \psi(Rx, Ry) ,$$

we get an elliptic equation

$$R^2 \omega \Delta \psi_R - \Delta^2 \psi_R = 0$$

with a large coefficient $R^2 \omega$. The arguments which will be used to establish (7.1) have common features with arguments to be given in §8. In order to minimize duplication we shall postpone the proof the the lemma until Section 9.

If $\varphi \not\equiv 0$ then by analyticity of φ we deduce that $\rho(R) > 0$ for all $R > 0$ and, by Lemma 2.1, there are only two possibilities: either (1.10) holds for some $\gamma > 0$, or else there exist sequences $R_j \to \infty$, $B_j \to \infty$ such that (1.11) holds. We wish to exclude the second case. Suppose (1.11) holds and introduce the sequence ψ_j by (1.12). Set

(7.2)
$$\widetilde{\Sigma}_{R, \eta} = \left\{ (x, y), \quad \frac{1}{2\eta} < \sqrt{x^2 + y^2} \leq 2\eta, \quad x > 0 \text{ and } 0 < y < A \frac{\log R}{R} \right\} .$$

Then

(7.3)
$$\left[\int_{\left(B_{2\eta} \backslash B_{\frac{1}{2\eta}} \right)^+ \backslash \widetilde{\Sigma}_{R_j, \eta}} |\nabla^2 \psi_j|^2 + \frac{1}{R_j^4} \int_{\left(B_{2\eta} \backslash B_{\frac{1}{2\eta}} \right)^+ \backslash \widetilde{\Sigma}_{R_j, \eta}} |\nabla^4 \psi_j|^2 \right]^{1/2}$$

$$+ \sup_{\widetilde{\Sigma}_{R_j, \eta}} \left[|\nabla \psi_j| + \frac{1}{R_j^3} |\nabla^4 \psi_j| \right] \leq C\eta^\alpha$$

for any $\eta \in (1, B_j)$. Also,

$$(7.4) \qquad\qquad \omega R_j^2 \Delta \psi_j = \Delta^2 \psi_j \text{ in } \mathbb{R}_+^2 \,,$$

$$(7.5) \qquad\qquad \psi_j(x, 0) = \frac{\partial}{\partial y} \psi_j(x, 0) = 0 \text{ if } x > 0 \,,$$

$$(7.6) \qquad\qquad \frac{\partial^2 \psi_j}{\partial x^2} - \frac{\partial^2 \psi_j}{\partial y^2} = 0 \text{ if } y = 0, \ x < 0 \,,$$

$$(7.7) \qquad\qquad 3 \frac{\partial^3 \psi_j}{\partial x^2 \partial y} + \frac{\partial^3 \psi_j}{\partial y^3} - \omega R_j^2 \frac{\partial \psi_j}{\partial y} = 0 \text{ if } y = 0, \ x < 0 \,.$$

From (7.3) and (7.5) we have

$$(7.8) \qquad\qquad \left[\int_{\left(B_{2\eta} \backslash B_{\frac{1}{2\eta}} \right)^+ \backslash \widetilde{\Sigma}_{R_j \eta}} \left| \nabla^2 \psi_j \right|^2 \right]^{1/2} \leq C \eta^\alpha$$

and

$$(7.9) \qquad |\psi_j(x, y)| \leq C \eta^\alpha \frac{\log R_j}{R_j} \,, \quad |\nabla \psi_j(x, y)| \leq C \eta^\alpha \quad \text{in } \widetilde{\Sigma}_{R_j, \eta} \,.$$

Using the Poincaré and Sobolev inequalities we deduce that, for any $M > 1$,

$$(7.10) \qquad\qquad |\psi_j|_{L^\infty \left((B_M \backslash B_{1/M})^+ \right)} \leq C \qquad (C = C_M) \,.$$

Furthermore, there is a function ψ^* to which a subsequence ψ_j is weakly convergent in $H^2_{\text{loc}}(\mathbb{R}_+^2)$. Since $R_j^2 \omega \to \infty$, we deduce from (7.4) that

$$(7.11) \qquad\qquad \Delta \psi^* = 0 \text{ in } \mathbb{R}_+^2 \,.$$

We claim that

$$(7.12) \qquad\qquad \frac{\partial \psi^*(x, 0)}{\partial y} = 0 \text{ if } x < 0 \,.$$

To prove it we multiply (7.4) by a test function h that vanishes in a neighborhood of the positive x-axis $\{y = 0, \ x \geq 0\}$ and then integrate over $\mathbb{R}_+^2$. By integration by parts,

$$(7.13)$$

$$R_j \omega \int_{\mathbb{R}_+^2} h \Delta \psi_j \quad = \quad \int_{\mathbb{R}_+^2} \Delta h \cdot \Delta \psi_j - \int_{-\infty}^0 h(x, 0) \, \partial_y \, \Delta \psi_j(x, 0) dx$$

$$+ \int_{-\infty}^0 (\partial_y h(x, 0)) \, \Delta \psi_j(x, 0) dx \,.$$

By (7.6)

$$\int_{-\infty}^0 (\partial_y h(x, 0)) \, \Delta \psi_j(x, 0) dx = 2 \int_{-\infty}^0 (\partial_y h(x, 0)) \frac{\partial^2 \psi_j(x, 0)}{\partial x^2} \, dx$$

$$= 2 \int_{-\infty}^0 \frac{\partial^3 h(x, 0)}{\partial x^2 \partial y} \, \psi_j(x, 0) dx \,,$$

and by (7.7),

$$-\int_{-\infty}^0 h(x,\,0)\partial_y\Delta\psi_j(x,\,0)dx = \int_{-\infty}^0 h(x,\,0)\left[\left(2\frac{\partial^3\psi_j}{\partial x^2\partial y} - \omega R_j^2\frac{\partial\psi_j}{\partial y}\right)(x,\,0)\right]dx$$

$$= 2\int_{-\infty}^0 \frac{\partial^2 h(x,\,0)}{\partial x^2}\,\frac{\partial\psi_j(x,\,0)}{\partial y}\,dx - \omega R_j^2\int_{-\infty}^0 h(x,\,0)\frac{\partial\psi_j(x,\,0)}{\partial y}\,dx\,.$$

Substituting these expressions into (7.13), we get

(7.14)

$$R_j^2\,\omega\int_{\mathbb{R}_+^2} h\Delta\psi_j = \int_{\mathbb{R}_+^2}\Delta h\cdot\Delta\psi_j - \omega R_j^2\int_{-\infty}^0 h(x,\,0)\,\frac{\partial\psi_j(x,\,0)}{\partial y}\,dx$$

$$+ 2\int_{-\infty}^0 \frac{\partial^3 h(x,\,0)}{\partial x^2\partial y}\,\psi_j(x,\,0)dx + 2\int_{-\infty}^0 \frac{\partial^2 h(x,\,0)}{\partial x^2}\,\frac{\partial\psi_j(x,\,0)}{\partial y}\,dx\,.$$

Note that by (7.8) and the trace theorem,

(7.15)
$$\frac{\partial\psi_j(x,\,0)}{\partial y} \longrightarrow \frac{\partial\psi^*(x,\,0)}{\partial y} \text{ in } L_{\mathrm{loc}}^2\left[-N,\,-\frac{1}{N}\right]\,,$$

(7.16)
$$\psi_j(x,\,0) \longrightarrow \psi^*(x,\,0) \text{ uniformly in } \left[-N,\,-\frac{1}{N}\right]$$

for any $N > 1$. Dividing both sides of (7.14) by R_j^2 and letting $j \to \infty$, we arrive at

$$\int_{-\infty}^0 h(x,\,0)\frac{\partial\psi^*(x,\,0)}{\partial y}\,dx = 0\,,$$

and (7.12) follows.

We shall next prove that

(7.17)
$$\psi^*(x,\,0) = 0 \text{ if } x > 0\,.$$

Indeed, from (7.8), (7.9) it follows that

$$\left(\int_{\left(B_{2\eta}\backslash B_{\frac{1}{2\eta}}\right)^+}|\nabla\psi_j|^p\right)^{1/p} \le C(\eta)\,.$$

for any $1 < \eta < B_j$, $p > 1$, so that the ψ_j are uniformly Hölder continuous in compact sets in $\{y \ge 0\}$, away from the origin. Since $\psi(x,\,0) = 0$ if $x > 0$, the assertion (7.17) follows.

From (7.8) we get

(7.18)
$$\left[\int_{\left(B_{2\eta}\backslash B_{\frac{1}{2\eta}}\right)^+}\left|\nabla^2\psi^*\right|^2\right]^{1/2} \le C\eta^\alpha$$

and, in particular,

(7.19)
$$\left[\int_{(B_{2\delta}\backslash B_\delta)^+}\left|\nabla^2\psi^*\right|^2\right]^{1/2} \le \frac{C}{\delta^\alpha} \quad \text{if } \delta < 1\,.$$

Notice that the function ψ_x^* satisfies the same boundary conditions (7.12), (7.17) as ψ^*. Using (7.18), we can apply elliptic estimates to $\psi_x^*(\eta x,\,\eta y)$ and conclude that

(7.20)
$$|\psi_x^*(x,\,y)| \le Cr^\alpha \quad \text{if } r \to \infty\,.$$

Similarly, by using (7.19) we derive the estimate

$$(7.21) \qquad\qquad |\psi_x^*(x,\,y)| \le Cr^{-\alpha} \quad \text{if } r \to 0 \,.$$

Since $n = 0,\ \alpha = \frac{1}{2} - \varepsilon$. We can expand ψ_x^* about $(0,\,0)$ in the form

$$\sum a_k r^{k/2} \sin \frac{k\theta}{2} \,.$$

From (7.21) it then follows that $k \ge 1$ if $a_k \ne 0$, so that

$$\psi_x^* = a_1 r^{1/2} \sin \frac{\theta}{2} + O\left(r^{3/2}\right) \quad \text{near } (0,\,0) \,.$$

Consider the function

$$w(x,\,y) = r^\mu \sin(\mu\theta + \beta) \qquad \left(\beta > 0\,,\ \mu = \frac{\frac{\pi}{2} - \beta}{\pi}\right),$$

where $\beta > 0$ is so small that $\mu > \alpha$. Then

$$w(x,\,0) > 0 \quad \text{if } x > 0\,, \quad w_y(x,\,0) = 0 \quad \text{if } x < 0\,,$$

and, for any small $\lambda > 0$,

$$\lambda w(x,\,y) > |\psi_x^*(x,\,y)| \quad \text{if } x^2 + y^2 \ge R_\lambda \,,$$

where $R_\lambda \to \infty$ if $\lambda \to 0$. By comparison we then deduce that

$$\pm\psi_x^*(x,\,y) < \lambda w(x,\,y) \quad \text{if } x^2 + y^2 \le R_\lambda \,,$$

so that $\psi_x^* \equiv 0$. Hence ψ^* is a function of y only, and since it is a harmonic function satisfying (7.12), (7.17), $\psi^* \equiv 0$. Consequently:

Lemma 7.2. *For any compact set B in $\overline{\mathbb{R}_+^2}$ such that $B \cap \{(x,\,0),\ x \ge 0\} = \phi$,*

$$(7.22) \qquad\qquad \psi_j \longrightarrow 0 \ \text{weakly in } H^2(B) \,,$$

and therefore also

$$(7.23) \qquad\qquad \psi_j \longrightarrow 0 \ \text{uniformly in } B\,.$$

8. Proof of Theorem 1.1 for $n = 0$ (continued)

Let $\zeta(x,\,y)$ be any cutoff function as illustrated in Figure 8.1, with $\partial^j \zeta(x,\,0)/\partial y^j = 0$ for $j = 1,\,2,\,3$; the support of $\{0 < \zeta < 1\}$ lies in the shaded region. Notice that the support of ζ has positive distance to the positive x-axis. Set $\widetilde{\psi}_j = \zeta\psi_j$. Then

$$(8.1) \qquad \begin{aligned} \omega\,R_j^2 \Delta\widetilde{\psi}_j - \Delta^2 \widetilde{\psi}_j &= \mu_j(x,\,y) \text{ in } \mathbb{R}_+^2 \,, \\ \mu_j &= O\left(R_j^2\psi_j,\ R_j^2\partial\psi_j,\ \psi_j,\ \partial\psi_j,\ \partial^2\psi_j,\ \partial^3\psi_j\right) \,, \end{aligned}$$

$$(8.2) \qquad \begin{aligned} \frac{\partial^2 \widetilde{\psi}_j}{\partial x^2} - \frac{\partial^2 \widetilde{\psi}_j}{\partial y^2} &= a_j(x) \text{ on } y = 0\,, \\ a_j &= O(\psi_j,\ \partial\psi_j) \,, \end{aligned}$$

$$(8.3) \qquad \begin{aligned} 3\frac{\partial^3 \widetilde{\psi}_j}{\partial x^2\partial y} + \frac{\partial^3 \widetilde{\psi}}{\partial y^3} - \omega\,R_j^2\frac{\partial\psi_j}{\partial y} &= b_j(x) \text{ on } y = 0\,, \\ b_j &= O\left(\partial^2\psi_j\right) \,, \end{aligned}$$
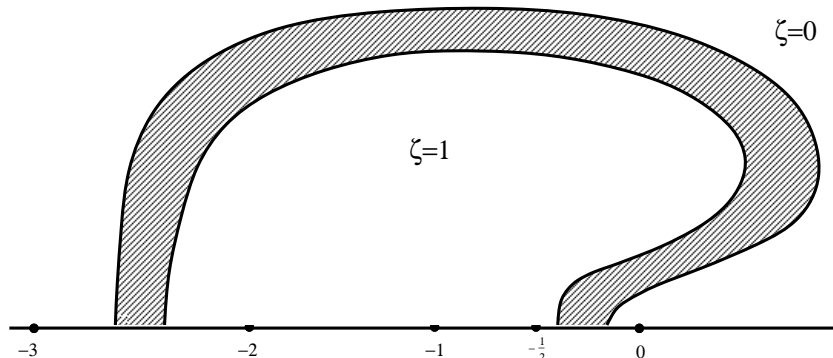
FIGURE 8.1

and O means a linear function in the indicated variables, with bounded smooth coefficients.

We wish to apply Lemma 3.1 to $\widetilde{\psi}_j$. This requires us to estimate the various terms on the right-hand side of (3.7). Consider first estimating $\int |\mu_j|^2$. Take the term with $\partial^3 \psi_j$ which appears in μ_j. The coefficient g_1 of this term is supported in the set where $0 < \zeta < 1$; this set is assumed to lie in, say, $B = (B_4 \backslash B_{1/4})^+ \backslash B^*$, where B^* is the sector $\{(x, y),\ x \geq 0,\ 0 \leq y \leq c_0 x\}$ for some c_0; see Figure 8.1. We take j so large that

$$\frac{A \log R_j}{R_j} \leq \frac{c_0}{2}.$$

By interpolation, for any $\varepsilon_1 > 0$,

$$(8.4) \qquad \int_B \left| g_1 \partial^3 \psi_j \right|^2 \leq \varepsilon_1 \int_B \left| \nabla^4 \psi_j \right|^2 + C_{\varepsilon_1} \sup_B |\psi_j|^2 .$$

Similarly the term $R_j^2 \partial \psi_j$ comes with a coefficient g_2 supported in B, and

$$(8.5) \qquad \int_{\mathbb{R}_+^2} \left| g_2 R_j^2 \partial \psi_j \right|^2 \leq \varepsilon_1 R_j^4 \int_B \left| \nabla^2 \psi_j \right|^2 + C_{\varepsilon_1} R_j^4 \sup_B |\psi_j|^2 .$$

The other terms in μ_j can be estimated similarly. Hence

$$(8.6) \qquad \int_{\mathbb{R}_+^2} |\mu_j|^2 \leq \varepsilon_1 \left[\int_B \left| \nabla^4 \psi_j \right|^2 + R_j^4 \left| \nabla^2 \psi_j \right|^2 \right] + C_{\varepsilon_1} R_j^4 \sup_B |\psi_j|^2 .$$

Next we estimate b_j; it has the form $g_3 \partial_x^2 \psi_j$ with g_3 supported in $B_0 \equiv B \cap \{y = 0\} = (-4, -\frac{1}{4})$. We have

$$
\begin{aligned}
\int_{-\infty}^0 R_j |b_j|^2 \ &\leq\ \int_{-4}^{-1/4} R_j \left| \partial^2 \left(g_3 \psi_j \right) \right|^2 + C \int_{-4}^{-1/4} R_j |\partial \psi_j|^2 \\
&\leq\ C \int_{-4}^{-1/4} R_j \left| \partial \psi_j \right|^2 + C R_j \int_{-4}^{-1/4} \left| \partial^{5/2} (g_3 \psi_j) \right|^2 .
\end{aligned}
$$

We shall use the interpolation formula [7, p. 49]

$$\|u\|_{H^{(1-\theta)s_1 + \theta s_2}} \leq C \left(\|u\|_{H^{s_1}} \right)^{1-\theta} \left(\|D^{s_2} u\|_{H^{s_2}} \right)^\theta$$
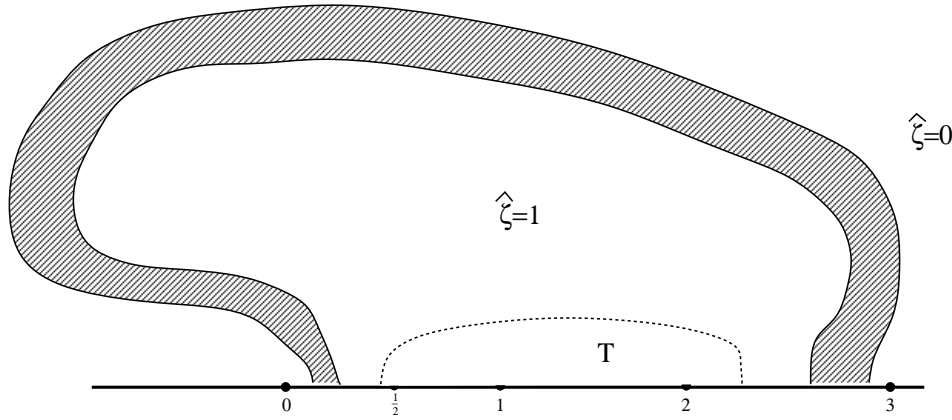
FIGURE 8.2

where $0 \leq \theta \leq 1$, $(1-\theta)s_1 + \theta s_2 = 1$, and u has compact support. Taking $u = \partial^{5/2}(g_3\psi_j)$, $s_1 = 0$, $s_2 = 1/2$, we find that

$$\int_{-\infty}^{0} R_j|b_j|^2 \leq C\int_{-4}^{-1/4} R_j|\partial\psi_j|^2 + \varepsilon_1 R_j^2 \int_{-4}^{-1/4} \left|D^2(g_3\psi_j)\right|^2$$
$$+ C_{\varepsilon_1}\int_{-4}^{-1/4} \left|D^3(g_3\psi_j)\right|^2.$$

The last two terms can be estimated analogously to (8.4), (8.5), whereas the term

$$\int_{-4}^{-1/4} R_j|\partial\psi_j|^2 \qquad \left(\text{or even } \int_{-4}^{-1/4} R_j^4|\partial\psi_j|^2\right)$$

can be estimated by means of the trace theorem by

$$\varepsilon_1 R_j^4 \int_B \left|\nabla^2\psi_j\right|^2 + C_{\varepsilon_1} R_j^4 \sup_B |\psi_j|^2.$$

