THE QUANTUM ANALOG OF A SYMMETRIC PAIR:
A CONSTRUCTION IN TYPE $(C_n, A_1 \times C_{n-1})$

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ABSTRACT. Let $\mathcal{I}$ be the ideal in the enveloping algebra of $\mathfrak{sp}(n, \mathbb{C})$ generated by the maximal compact subalgebra of $\mathfrak{sp}(n-1, 1)$. In this paper we construct an analog of $\mathcal{I}$ in the quantized enveloping algebra $\mathfrak{U}$ corresponding to a type C_n diagram at generic q. We find generators for $\mathcal{I}$ and explicit bases for $\mathfrak{U}/\mathcal{I}$.

1. INTRODUCTION

The representation theory of the quantized enveloping algebras resembles, at least in the generic case, the corresponding theory of their classical counterparts. There are however notions that are not readily generalizable though they are important in the nondeformed picture. In particular one would like to define a quantum analogue of a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is the complexification of a semisimple real Lie algebra $\mathfrak{g}_0$ and $\mathfrak{k}$ the complexification of a maximal compact subalgebra of $\mathfrak{g}_0$. This notion has an obvious quantization only in the hermitian symmetric case.

In this paper we give a construction of such a quantization in the case where $\mathfrak{g} \simeq \mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{k} \simeq \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sp}(n-1, \mathbb{C})$. This pair arises from the semisimple group $Sp(n-1, 1)$. As a consequence of our construction we prove two results (see § 8): in the first one we deal with a behavior that appears only in the deformed picture, while the second one extends a well known classical situation.

We now describe our results more precisely. Let $\mathfrak{U}$ be the quantized enveloping algebra corresponding to a diagram of type C_n. The difficulty in finding a deformation of $\mathfrak{k}$ is in finding generators for the $\mathfrak{sl}(2, \mathbb{C})$-part. This is because the $\mathfrak{sp}(n-1, \mathbb{C})$-part corresponds to a subdiagram of the Dynkin diagram of $\mathfrak{g}$, so it has an obvious analog in $\mathfrak{U}$ that we denote by $\mathfrak{M}_0$. It is then natural to require that a deformation $\mathfrak{K}$ of $\mathfrak{k}$ should contain $\mathfrak{M}_0$.

In the classical setting, the left ideal in the enveloping algebra of $\mathfrak{g}$ generated by $\mathfrak{k}$ is precisely the annihilator of the $\mathfrak{k}$-invariant vectors in the finite dimensional representations of $\mathfrak{g}$, so in order to define a deformation of $\mathfrak{k}$ we need to identify its spherical vectors in the finite dimensional representations of $\mathfrak{U}$.

Any finite dimensional irreducible representation $F(\lambda)$ of $\mathfrak{g}$ has a deformation $V(\lambda)$ that turns out to be an irreducible finite dimensional representation of $\mathfrak{U}$, and essentially all finite dimensional irreducible representations of $\mathfrak{U}$ are obtained this way. In order to describe the quantum analogs of the $\mathfrak{k}$-spherical vectors we only require that the deformations of the irreducible representations that are of class one for the pair $(\mathfrak{g}, \mathfrak{k})$ should be of class one for $(\mathfrak{U}, \mathfrak{K})$. Since $\mathfrak{M}_0$ should be contained

in $\mathfrak{K}$, it is easy to deduce from a multiplicity one argument that we must choose as spherical vectors for $\mathfrak{K}$ the $\mathfrak{M}_0$-invariant vectors in the deformations of the class one representations of the pair $(\mathfrak{g}, \mathfrak{k})$.

In our work we construct explicitly a finite set K of elements of $\mathfrak{U}$ that annihilate these vectors (see (4.2)). We observe that these elements are deformations of a set of generators for $\mathfrak{k}$. We define $\mathfrak{K}$ as the algebra generated by K, and then show that the annihilator $\mathcal{I}$ of the $\mathfrak{M}_0$-invariant vectors that occur in the deformations of the class one representations of $(\mathfrak{g}, \mathfrak{k})$ is exactly $\mathcal{J} = \mathfrak{U}(\mathfrak{K} \cap \mathcal{A})$ (here $\mathcal{A}$ is the augmentation ideal). An immediate consequence of our work is the fact that the annihilator of all $\mathfrak{K}$-spherical vectors is precisely $\mathcal{J}$.

Most of the paper is dedicated to the proof that the ideals $\mathcal{I}$ and $\mathcal{J}$ are indeed the same: the techniques are quite general and have applications that go beyond the limited scope of this paper.