The other terms in $D^{1/2}b_j$, $D^{3/2}a_j$, $R_j^{3/2}a_j$ can be estimated similarly. If we now apply Lemma 3.1 we obtain

(8.7)
$$R_j^4 \int_{\mathbb{R}_+^2} \left|\nabla^2\widetilde{\psi}_j\right|^2 + \int_{\mathbb{R}_+^2} \left|\nabla^4\widetilde{\psi}_j\right|^2 \leq \varepsilon_1\left[R_j^4 \int_B \left|\nabla^2\psi_j\right|^2 + \int_B \left|\nabla^4\psi_j\right|^2\right]$$
$$+ C_{\varepsilon_1} R_j^4 \sup_B |\psi_j|^2 \qquad \text{for any } \varepsilon_1 > 0.$$

Using (7.3) and Lemma 7.2, we then deduce that

(8.8)
$$R_j^4 \int_{\mathbb{R}_+^2} \left|\nabla^2\widetilde{\psi}_j\right|^2 + \int_{\mathbb{R}_+^2} \left|\nabla^4\widetilde{\psi}_j\right|^2 = o(R_j^4) \qquad \text{if } j \to \infty.$$

Next we work with a cutoff function $\widehat{\zeta}$ as illustrated in Figure 8.2. Set $\widehat{\psi}_j = \widehat{\zeta}\psi_j$. We want to estimate $\widehat{\psi}_j$ in the domain T bounded by the broken curve and the x-axis. We can write

(8.9)
$$\omega R_j^2 \Delta\widehat{\psi}_j - \Delta^2\widehat{\psi}_j = \widehat{\mu}_j \text{ in } \mathbb{R}_+^2,$$

where $\widehat{\mu}_j$ has the same structure as μ_j in (8.1). We also have

$$(8.10) \qquad\qquad \widehat{\psi}_j(x,\, 0) = \partial_y\, \widehat{\psi}_j(x,\, 0) = 0 \text{ if } x \in \mathbb{R}\,.$$

We shall use the Green function $V(x,\, y,\, \theta)$ constructed in Section 6 (see (6.6), (6.11)) and the estimates obtained for G_1 (or Φ) and G_2 in Sections 5 and 6, respectively. Introducing

$$\widetilde{V}_R(\xi,\, \eta,\, \theta) = \frac{1}{R^2}\, V(R\xi,\, R\eta,\, R\theta)\,,$$

we have

$$(8.11) \qquad\qquad \omega R^2\, \Delta \widetilde{V}_R - \Delta^2\, \widetilde{V}_R = \delta(\xi)\, \delta(\eta - \theta)\,,$$

so that

$$(8.12) \qquad\qquad \widehat{\psi}_j(\xi,\, \eta) = \int_{\mathbb{R}_+^2} \widetilde{V}_{R_j}\, (\xi - \lambda,\, \eta,\, \theta)\, \widehat{\mu}_j(\lambda,\, \theta) d\lambda\, d\theta\,.$$

The support of $\widehat{\mu}_j$ is continued in the set $\left\{0 < \widehat{\zeta} < 1\right\}$ which is a subset of, say, $(B_4 \backslash B_{1/8})^+$. We can choose $\widehat{\zeta}$ and T such that

$$(8.13) \qquad \begin{aligned} &\text{if } (\xi,\, \eta) \in T \text{ and } 0 < \widehat{\xi}(\lambda,\, \theta) < 1, \text{ then} \\ &2c_0 \le |\xi - \lambda|^2 + \theta^2\,, \quad \eta \le \eta_0 \text{ and } \eta_0 \ll c_0\,, \end{aligned}$$

where $c_0,\, \eta_0$ are as in (6.15).

By (7.3), Lemma 7.2 and interpolation one can show (as in the proof of (8.6) with $\mathbb{R}_2^+$ replaced by $\mathbb{R}_2^+ \backslash \widetilde{\Sigma}_{R_j,\, \eta_0}$, $\eta_0 = 4$) that

$$(8.14) \qquad\qquad \frac{1}{R_j^4} \int_{\mathbb{R}_+^2 \backslash \widetilde{\Sigma}_{R_j,\, \eta_0}} |\widehat{\mu}_j|^2 \longrightarrow 0 \quad \text{if } j \longrightarrow \infty\,.$$

Since $\widehat{\mu}_j(\lambda,\, \theta) = 0$ if $0 < \widehat{\xi}(\lambda,\, \theta) < 1$, in estimating $\nabla^k \widehat{\psi}_j(\xi,\, \eta)$ we only need to estimate $\nabla^k \tilde{V}_{Rj}(\xi - \lambda,\, \eta,\, \theta)$ at points of the set

$$(8.15)$$
$$Z = \left\{(\xi - \lambda,\, \eta,\, \theta)\,; \quad 2c_0 < |\xi - \lambda|^2 + \theta^2,\quad \eta \le \eta_0\right\} \quad \text{where } \eta_0 \ll c_0\,.$$

Setting $\widetilde{\xi} = \xi - \lambda$, $R = R_j$ we have

$$(8.16)$$
$$\begin{aligned} \nabla \widetilde{V}_R(\widetilde{\xi},\, \eta,\, \theta) &= \frac{1}{R} \left\{ \nabla_{xy}\varphi\left(R\widetilde{\xi},\, R(\eta - \theta)\right) + \nabla_{xy}\varphi\left(R\widetilde{\xi},\, R(\eta + \theta)\right) \right\} \\ &\quad + \frac{1}{R}\, \nabla_{xy}G_1\left(R\widetilde{\xi},\, R\eta,\, R\theta\right) + \frac{1}{R}\nabla_{xy}G_2\left(R\widetilde{\xi},\, R\eta,\, R\theta\right) \end{aligned}$$

and

$$(8.17)$$
$$\frac{1}{R}\left[\left|\nabla_{xy}^j\varphi\left(R\widetilde{\xi},\, R(\eta - \theta)\right)\right| + \left|\nabla_{xy}^j\varphi\left(R\widetilde{\xi},\, R(\eta + \theta)\right)\right|\right] \le \frac{C}{R^2}$$

for $1 \le j \le 4$. By (8.15), the points $(R\widetilde{\xi},\, R\eta,\, R\theta)$ satisfy the condition in (6.15). Hence, by Lemma 6.1 we have

$$(8.18) \qquad\qquad \frac{1}{R}\left|\nabla_{xy}G_2\left(R\widetilde{\xi},\, R\eta,\, R\theta\right)\right| \le \frac{C}{R^2} \text{ in } Z\,.$$

Next, by (6.13) and Lemma 5.1,

$$
(8.19) \qquad \left| \frac{1}{R} \nabla_{xy} G_1 \left(R\widetilde{\xi},\, R\eta,\, R\theta \right) \right| \leq \frac{C}{R^2} \text{ if } \eta > A \frac{\log R}{R} \,.
$$

To estimate derivatives of G_1 for $\eta < A(\log R)/R$ we use Lemma 5.2. Accordingly,

$$
\frac{\partial}{\partial x} G_1 \left(R\widetilde{\xi},\, R\eta,\, R\theta \right) - \frac{2\Gamma}{R} \frac{\widetilde{\xi}^2}{1 + \widetilde{\xi}^2} \;=\; O\!\left(\frac{1}{R^2} \right),
$$

$$
\frac{\partial}{\partial y} G_1 \left(R\widetilde{\xi},\, R\eta,\, R\theta \right) - \frac{2\Gamma}{R} \left(1 - e^{-\sqrt{\omega}\, R\eta} \right) \frac{1}{1 + \widetilde{\xi}^2} \;=\; O\!\left(\frac{1}{R^2} \right).
$$

Since G_1 is a solution in some domain containing Z of $\omega \Delta u - \Delta^2 u = 0$, we can use elliptic regularity [1], [2] to deduce that

$$
(8.20)
$$

$$
\left| \nabla_{\widetilde{\xi}\eta}^{k-1} \nabla_{xy} G_1 \left(R\widetilde{\xi},\, R\eta,\, R\theta \right) \right| \leq \frac{C}{R} \text{ in } Z, \text{ for } 1 \leq k \leq 4, \text{ if } \eta < A \frac{\log R}{R} \,.
$$

The proofs of (8.18), (8.19) can be extended to derive the same estimates for higher derivatives (by using Lemma 6.1 for $j \geq 2$ and Lemma 5.3). Combining these estimates with (8.20) and (8.17), we deduce from (8.16) that

$$
(8.21) \qquad \left| \nabla^k \widetilde{V}_R \left(\widetilde{\xi},\, \theta,\, \eta \right) \right| \leq \frac{C}{R^2} \quad \text{in } Z \quad (1 \leq k \leq 4)
$$

for $R = R_j$.

From (8.12),

$$
(8.22)
$$

$$
\begin{aligned}
D^k \widehat{\psi}_j(\xi,\, \eta) \;=\; & \int_{\mathbb{R}_+^2 \setminus \widetilde{\Sigma}_{R_j,\, \eta_0}} D^k \widetilde{V}_{R_j} \left(\xi - \lambda,\, \eta,\, \theta \right) \cdot \widehat{\mu}_j(\lambda,\, \theta) d\lambda \, d\theta \\
& + \int_{\widetilde{\Sigma}_{R_j,\, \eta_0}} D^k \widetilde{V}_{R_j} \left(\xi - \lambda,\, \eta,\, \theta \right) \cdot \widehat{\mu}_j(\lambda,\, \theta) d\lambda \, d\theta \equiv J_{1k} + J_{2k}
\end{aligned}
$$

for $1 \leq k \leq 4$. By (8.14) and (8.21) we have

$$
(8.23) \qquad |J_{1k}| \leq \frac{C}{R_j^2} \int_{\mathbb{R}_+^2 \setminus \widetilde{\Sigma}_{R_j,\, \eta_0}} |\widehat{\mu}_j| = o(1) \text{ as } j \to \infty \,.
$$

Next, from (7.3) we have

$$
|\nabla \psi_j| + \frac{1}{R_j^3} \left| \nabla^4 \psi_j \right| \leq C \eta^\alpha \text{ in } \widetilde{\Sigma}_{R_j,\, \eta} \,,
$$

and we can use this inequality to estimate all the terms in $\widehat{\mu}_j$. We obtain

$$
|\widehat{\mu}_j| = O\left(R_j^2 \right)\,,
$$

so that

$$
\int_{\widetilde{\Sigma}_{R_j,\, \eta_0}} |\widehat{\mu}_j| \leq C R_j^2 \frac{\log R_j}{R_j} \,.
$$

It follows that

$$
(8.24) \qquad |J_{2k}| \leq \frac{C}{R_j^2} \int_{\widetilde{\Sigma}_{R_j,\, \eta_0}} |\widehat{\mu}_j| \leq C \frac{\log R}{R_j^3} \to 0 \text{ if } j \to \infty \,.
$$

Combining this with the estimate of J_{1k}, we get from (8.22)

$$(8.25) \qquad \left| D^k \widehat{\psi}_j(\xi,\eta) \right| \to 0 \text{ in } T, \text{ as } j \to \infty .$$

We can choose ζ and $\widehat{\zeta}$ so that the union of the sets where $\zeta = 1$ and $\widehat{\zeta} = 1$ contains $(B_2 \backslash B_{1/2})^+$. Then (8.8) and (8.25) imply that the left-hand side of (1.13) converges to zero as $j \to \infty$, which is a contradiction. Therefore (1.10) must hold for some $\gamma > 0$.

From (1.10) it follows that

$$(8.26) \int_{(B_{2R}\backslash B_{R/2})^+ \backslash \Sigma_R} \left| \nabla^2 \varphi \right|^2 \le \frac{C}{R^{4+2\gamma}} , \qquad \left| \nabla \varphi \right| + \left| \nabla^4 \varphi \right| \le \frac{C}{R^{1+\gamma}} \text{ in } \Sigma_R .$$

Since $\varphi(x,0) = 0$ if $x > 0$, from the second inequality in (8.26) it follows that

$$(8.27) \qquad |\varphi(x,y)| \le \frac{C \log R}{R^{1+\gamma}} \quad \text{in } \Sigma_R .$$

Setting

$$\psi(x,y) = \varphi(Rx, Ry) ,$$

the first inequality in (8.26) gives

$$\left(\int_{(B_2 \backslash B_{1/2})^+ \backslash \widetilde{\Sigma}} |\nabla^2 \psi|^2 \right)^{1/2} \le \frac{C}{R^{1+\gamma}} \qquad \left(\widetilde{\Sigma} = \frac{1}{r} \Sigma_R \right)$$

and, therefore, by (8.27) and the Poincaré and Sobolev inequalities,

$$|\psi| \le \frac{C \log R}{R^{1+\gamma}} + \frac{C}{R^{1+\gamma}} \qquad \text{in } (B_2 \backslash B_{1/2})^+ ,$$

or

$$|\varphi(x,y)| \le \frac{C \log R}{R^{1+\gamma}} \qquad \text{in } (B_{2R} \backslash B_{R/2})^+ .$$

Using elliptic estimates [1], [2], we obtain the same bound for the derivatives of φ; in particular,

$$(8.28) \qquad |D^j \varphi(x,y)| \le \frac{C \log R}{R^{1+\gamma}} \qquad \text{in } (B_{2R} \backslash B_{R/2})^+ \quad (0 \le j \le 3) .$$

Since $\varphi = O(r^{\frac{1}{2}+\varepsilon})$ as $r \to 0$, we can deduce similarly to the local regularity results established in [4] and [5] that φ is actually $O(r^{\frac{3}{2}} |\log r|)$ as $r \to 0$ (In fact, the proof can be established by slight changes in the proof of Theorem 5.2 of [5].) Consequently,

$$(8.29) \qquad D^\sigma \varphi = O(r^{\frac{3}{2} - |\sigma| - \delta}) \quad \text{as } r \to 0$$

for $0 \le |\sigma| \le 3$ and arbitrarily small $\delta > 0$.

We now multiply (1.1) by φ and integrate over $B_R^+ \backslash B_{1/R}^+$. By integration by parts we obtain (cf. [4], [5])

$$(8.30) \int_{B_R^+ \backslash B_{1/R}^+} \left[\left| \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right|^2 + 4 \left| \frac{\partial^2 \varphi}{\partial x \partial y} \right|^2 \right] + 2(Re\,\omega) \int_{B_R^+ \backslash B_{1/R}^+} |\nabla \varphi|^2 = I_R$$

where I_R is a sum of boundary integrals on $\partial B_R \cap \{y > 0\}$ and $\partial B_{1/R} \cap \{y > 0\}$ which, by (8.28) and (8.29), converge to zero as $R \to \infty$. It follows that

$$\varphi_{xx} = \varphi_{yy}, \qquad \varphi_{xy} = 0 \text{ in } \mathbb{R}_+^2 \ ,$$

so that $\varphi \equiv 0$.

9. Completion of the proof of Theorem 1.1

We begin with:

Proof of Lemma 7.1. Take any sequence $R_j \nearrow \infty$ and define

$$(9.1) \qquad \qquad \psi_j(x, \, y) = \frac{\varphi(R_j x, \ R_j y)}{R_j^\alpha} \ .$$

Then for any positive constants $c_0, \ C_0$,

$$(9.2) \qquad \qquad |\psi_j(x, \, y)| \leq C \quad \text{if } c_0 \leq x^2 + y^2 \leq C_0 \ .$$

Consider $\widehat{\psi}_j = \widehat{\xi} \psi_j$, where $\widehat{\xi}$ is a cutoff function as in Figure 8.2. We want to estimate $\widehat{\psi}_j$ in T, using the representation (8.12). Recall that $\widehat{\mu}_j$ is a linear combination of

$$R_j^2 \psi_j, \ R_j^2 \partial \psi_j, \ \psi_j, \ \partial \psi_j, \ \partial^2 \psi_j, \ \partial^3 \psi_j.$$

We can move the derivatives from $\partial_j^k \psi$ to $\widetilde{V}_{Rj}$ (in the integral in (8.12)). Noting that all the boundary integrals vanish and recalling that

$$|\partial_k \widetilde{V}_{Rj}(\xi - \lambda, \, \eta, \, \sigma)| \leq \frac{C}{R^2} \ ,$$

we obtain, upon using (9.2),

$$|D^l \widehat{\psi}_j(x, \, y)| \leq C \quad \text{in } T \ , \quad |l| \leq 4 \ ;$$

hence, in particular,

$$(9.3) \qquad \qquad \sup_T \left[|\nabla \psi_j| + \frac{l}{R_j^3} |\nabla^3 \psi_j| \right] \leq C \ .$$

We now turn to $\widetilde{\psi}_j = \xi \psi_j$, where ξ is a cutoff function as in Figure 8.1. By Lemma 3.1 and (9.2), we can estimate the L^2-norm of $|\nabla^4 \widetilde{\psi}_j| + R^2 |\nabla^2 \widetilde{\psi}_j|$ by

$$\varepsilon_1 \left\| \left[|\nabla^4 \psi_j| + R_j^2 |\nabla^2 \psi_j| \right] \right\|_{L^2\left[(B_4 \backslash B_{1/4})^+ \backslash B^* \right]} + C_{\varepsilon_1} R_j^2$$

for any small $\varepsilon_1 > 0$. This together with (9.3) give a bound on $\rho(R_j)$ in terms of $\varepsilon_1[\rho(2R_j) + \rho(\frac{1}{2} R_j)]$. This bound however does not enable us to control $\rho(R_j)$. We shall therefore proceed differently, making use of Lemma 3.2.

Note that $\partial^{-3} \widetilde{\mu}_j$ is a linear combination of

$$R_j^2 \partial^{-3} \psi_j, \ R_j^2 \partial^{-3} \partial \psi_j, \ \partial^{-3} \psi_j, \dots, \partial^{-3} \partial^3 \psi_j$$

where $\partial^{-l} \psi_j$ is any expression $\partial_x^{-l_1} \partial_y^{-l_2} \psi_j$, $l_1 + l_2 = l$. Consider, for simplicity, first the case where $a_j = b_j = 0$. Using Lemma 3.2 with appropriate m_1, m_2 such that $m_1 + m_2 = 3$, we obtain

$$\|\nabla \widetilde{\psi}_j\|_{L^2} + R_j^2 \|\nabla^{-1} \widetilde{\psi}_j\|_{L^2} \leq C \left[R_j^2 \|\nabla^{-2} \psi_j\|_{L^2(G)} + \|\psi_j\|_{L^2(G)} \right] \ ,$$

where G is the region $(B_4 \backslash B_{1/4})^+ \backslash B^*$. Since $\psi_j(x, 0) = 0$ if $x > 0$ and $|\psi_j| \leq C$,

$$\|\nabla^{-2}\psi_j\|_{L^2(G)} \leq C \ ,$$

and so

(9.4) $$\|\nabla \widetilde{\psi}_j\|_{L^2} + \|R_j^2 \nabla^{-1} \widetilde{\psi}_j\|_{L^2} \leq CR_j^2 \ .$$

The same estimates can be easily obtained also if $a_j \not\equiv 0$, $b_j \not\equiv 0$.

We next take a cutoff function ξ_1 supported on the set where $\xi = 1$ and define $\widetilde{\widetilde{\psi}}_j = \xi_1 \psi_j$. Using Lemma 3.2 with appropriate m_1, m_2 such that $m_1 + m_2 = 2$, we get

(9.5) $$\|\nabla^2 \widetilde{\widetilde{\psi}}_j \|_{L^2} + R_j^2 \| \widetilde{\widetilde{\psi}}_j \|_{L^2} \leq C \left[R_j^2 \|\nabla^{-1}\psi_j\| + \|\nabla \psi_j\|_{L^2(G_1)} \right] \leq CR_j^2 \ ,$$

by (9.4). Procceding similarly to apply Lemma 3.2 with $m_1 + m_2 = 1$ to $\xi_2 \psi_j$ with another cutoff function ξ_2 and using (9.5), we arrive at the estimate

$$\|\nabla^3 \psi_j\|_{L^2(G_0)} + R_j^2 \|\nabla \psi_j\|_{L^2(G_0)} \leq CR_j^2 \ ,$$

where G_0 can be taken to be, say, $(B_3 \backslash B_{1/3})^+ \backslash B^*$ with a slightly different B^*. This estimate enables us to apply Lemma 3.2 and obtain

$$\|\nabla^4 \widetilde{\psi}_j\|_{L^2} + R_j^2 \|\nabla^2 \widetilde{\psi}_j\|_{L^2} \leq CR_j^2 \ ,$$

where $\widetilde{\psi}_j = \widetilde{\xi}\psi_j$, with another cutoff function supported on $(B_3 \backslash B_{1/3})^+ \backslash B^*$. Combining this with (9.3), we find that

$$\rho(R_j) \leq CR_j^\alpha \ ,$$

and the proof of Lemma 7.1 is complete.   □

This also completes the proof of Theorem 1.1 in case $n = 0$.

To prove the theorem in case $n = 1$ we establish:

Lemma 9.1. *If φ satisfies the assumptions of Theorem 1.1 for $n = 1$, then φ_x satisfies the assumptions of Theorem 1.1 for $n = 0$.*

Proof. We need to prove that

(9.6) $$|\varphi_x(x, y)| \leq Cr^{\frac{1}{2}-\varepsilon} \quad \text{if } r \to \infty \ ,$$

(9.7) $$|\varphi_x(x, y)| \leq Cr^{\frac{1}{2}+\varepsilon} \quad \text{if } r \to 0 \ .$$

We get (9.7) by applying elliptic estimates [1], [2] to $R^{-(\frac{1}{2}+\varepsilon)}\varphi(Rx, Ry)$ for R small. To prove (9.6) we consider the function

$$\psi(x, y) = \frac{1}{R^{\frac{3}{2}-\varepsilon}} \ \varphi(Rx, Ry) \quad \text{for } R \text{ large .}$$

It satisfies

$$\omega R^2 \Delta \psi = \Delta^2 \psi \ , \quad |\psi| \leq C \quad \text{in } \mathbb{R}_+^2 \ ,$$

but since R is large we cannot apply standard elliptic estimates. Instead, as before, we shall work with

$$\widetilde{\psi} = \xi \psi \ , \quad \widehat{\psi} = \widehat{\xi}\psi \ ,$$

where ξ, $\widehat{\xi}$ are cutoff functions indicated in Figures 8.1 and 8.2, respectively. Using the representation of $\widehat{\psi}$ by Green's function (as in (8.12)), we obtain (cf. the proof of Lemma 7.1)

$$(9.8) \qquad\qquad |D^l\widehat{\psi}| \leq C \quad \text{in } T \quad (|l| \leq 4) \ .$$

Next we estimate derivatives of $\widetilde{\psi}$ by using Lemma 3.2 (cf. the proof of Lemma 7.1) and obtain

$$\|\nabla^2\widetilde{\psi}_x\|_{L^2} \leq C \ .$$

Since $|\widetilde{\psi}| \leq C$, we can use Sobolev's inequality to conclude that

$$|\widetilde{\psi}_x| \leq C \ .$$

Combining this with (9.8), we deduce that $|\psi_x| \leq C$ in some neighborhood of $\{x^2 + y^2 = 1,\ y \geq 0\}$, and this yields the assertion (9.6). $\qquad\qquad\square$

From Lemma 9.1 and Theorem 1.1 for $n = 0$ it follows that $\varphi_x \equiv 0$, from which we easily deduce that also $\varphi \equiv 0$. Thus Theorem 1.1 holds for $n = 1$. By repeating the above process we can establish Theorem 1.1 for all positive integers n.

<center>PART II. THE CASE $\omega = 0$</center>

<center>10. THE MAIN RESULTS</center>

In this part we prove two theorems.

Theorem 10.1. *Let φ be a function in $H^2(\mathbb{R}^2_+ \cap B_R)$ for any $R > 0$, which is a solution of*

$$(10.1) \qquad\qquad \Delta^2\varphi = 0 \quad in\ \mathbb{R}^2_+ \,,$$

satisfying the boundary conditions (1.2), (1.3) and

$$(10.2) \qquad\qquad 3\frac{\partial^3\varphi}{\partial x^2\partial y} + \frac{\partial^3\varphi}{\partial y^3} = 0\ if\ y = 0,\ x < 0\,.$$

Suppose that, for some $\varepsilon > 0$,

$$(10.3) \qquad\qquad |\varphi(x,\,y)| \leq \begin{cases} Cr^\alpha & if\ r \to \infty\,, \\ Cr^\beta & if\ r \to 0\,, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ and

$$(10.4)$$
$$\alpha = n + \frac{1}{2} - \varepsilon\,, \quad \beta = n - \frac{1}{2} + \varepsilon\,, \quad n\ a\ nonnegative\ integer\,.$$

Then $\varphi \equiv 0$.

Theorem 10.2. *Let φ be a function in $H^2(\mathbb{R}^2_+ \cap B_R)$ for any $R > 0$, which is a solution of (10.1) satisfying the boundary conditions (1.2), (1.3) and*

$$(10.5) \qquad\qquad \mu f - \beta_0\frac{df}{dx} = \frac{\partial\varphi}{\partial x}\ if\ y = 0,\ x < 0,$$

$$(10.6) \qquad\qquad 3\frac{\partial^3\varphi}{\partial x^2\partial y} + \frac{\partial^3\varphi}{\partial y^3} + \sigma\frac{d^3 f}{dx^3} = 0\ if\ y = 0,\ x < 0\,,$$

where μ, β_0, σ are constants, $\beta_0 > 0$, $\sigma > 0$, and

$$(10.7) \qquad Re\,\mu \geq 0, \quad \frac{\mu}{\sigma \pm 2\beta_0 i} \notin (-\infty, 0)\,.$$

Suppose that, for some $\varepsilon > 0$,

$$(10.8) \qquad |\varphi(x,\,y)| \leq \begin{cases} Cr^\alpha & \text{if } r \to \infty\,, \\ Cr^\beta & \text{if } r \to 0\,, \end{cases}$$

and

$$(10.9) \qquad |f(x)| \leq C(1 + |x|^\alpha) \quad \text{if } -\infty < x < 0\,,$$

where
(10.10)

$$\alpha = n + \frac{1}{2} - \varepsilon\,, \quad \beta = n - \rho_0 + \varepsilon\,, \quad n \text{ a nonnegative integer}\,,$$

and

$$(10.11) \qquad \rho_0 = \frac{1}{2\pi i}\log\frac{\sigma + 2\beta_0 i}{\sigma - 2\beta_0 i}\,, \quad 0 < \rho_0 < \frac{1}{2}\,.$$

Then $\varphi \equiv 0$, $f \equiv 0$.

Note that if (10.8) or (10.9) holds for some $\varepsilon > 0$ then it holds also for any smaller $\varepsilon > 0$. Hence without loss of generality we shall assume that $0 < \varepsilon < \rho_0$; this will simplify several technical arguments in subsequent proofs.

Remark 10.1. For the linearized stationary coating problem in $\mathbb{R}_+^2$ the homogeneous system is given by (10.1), the boundary conditions (1.2), (1.3) and

$$(10.12) \qquad 3\frac{\partial^3\varphi}{\partial x^2 \partial y} + \frac{\partial^3\varphi}{\partial y^3} + \kappa\frac{\partial^3\varphi}{\partial x^3} = 0 \quad \text{at } y = 0,\ x < 0\,,$$

where $\kappa > 0$ (see [3]). If $\kappa = 0$ then the last condition reduces to (10.2). For $\kappa > 0$ Theorem 10.1 was proved by Kondrat′ev [6], Maz′ya and Plamenevskii [8], [9] for $\alpha = \beta$ and any α which avoids the sequence of eigenvalues

$$(10.13) \qquad \left\{\frac{2n+1}{2}\,,\ \frac{2n+1}{2} - \rho\,;\quad n = 1, 2, \dots\right\},$$

where ρ is given by

$$\rho = \frac{1}{2\pi i}\,\log\frac{1 + 2i\kappa}{1 - 2i\kappa}\,, \qquad 0 < \rho < \frac{1}{2}\,.$$

In case $\kappa = 0$ this sequence reduces to

$$(10.14) \qquad \left\{\frac{3}{2}\,,\ \frac{5}{2}\,,\ \dots\,,\ \frac{2n+1}{2}\,,\ \dots\right\}.$$

Our method of proof is entirely different from the method in [6], [8], [9]; it is based on the Wiener-Hopf approach.

Remark 10.2. We need to explain the meaning of the boundary conditions (10.5), (10.6). From (10.5) we can solve f in terms of φ. This allows us to formally rewrite (10.6) in the form

(10.15)

$$3\frac{\partial^3\varphi}{\partial x^2 \partial y} + \frac{\partial^3\varphi}{\partial x^3} + \frac{\sigma}{\beta_0}\,\frac{\partial^3\varphi}{\partial x^3} + c_1\frac{\partial^2\varphi}{\partial x^2} + c_2\frac{\partial\varphi}{\partial x} + c_3 f = 0\,,$$

where c_1, c_2, c_3, are constants. By elliptic regularity [1], [2] (for the boundary condition (1.3), (10.15)) it follows that φ is smooth (say $C^{2+\alpha}$) for $x < 0$, $y \geq 0$, and we can then bootstrap the regularity of f, and then also that of φ, etc. Thus the solution φ is C^∞ for $x < 0$, $y \geq 0$, and (10.5), (10.6) hold in the classical sense.

Remark 10.3. In Section 12 we give the proof of Theorem 10.1 in the special case $n = 0$; some auxiliary results are established in Section 11. One of the auxiliary results is the esimate

(10.16) $|D_j\varphi(x,\,y)| \leq Cr^{\alpha-j}$ if $r \to \infty$

for any $j > 0$. In the case of Theorem 10.2, the derivation of this estimate is quite difficult and requires arguments of the type used in proving Lemma 9.1. This in turn relies on a priori estimates for solutions of $\Delta^2\varphi = \mu$ in $\mathbb{R}^2_+$ satisfying "strange" boundary conditions, more complicated than those in Section 3. The derivation of these a priori estimates is given in Section 13. In Section 14 we establish auxiliary results needed for the proof of Theorem 10.2 (including the inequality (10.16)), and in Section 15 we give the proof of Theorem 10.2 for $n = 0$. The proofs of Theorems 10.1, 10.2 for $n \geq 1$ follow from the special case $n = 0$ by applying the results for $n = 0$ to x-derivatives of φ, as in the case of Theorem 1.1. Note that if $n = 0$ then

(10.17) $0 < \alpha < 1 \,,\quad -1 < \beta < 0 \,.$

These inequalities will be implicitly used throughout the proofs of Theorems 10.1, 10.2 for the case $n = 0$.

Outline of the proofs of Theorems 10.1, 10.2. We define the Fourier transform $\widehat{\varphi}(k, y)$ of $\varphi(x, y)$ in such a way that it is a well-defined complex-valued function for all $k \neq 0$, and it coincides (for $k \neq 0$) with the Fourier transform of φ in the distribution sense. We use the boundary conditions to deduce that the function

(10.18) $\Phi(k) = \begin{cases} (k_+)^{\frac{1}{2}}\widehat{\varphi}_y(k,0) & \text{for} \quad Im\ k > 0 \,, \\[2mm] \dfrac{|k|}{(k_-)^{\frac{1}{2}}}\widehat{\varphi}_y(k,0) & \text{for} \quad Im\ k < 0 \end{cases}$

is analytic and can be extended analytically to $Im\ k = 0$, $k \neq 0$. Finally we analyze its behavior at $k = 0$ and $k = \infty$ and deduce that it is regular at these points and vanishes at 0. Hence $\Phi(k) \equiv 0$, and so $\widehat{\varphi}_y(k,0) \equiv 0$ if $k \neq 0$. Formula (10.18) is an example of the Wiener-Hopf decomposition. A similar decomposition can also be obtained for $\widehat{\varphi}(k,\,0)$. However, in the proof of Theorem 10.2 the Wiener-Hopf decomposition for $\widehat{\varphi}$ takes a very complicated form. In any case, this second decomposition enables us to show that $\widehat{\varphi}(k,0) \equiv 0$ if $k \neq 0$. It follows that $\widehat{\varphi}(k,y) \equiv 0$ if $y > 0, k \neq 0$, and thus the Fourier transform of φ, in the sense of distributions, must be a linear combination of derivatives of Dirac's measure $\delta(k)$. From this we easily deduce that $\varphi \equiv 0$.

11. Auxiliary results needed for proving Theorem 10.1

Lemma 11.1. *Let φ be as in Theorem 10.1, but assume that* (*10.3*) *holds for some unrestricted real numbers α, β. Then for any positive integer j_0 there is a constant C such that, for all $(x, y) \in \mathbb{R}^2_+$,*

(11.1) $|D^j\varphi(x,y)| \leq \begin{cases} Cr^{\alpha-j} & \text{if } r > 1 \,, \\[2mm] Cr^{\beta-j} & \text{if } r < 1 \,, \end{cases}$

for $0 \leq j \leq j_0$.

Proof. Consider the function

$$\varphi_R(x,y) = \frac{1}{R^\alpha} \; \varphi(Rx, Ry)$$

in $(B_2 \backslash B_{1/2}) \cap \mathbb{R}_+^2$, for any $R > 1$. By elliptic estimates [1],

$$|D^j \varphi_R(x,y)| \leq C \quad \text{in } (B_{3/2} \backslash B_{3/4}) \cap \mathbb{R}_+^2$$

for $0 \leq j \leq j_0$, and taking (x,y) with $x^2 + y^2 = 1$ we get (11.1) for r near ∞. The proof of (11.1) for r near 0 is obtained by working with $\widetilde{\varphi}_R(x,y) = R^{-\beta} \varphi(Rx, Ry)$, $R < 1$. $\qquad\square$

In the rest of this section we assume that φ is as in Theorem 10.1, except that (10.3) holds with some α, β satisfying (10.17).

Let $\xi(x) \in C^\infty (-\infty, \infty)$, $\xi(x) = 1$ if $|x| < \frac{1}{2}$, $\xi(x) = 0$ if $|x| > 1$. We define the Fourier transform $\widehat{\varphi}(k,y)$ of $\varphi(x,y)$ for $k \neq 0$ by the following formula:

(11.2)

$$\begin{aligned}
\widehat{\varphi}(k,y) = \quad &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \; \xi(x)\varphi(x,y)dx \\
&+ \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \; [(1 - \xi(x))\varphi(x,y)] \, dx \\
\equiv \quad &\widehat{\varphi}_1(k,y) + \widehat{\varphi}_2(k,y) \; .
\end{aligned}$$

If $\varphi(x,y)$ decays at $x = \pm\infty$, then $\widehat{\varphi}(k,y)$ coincides with the usual Fourier transform.

Using (11.1) we find that the integral defining $\widehat{\varphi}_2(k,y)$ is absolutely convergent and

$$|\widehat{\varphi}_2(k,y)| \leq \frac{C}{|k|^3} \; .$$

Since $|\widehat{\varphi}_1(k,y)| \leq C$, we conclude that

(11.3) $$|\widehat{\varphi}(k,y)| \leq C + \frac{C}{|k|^3} \quad \text{for all } y > 0 \; .$$

Similarly we can derive the same bound for $D_y\widehat{\varphi}(k,y)$. Furthermore,

(11.4) $\widehat{\varphi}(k,y)$ and $\widehat{\varphi}_y(k,y)$ are continuous in (k,y) for all $k \neq 0$, $y \geq 0$.

Lemma 11.2. *The following estimates hold:*

(11.5) $$|\widehat{\varphi}(k,0)| \leq \frac{C}{|k|^{\alpha+1}} \; ,$$

(11.6) $$|\widehat{\varphi}_y(k,0)| \leq \frac{C}{|k|^\alpha} \quad \text{for } k \text{ near } 0, \, k \neq 0 \; .$$

Proof. Set

$$h(x) = \frac{\partial^2}{\partial x^2} \; [(1 - \xi(x)) \; \varphi(x,0)] \; .$$

Then, for any $0 \leq \delta \leq \alpha$,

$$|\sqrt{2\pi}(ik)^2 \widehat{\varphi}_2(k,0)| = |\int_{-\infty}^{\infty} h(x)(e^{-ikx} - 1)dx|$$

$$\leq C \int_{\frac{1}{2}<|x|<\frac{1}{|k|}} |h(x)| \, |kx|^{1-\delta} dx + \int_{|x|>\frac{1}{|k|}} 2|h(x)|dx$$

$$\leq C \int_{\frac{1}{2}<|x|<\frac{1}{|k|}} |k|^{1-\delta} |x|^{1-\delta} |x|^{\alpha-2} \, dx + C \int_{|x|>\frac{1}{|k|}} |x|^{\alpha-2} dx$$

$$\leq C \left(\frac{1}{k}\right)^{\alpha-1} ,$$

by using Lemma 11.1 and taking $\delta = 0$ in the last inequality. Hence

$$|\widehat{\varphi}_2(k,0)| \leq \frac{C}{|k|^{\alpha+1}}$$

and, since $|\widehat{\varphi}_1(k,0)| \leq C$, (11.5) follows. The proof of (11.6) is similar. $\square$

Lemma 11.3. *The following estimates hold:*

(11.7)
$$|\widehat{\varphi}(k,0)| \leq \frac{C}{|k|^{\beta+1}} ,$$

(11.8)
$$|\widehat{\varphi}_y(k,0)| \leq \frac{C}{|k|^{\beta}} \quad for \ |k| \ near \ \infty .$$

Proof. Set

$$h(x) = \xi(x)\varphi(x, 0) .$$

By integration by parts,

$$\sqrt{2\pi} \, \widehat{\varphi}_1(k,0) = \int_{-\infty}^{\infty} h(x)e^{-ikx} = \frac{1}{2} \int_{-\infty}^{\infty} h(x) \left[e^{-ikx} - e^{-ik(x-\pi/k)}\right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[h(x) - h(x + \frac{\pi}{k})\right] e^{-ikx} .$$

Using Lemma 11.1 we get,

$$|\widehat{\varphi}_1(k,0)| \leq C \left\{ \int_{|x|<\frac{1}{|k|}} |x|^{\beta} dx + \int_{\frac{1}{|k|}<|x|<2} \left|\frac{\pi}{k}\right| |x|^{\beta-1} dx \right\} \leq \frac{C}{|k|^{\beta+1}} .$$

Since $|\widehat{\varphi}_2(k,0)| \leq C/|k|^3$, (11.7) follows. The proof of (11.8) is similar. $\square$

Lemma 11.4. *The function $\widehat{\varphi}$ satisfies*

(11.9)
$$\left[D_y^2 + (ik)^2\right]^2 \widehat{\varphi}(k,y) = 0 \quad if \ k \neq 0 .$$

Formally we expect the Fourier transform to satisfy (11.9). Since however $\widehat{\varphi}$ is not the standard Fourier transform, (11.9) requires a proof.