As an outcome we have the two applications we mentioned before. The first consequence, loosely stated, says that the maximal ideal in $\mathfrak{U}$ that annihilates the $\mathfrak{K}$-invariant vectors and that has the coideal property is essentially $\mathfrak{U}(\mathfrak{M}_0 \cap \mathcal{A})$. In effect the result (Theorem 8.1) is stronger and more precise, and shows the obstructions to finding good analogs for the notion of a symmetric pair. The second application says that what should be interpreted as the algebra of invariant differential operators on the symmetric space is abelian (Theorem 8.5).

The paper is organized as follows. In § 2 we set up most of the notation and describe the construction of a PBW basis for $\mathfrak{U}$. In § 3 we reformulate some elementary linear algebra in the language of filtrations. In § 4 we give full details of the construction of the left ideals $\mathcal{I}$ and $\mathcal{J}$ as outlined in this introduction. Various bases for $\mathfrak{U}$ and for the $\mathfrak{U}$-module $\mathfrak{U}/\mathcal{J}$ are then found in § 5. In § 6 we show that $\mathfrak{U}/\mathcal{I}$ and $\mathfrak{U}/\mathcal{J}$ both have Gelfand-Kirillov dimension equal to $\dim \mathfrak{g} - \dim \mathfrak{k}$. Finally, in § 7, this fact is used to establish the equality between $\mathcal{I}$ and $\mathcal{J}$. In § 8 we give the applications we alluded to above.

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2. Preliminaries

In this section we recall some of the basic constructions for quantized algebras needed in this paper.

Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, let $\mathfrak{t}$ be the complexification of a Cartan subalgebra $\mathfrak{t}_0$ of a compact real form of $\mathfrak{g}$. Let $(,)$ be the Killing form. The form $(,)$ induces an inner product on the real space $(\sqrt{-1}\mathfrak{t}_0)^*$ by setting $(\alpha, \beta) = (h_\alpha, h_\beta)$, where $h_\alpha \in \mathfrak{t}$ denotes the element such that $\alpha(h) = (h, h_\alpha)$.

It is known ([3]) that one can find an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of $(\sqrt{-1}\mathfrak{t}_0)^*$ such that the root system of $\mathfrak{g}$ is the set

$$R = \{\pm e_i \pm e_j | i \neq j\} \cup \{\pm 2e_i | i = 1, \dots, n\}.$$

We choose the set

$$\Delta = \{\alpha_i = e_i - e_{i+1} | i = 1, \dots, n-1\} \cup \{\alpha_n = 2e_n\}$$

as a set of simple roots for R and let R^+ be the set of positive roots corresponding to our choice.

If $1 \leq i, j \leq n$, we set $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. Let A denote the matrix (a_{ij}). We let $\mathfrak{U} = \mathfrak{U}_q(A)$ denote the quantized enveloping algebra associated to the matrix

A. This is the algebra over the field $\mathbb{C}(q)$ with generators E_i, F_i, K_i and K_i^{-1} $(i = 1, \ldots, n)$ satisfying the relations described in § 1.1 of [8]. As is now customary we denote by $\mathfrak{U}^-$, $\mathfrak{U}^+$ and $\mathfrak{U}^0$ the subalgebras generated by the F_i, E_i and K_i, K_i^{-1} respectively.

We let Ω be the $\mathbb{C}$-algebra antihomomorphism of $\mathfrak{U}$ defined by

$$(2.1) \qquad \Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_i) = K_i^{-1}, \quad \Omega(q) = q^{-1}.$$

Let $\mathcal{W}$ denote the Weyl group associated to R and let $s_i = s_{\alpha_i}$ $(i = 1, \ldots, n)$ be the simple reflections. Let w_0 be the longest element of $\mathcal{W}$. We choose a reduced expression for w_0 in the following way:

$$(2.2) \qquad w_0 = s^{(1)} s^{(2)} \ldots s^{(n-1)} s_n,$$

where

$$(2.3) \qquad s^{(i)} = s_i \ldots s_n \ldots s_i.$$

This choice is fixed for the rest of the paper. From it we define a convex ordering on R^+ so that we can write

$$R^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$$

where $N = |R^+| \ (= n^2)$.

If $i = 1, \ldots, n$ we let T_i denote the automorphism of $\mathfrak{U}$ defined in [8, Theorem 3.1]. Following [8, 2] we define the root vectors in $\mathfrak{U}$ by means of the automorphisms T_i. We write the fixed reduced expression (2.2) as a product of simple reflections:

$$w_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$$

and, for $\beta_j \in R^+$, we set

$$X_{\beta_j} = T_{i_1} \ldots T_{i_{j-1}}(E_{i_j}) \text{ and } X_{-\beta_j} = \Omega(X_{\beta_j}).$$

By [2], § 9.5, if α_i is a simple root, then $X_{\alpha_i} = E_i$ and $X_{-\alpha_i} = F_i$.