Proof. We have

$$\sqrt{2\pi} \, D_y^4 \widehat{\varphi}_2 = \frac{1}{(ik)^3} \, D_y^4 \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \, ((1-\xi)\varphi)dx$$

$$= \int_{-\infty}^{\infty} e^{-ikx} D_y^4((1-\xi)\varphi)dx$$

by integration by parts (using (11.1)). Hence

$$\sqrt{2\pi}\, D_y^4 \widehat{\varphi} = \int_{-\infty}^{\infty} e^{-ikx} D_y^4 \left[(1-\xi)\varphi + \xi\varphi\right] dx = \int_{-\infty}^{\infty} e^{-ikx} D_y^4 \varphi(x,y) dx \ .$$

Again, by integration by parts,

$$2\sqrt{2\pi}\,(ik)^2 D_y^2 \widehat{\varphi}_2 = \int_{-\infty}^{\infty} 2e^{-ikx} D_y^2 \frac{\partial^2}{\partial x^2} \left((1-\xi)\varphi\right) dx$$

so that

$$2\sqrt{2\pi}\,(ik)^2 D_y^2 \widehat{\varphi} = \int_{-\infty}^{\infty} 2e^{-ikx} \frac{\partial^4}{\partial x^2 \partial y^2} \left[(1-\xi)\varphi + \xi\varphi\right] dx$$

$$= \int_{-\infty}^{\infty} 2e^{-ikx} \frac{\partial^4 \varphi(x,y)}{\partial x^2 \partial y^2} dx \ .$$

Finally, by integration by parts,

$$\sqrt{2\pi}\,(ik)^4 \widehat{\varphi}_2 = \int_{-\infty}^{\infty} (ik)e^{-ikx} \frac{\partial^3}{\partial x^3} \left((1-\xi)\varphi\right) dx = \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^4}{\partial x^4} \left((1-\xi))\varphi\right) dx \ ,$$

so that

$$\sqrt{2\pi}\,(ik)^4 \widehat{\varphi} = \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^4 \varphi(x,y)}{\partial x^4} dx \ .$$

Combining these results and recalling that $\triangle^2 \varphi = 0$, we get (11.9). $\qquad\square$

We may consider $\varphi(x,y)$ for each $y > 0$, as a tempered distribution. Its Fourier transform $F_y(\varphi)$ is then defined, as a distribution, by the formula

$$\langle F_y(\varphi), \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ \varphi(x,y) \int_{-\infty}^{\infty} \psi(k)e^{-ikx} dk$$

for any test function $\psi(k)$. The connection between $F_y(\varphi)$ and $\widehat{\varphi}$, as defined in (11.2), is given by the following lemma:

Lemma 11.5. *For any $k \neq 0$*

(11.10) $$F_y(\varphi) = \widehat{\varphi}(k,y) \ .$$

Proof. For simplicity we drop the parameter y. The assertion (11.10) means that for any test function $\psi(k)$ with compact support in $\mathbb{R}\backslash\{0\}$,

(11.11) $$\langle \widehat{\varphi}, \psi \rangle = \langle F(\varphi), \psi \rangle \ ,$$

where, by definition,

$$\langle \widehat{\varphi}, \psi \rangle = \int_{-\infty}^{\infty} dk\ \widehat{\varphi}(k)\psi(k) \ .$$

Now, by (11.2)

$$\sqrt{2\pi}\,\langle \widehat{\varphi}, \psi \rangle = \int_{-\infty}^{\infty} dx\ \varphi(x)\xi(x) \int_{-\infty}^{\infty} \psi(k)e^{-ikx} dk$$

$$+ \int_{-\infty}^{\infty} dx\ \varphi(x)(1-\xi(x)) \int_{-\infty}^{\infty} \psi(k) \frac{1}{(ik)^3} \frac{\partial^3}{\partial x^3} \left(e^{-ikx}\right) dx \ .$$

The inner integral in the last term is equal to

$$\int_{i\infty}^{\infty} \psi(k)e^{-ikx} \ ,$$

and therefore

$$\langle \widehat{\varphi}, \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; \varphi(x) \int_{-\infty}^{\infty} \psi(k) e^{-ikx} = \langle F(\varphi), \psi \rangle \; ,$$

and (11.10) follows.   □

12. Proof of Theorem 10.1 for $n = 0$

By Lemma 11.2, $\widehat{\varphi}(k, y)$ is a linear combination of

$$e^{|k|y}, \; e^{-|k|y}, \; ye^{|k|y}, \; ye^{-|k|y} \; .$$

But since, by (11.3), $\widehat{\varphi}(k, y)$ is bounded in y, as $y \to \infty$, we conclude that

$$\widehat{\varphi}(k, y) = A(k) e^{-|k|y} + B(k) e^{-|k|y} y \; ,$$

so that, for all $k \neq 0$,

$$(12.1) \qquad\qquad (D_y^2 + 2|k|D_y + k^2) \, \widehat{\varphi}(k, y) = 0 \qquad \text{for } y \geq 0 \; .$$

From the boundary conditions $\varphi(x, 0) = \varphi_y(x, 0) = 0$ for $x > 0$ and (11.1) it follows that

(12.2) $\widehat{\varphi}(k, 0)$ and $\widehat{\varphi}_y(k, 0)$ (defined by means of (11.2)) are analytic functions in $Im \; k > 0$, continuous in $Im \; k \geq 0, \; k \neq 0$.

We also observe that the proofs of Lemma 11.2 and (11.3) extend to $Im \; k \geq 0$; thus,

(12.3) the inequalities (11.5)–(11.8) hold for all complex k with $Im \; k \geq 0, \; k \neq 0$.

We next wish to obtain results similar to (12.2), (12.3) for $Im \; k \leq 0$, by using the boundary conditions (1.3), (10.2). Formally

$$((ik)^2 - D_y^2)\widehat{\varphi}(k, 0) \text{ and } 3(ik)^2\widehat{\varphi}_y(k, 0) + D_y^3\widehat{\varphi}(k, 0)$$

are analytic for $Im \; k < 0$. However this proof is not straightforward because of singularities of derivatives of φ at the origin.

To overcome this difficulty we shall work with integrals of $\widehat{\varphi}$. We begin by introducing

$$(12.4) \qquad\qquad G(x, y) = \int_0^x \varphi(\xi, y) d\xi \; ,$$

and its Fourier transform defined by

(12.5)

$$\widehat{G}(k, y) = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \xi(x) G(x, y) dx$$

$$+ \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \left[(1 - \xi(x)) G(x, y) \right] dx \; .$$

By Lemma 11.1, the second integral is absolutely convergent.

By integration by parts in the first integral on the right-hand side of (12.5) we find that

$$\sqrt{2\pi}\,(ik)\widehat{G}(k,y) = \int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\partial \xi}{\partial x}\,G + \xi\frac{\partial G}{\partial x}\right)$$
$$+ \frac{1}{(ik)^3}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^3}{\partial x^3}\left[\frac{\partial}{\partial x}\,(1-\xi)\cdot G + (1-\xi)\frac{\partial G}{\partial x}\right],$$

and since $G_x = \varphi$ we get

(12.6) $$(ik)\widehat{G}(k,y) = \widehat{\varphi}(k,y).$$

Similarly

(12.7) $$(ik)\widehat{G}_y(k,y) = \hat{\varphi}_y(k,y).$$

Applying $(ik)^2 - D_y^2$ to $\widehat{G}(k,y)$ at $y = 0$ and using integration by parts, we get

(12.8)
$$\sqrt{2\pi}\,((ik)^2 - D_y^2)\widehat{G}(k,0) = \int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(\xi G)(x,0)dx$$
$$+ \frac{1}{ik}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^3}{\partial x^3}\,((1-\xi)G)(x,0)dx$$
$$+ \frac{1}{ik}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y^2}((1-\xi)G)\right](x,0)dx$$
$$= \int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(\xi G)(x,0)dx$$
$$+ \frac{1}{ik}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)((1-\xi)G)\right]dx.$$

Observe that

(12.9) $$\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)G\bigg|_{y=0} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\varphi\bigg|_{y=0} = 0$$

if $x < 0$. Hence, we can rewrite the first integral on the right-hand side of (12.8) in the form

$$\int_{0}^{\infty} e^{-ikx}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(\xi G)(x,0)$$
$$+ \frac{1}{ik}\int_{-\infty}^{0} e^{-ikx}\frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(\xi G)\right](x,0) + \frac{c_0}{ik},$$

where

$$c_0 = \text{const.} = \left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^2}\right)G(x,0).$$

Adding this to the second integral on the right-hand side of (12.8) and using also (12.9), we see that

(12.10)

$$\sqrt{2\pi}\,((ik)^2 - D_y^2)\widehat{G}(k,0) = \frac{c_0}{ik} + \int_0^\infty e^{-ikx}\left\{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)(\xi G)(x,0)\right.$$
$$\left. + \frac{1}{ik}\frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)((1-\xi)G)\right](x,0)\right\}\ .$$

It follows that

$$((ik)^2 - D_y^2)\widehat{G}(k,0) \text{ is analytic in } \{Im\ k < 0\}\ .$$

Recalling (12.6), (12.7), we conclude that

$$(ik)^2\widehat{\varphi}(k,0) - D_y^2\widehat{\varphi}(k,0)$$

is analytic in $\{Im\ k < 0\}$. Comparing with (12.1) at $y = 0$, it follows that

(12.11) $|k|\widehat{\varphi}_y(k,0)$ is analytic in $\{Im\ k < 0\}$.

The proof also shows that

(12.12) $|k|\widehat{\varphi}_y(k,0)$ is continuous in $\{Im\ k \le 0, k \ne 0\}$

and, since the right-hand side of (12.10) is bounded, as $k \to \infty$, by comparing once again with (12.1) we deduce that

(12.13) $|\widehat{\varphi}_y(k,0)| \le C$ if $Im\ k \le 0,\ |k| \to \infty$.

Next we introduce the double integral

(12.14) $$H(x,y) = \int_0^x dx \int_0^\lambda \varphi(\lambda, y)d\lambda = \int_0^y G(x,y)dx$$

and its Fourier transform

(12.15)

$$\widehat{H}(k,y) = \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ikx}\xi(x)H(x)dx$$
$$+ \frac{1}{\sqrt{2\pi}}\frac{1}{(ik)^5}\int_{-\infty}^\infty e^{-ikx}\frac{\partial^5}{\partial x^5}\,[(1-\xi(x))H(x,y)]dx\ .$$

By the method of proof of (12.6) and (12.7) we get

(12.16)
$$(ik)^2\widehat{H}(k,y) = \widehat{\varphi}(k,y)\ ,$$
$$(ik)^2\widehat{H}_y(k,y) = \widehat{\varphi}_y(k,y)\ .$$

By the boundary condition (10.2),

(12.17) $$\frac{\partial^2}{\partial x^2}\,(3H_{xxy} + H_{yyy}) = 3\varphi_{xxy} + \varphi_{yyy} = 0 \quad \text{if } y = 0,\ x < 0\ ,$$

so that

(12.18) $$\frac{\partial}{\partial x^2}\,(3H_{xxy} + H_{yyy}) = \text{const.} = c_1 \quad \text{if } y = 0,\ x < 0\ .$$

Using integration by parts we proceed to compute at $y = 0$, $x < 0$:

(12.19)
$$\sqrt{2\pi}\,[3(ik)^2\widehat{H}_y + \widehat{H}_{yyy}] = \int_{-\infty}^{\infty} e^{-ikx}\left[3\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3}\right](\xi H)$$
$$+ \frac{1}{(ik)^2}\int_{-\infty}^{\infty} e^{-ikx}3\frac{\partial^6}{\partial x^5 \partial y}\ [(1-\xi)H]$$
$$+ \frac{1}{ik}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^5}{\partial x^2 \partial y^3}\ [(i-\xi)H]$$
$$= \int_{-\infty}^{\infty} e^{-ikx}\left[\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3}\right](\xi H)$$
$$+ \frac{1}{ik}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^2}{\partial x^2}\left[\left(3\frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3}\right)((1-\xi)H)\right]\ .$$

As before, we perform two integrations by parts in the part $\int_{-\infty}^{0}$ of the first integral on the right-hand side of (12.19) and combine it with the part $\int_{-\infty}^{0}$ of the second integral. Using (12.17), (12.18), the sum of these integrals becomes

$$\frac{c_1}{(ik)^2}\ .$$

The remaining portions $\int_{0}^{\infty}$ of the first two integrals on the right-hand side of (12.19) are bounded analytic functions in $\{Im\ k < 0\}$. Thus we conclude that

$$3(ik)^2\widehat{H}_y + \widehat{H}_{yyy}\quad \text{is analytic in}\ \{Im\ k < 0\}\ .$$

Using (12.16) we see that

$$3(ik)^2 D_y\widehat{\varphi} + D_y^3\widehat{\varphi}\quad \text{is analytic in}\ \{Im\ k\} < 0\ .$$

If we divide this operator (applied to $\widehat{\varphi}$) by $D_y^2 + 2|k|D_y + k^2$ and use (12.1), the remainder satisfies the same analyticity property. Thus

(12.20) $|k|\widehat{\varphi}(k,0)$ is analytic in $\{Im\ k < 0\}$;
it is clearly also continuous if $Im\ k \le 0,\ k \ne 0$.

The above proof also shows that

(12.21) $|\widehat{\varphi}(k,0)| \le \dfrac{C}{|k|^3}\quad \text{if}\ Im\ k \le 0,\ |k| \to \infty\ .$

We finally note that, by the proof of Lemma 11.2,

(12.22)   inequalities (11.5), (11.6) hold for $Im\ k \le 0$, and $|k|$ small.

We are now in position to apply the Wiener-Hopf technique, using (12.2), (12.3), (12.22), (12.11), (12.12), (12.13), and (12.21).

We cut the complex plane by the negative real axis and define two branches of $\sqrt{k}$ in the complex plane:

$$(k)_{+}^{1/2} = |k|^{1/2}e^{i\theta/2}$$

and

$$(k)_{-}^{1/2} = |k|^{1/2}e^{-i\theta/2}\ ,$$

where $k = |k|e^{i\theta},\ -\pi < \theta < \pi$.

Consider the function

$$
(12.23) \qquad \Phi(k) = \begin{cases} (k)_+^{1/2} \, \widehat{\varphi}_y(k,0) & \text{if } Im \ k \geq 0 \ , \\[2ex] \dfrac{|k|}{(k)_-^{1/2}} \, \widehat{\varphi}_y(k,0) & \text{if } Im \ k \leq 0 \ . \end{cases}
$$

By what we have proved above, $\Phi(k)$ is holomorphic in $\{Im \ k \neq 0\}$, continuous for all $k \neq 0$, and (since $\alpha = \frac{1}{2} - \varepsilon, \ \beta = -\frac{1}{2} + \varepsilon$)

$$
|\Phi(k)| \leq \frac{C|k|^{1/2}}{|k|^\alpha} = C|k|^\varepsilon \qquad \text{for } k \text{ near } 0 \ ,
$$

$$
|\Phi(k)| \leq \frac{C|k|^{1/2}}{|k|^\beta} = C|k|^{1-\varepsilon} \qquad \text{if } Im \ k \geq 0 \ , \ |k| \text{ near } \infty \ .
$$

By standard results for holomorphic functions, it follows that $\Phi(k)$ is holomorphic at $k = 0$ and at $k = \infty$, and $\Phi(0) \equiv 0$. Hence, by Liouville's theorem, $\Phi(k) \equiv 0$, and thus

$$
(12.24) \qquad \hat{\varphi}_y(k,0) = 0 \ .
$$

Similarly, by working with

$$
(12.25) \qquad Q(k) = \begin{cases} (k)_+^{1/2} k \widehat{\varphi}(k,0) & \text{if } Im \ k \geq 0 \ , \\[2ex] (k)_-^{1/2} k \widehat{\varphi}(k,0) & \text{if } Im \ k \leq 0 \ , \end{cases}
$$

we deduce that $Q(k) \equiv 0$ and therefore

$$
(12.26) \qquad \widehat{\varphi}(k,0) \equiv 0 \ .
$$

In view of (12.24), (12.25), the only solution to (12.1) is $\widehat{\varphi}(k,y) \equiv 0$.

By Lemma 11.5, the distribution $F_y(\varphi)$ coincides with $\widehat{\varphi}(k,y)$ for all $k \neq 0$. It follows that $F_y(\varphi) = 0$ if $k \neq 0$, and so $F_y(\varphi)$ must be a finite linear combination of derivatives of Dirac's function. Hence

$$
\varphi(x,y) = \sum_{j=0}^{N} a_j(y) x^j \ .
$$

Since $\varphi = O(r^{\frac{1}{2} - \varepsilon})$ if $r \to \infty$, $N = 0$. Using the equation $\triangle^2 \varphi = 0$, we deduce that a_0 is a linear function in y, and the boundary conditions in (1.2) then show that $\varphi \equiv 0$.

13. A PRIORI ESTIMATES FOR ANOTHER ELLIPTIC SYSTEM WITH "STRANGE" BOUNDARY CONDITIONS

In this section we consider the elliptic equation

$$
(13.1) \qquad \epsilon z \Delta W - \Delta^2 W = \mu(x,y) \quad \text{in} \quad \mathbb{R}_+^2 \ ,
$$

where $\epsilon, \ z$ are parameters satisfying

$$
(13.2) \qquad 0 < \epsilon \leq 1, \qquad \text{Re } z \geq 0,
$$

subject to the boundary conditions at $y = 0$:

$$(13.3) \qquad zF + \beta_0 \frac{\partial F}{\partial x} - \frac{\partial W}{\partial x} = a(x),$$

$$(13.4) \qquad \frac{\partial^3 W}{\partial x^2 \partial y} + \frac{\partial^3 W}{\partial y^3} + \sigma \frac{\partial^3 F}{\partial x^3} - \epsilon z \frac{\partial W}{\partial y} = b(x),$$

$$(13.5) \qquad \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} = c(x).$$

We introduce the "norms"

(13.6)

$$\|(a, b, c, \mu)\|_0 = \left\{ \int_{-\infty}^{\infty} dx \int_0^{\infty} dy |\mu(x, y)|^2 \right.$$

$$\left. + \int_{-\infty}^0 dx [|D_x^{5/2} a|^2 + |D_x^{1/2} b|^2 + |\epsilon z|^{1/2} |b|^2 + |D_x^{3/2} c|^2 + |\epsilon z|^{3/2} |c|^2] \right\}^{1/2},$$

(13.7)

$$\||(W, F)\||_0 = \left\{ \int_{-\infty}^{\infty} dx \int_0^{\infty} dy [|\nabla^4 W|^2 + |\epsilon z|^2 |\nabla^2 W|^2] \right.$$

$$+ \int_{-\infty}^0 dx [|\epsilon z|^{1/2} |\nabla^3 W(x, 0)|^2 + |\epsilon z|^2 |\nabla^{3/2} W(x, 0)|^2$$

$$+ |\epsilon z|^{3/2} |\nabla^2 W(x, 0)|^2 + |\epsilon z|^{2+1/2} [\partial_y W(x, 0)|^2]$$

$$\left. + \int_{-\infty}^0 dx [|z| |D_x^3 F|^2 + |z|^2 |D_x^{5/2} F|^2] \right\}^{1/2},$$

(13.8)

$$\|W\|_0 = \left\{ \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \left[\left| \frac{\partial^4 W}{\partial x^4} \right|^2 + \left| \frac{\partial^4 W}{\partial y^4} \right|^2 \right. \right.$$

$$\left. \left. + |\epsilon z|^2 \left| \frac{\partial^2 W}{\partial x^2} \right|^2 + |\epsilon z|^2 \left| \frac{\partial^2 W}{\partial y^2} \right|^2 \right] \right\}^{1/2}.$$

Our aim is to establish a priori estimates for W, F in terms of the data a, b, c, μ. For our future applications it suffices to consider the case where

$$(13.9) \qquad a, b, c \quad \text{and} \quad F \quad \text{vanish if} \quad |x| > R_0$$

for some fixed number R_0, and

$$(13.10) \qquad \mu(x, y) = 0, \quad W(x, y) = 0 \quad \text{if} \quad x^2 + y^2 > R_0^2 .$$

Our main result is:

Lemma 13.1. *There exists a constant C_0 depending only on R_0 such that, for any solution (W, F) of (13.1)–(13.5) satisfying (13.9), (13.10),*

$$(13.11) \qquad \||(W, F)\||_0 \le C_0 \|(a, b, c, \mu)\|_0 .$$

It is assumed here that the norm $\|(a, b, c, \mu)\|_0$ is finite.

Lemma 13.1 is of the same type as Lemma 3.1. The two lemmas are different from one another since, although the assumptions of Lemma 3.1 may be viewed as contained in the more general assuptions of Lemma 13.1, the assertion in Lemma 3.1 is somewhat stronger than the assertion we obtain by applying Lemma 13.1. The proof of Lemma 13.1 follows by the same method used to prove Lemma 3.1, but the technical details are more complicated. We also note that Lemma 13.1 can be extended to include estimates of $\partial_x^{-m_1} \partial_y^{-m_2} \nabla^j W$; this extension can be established by observations similar to those made in the proof of Lemma 3.2. Setting $\mu(x, y) = f(x, y)$, $z = \mu R$ (Re $\mu \geq 0$, $R > 0$) and taking $\varepsilon \to 0$, we obtain the following result:

Lemma 13.2. *Let $(\Phi(x, y), G(x))$ be a solution to*

$$(13.12) \qquad \qquad \Delta^2 \Phi = f \quad in \ \mathbb{R}_+^2$$

with the boundary conditions at $y = 0$:

$$(13.13) \qquad \qquad \mu R G + \beta_0 \frac{dG}{dx} - \frac{\partial \Phi}{\partial x} = a(x) \ ,$$

$$(13.14) \qquad \qquad \frac{\partial^3 \Phi}{\partial x^2 \partial y} + \frac{\partial^3 \Phi}{\partial y^3} + \sigma \frac{d^3 G}{dx^3} = b(x) \ ,$$

$$(13.15) \qquad \qquad \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} = c(x) \ ,$$

and assume that $\Phi(x, y) = 0$, $G(x) = 0$ if $x^2 + y^2 > R_0^2$. Then

$$(13.16)$$
$$\sum_{j_1 + j_2 = 4} \|\partial_x^{j_1 - m_1} \partial_y^{j_2 - m_2} \Phi\|_{L^2(\mathbb{R}_+^2)} + R^{1/2} \|\partial_x^{-m} D_x^{3/2} G\|_{L^2(\mathbb{R})} + \|\partial_x^{-m} D_x^{5/2} G\|_{L^2(\mathbb{R})}$$

$$\leq C \|f\|_{L^2(\mathbb{R}_+^2)} + C \left[\|\partial_x^{-m} D_x^{5/2} a\|_{L^2(\mathbb{R})} + \|\partial_x^{-m} D_x^{1/2} b\|_{L^2(\mathbb{R})} + \|\partial_x^{-m} D_x^{3/2} c\|_{L^2(\mathbb{R})} \right]$$

for any integers $m_1 \geq 0$, $m_2 \geq 0$, $m = m_1 + m_2 \leq 3$, where C is a constant depending only on R_0.

In this paper we shall only need to use Lemma 13.2; Lemma 13.1 will be used in our forthcoming paper [5], which deals with the linearized coating flow problem.

For the sake of clarity, before giving the complete proof of Lemma 13.1 we establish a part of its estimate:

Lemma 13.3. *Under the assumptions of Lemma 13.1,*

$$(13.17) \qquad \qquad \|W\|_0 \leq C \|(a, b, c, \mu)\|_0 \ ,$$

where C is a constant depending only on R_0.

Remark 13.1. For fixed ϵ, z the estimate (13.17) follows from general estimates for elliptic equations with complex coefficients [1], [2]. The novelty of the estimate (13.17) (and of (13.11)) is that it provides a very precise dependence on the parameters ϵ, z.

Proof of Lemma 13.3. We begin by extending μ into all of $\mathbb{R}^2$ by

$$(13.18) \qquad \qquad \mu(x, -y) = \mu(x, y)$$

and constructing a special solution $\tilde{W}$ of

(13.19) $$\epsilon z \Delta \tilde{W} - \Delta^2 \tilde{W} = \mu \quad \text{in} \quad \mathbb{R}^2.$$

In this section, the Fourier transform $\hat{f}(k_1, k_2)$ of a function $f(x, y)$ is defined by

$$\hat{f}(k_1, k_2) = \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-ik_1 x - ik_2 y} dx dy.$$

Taking the Fourier transform in (13.19), we find that

(13.20) $$\hat{\tilde{W}}(k_1, k_2) = -\frac{\hat{\mu}(k_1, k_2)}{[\epsilon z + (k_1^2 + k_2^2)](k_1^2 + k_2^2)}.$$

Later on we shall estimate $\tilde{W}$, but for now we just observe that, by (13.18), $\tilde{W}(x, -y) = \tilde{W}(x, y)$ and, consequently,

(13.21) $$\frac{\partial^m \tilde{W}}{\partial y^m}(x, 0) = 0 \quad \text{for} \quad m = 1, 3.$$

Set

$$W = \tilde{W} + \phi.$$

Then ϕ satisfies

(13.22) $$\epsilon z \Delta \phi - \Delta^2 \phi = 0 \quad \text{in} \quad \mathbb{R}_2^+$$

and, on $y = 0$,

(13.23) $$zF + \beta_0 \frac{\partial F}{\partial x} - \frac{\partial \phi}{\partial x} = a + \frac{\partial \tilde{W}}{\partial x} \equiv \tilde{a},$$

(13.24)
$$3\frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial^3 \phi}{\partial y^3} + \sigma \frac{\partial^3 F}{\partial x^3} - \epsilon z \frac{\partial \phi}{\partial y} = b - \frac{\partial^3 \tilde{W}}{\partial x^2 \partial y} - \frac{\partial^3 \tilde{W}}{\partial y^3} + \epsilon z \frac{\partial \tilde{W}}{\partial y} \equiv \tilde{b},$$

(13.25) $$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = c - \frac{\partial^2 \tilde{W}}{\partial x^2} + \frac{\partial^2 \tilde{W}}{\partial y^2} \equiv \tilde{c}.$$

Taking formally the Fourier transform in the x-variables, we get

(13.26) $$\epsilon z(-k^2 + D_y^2)\hat{\phi} = (-k^2 + D_y^2)^2 \hat{\phi}, \quad 0 < y < \infty,$$

and, at $y = 0$,

(13.27) $$z\hat{F} + i\beta_0 k\hat{F} - ik\hat{\phi} = \hat{\tilde{a}},$$

(13.28) $$-3k^2 \frac{\partial \hat{\phi}}{\partial y} + \frac{\partial^3 \hat{\phi}}{\partial y^3} - i\sigma k^3 \hat{F} - \epsilon z \frac{\partial \hat{\phi}}{\partial y} = \hat{\tilde{b}},$$

(13.29) $$-k^2 \hat{\phi} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = \hat{\tilde{c}};$$

here

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx.$$

Note that $\tilde{a}, \tilde{b}, \tilde{c}$ and their Fourier transforms $\hat{\tilde{a}}, \hat{\tilde{b}}, \hat{\tilde{c}}$ depend on the parameter z.

From the assumptions of the lemma and (13.20) it follows that the passage from (13.22)–(13.25) to (13.26)–(13.29) can be justified rigorously; furthermore, $\hat{\phi}$ and $\hat{F}, \hat{\tilde{a}}, \hat{\tilde{b}}, \hat{\tilde{c}}$ have all the boundedness properties that will be required in the sequel.

Rewriting (13.26) in the form

$$(-k^2 + D_y^2)(-k^2 - \epsilon z + D_y^2)\hat{\phi} = 0$$

we see, by virtue of the boundedness properties of $\hat{\phi}$ as $y \to \infty$, that $\hat{\phi}$ must have the form

(13.30) $$\hat{\phi} = A(k)e^{-|k|y} + B(k)e^{-\sqrt{k^2+\epsilon z}y}, \quad y > 0 ,$$

where the square root is chosen to have real part ≥ 0. Therefore

(13.31) $$(D_y + |k|)(D_y + \sqrt{k^2 + \epsilon z})\hat{\phi} = 0.$$

This equation allows us to simplify the boundary conditions by reducing $D_y^3\hat{\phi}$ and $D_y^2\hat{\phi}$ to first order differential operators:

If we divide D_y^3 by

$$(D_y + |k|)(D_y + \sqrt{k^2 + \epsilon z}) \equiv D_y^2 + (|k| + \sqrt{k^2 + \epsilon z})D_y + |k|\sqrt{k^2 + \epsilon z}$$

and use (13.31), we find that

$$D_y^3\hat{\phi} = (2k^2 + |k|\sqrt{k^2 + \epsilon z} + \epsilon z)D_y\hat{\phi} + [(|k| + \sqrt{k^2 + \epsilon z})|k|\sqrt{k^2 + \epsilon z}]\hat{\phi}.$$

Similarly

$$D_y^2\hat{\phi} = -(|k| - \sqrt{k^2 + \epsilon z})D_y\hat{\phi} - |k|\sqrt{k^2 + \epsilon z}\hat{\phi}.$$

Substituting this into (13.28), (13.29) we find (after some minor simplifications) that the system (13.27)–(13.29) becomes

(13.32)

$$\begin{pmatrix} -ik & 0 & z + i\beta_0 k \\ |k|(|k| + \sqrt{k^2 + \epsilon z})\sqrt{k^2 + \epsilon z} & |k|\sqrt{k^2 + \epsilon z} - k^2 & -i\sigma k^3 \\ |k|\sqrt{k^2 + \epsilon z} - k^2 & |k| + \sqrt{k^2 + \epsilon z} & 0 \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \hat{\phi}_y \\ \hat{F} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\tilde{a}} \\ \hat{\tilde{b}} \\ \hat{\tilde{c}} \end{pmatrix}$$

Solving for $\hat{F}, \hat{\phi}, \hat{\phi}_y$ we find, after some simple calculations, that

(13.33)

$$\hat{F} = \frac{1}{\Delta}\left\{ \hat{\tilde{a}}[(4k^2 + \epsilon z)|k|\sqrt{k^2 + \epsilon z} + k^2\epsilon z] \right.$$
$$\left. + \hat{\tilde{b}}[ik(|k| + \sqrt{k^2 + \epsilon z})] - \hat{\tilde{c}}\,[ik\sqrt{k^2 + \epsilon z} - k^2] \right\},$$

(13.34)

$$\hat{\phi} = \frac{1}{\Delta}\left\{ \hat{\tilde{a}}\,[i\sigma k^3(|k| + \sqrt{k^2 + \epsilon z}] \right.$$
$$\left. + \hat{\tilde{b}}\,[(z + i\beta_0 k)(|k| + \sqrt{k^2 + \epsilon z})] - \hat{\tilde{c}}\,[(z + i\beta_0 k)(|k|\sqrt{k^2 + \epsilon z} - k^2)] \right\},$$

(13.35)

$$\phi_y = \frac{1}{\Delta} \left\{ \hat{\hat{a}} \left[-i\sigma k^3 (|k|\sqrt{k^2 + \epsilon z} - k^2) \right] - \hat{\hat{b}} [(z + i\beta_0 k)(|k|\sqrt{k^2 + \epsilon z} - k^2)] \right.$$
$$\left. + \hat{\hat{c}} \left[\sigma k^4 + (z + i\beta_0 k)|k|(|k| + \sqrt{k^2 + \epsilon z})\sqrt{k^2 + \epsilon z} \right] \right\} ,$$

where $\Delta = \Delta(k; z)$ is the determinant of the coefficients, given by

(13.36)

$$\Delta(k; z) = |k|(z + i\beta_0 k)[(4k^2 + \epsilon z)\sqrt{k^2 + \epsilon z} + |k|\epsilon z] + \sigma k^4 (|k| + \sqrt{k^2 + \epsilon z}).$$

We need to analyze the zeros of $\Delta(k; z)$. To do that we introduce

(13.37)
$$\omega = \frac{\epsilon z}{k^2}, \quad \lambda = \frac{\sigma}{\beta_0} > 0, \quad Q = \frac{|k|}{\beta_0 \epsilon} \geq 0.$$

Then

(13.38)
$$\Delta(k; z) = \beta_0 |k|^5 \left\{ (Q\omega + i \operatorname{sgn}(k))[(4 + \omega/\sqrt{1 + \omega} + \omega] + \lambda(1 + \sqrt{1 + \omega}) \right\}$$

$$\equiv \beta_0 |k|^5 g(\omega, \operatorname{sgn}(k)) ,$$

where, as above, the square root $\sqrt{\zeta}$ is taken to have real part ≥ 0; this implies that

$$\sqrt{\zeta}^* = \sqrt{\zeta^*}$$

where "*" means the complex conjugate. It follows that

$$g(\omega; +1)^* = g(\omega^*; -1) ,$$

so that it suffices to analyze the zeros of $\Delta(k; z)$ in case $\operatorname{sgn}(k) = 1$, i.e., for the function

(13.39)
$$g(\omega; +1) = (Q\omega + i)[(4 + \omega)\sqrt{1 + \omega} + \omega] + \lambda(1 + \sqrt{1 + \omega}).$$

Setting

(13.40)
$$\theta = \sqrt{1 + \omega} \quad (\omega = \theta^2 - 1),$$

we have

(13.41)
$$g(\omega; +1) \equiv P_Q(\theta) = [Q(\theta^2 - 1) + i][\theta^3 + \theta^2 + 3\theta - 1] + \lambda(1 + \theta).$$

Thus we need to analyze the zeros of $P_Q(\theta)$ for fixed $\lambda > 0$ (λ is of order 1) and all $0 \leq Q < \infty$. Although θ here varies only in the region

$$\{\theta = x + iy, \quad x^2 - y^2 \geq 1\},$$

for technical reasons we shall need to consider the zeros of $P_Q(\theta)$ for all $\theta \in \mathbb{C}$.

Lemma 13.4. $P_Q(\theta)$ *does not vanish on the imaginary axis, for any* $Q > 0$.

Proof. Suppose $P_Q(\eta i) = 0$ for some $\eta \in \mathbb{R}$. Taking the real and imaginary parts of this equation, we get

$$Q(\eta^2 + 1)^2 + \eta(\eta^2 - 3) + \lambda = 0,$$
$$Q(\eta^2 + 1)\eta(\eta^2 - 3) - (\eta^2 + 1) + \lambda\eta = 0.$$

Dividing by Q, we obtain a linear system of equations for λ/Q and $1/Q$, and we can solve for λ/Q:

$$\frac{\lambda}{Q} = \frac{\begin{vmatrix} -(\eta^2+1)^2 & \eta(\eta^2-3) \\ -(\eta^2+1)\eta(\eta^2-3) & -(\eta^2+1) \end{vmatrix}}{\begin{vmatrix} 1 & \eta(\eta^2-3) \\ \eta & -(\eta^2+1) \end{vmatrix}} = -(\eta^2+1)\frac{(\eta^2+1)^2+\eta^2(\eta^2-3)^2}{(\eta^2-1)^2} < 0,$$

a contradiction, since $\lambda/Q > 0$. $\qquad\square$

Lemma 13.5. $P_Q(\theta)$ *does not vanish if* $\theta = \sqrt{1+\Gamma i}$, $\Gamma \in \mathbb{R}$, *for any* $Q > 0$.

Proof. Write $\theta = x + iy$. Then $x^2 - y^2 = 1$, so that

$$\theta = \cosh A + i\sinh A \quad \text{for some} \quad A \in \mathbb{R}.$$

If $P_Q(\theta) = 0$ then

$$-(4\sinh A + \Gamma\cosh A + \Gamma)B + \lambda(\cosh A + 1) = 0,$$
$$(4\cosh A - \Gamma\sinh A)B + \lambda\sinh A = 0 ,$$

where $B = Q\Gamma + 1 \in \mathbb{R}$. Since this linear homogeneous system has a nontrivial solution (B, λ), the determinant must vanish, and this yields, after simple computation,

$$0 = 4(\cosh^2 A + \sinh^2 A) + 4\cosh\ A$$

which is a contradiction (since the right-hand side is positive). $\qquad\square$

Lemma 13.6. *There are precisely two zeros of* $P_Q(\theta)$ *in* $\{Re\ \theta > 0\}$, θ_1 *and* θ_2; *both lie outside the set* $\{x^2 - y^2 \geq 1\}$; *further, as* $Q \to \infty$,

$$\theta_1 \to \tilde{\theta} \in (0,1)$$

and

$$(13.42) \qquad\qquad \theta_2 \approx 1 - \frac{1}{2Q}\left(i + \frac{\lambda}{2}\right).$$

Proof. As $Q \to \infty$ the zeros of $P_Q(\theta)$ approach solutions of

$$(\theta^2 - 1)(\theta^3 + \theta^2 + 3\theta - 1) = 0.$$

Thus θ converges either to ± 1 or to one of the three zeros of

$$h(\theta) = \theta^3 + \theta^2 + 3\theta - 1.$$

This polynomial is the same as the polynomial $h(\theta)$ in (3.11). As proved in Section 3, h has only one zero $\tilde{\theta}$ in the half-plane Re $\theta > 0$, and $0 < \tilde{\theta} < 1$.

So far we have proved that for $Q \to \infty$ there are precisely two zeros of $P_Q(\theta)$ in $\{Re\ \theta \geq 0\}$, θ_1 and θ_2, and

$$\theta_1 \to \tilde{\theta} \quad \text{and} \quad \theta_2 \to 1.$$

Observe that, for $\theta_2 \approx 1$, the equation $P_Q(\theta_2) = 0$ can be approximated by

$$(2Q(\theta_2 - 1) + i)4 + 2\lambda = 0,$$

which gives (13.42). The zero θ_2 lies outside the region $\{x^2 - y^2 \geq 1\}$. Since by Lemmas 13.4 and 13.5 the zeros of $P_Q(\theta)$ do not cross the imaginary axis as well as the hyperbola $\{x^2 - y^2 = 1\}$, the proof of Lemma 13.6 is complete. $\qquad\square$

Lemma 13.7. *As* $Q \to 0$, $\theta_1 \to \hat{\theta}$, *where* $\hat{\theta}$ *lies outside the set* $\{x^2 - y^2 \geq 1\}$, $|\theta_2| \to \infty$, *and, furthermore,*

$$(13.43) \qquad\qquad Re\{\theta_2^2 - 1\} = -4\lambda Q + O(Q^{3/2}).$$

Proof. As $Q \to 0$, three zeros of $P_Q(\theta)$ will converge to the zeros of

$$g(\theta) \equiv i(\theta^3 + \theta^2 + 3\theta - 1) + \lambda(1 + \theta)$$

and two zeros will escape to infinity. These latter zeros satisfy

$$(Q(\theta^2 - 1) + i)\theta^3 + \lambda\theta \approx 0,$$

from which we deduce that

$$(13.44) \qquad\qquad \theta \approx \frac{1}{Q^{1/2}}(-i)^{1/2} \qquad \text{for} \quad \theta = \theta_2.$$

To derive (13.43) we use the relation $P_Q(\theta) = 0$:

$$Q(\theta^2 - 1) + i = -\frac{\lambda(1 + \theta)}{\theta^3 + \theta^2 + 3\theta - 1} = -\frac{\lambda}{\theta^2(1 + \theta)}[1 - \frac{3}{\theta^2} + O(\frac{1}{\theta^3})],$$

or

$$(13.45) \qquad\qquad \theta^2 - 1 = -\frac{i}{Q} - \frac{\lambda}{Q\theta^2} + \frac{3\lambda}{Q\theta^4} + O(\frac{1}{Q\theta^5}).$$

Substituting θ from (13.40) into the right-hand side of (13.45), we get

$$\theta^2 - 1 = -\frac{i}{Q} + \frac{\lambda}{i} + O(Q),$$

and, multiplying by Q, we find that

$$Q\theta^2 = (-i)[1 + (\lambda + i)Q + O(Q^2)].$$

Substituting this expression for $Q\theta^2$ into the right-hand side of (13.45), we arrive at the relation

$$\theta^2 - 1 = -\frac{i}{Q} - \lambda i + \lambda^2 iQ - 4\lambda Q + O(Q^{3/2}),$$

from which (13.43) follows.