We set $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ to be the root lattice for R and let P denote the lattice of integral weights. Let P^+ denote the set of dominant weights. We recall ([1]) that in our setting $P = \sum_1^n \mathbb{Z}e_i$ and

$$P^+ = \{\sum_{i=1}^n a_i e_i \in P \mid a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

If $\lambda \in P^+$ we let $V(\lambda)$ denote the finite dimensional irreducible $\mathfrak{U}$-module of highest weight λ as defined in [7] or [4].

We will use the root vectors X_α and the K_λ defined above to determine a PBW-basis for $\mathfrak{U}$: if $I = (i_1, \ldots, i_N) \in \mathbb{N}^N$ we denote $X^I = \prod X_{\beta_1}^{i_1} \ldots X_{\beta_N}^{i_N}$ and $Y^I = \Omega(X^I)$. If $\lambda = \sum_1^n n_i \alpha_i \in Q$ then we set $K_\lambda = \prod_1^n K_i^{n_i}$, in particular $K_i = K_{\alpha_i}$. We call the monomial $Y^J K_\lambda X^I$ a standard monomial for $\mathfrak{U}$.

The PBW-Theorem (see [8]) for quantized enveloping algebras states that the standard monomials form a basis for $\mathfrak{U}$.

If V is a $\mathfrak{U}^0$-module, the weight space corresponding to a weight $\lambda \in P$ is the space

$$V_\lambda = \{m \in V \mid K_\alpha \cdot v = q^{(\alpha, \lambda)} v \ \forall \alpha \in Q\}.$$

We say that the module V is $\mathfrak{U}^0$-admissible (or simply admissible) if

$$V = \bigoplus_{\lambda \in P} V_\lambda.$$

Let Ad denote the adjoint action of $\mathfrak{U}^0$ on $\mathfrak{U}$ that is defined by setting $Ad(K_i)u = K_i u K_i^{-1}$. We observe that $\mathfrak{U}$ itself is an admissible module, so we will denote by $\mathfrak{U}_\lambda$ its weight spaces.

If a module V is admissible then one can define the character of V to be the formal series

$$\chi(V) = \sum_\lambda (\dim_{\mathbb{C}(q)} V_\lambda) e^\lambda.$$

It is known that the modules $V(\lambda)$ are admissible representations of $\mathfrak{U}$ and are exactly the irreducible elements in the category of finite dimensional admissible $\mathfrak{U}$-modules. Furthermore, if V is an admissible finite dimensional representation of $\mathfrak{U}$ then V is completely reducible.

On the other hand, from the viewpoint of $\mathfrak{g}$, if $\lambda \in P^+$ we let $F(\lambda)$ denote the irreducible finite dimensional representation of $\mathfrak{g}$ of highest weight λ. It is well known that $F(\lambda)$ decomposes into a sum of weight spaces $F(\lambda)_\mu$ with $\mu \in P$, so one can define the character of $F(\lambda)$ as

$$\chi(F(\lambda)) = \sum_{\mu \in P} (\dim_{\mathbb{C}} F(\lambda)_\mu) e^\mu.$$

We will make frequent use of an important theorem of Lusztig ([7, Theorem 4.12]):

Theorem 2.1. *If $\lambda \in P^+$, then*

$$\dim_{\mathbb{C}(q)} V(\lambda)_\mu = \dim_{\mathbb{C}} F(\lambda)_\mu \qquad \forall \, \mu \in P,$$

so, in particular, $\chi(V(\lambda)) = \chi(F(\lambda))$.

3. GENERALITIES ON FILTRATIONS

Many of the arguments and calculations in the paper can be seen most clearly as general facts on filtered vector spaces and algebras. So for the moment we work in this generality.

Let I be a well ordered set and let V be a vector space; suppose that we have a filtration $\mathcal{F}$ of V over I. This means that we associate to $i \in I$ a subspace $\mathcal{F}_i(V)$ of V in such a way that, if $i < j$, then $\mathcal{F}_i(V) \subset \mathcal{F}_j(V)$ and $\bigcup_{i \in I} \mathcal{F}_i(V) = V$. If $i \in I$, we set $\mathcal{F}_{<i}(V) = \sum_{j<i} \mathcal{F}_j(V)$ (if i is the minimum of I we set $\mathcal{F}_{<i}(V) = 0$). We set $\mathcal{F}^i(V) = \mathcal{F}_i(V)/\mathcal{F}_{<i}(V)$. If there is no chance for confusion we will denote $\mathcal{F}_i(V)$ by V_i, $\mathcal{F}_{<i}(V)$ by $V_{<i}$, and $\mathcal{F}^i(V)$ by V^i.