It remains to show that $\hat{\theta}$ does not lie on the hyperbola $x^2 - y^2 = 1$. In fact, if this were the case then $g(\eta i) = 0$, and one can proceed to derive a contradiction precisely as in the proof of Lemma 13.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Expressing $\Delta(k; z)$ in terms of $P_Q(\theta)$ (by (13.38), (13.41)) and expressing the parameters ϵ, z in terms of Q, θ (by (13.37), (13.40)), we can rewrite the representation (13.33)–(13.35) in the following form:

$$(13.46)$$

$$\hat{F} = \frac{1}{P_Q(\theta; \pm 1)} \left[\frac{1}{|k|}(\theta^3 + \theta^2 + 3\theta - 1)\hat{\tilde{a}} + \frac{i(\text{sgn}k)}{|k|^3}(1 + \theta)\hat{\tilde{b}} \right.$$
$$\left. + \frac{i(\text{sgn}k)}{|k|^2}(1 - \theta)\hat{\tilde{c}} \right],$$

$$(13.47)$$

$$\hat{\phi} = \frac{1}{P_Q(\theta; \pm 1)} \left[\frac{i(\text{sgn}k)}{|k|}\sigma(1 + \theta)\hat{\tilde{a}} + \frac{\beta_0}{|k|^3}(Q(\theta^2 - 1) + i(\text{sgn}k))(\theta + 1)\hat{\tilde{b}} \right.$$
$$\left. + \frac{\beta_0}{|k|^2}(Q(\theta^2 - 1) + i(\text{sgn}k))(1 - \theta)\hat{\tilde{c}} \right],$$

(13.48)

$$\hat{\phi}_y = \frac{1}{P_Q(\theta;\pm 1)}\left[\sigma i(\mathrm{sgn}k)(1-\theta)\hat{\hat{a}} + \frac{\beta_0}{|k|^2}(Q(\theta^2-1)+i(\mathrm{sgn}k))(1-\theta)\hat{\hat{b}}\right.$$
$$\left. +\frac{1}{|k|}\left[\sigma + \beta_0(Q(\theta^2-1)+i(\mathrm{sgn}k))(1+\theta)\theta\right]\hat{\hat{c}}\right];$$

here $P_Q(\theta;+1) = P_Q(\theta)$ if $\mathrm{sgn}k = 1$ and $P_Q(\theta;-1) = P_Q(\theta^*)^*$ if $\mathrm{sgn}k = -1$.

We wish to use (13.46)–(13.48) to estimate $\hat{F}, \hat{\phi}, \hat{\phi}_y$. By Lemma 13.6, $P_Q(\theta;\pm 1)$ does not vanish for all values of $\theta = \sqrt{1+\omega}$ when the parameter z satisfies $\mathrm{Re}\ z \geq 0$. However, $P_Q(\theta) \to 0$ if $Q \to \infty$ and $\theta \to 1$, or if $Q \to 0$ and $\theta \sim \theta_2$ (θ_2 as in Lemma 13.7).

In the case $Q \to 0$, $P_Q(\theta)$ goes to zero "too fast" with $\theta \approx \theta_2$, and we cannot directly control $1/P_Q(\theta)$ in (13.46)–(13.48). We shall therefore assume for the time being that $|k| \geq \delta$, so that

(13.49) $$Q \geq \frac{\delta}{\beta_0\epsilon} \geq \frac{\delta}{\beta_0} \equiv \delta_0 \qquad (\delta_0 > 0).$$

Later on we shall deal with the case $0 < Q < \delta_0$.

Consider now the case $Q \to \infty$ and $\theta = 1 + re^{i\psi}$, $r \to 0$, $\mathrm{Re}\ e^{i\psi} \geq 0$. Then

$$P_Q(\theta) \approx (2rQe^{i\psi} + i)4 + 2\lambda.$$

If $Qr \gg 1$ then $|P_Q(\theta)| \sim Qr$, whereas if $Qr \ll 1$ or $Qr \sim 1$ then $|P_Q(\theta)| \sim 1$. It follows that

$$C_1(|\theta-1|Q+1) \leq |P_Q(\theta)| \leq C_2(|\theta-1|Q+1) \quad \text{if } Q \geq \delta_0 ,$$

where $C_1 > 0$, $C_2 > 0$. If $|\theta-1| \geq \sigma_0 > 0$, then we have

$$C_1 Q|\theta|^5 \leq |P_Q(\theta)| \leq C_2 Q|\theta|^5, \quad \text{if} \quad Q \geq \delta_0 > 0.$$

Combining these estimates we can state:

Lemma 13.8. *For any $\delta_0 > 0$ there exist positive constants C_1, C_2 such that if $Q \geq \delta_0$ then*

(13.50) $$C_1|\theta|^4(|\theta-1|Q+1) \leq |P_Q(\theta)| \leq C_2|\theta|^4(|\theta-1|Q+1).$$

Introduce a cutoff function

(13.51) $$\lambda_\delta(k) = \begin{cases} 0 & \text{if } |k| \leq \delta , \\ 1 & \text{if } |k| > 2\delta. \end{cases}$$

Lemma 13.9. *There is a constant C depending on δ such that*

(13.52) $$\lambda_\delta(k)|k^3\hat{F}|^2 \leq \frac{C}{\theta^2}\lambda_\delta(k)[|k^2\hat{\hat{a}}|^2 + |\hat{\hat{b}}|^2 + |k\hat{\hat{c}}|^2],$$

(13.53) $$\lambda_\delta(k)|k^3\hat{\phi}|^2 \leq C\lambda_\delta(k)[\frac{|k^2\hat{\hat{a}}|^2}{|\theta|^8} + \frac{|\hat{\hat{b}}|^2}{|\theta|^4} + \frac{|k\hat{\hat{c}}|^2}{|\theta|^2}],$$

(13.54) $$\lambda_\delta(k)|k^2\hat{\phi}_y|^2 \leq C\lambda_\delta(k)[\frac{|k^2\hat{\hat{a}}|^2}{|\theta|^8} + \frac{|\hat{\hat{b}}|^2}{|\theta|^4} + \frac{|k\hat{\hat{c}}|^2}{|\theta|^2}].$$

Proof. To prove (13.52) we simply use (13.46) and Lemma 13.8. The proofs of (13.53) and (13.54) are similar, except that we have to be a bit more careful with the growth of the coefficients of $\hat{\hat{a}}, \hat{\hat{b}}, \hat{\hat{c}}$ and of P_Q with respect to θ, for $|\theta|$ large. $\quad\square$

The functions $\hat{\phi}$, $\hat{\phi}_y$ in Lemma 13.8 are actually the boundary values at $y = 0$ of $\hat{\phi}(k, y)$, $\hat{\phi}_y(k, y)$. We can write

$$(13.55) \qquad \hat{\phi}(k, y) = A(k)e^{-|k|y} + B(k)e^{-\sqrt{k^2 + \epsilon z}y},$$

so that

$$\hat{\phi}_y(k, y) = -|k|A(k)e^{-|k|y} - \sqrt{k^2 + \epsilon z}B(k)e^{-\sqrt{k^2 + \epsilon z}y}.$$

It then follows that $A(k)$ and $B(k)$ are determined by

$$\hat{\phi}(k, 0) = A(k) + B(k),$$
$$\hat{\phi}_y(k, 0) = -|k|A(k) - \sqrt{k^2 + \epsilon z}B(k),$$

or

$$A(k) = \frac{\hat{\phi}(k, 0)\sqrt{k^2 + \epsilon z} + \hat{\phi}_y(k, 0)}{\sqrt{k^2 + \epsilon z} - |k|},$$
$$B(k) = -\frac{|k|\hat{\phi}(k, 0) + \hat{\phi}_y(k, 0)}{\sqrt{k^2 + \epsilon z} - |k|}.$$

Substituting this into (13.55), we get:

Lemma 13.10.

$$\hat{\phi}(k, y) = \hat{\phi}(k, 0)\frac{\sqrt{k^2 + \epsilon z}e^{-|k|y} - |k|e^{-\sqrt{k^2 + \epsilon z}y}}{\sqrt{k^2 + \epsilon z} - |k|}$$

$$(13.56)$$

$$+ \hat{\phi}_y(k, 0)\frac{e^{-|k|y} - e^{\sqrt{k^2 + \epsilon z}y}}{\sqrt{k^2 + \epsilon z} - |k|}$$

where $\hat{\phi}(k, 0), \hat{\phi}_y(k, 0)$ satisfy the inequalities (13.53), (13.54).

It will be convenient to introduce the kernels

$$K_1(k; y) = \frac{e^{-|k|y} - e^{-\sqrt{k^2 + \epsilon z}y}}{\sqrt{k^2 + \epsilon z} - |k|},$$
$$K_2(k; y) = \frac{\sqrt{k^2 + \epsilon z}e^{-|k|y} - |k|e^{-\sqrt{k^2 + \epsilon z}y}}{\sqrt{k^2 + \epsilon z} - |k|},$$
$$K_3(k; y) = \frac{\sqrt{k^2 + \epsilon z}e^{-\sqrt{k^2 + \epsilon z}y} - |k|e^{-|k|y}}{\sqrt{k^2 + \epsilon z} - |k|},$$
$$K_4(k; y) = \epsilon z\frac{e^{-\sqrt{k^2 + \epsilon z}y}}{\sqrt{k^2 + \epsilon z} - |k|}.$$

Note that

$$(13.57) \qquad \frac{\partial K_1}{\partial y} = K_3, \qquad \frac{\partial K_2}{\partial y} = -k^2\theta K_1,$$
$$\frac{\partial K_3}{\partial y} = k^2 K_1 - K_4, \qquad \frac{\partial K_4}{\partial y} = -|k|\theta K_4,$$

since $\epsilon z = k^2\omega$ and $\theta = \sqrt{1 + \omega}$. By direct computation,

$$\int_0^\infty |K_1(y; k)|^2 dy = \frac{1}{|k|^3|\theta - 1|^2}\int_0^\infty |e^{-\xi} - e^{-\theta\xi}|^2 d\xi$$

$$= \frac{1}{|k|^3|\theta - 1|^2}\left[\frac{1}{2} - \frac{1}{1 + \theta} - \frac{1}{1 + \overline{\theta}} + \frac{1}{2\text{Re}(\theta)}\right].$$

One can verify that the expression in brackets vanishes quadratically at $\theta = 1$. Consequently

$$(13.58) \qquad \int_0^\infty |K_1(y;k)|^2 dy \leq \frac{C}{|\theta|^2 |k|^3}.$$

Similarly

$$(13.59) \qquad \int_0^\infty |K_2(y;k)|^2 dy \leq \frac{C}{|k|},$$

$$(13.60) \qquad \int_0^\infty |K_3(y;k)|^2 dy \leq \frac{C}{|\theta||k|},$$

and

$$(13.61) \qquad \int_0^\infty |K_4(y;k)|^2 dy \leq C|\theta||k|.$$

From (13.56) we have

$$(13.62) \qquad \hat{\phi}(k,y) = \hat{\phi}(k,0)K_2(k;y) + \hat{\phi}_y(k,0)K_1(k;y)$$

and therefore, upon using (13.58), (13.59),

$$\int_0^\infty |\hat{\phi}(k,y)|^2 \leq \frac{C}{|k|}|\hat{\phi}(k,0)|^2 + \frac{C}{|\theta|^2|k|^3}|\hat{\phi}_y(k,0)|^2.$$

Using (13.53), (13.54), we then find that

(13.63)

$$\lambda_\delta(k) \int_0^\infty |k^4 \hat{\phi}(k,y)|^2 dy$$
$$\leq C\lambda_\delta(k) \left[||k|^{3-1/2}\hat{\hat{a}}|^2 + \||k|^{1-1/2}\hat{\hat{b}}|^2 + \||k|^{2-1/2}\hat{\hat{c}}|^2 \right].$$

Similarly

(13.64)

$$\lambda_\delta(k)|\epsilon z|^2 \int_0^\infty |k^2 \hat{\phi}(k,y)|^2 dy$$
$$\leq C\lambda_\delta(k) \left[\||k|^{3-1/2}\hat{\hat{a}}|^2 + \||k|^{1-1/2}\hat{\hat{b}}|^2 + ||k|^{2-1/2}\hat{\hat{c}}|^2 \right],$$

where we have used (13.53), (13.54) and the relation

$$\frac{\epsilon z}{k^2} = \theta^2 - 1.$$

Next we consider $\partial^2 \hat{\phi}/\partial y^2$. Differentiating (13.62) twice in y and using the relations (13.57), we deduce that

$$\int_0^\infty \left| \frac{\partial^2 \hat{\phi}}{\partial y^2} \right|^2 dy \leq 2|k|^4|\theta|^2|\hat{\phi}(k,0)|^2 \int_0^\infty |K_3(y;k)|^2 dy$$
$$+ 2|\phi_y(k,0)|^2 \left[\int_0^\infty |k|^4 |K_1(y;k)|^2 dy + \int_0^\infty |K_4(y;k)|^2 dy \right].$$

Using (13.58), (13.60) and (13.61) to estimate the last integrals, we get

$$(13.65) \qquad \int_0^\infty \left| \frac{\partial^2 \hat{\phi}}{\partial y^2} \right|^2 dy \leq C|\theta|[|k|^3|\hat{\phi}(k,0)|^2 + |k||\hat{\phi}_y(k,0)|^2],$$

and using the estimates (13.53), (13.54), we get

$$\lambda_\delta(k)|\epsilon z|^2 \int_0^\infty \left|\frac{\partial^2\hat{\phi}}{\partial y^2}\right|^2 dy$$

$$\leq C\lambda_\delta(k)|\epsilon z|^2|\theta|\left[\frac{|k|}{|\theta|^8}|\hat{\hat{a}}|^2 + \frac{1}{|\theta|^4}\frac{1}{|k|^3}|\hat{\hat{b}}|^2 + \frac{1}{|\theta|^2}\frac{1}{|k|}|\hat{\hat{c}}|^2\right].$$

Since

$$\left|\frac{\epsilon z}{k^4}\right| = |\theta^2 - 1|^2 \leq C|\theta|^4,$$

we obtain

$$\lambda_\delta(k)|\epsilon z|^2 \int_0^\infty \left|\frac{\partial^2\hat{\phi}}{\partial y^2}\right|^2 dy$$

$$\leq C\lambda_\delta(k)\left[\||k|^{3-1/2}\hat{\hat{a}}|^2 + |\theta|\||k|^{1-1/2}\hat{\hat{b}}|^2 + |\theta|^3\||k|^{2-1/2}\hat{\hat{c}}|^2\right].$$

Observing that

$$|\theta| = |\sqrt{1+\omega}| \leq C\frac{|k| + |\epsilon z|^{1/2}}{|k|^{1/2}}.$$

we then find that
(13.66)

$$\lambda_\delta(k)|\epsilon z|^2 \int_0^\infty |\frac{\partial^2\hat{\phi}}{\partial y^2}|^2 dy$$
$$\leq C\lambda_\delta(x)\left[\||k|^{3-1/2}\hat{\hat{a}}|^2 + (|k| + |\epsilon z|^{1/2})|\hat{\hat{b}}|^2 + (|k|^3 + |\epsilon z|^{3/2})|\hat{\hat{c}}|^2\right].$$

Finally, we use the differential equation (13.26) to estimate $\partial^4\hat{\phi}/\partial y^4$ by the other terms, which have already been estimated in (13.63)–(13.66). The final result is then:

Lemma 13.11. *For any $\delta > 0$ there is a constant C such that*
(13.67)

$$\lambda_\delta(k)\int_0^\infty\left[|k|^8|\hat{\phi}(k,y)|^2 + |\epsilon z|^2|k|^4|\hat{\phi}(k,y)|^2 + |\epsilon z|^2\left|\frac{\partial^2\hat{\phi}}{\partial y^2}\right|^2 + \left|\frac{\partial^4\hat{\phi}}{\partial y^4}\right|^2\right]dy$$

$$\leq C\lambda_\delta(k)\left[\||k|^{3-1/2}\hat{\hat{a}}|^2 + \||k|^{1-1/2}\hat{\hat{b}}|^2 + \||\epsilon z|^{1/4}\hat{\hat{b}}|^2 + \||k|^{2-1/2}\hat{\hat{c}}|^2 + \||\epsilon z|^{1-1/4}\hat{\hat{c}}|^2\right].$$

From (13.20), (13.18),
(13.68)

$$\int_{\mathbb{R}^2} |\hat{\hat{W}}(k_1,k_2)|^2[|\vec{k}|^4 + |\epsilon z||\vec{k}|^2] \leq C\int_{\mathbb{R}^2_+} |\mu(x_1,x_2)|^2 dx_1 dx_2 ,$$

where $\vec{k} = (k_1, k_1)$ Observe that the function

$$U = W - \tilde{W} - \varphi$$

is in $H^4(R_+^2)$, since the same is true of ϕ, by (13.67), and of W, by assumption.

Since U satisfies the homogeneous boundary conditions (13.3)–(13.5), we can proceed similarly to (8.30) to derive an energy equality, from which it follows that $U \equiv 0$. Hence
(13.69)

$$W = \tilde{W} + \phi,$$

and from Lemma 13.11 and (13.68) we then get:

Lemma 13.12. *For any* $\delta > 0$ *there is a constant* C *such that*

(13.70)

$$
\lambda_\delta(k) \int_0^\infty [|k|^8 |\hat{W}(k,y)|^2 + |\epsilon z|^2 |k|^4 |\hat{W}(k,y)|^2 + |\epsilon z|^2 |\frac{\partial^2 \hat{W}}{\partial y^2}|^2 + |\frac{\partial^4 \hat{W}}{\partial y^4}|^2] dy
$$
$$
\leq C\lambda_\delta(k)[\||k|^{3-1/2}\hat{\tilde{a}}|^2 + \||k|^{1-1/2}\hat{\tilde{b}}|^2 + \||\epsilon z|^{1/4} \hat{\tilde{b}}|^2 + \||k|^{3/2}\hat{\tilde{c}}|^2 + \||\epsilon z|^{1-1/4} \hat{\tilde{c}}|^2].
$$

Recall that $\tilde{a}, \tilde{b}, \tilde{c}$ were defined in (13.23)–(13.25). We want to use these relations to prove that in the right-hand side of (13.70) we can replace $\tilde{a}, \tilde{b}, \tilde{c}$ by a, b, c provided the term

$$
C \int_{-\infty}^\infty |\hat{\mu}(k_1, k_2)|^2 dk_2
$$

is added to the expression in brackets.

Introduce

$$
\hat{\tilde{W}}(k_1, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{\tilde{W}}(k_1, k_2) e^{ik_2 y} dk_2.
$$

Then, by (13.20),

(13.71)

$$
|\frac{\partial^m}{\partial y^m}\hat{\tilde{W}}(k_1, y = 0)|
$$
$$
\leq C \left[\int_{-\infty}^\infty |\hat{\mu}(k_1, k_2)|^2 dk_2\right]^{\frac{1}{2}} \left[\int_{-\infty}^\infty \frac{k_2^{2m} dk_2}{[\epsilon z + (k_1^2 + k_2^2)]^2 (k_1^2 + k_2^2)^2}\right]^{\frac{1}{2}}
$$

for $m = 0, 1, 2, 3$.

From (13.71), with $m = 0$, we get

(13.72)
$$
\||k_1|^{4-1/2}\hat{\tilde{W}}(k_1, y = 0)|^2 \leq C \int_{-\infty}^\infty |\hat{\mu}(k_1, k_2)|^2 dk_2
$$

since

$$
k_1^{4-1/2} \int_{-\infty}^\infty \frac{dk_2}{|\epsilon z + (k_1^2 + k_2^2)|^2 (k_1^2 + k_2^2)^2} \leq C
$$

as can easily be seen by substituting $k_2 = |k_1|\lambda, dk_2 = |k_1| d\lambda$. It follows that

(13.73)
$$
\||k|^{3-1/2}\hat{\tilde{a}}|^2 \leq C\||k|^{3-1/2}\hat{a}|^2 + C \int_{-\infty}^\infty |\hat{\mu}(k_1, k_2)|^2 dk_2.
$$

Next we use (13.71) with $m = 2$ and substitute $k_2 = |k_1|\lambda$. We obtain

$$
|\frac{\partial^2 \hat{\tilde{W}}}{\partial y^2}(k_1, y = 0)|^2 \leq C \int_{-\infty}^\infty |\hat{\mu}(k_1, k_2)|^2 dk_2 \cdot I ,
$$

where

(13.74)
$$
I = \frac{1}{|k_1|^3} \int_{-\infty}^\infty \frac{\lambda^4 d\lambda}{(1 + \lambda^2)^2 \left|\frac{\epsilon z}{k_1^2} + (1 + \lambda^2)\right|^2} \leq \frac{C}{|k_1|^3}
$$

since Re $z \geq 0$. If we set $\tau = \epsilon z/k_1^2$ and substitute $\lambda = |\tau|^{1/2}\xi$, we also get the bound

(13.75)
$$
I \leq \frac{C}{|\epsilon z|^{3/2}}.
$$

The same bound can be established for

$$|k_1^2 \hat{\tilde{W}}(k_1, y = 0)|^2.$$

Hence, recalling the definition of $\tilde{c}$ in (13.25), we conclude (using both bounds (13.74) and (13.75)) that

(13.76)

$$||k|^{3/2}\hat{\tilde{c}}|^2 + ||\epsilon z|^{1-1/4}\hat{\tilde{c}}|^2$$

$$\leq C[||k|^{3/2}\hat{c}|^2 + ||\epsilon z|^{1-1/4}\hat{c}|^2] + C \int_{-\infty}^{\infty} |\hat{\mu}(k, k_2)|^2 dk_2.$$

Note that by (13.24) and (13.21)

$$\tilde{b} = b.$$

Using this and (13.76), (13.73) in (13.70), we arrive at the following result:

Lemma 13.13. *For any $\delta > 0$ there is a constant C such that*

(13.77)

$$\lambda_\delta(k) \int_0^\infty \left[|k|^8 |\hat{W}(k,y)|^2 + |\epsilon z|^2 |k|^4 |\hat{W}(k,y)|^2 + |\epsilon z|^2 \left| \frac{\partial^2 \hat{W}}{\partial y^2} \right|^2 + \left| \frac{\partial^4 \hat{W}}{\partial y^4} \right| \right] dy$$

$$\leq C\lambda_\delta(k) \left\{ \left[|k^{5/2}\hat{a}|^2 + ||k|^{1/2}\hat{d}|^2 ||\epsilon z|^{1/4}\hat{b}|^2 + ||k|^{3/2}\hat{c}|^2 + ||\epsilon z|^{3/4}\hat{c}|^2 \right] \right.$$

$$\left. + \int_{-\infty}^{\infty} |\hat{\mu}(k, k_2)|^2 dk_2 \right\}.$$

Using the uncertainty principle (Lemma 3.3), we can replace $\lambda_\delta(k)$ on the left-hand side of (13.77) by 1, provided $\delta \leq \pi/(4R_0)$. Taking the inverse Fourier transform in k, the proof of Lemma 13.3 follows.

Having completed the proof of Lemma 13.3, we now proceed to prove Lemma 13.1.

Proof of Lemma 13.1. Set

(13.78) $$M = \|(a, b, c, \mu)\|_0.$$

Since W has compact support (see (13.10)) we can estimate lower derivatives of W in terms of higher derivatives, by integrating along the x-direction. Using Lemma 13.3 we get

(13.79) $$(\int_{\mathbb{R}^2_+} |W|^2)^{1/2} \leq C_0 M, \quad |\epsilon z| (\int_{\mathbb{R}^2_+} |W|^2)^{1/2} \leq C_0 M.$$

By interpolation and Lemma 13.3 we then deduce that

(13.80) $$(\int_{\mathbb{R}^2_+} |D_y^k W|^2)^{1/2} \leq C_0 M \quad \text{for} \quad 0 \leq k \leq 3.$$

Let $\zeta = \zeta(y)$ be a C^∞ function, $\zeta(y) = 1$ if $y > -1$, $\zeta(y) = 0$ if $y < -2$. Extend $W(x, y)$ to $y < 0$ by

$$W(x, y) = \sum_{j=0}^{3} \lambda_j W(x, -\frac{y}{j})\zeta(y)$$

so that
$$\partial_y^k W(x, 0+) = \partial_y^k W(x, 0-) \quad \text{for} \quad 0 \le k \le 3,$$

by choosing the λ_j to satisfy
$$\sum_{j=0}^{3} \lambda_j (-\frac{1}{j})^k = 1 \quad \text{if} \quad 0 \le k \le 3.$$

By (13.80) and Lemma 13.3
$$[\int_{\mathbb{R}^2} (|D_x^4 W|^2 + |D_y^4 W|^2]^{1/2} \le C_0 M.$$

The Fourier transform $\hat{W}(k_1, k_2)$ of W then satisfies
$$\{\int_{\mathbb{R}^2} [|k_1|^4 |\hat{W}(k_1, k_2)| + |k_2|^4 |\hat{W}(k_1, k_2)|]^2\}^{1/2} \le C_0 M.$$

Since
$$k_1^2 k_2^2 + |k_1^3 k_2| + |k_1 k_2^3| \le C(k_1^4 + k_2^4),$$

it follows that

(13.81)
$$(\int_{\mathbb{R}^2_+} |\nabla^4 W|^2)^{1/2} \le C_0 M.$$

Similarly we can use the estimate
$$|\epsilon z| \{\int_{\mathbb{R}^2_+} [|D_x^2 W|^2 + |D_y^2 W|^2]\}^{1/2} \le C_0 M,$$

to deduce that

(13.82)
$$|\epsilon z| (\int_{\mathbb{R}^2_+} |\nabla^2 W|^2)^{1/2} \le C_0 M$$

and, by the trace embedding, also

(13.83)
$$|\epsilon z| (\int_{-\infty}^{0} |\nabla^{3/2} W(x, 0)|^2)^{1/2} \le C_0 M.$$

We shall now apply the interpolation formulas [7, p. 49]
$$\|D^{3/2} u\|_{L^2(G)} \le C \|u\|_{L^2(G)}^{1/2} \|u\|_{H^2(G)}^{1/2},$$
$$\|D^{1/2} u\|_{L^2(G)} \le C \|u\|_{L^2(G)}^{3/4} \|u\|_{H^2(G)}^{1/4},$$

where $G = \mathbb{R}^2_+ \cap \{x^2 + h^2 < R_0^2\}$ and $u = D^2 W$. Using (13.81), (13.82) we get
$$|\epsilon z|^{1/4} \|D^{1/2} D^3 W\|_{L^2(G)} \le C_0 M,$$
$$|\epsilon z|^{3/4} \|D^{1/2} D^2 W\|_{L^2(G)} \le C_0 M,$$

and then, by the trace embedding,

(13.84)
$$|\epsilon z|^{1/4} [\int_{-\infty}^{0} |D^3 W(x, 0)|^2]^{1/2} \le C_0 M,$$

(13.85)
$$|\epsilon z|^{3/4} [\int_{-\infty}^{0} |D^2 W(x, 0)|^2]^{1/2} \le C_0 M.$$

We now turn to F. From (13.81), (13.82) and the trace embedding we have

$$[\int_{-\infty}^{0} |D^{7/2}W(x,0)|^2]^{1/2} \le C_0 M,$$

$$|\epsilon z|[\int_{-\infty}^{0} [DW(x,0)|^2]^{1/2} \le C_0 M.$$

Using (13.4) we conclude that

(13.86)
$$[\int_{-\infty}^{0} |D_x^{7/2}F|^2]^{1/2} \le C_0 M$$

and then, by (13.3),

(13.87)
$$|z|[\int_{-\infty}^{0} |D_x^{5/2}F|^2]^{1/2} \le C_0 M.$$

Using the inequality

$$|z|^{1/2}|k|^3 \le C(|z||k|^{5/2} + |k|^{7/2}) \ ,$$

we conclude from (13.86), (13.87) that

$$|z|^{1/2}[\int_{\infty}^{0} |D_x^3 F|^2] \le C_0 M.$$

To complete the proof of Lemma 13.1 it remains to establish the bound

(13.88)
$$|\epsilon z|^{1+1/4}[\int_{-\infty}^{0} |\partial_y W(x,0)|^2]^{1/2} \le C_0 M.$$

To prove it we use (13.52). Recalling the definition of θ in (13.40), (13.37), we obtain

(13.89)
$$\lambda_\delta(k)|k^3 \hat{F}(k)|^2 |\epsilon z|^{1/2} \le C \frac{|\epsilon z/k^2|^{1/2}}{|1 + \epsilon z/k^2|} \lambda_\delta(k) B(k) \ ,$$

where $B(k)$ is equal to $|k|$ times the expression in brackets in (13.52). Its L^2-norm is bounded by CM^2 (by (13.6), (13.78)). Using the uncertainty principle (Lemma 13.3) we then get, from (13.89),

$$|\epsilon z|^{1/4}[\int_{-\infty}^{0} |D_x^3 F|^2]^{1/2} \le C_0 M.$$

Using this and (13.84) in (13.4), the assertion (13.88) follows. $\qquad \square$

14. AUXILIARY RESULTS NEEDED FOR PROVING THEOREM 10.2

Lemma 14.1. *Let φ, f be as in Theorem 10.2, but assume that (10.8), (10.9) hold for some unrestricted real numbers α, β. Then for any positive integer j there is a positive constant C such that for all $(x, y) \in \mathbb{R}_+^2$,*

(14.1)
$$|D^j \varphi(x,y)| \le \begin{cases} Cr^{\alpha-j} & \text{if } r > 1 \ , \\ Cr^{\beta-j} & \text{if } r < 1 \ , \end{cases}$$

and

(14.2)
$$|D^j f(x)| \le C(1 + |x|)^{\alpha-j} \quad \text{if } -\infty < x < -1 \ .$$

Proof. The proof of (14.1) for $r < 1$ is similar to the corresponding proof in Lemma 11.1; we use scaling and elliptic estimates (which can be justified by Remark 10.2). To consider the case $r > 1$ we introduce

$$\psi(x, y) = \varphi(Rx, Ry) , \quad g(x) = f(Rx)$$

for $R > 1$. Then

$$\Delta^2 \psi = 0 \quad \text{in } \mathbb{R}^2_+ ,$$

(14.3) $$\qquad \mu R g + \beta_0 \frac{dg}{dx} - \frac{\partial \psi}{\partial x} = 0 \quad \text{if } y = 0 , \quad x < 0 ,$$

and ψ, g satisfy also the boundary conditions (1.2), (1.3), (10.6). Thus the only difficulty that arises in estimating the derivation of ψ, g is due to the large coefficient μR of g in (14.3). We shall overcome this difficulty by the same idea that was used in the proofs of Lemmas 7.1 and 9.1. We introduce a cutoff function ξ such that $\xi = 1$ in $(B_2 \backslash B_1)^+$ and set

$$\Phi = \xi \psi , \quad G = \xi g .$$

Then Φ, G satisfy a system as in Lemma 13.2, where f is a linear combination of the first three derivatives of ψ. Applying Lemma 13.2 step-by-step with $m = 3, 2, 1, 0$, and using the assumptions (10.8) (for $r > 1$) and (10.9), we establish the bound

$$|D^j \Phi| \leq CR^\alpha , \quad |D^j G| \leq CR^\alpha$$

for any j, from which assertions (14.1) (for $r > 1$) and (14.2) follow. $\square$

In the rest of this section we assume that α, β satisfy (10.17).
Define functions

(14.4) $$\qquad \Phi(x, y) = \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \varphi(x_3, y) ,$$

(14.5) $$\qquad F(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 f(x_3) .$$

Then

(14.6) $$\qquad |D^j \Phi(x, y)| \leq Cr^{\alpha+3-j} \quad \text{if } r \to \infty ,$$

(14.7) $$\qquad |D^j F(x)| \leq C(1 + |x|)^{\alpha+3-j} \quad \text{for all } -\infty < x < -1 .$$

By (10.6),

(14.8) $$\qquad \frac{\partial^3}{\partial x^3} \left[\left(3D_x^2 D_y + D_y^3 \right) \Phi + \sigma F_{xxx} \right] = 0 \quad \text{if } y = 0, \ x < 0 .$$

Hence

$$\left(3D_x^2 D_y + D_y^3 \right) \Phi + \sigma F_{xxx} = Ax^3 + Bx + \gamma \quad \text{if } y = 0, \ x < 0 .$$

Since, by (14.6) and (14.7), the left-hand side is $O(|x|^\alpha)$ as $x \to -\infty$, A and B must be equal to zero, and so

(14.9) $$\qquad \left(3D_x^2 D_y + D_y^3 \right) \Phi(x, 0) + \sigma F_{xxx}(x) = \gamma \quad \text{if } x < 0 ,$$

and $F_{xxx} = f$. We shall extend $f(x)$ to $x \geq 0$ as a distribution, by

(14.10) $$\qquad \sigma f(x) = \gamma - \left(3D_x^2 D_y + D_y^3 \right) \Phi(x, 0) , \quad x \geq 0 .$$

Then, defining $F(x)$ by (14.5) also for $x \geq 0$, (14.9) holds for all $x \in \mathbb{R}$, and then so do (14.7) for $|x| > 1$ and (14.8).

Next we introduce a convenient definition of the Fourier transforms for Φ and F:

$$
(14.11) \quad
\begin{aligned}
\widehat{\Phi}(k, y) = \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \xi(x) \Phi(x, y) dx \\
& + \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^6} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^6}{\partial x^6} \left[(1 - \xi(x)) \Phi(x, y) \right] dx \ ,
\end{aligned}
$$

$$
(14.12) \quad
\begin{aligned}
\widehat{F}(k) = \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \xi(x) f(x) dx \\
& + \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^6} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^6}{\partial x^6} \left[(1 - \xi(x)) f(x) \right] dx
\end{aligned}
$$

for $k \neq 0$, where $\xi(x)$ is the same cutoff function as in (11.2). By (14.6), (14.7), the integrals in (14.11), (14.12) are absolutely convergent.

We also define the Fourier transforms

$$
(14.13) \quad
\begin{aligned}
\widehat{\varphi}(k, y) = \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \xi(x) \varphi(x, y) dx \\
& + \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \left[(1 - \xi(x)) \varphi(x, y) \right] dx \ ,
\end{aligned}
$$

$$
(14.14) \quad
\begin{aligned}
\widehat{f}(k) = \quad & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \xi(x) f(x) dx \\
& + \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \left[(1 - \xi(x)) f(x) \right] dx \ ,
\end{aligned}
$$

for $k \neq 0$; by (14.1), (14.2), these integrals are absolutely convergent.

As in the proof of (12.6),

$$
(14.15) \qquad (ik)^3 \widehat{\Phi}(k, y) = \widehat{\varphi}(k, y) \ , \ \ (ik)^3 \widehat{\Phi}_y(k, y) = \varphi_y(k, y) \ ;
$$

similarly

$$
(14.16) \qquad\qquad\qquad (ik)^3 \widehat{F}(k, y) = \widehat{f}(k) \ .
$$

By integration by parts,

$$
\begin{aligned}
& \left[3(ik)^3 D_y + D_y^3 \right] \widehat{\Phi}(k, 0) + \sigma(ik)^3 \widehat{F}(k) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[\left(3 \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3} \right) (\xi\Phi) + \sigma \frac{\partial^3}{\partial x^3} (\xi F) \right] dx \\
&\quad + \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \left[\left(3 \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3} \right) ((1 - \xi)\Phi) + \sigma \frac{\partial^3}{\partial x^3} F \right] dx \ .
\end{aligned}
$$

Integrating by parts three times in the first integral and using (14.8) (which holds for all $x \in \mathbb{R}$, by (14.10)), we find that the sum of the right-hand side vanishes. The same is true of the left-hand side, and so, by (14.15), (14.16),

$$
\left[3(ik)^2 D_y + D_y^3 \right] \widehat{\varphi}(k, 0)(ik)^3 + \sigma(ik)^3 \widehat{f}(k)(ik)^3 = 0 \ .
$$

Using (12.1) to reduce the order of the differential operator on the left-hand side, we get

$$
(14.17) \qquad\qquad 2|k|k^2 \widehat{\varphi}(k, 0) = i\sigma k^3 \widehat{f}(k) \quad \text{if } k \neq 0 \ .
$$

Lemma 14.2. *If f is extended by (14.10), then (14.17) holds,*

$$(14.18) \qquad \widehat{\varphi}(k,0) \quad \text{is analytic in } \{Im \ k > 0\} \ ,$$

$$(14.19) \qquad \widehat{\varphi}_y(k,0) \quad \text{is analytic in } \{Im \ k > 0\} \ ,$$

$$(14.20) \qquad |k|\widehat{\varphi}_y(k,0) \quad \text{is analytic in } \{Im \ k < 0\} \ ,$$

and

(14.21)

$$\mu\widehat{f}(k) + \beta_0(ik)\widehat{f}(k) - (ik)\widehat{\varphi}(k,0) \quad \text{is analytic in } \{Im \ k < 0\} \ ,$$

and all these functions are continuous up to Im $k = 0$, $k \neq 0$.

Proof. The proofs of (14.18)–(14.20) proceed exactly as in the case of Theorem 10.1 (see §12). Thus it remains to establish (14.21); here we shall use the boundary condition (10.5).

From the definitions of $\widehat{f}$, $\widehat{\varphi}$,

$$\mu\widehat{f} + (ik)\widehat{f}(k) - (ik)\widehat{\varphi}(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[\mu\xi f + \frac{\partial}{\partial x}(\xi f) - \frac{\partial}{\partial x}((\xi\varphi(x,0)) \right]$$

$$+ \frac{1}{\sqrt{2\pi}} \frac{1}{(ik)^3} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^3}{\partial x^3} \left[\mu(1-\xi)f + \frac{\partial}{\partial x}((1-\xi)f) - \frac{\partial}{\partial x}((1-\xi)\varphi(x,0)) \right] \ .$$

We break up each of the two integrals into $\int_{-\infty}^{0} + \int_{0}^{\infty}$. In the second integral $\int_{-\infty}^{0}$ the integrand vanishes if $|x| > 1$ (by (10.5)) and therefore we can integrate by parts three times. We then find that the sum of the two integrals $\int_{-\infty}^{0}$ is equal to zero (again by (10.5)). Finally, each of the integrals $\int_{0}^{\infty}$ has analytic extension to $\{Im \ k < 0\}$. $\qquad\square$

15. Proof of Theorem 10.2 for $n = 0$

Consider first the case $\mu \neq 0$. From (14.17), (14.21) we have

$$2k^2|k|(\mu + \beta_0(ik))\widehat{\varphi}(k,0) = \sigma k^3 i(\mu\widehat{f} + \beta_0 ik\widehat{f})$$

$$= -\sigma k^4\widehat{\varphi}(k,0) + \text{ analytic function in } \{Im \ k < 0\} \ .$$

Therefore

(15.1)

$$\left[\sigma k + 2(sgnk)^{-1}(\mu + \beta_0 ik) \right] \widehat{\varphi}(k,0) \quad \text{is analytic in } \{Im \ k < 0\} \ .$$

Introduce the function

$$(15.2) \qquad p(k) = \frac{\sigma k + 2(sgnk)(\mu + \beta_0 ik)}{(\sigma + 2\beta_0 i)k}$$

for k real, $k \neq 0$ $((sgnk)^{-1} = sgn \ k$ for such k's) and note that

$$\lim_{k \to +\infty} p(k) = 1 \ , \qquad \lim_{k \to -\infty} p(k) = \frac{\sigma - 2\beta_0 i}{\sigma + 2\beta_0 i} \ .$$

Define the number ρ_0 by (10.11) or, equivalently, by

$$(15.3) \qquad e^{-2\pi\rho_0 i} = \frac{\sigma - 2\beta_0 i}{\sigma + 2\beta_0 i} \ , \qquad 0 < \rho_0 < \frac{1}{2} \ ,$$

and set

$$(15.4) \qquad r_+(k) \equiv (k)_+^{-\rho_0} \equiv \begin{cases} (k)^{-\rho_0} , & k > 0 , \\[2ex] e^{-\pi\rho_0 i}(-k)^{-\rho_0} , & k < 0 , \end{cases}$$

$$(15.5) \qquad r_-(k) \equiv (k)_-^{-\rho_0} \equiv \begin{cases} (k)^{-\rho_0} , & k > 0 , \\[2ex] e^{\pi\rho_0 i}(-k)^{-\rho_0} , & k < 0 . \end{cases}$$

Then

$$\frac{r_+(k)}{r_-(k)} = e^{-2\pi\rho_0 i} = \frac{\sigma - 2\beta_0 i}{\sigma + 2\beta_0 i} \quad \text{if } k < 0 ,$$

$$\frac{r_+(k)}{r_-(k)} = 1 \quad \text{if } k > 0 .$$

The function

$$(15.6) \qquad \lambda(k) = \frac{r_-(k)}{r_+(k)} \, p(k)$$

satisfies

$$(15.7) \qquad \lambda(k) = \begin{cases} 1 + \dfrac{2\mu}{\sigma + 2\beta_0 i} \dfrac{1}{k} , & k > 0 , \\[3ex] 1 - \dfrac{2\mu}{\sigma - 2\beta_0 i} \dfrac{1}{k} , & k < 0 , \end{cases}$$

and, in particular, $\lambda(k) \to 1$ if $k \to \pm\infty$. By (10.7), $\lambda(k) \neq 0$ for all $k \neq 0$. It is easily seen that in each interval $\{0 < k < \infty\}$, $\{-\infty < k < 0\}$ the phase of $\lambda(k)$ varies less than π. We can define a branch of $\log\lambda(k)$ in each of these intervals by taking $\log\lambda(k) \to 0$ if $k \to \pm\infty$. Then $\arg\lambda(k) \in (-\pi, \pi)$ and $\log\pi(k)$ is well defined for $k \in \mathbb{R}\backslash\{0\}$; it may have a jump at $k = 0$.

For any $z \in \mathbb{C}$, $Im\, z \neq 0$, define

$$(15.8) \qquad g(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log\lambda(k)}{k - z} \, dk .$$

In view of (15.6), the integral (taken in the sense of the principal value) is convergent and it defines a holomorphic function. Since $\log\lambda(k)$ is Hölder continuous for $k \neq 0$, we have the well-known jump relation

$$(15.9) \qquad g(k + 0, 0) - g(k - 0, 0) = \log\lambda(k) \quad \text{if } k \in \mathbb{R}\backslash\{0\} .$$

Set

$$(15.10) \qquad e^{g(k)} = \begin{cases} q_+(k) & \text{if } Im\, k > 0 , \\[2ex] q_-(k) & \text{if } Im\, k < 0 . \end{cases}$$

Extending $r_+(k)$ and $r_-(k)$ as analytic functions into $Im\, k > 0$ and $Im\, k < 0$ by the second parts of (15.4) and (15.5), respectively, we form the new functions

$$(15.11) \qquad \begin{aligned} M_+(k) &= r_+(k)q_+(k) \quad \text{for } Im\, k > 0 , \\[1ex] M_-(k) &= r_-(k)q_-(k) \quad \text{for } Im\, k < 0 ; \end{aligned}$$

they are analytic, and can be extended continuously to $Im\ k = 0$, $k \neq 0$. We also have, by (15.6), (15.9),

$$(15.12) \qquad\qquad p(k) = \frac{M_+(k)}{M_-(k)} \ , \quad k \in \mathbb{R}\backslash 0 \ .$$

From (15.7), (15.8) we deduce that $g(z) \to 0$ if $|z| \to \infty$; consequently, by (15.10)

$$(15.13) \qquad\qquad |q_{\pm}(z)| \to 1 \quad \text{if } |z| \to \infty \ .$$

We proceed to study the behavior of $q_{\pm}(z)$ as $z \to 0$. Writing

$$\int_{-\infty}^0 \frac{\log\left(1 - \dfrac{2\mu}{\sigma - 2\beta_0 i}\dfrac{1}{k}\right)}{k-z} = -\int_0^\infty \frac{\log\left(1 + \dfrac{2\mu}{\sigma - 2\beta_0 i}\dfrac{1}{u}\right)}{u+z}\ du \ ,$$

we have

(15.14)

$$\log q(z) = \frac{1}{2\pi i}\int_0^\infty \left[\frac{\log\left(1 + \dfrac{2\mu}{\sigma + 2\beta_0 i}\dfrac{1}{k}\right)}{k-z} - \frac{\log\left(1 + \dfrac{2\mu}{\sigma - 2\beta_0 i}\dfrac{1}{k}\right)}{k+z}\right]\ dk$$

$$= \frac{1}{2\pi i}\int_0^\infty \left[\frac{\log\left(k + \dfrac{2\mu}{\sigma + 2\beta_0 i}\right)}{k-z} - \frac{\log\left(k + \dfrac{2\mu}{\sigma - 2\beta_0 i}\right)}{k+z}\right]\ dk$$

$$+ \frac{1}{2\pi i}\int_0^\infty \left(\log\frac{1}{k}\right)\frac{2z}{(k-z)(k+z)}\ dk \equiv L_1 + L_2$$

for $Im\ z \neq 0$, where $q = q_{\pm}$ in $Im\ z \gtrless 0$.

In L_2 we substitute $k = -|z|\xi$, $z = |z|w$ to obtain

$$L_2 = \frac{w}{\pi i}\int_0^\infty \log\frac{1}{|z|}\ \frac{d\xi}{(\xi-w)(\xi+w)} + \frac{w}{\pi i}\int_0^\infty \left(\log\frac{1}{\xi}\right)\frac{d\xi}{(\xi-w)(\xi+w)}$$

$$\equiv L_{21} + L_{22}\ .$$

L_{22} is obviously bounded. To evaluate L_{21} we compute

$$J \equiv \int_0^\infty \frac{d\xi}{(\xi-w)(\xi+w)} = \frac{1}{2w}\int_0^\infty \left(\frac{1}{\xi-w} - \frac{1}{\xi+w}\right)d\xi$$

$$= \frac{1}{2w}\left[\text{Log}(\xi-w) - \text{Log}(\xi-w)\right]_0^\infty$$

where

$$\text{Log}\ \xi = \log|\xi| + i\,\text{Arg}(\xi)\ , \quad \text{Arg}(\xi) \in (-\pi, \pi)\ ;$$

note that the upper limit vanishes.

Suppose first that $Im\ w > 0$, i.e., $\text{Arg}(w) \in (0, \pi)$. Then, as $\xi \to 0$,

$$\text{Log}(\xi - w) \to \log|w| + i\,\text{Arg}(-w) = \log|w| + i[-\pi + \text{Arg}(w)]\ ,$$

and

$$\text{Log}(\xi + w) \to \log|w| + i\,\text{Arg}(w)\ ,$$

and therefore

$$J = \frac{\pi i}{2w} \ .$$

Hence

(15.15) $$L_2 = \frac{1}{2} \log \frac{1}{|z|} + O(1) \quad \text{as } |z| \to 0 \ , \quad Im \, z > 0 \ .$$

On the other hand, if $Im \, w < 0$, then $J = -\pi i/(2w)$, so that

(15.16) $$L_2 = -\frac{1}{2} \log \frac{1}{|z|} + O(1) \quad \text{as } |z| \to 0 \ , \quad Im \, z < 0 \ .$$

Next we evaluate L_1 as $z \to 0$. Observe that the integrand vanishes, at $z = 0$, for all $k \neq 0$, and the part of the integral from any $k = \delta$ $(\delta > 0)$ to $k = \infty$ is uniformly convergent. Hence

$$L_1 = \frac{1}{2\pi i} \int_0^\delta \left[\frac{\log\left(k + \frac{2\mu}{\sigma + 2\beta_0 i}\right)}{k - z} - \frac{\log\left(k + \frac{2\mu}{\sigma - 2\beta_0 i}\right)}{k + z} \right] dk + O(1)$$

$$= \frac{1}{2\pi i} \left[\text{Log}\left(\frac{2\mu}{\sigma + 2\beta_0 i}\right) \int_0^\delta \frac{dk}{k - z} - \text{Log}\left(\frac{2\mu}{\sigma - 2\beta_0 i}\right) \int_0^\delta \frac{dk}{k + z} \right]$$

$$+ o\left(\log \frac{1}{|z|}\right) \ .$$

The last expression in brackets is equal to

$$\log|z| \left[-\text{Log}\frac{2\mu}{2\sigma + \beta_0 i} + \text{Log}\frac{2\mu}{\sigma - 2\beta_0 i} \right] + O(1) \ ,$$

and therefore

$$L_1 = \frac{1}{2\pi i} \log|z| \left[i\,\text{Arg}\frac{2\mu}{\sigma - 2\beta_0 i} - i\,\text{Arg}\frac{2\mu}{\sigma + 2\beta_0 i} \right] + o\left(\log \frac{1}{|z|}\right) \ .$$

Set

$$\gamma_1 = \frac{2\mu}{\sigma - 2\beta_0 i} \ , \quad \gamma_2 = \frac{2\mu}{\sigma + 2\beta_0 i} \ ,$$

so that

$$\gamma_1 = \gamma_2 e^{-2\pi \rho_0 i} \ .$$

Clearly γ_1 and γ_2 belong respectively to the fourth and first quadrants, so that

$$\text{Arg}(\gamma_1) = \text{Arg}(\gamma_2) - 2\pi \rho_0 \ .$$

Consequently

$$L_1 = \frac{1}{2\pi} \left(\log|z|\right) 2\pi \rho_0 [1 + o(1)] \ .$$

Combining this with (15.15), (15.16) and (15.14), we get

(15.17)
$$q_+(z) = C_+ |z|^{\rho_0 - 1/2}(1 + o(1)) \quad \text{as } z \to 0 \ ,$$
$$q_-(z) = C_- |z|^{\rho_0 + 1/2}(1 + o(1)) \quad \text{as } z \to 0 \ ,$$

where $C_\pm$ are constants. Recalling (15.11) and the definitions of $r_\pm(k)$ and $M_\pm(k)$, we see that

(15.18)
$$M_+(k) \simeq C_+|k|^{-1/2} \, , \quad M_-(k) \simeq C_-|k|^{1/2} \quad \text{as } |k| \to 0$$

with other constants $C_\pm$. From (15.13) we also get

(15.19)
$$M_+(k) \simeq \widetilde{C}_+|k|^{-\rho_0} \, , \quad M_-(k) \simeq \widetilde{C}_-|k|^{-\rho_0} \quad \text{as } |k| \to \infty \, ,$$

where $\widetilde{C}_\pm$ are constants.

We now introduce the Wiener-Hopf decomposition

$$Q(k) = \begin{cases} M_+(k)\widehat{\varphi}(k,0) & \text{for } Im \ k > 0 \, , \\ M_-(k)p(k)\widehat{\varphi}(k,0) & \text{for } Im \ k < 0 \, . \end{cases}$$

By (14.18) and (15.1), (15.2), $Q(k)$ is analytic in the set $\{Im \ k \neq 0\}$; furthermore, it has continuous extension to $Im \ k = 0$, $k \neq 0$.

By Lemmas 11.2, 11.3 (which hold for the present case, with the same proof) and by (15.18),

(15.20)
$$|kQ(k)| \leq \begin{cases} \dfrac{C|k|^{-1/2}}{|k|^\alpha} \leq C|k|^{\varepsilon-1} & \text{if } k \to 0 \, , \ Im \ k > 0 \\[2ex] \dfrac{C|k|^{-\rho_0}}{|k|^\beta} \leq C|k|^{-\varepsilon} & \text{if } |k| \to \infty \, , \ Im \ k > 0 \, ; \end{cases}$$

We conclude that $kQ(k)$ is holomorphic at $k = 0$ and at $k = \infty$, and it vanishes as $|k| \to \infty$. By Liouville's theorem it follows that $kQ(k) \equiv 0$, and therefore $\widehat{\varphi}(k,0) \equiv 0$ if $k \in \mathbb{R}$, $k \neq 0$.

The proof that $\widehat{\varphi}_y(k,0) \equiv 0$ if $k \in \mathbb{R}$, $k \neq 0$ is the same as in §12, and we can now proceed as in §12 to complete the proof of Theorem 10.2.

So far we have assumed that $\mu \neq 0$. If $\mu = 0$ then $\lambda(k) \equiv 1$ (see (15.7)) and $q_\pm(z) \equiv 1$. It follows that (15.19) holds for both $|k| \to \infty$ and $|k| \to 0$, but the rest of the proof remains unchanged.

Completion of the proofs of Theorems 10.1, 10.2. To prove Theorem 10.1 for $n = 1$ we note that, by Lemma 11.1, we can apply Theorem 10.1 for $n = 0$ to φ_x. We then conclude that $\varphi_x \equiv 0$, from which it easily follows that $\varphi \equiv 0$. The proof of Theorem 10.1 for $n = 2, 3, \ldots$ follows by the same argument. Theorem 10.2 for $n \geq 1$ can be proved in the same way, using the estimates of Lemma 14.2.

16. Another Liouville type theorem

The techniques developed in this paper can be used to establish other Liouville type theorems. We illustrate this in one simple example:

Theorem 16.1. *If*

(16.1)
$$\omega\Delta\varphi - \Delta^2\varphi = 0 \quad in \ \mathbb{R}_+^2 \quad (Re \ \omega \geq 0 \, , \quad \omega \neq 0) \, ,$$

(16.2)
$$\varphi(x,\, 0) = \partial_y\varphi(x,\, 0) \quad if \ x \in \mathbb{R}$$

and

(16.3) $$|\varphi(x,\, y)| \le Cr^\alpha \quad if \ r > 1$$

for some $\alpha < 1$, then $\varphi \equiv 0$.

Proof. We claim that

(16.4) $$|D^j \partial_x \varphi(x,\, y)| \le Cr^{\alpha - j} \quad \text{if } r > 1 \ .$$

Indeed, we shall prove this by the method used to prove Lemmas 7.1 and 9.1. Suppose $\widehat{\xi}$ is a cutoff function, $\widehat{\xi} = 1$ in $B_2 \backslash B_{1/2}$, and let $\widehat{\varphi}_j = \widehat{\xi}\varphi_j$, where

$$\varphi_j(x,\, y) = \varphi(R_j x,\, R_j y) \quad (R_j \nearrow \infty) \ .$$

Then we can prove, by the argument given in the proof of Lemma 7.1, that

(16.5) $$|D^l \partial_x \widehat{\varphi}_j| \le CR_j^\alpha \ ,$$

provided the following estimates hold:

(16.6) $\quad |D^l \partial_x \widetilde{V}_{R_j}| \le \dfrac{C}{R_j^2} \quad$ for $0 \le y \le y_0 \quad (y_0 > 0$ and independent of $j)$,

where C is a constant depending on l. The proof of (16.6) follows directly from the estimates in §5. (Note that in Section 9 we proved that

$$|D^l \widetilde{V}_{R_j}| \le \frac{C}{R_j^2}$$

holds in a rather restrictive domain (cf. Figure 8.2); the difference between this inequality and (16.6) is that ∂_x, which occurs in (16.6), eliminates the boundary layer effect at $y = 0$ manifested in (5.10).)

From (16.5) we immediately deduce the assertion (16.4). We can now define the Fourier transform by

$$\widehat{\varphi}(k,\, y) = \frac{1}{\sqrt{2\pi}} \, \frac{1}{(ik)^2} \int_{-\infty}^{\infty} \left[\frac{\partial^2}{\partial x^2} \, \varphi(x,\, y) \right] e^{-ikx} dx \ , \quad k \ne 0 \ ,$$

and derive the equation

$$\left[\omega(D_y^2 + (ik)^2) - (D_y^2 + (ik)^2)^2 \right] \widehat{\varphi}(k,\, y) = 0 \ .$$

From this and the bound (16.4) we obtain:

$$\widehat{\varphi}(k,\, y) = A(k)e^{-|k|y} + B(k)e^{-\sqrt{k^2 + \omega}\,y} \ , \quad k \ne 0 \ ,$$

and, since

$$\frac{\partial^2 \varphi(x,\, 0)}{\partial x^2} = \frac{\partial}{\partial y} \, \frac{\partial^2 \varphi(x,\, 0)}{\partial x^2} = 0 \ ,$$

we have $(ik)^2 A(k) = (ik)^2 B(k) = 0$, so that $\widehat{\varphi}(k,\, y) = 0$ if $k \ne 0$, and the assertion $\varphi \equiv 0$ easily follows using the growth condition (16.3) with $\alpha < 1$. $\qquad \square$

We can also prove the theorem by using the method of Part I. But more simply, we can use (16.4) to derive from (8.30) an energy equality for φ_x, which implies that $\varphi_x \equiv 0$, so that φ is a function of y. The only such function possible is

$$C_0 \left[\left(e^{-\sqrt{\omega}y} - 1 \right) + \sqrt{\omega}y \right] \qquad (C_0 \ \text{constant}) \ ,$$

but in view of (16.3) (with $\alpha < 1$) $C_0 = 0$.

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