The graded space associated to the filtration $\mathcal{F}$ is

$$Gr_{\mathcal{F}}(V) = \bigoplus_{i \in I} V^i = \bigoplus_{i \in I} V_i/V_{<i}.$$

As usual one can define the degree map $d_{\mathcal{F}} : V \to I$ by setting

$$d_{\mathcal{F}}(v) = \inf\{i \in I \mid v \in V_i\}$$

and the symbol map $s_{\mathcal{F}} : V \to Gr_{\mathcal{F}}(V)$ by

$$s_{\mathcal{F}}(v) = v + V_{<d_{\mathcal{F}}(v)}.$$

Lemma 3.1. *If $\{v_j\}_{j \in J}$ is a subset of V such that $\{s_{\mathcal{F}}(v_j)\}$ is a basis of $Gr_{\mathcal{F}}(V)$, then $\{v_j\}$ is a basis of V.*

Proof. If $\sum c_j v_j = 0$ set $i = \sup\{d_{\mathcal{F}}(v_j) \mid c_j \neq 0\}$.

Since $\sum_{d_{\mathcal{F}}(v_j)=i} c_j v_j = -\sum_{d_{\mathcal{F}}(v_j)<i} c_j v_j$, we have that

$$0 = \sum_{d_{\mathcal{F}}(v_j)=i} c_j v_j + V_{<i} = \sum_{d_{\mathcal{F}}(v_j)=i} c_j s_{\mathcal{F}}(v_j);$$

thus $c_j = 0$ for all j such that $d_{\mathcal{F}}(v_j) = i$. This contradicts the definition of i.

Let us prove that the v_j generate V. Suppose not and set

$$i = \inf\{d(v) \mid v \text{ is not a linear combination of the } v_j\}$$

and choose v that is not a linear combination of the $\{v_j\}$ such that $d(v) = i$. Since $s_{\mathcal{F}}(v) \in V_i/V_{<i}$, we can write $s_{\mathcal{F}}(v) = \sum c_j s_{\mathcal{F}}(v_j)$ with $s_{\mathcal{F}}(v_j) \in V_i/V_{<i}$. This says that $v - \sum c_j v_j \in V_{<i}$; hence, by the minimality of i,

$$v - \sum c_j v_j = \sum d_r v_r,$$

but then v is a linear combination of the v_j. $\qquad\square$

If $\mathcal{B}$ is a basis of V and $\delta : \mathcal{B} \to I$ is a map in our well ordered set, then we can define a filtration $\mathcal{F}(\delta, \mathcal{B})$ of V by setting V_i to be the subspace generated by $\{w \in \mathcal{B} \mid \delta(w) \leq i\}$. Notice that, if we set $\mathcal{F} = \mathcal{F}(\delta, \mathcal{B})$, then $d_{\mathcal{F}}(w) = \delta(w)$ for all $w \in \mathcal{B}$. More precisely, if $v \in V$ and we write v as a linear combination of elements of $\mathcal{B}$, i. e. $v = \sum c_r w_r$, then $d_{\mathcal{F}}(v) = \sup\{\delta(w_r) \mid c_r \neq 0\}$. It is then clear that

$$(3.1) \qquad s_{\mathcal{F}}(v) = \sum_{\delta(w_r)=d_{\mathcal{F}}(v)} c_r w_r + V_{<d_{\mathcal{F}}(v)} = \sum_{\delta(w_r)=d_{\mathcal{F}}(v)} c_r s_{\mathcal{F}}(w_r).$$

Conversely, given a filtration $\mathcal{F}$ on V and a basis $\mathcal{B}$ of V, then setting $\delta = d_{\mathcal{F}}|_{\mathcal{B}}$ one can construct the filtration $\mathcal{F}(\delta, \mathcal{B})$. In general $\mathcal{F}$ differs from $\mathcal{F}(\delta, \mathcal{B})$, indeed we have:

Lemma 3.2. *If $\mathcal{F}$ is a filtration on V, $\mathcal{B}$ is a basis of V, and $\delta = d_{\mathcal{F}}|_{\mathcal{B}}$, then $\mathcal{F} = \mathcal{F}(\delta, \mathcal{B})$ if and only if $\{s_{\mathcal{F}}(w) \mid w \in \mathcal{B}\}$ is a basis of $Gr_{\mathcal{F}}(V)$.*

Proof. If $\mathcal{F} = \mathcal{F}(\delta, \mathcal{B})$ then (3.1) says that $\{s_{\mathcal{F}}(w) \mid w \in \mathcal{B}\}$ generates $Gr_{\mathcal{F}}(V)$. If $\sum c_r s_{\mathcal{F}}(w_r) = 0$, then we can assume that $d_{\mathcal{F}}(w_r) = \delta(w_r) = i$ for all r. This implies that $\sum c_r w_r \in V_{<i}$, but then $\sum c_r w_r = \sum d_s w_s$ with $\delta(w_s) < i$ for all s, and this implies that $c_r = 0$ for all r.

Suppose now that $\{s_{\mathcal{F}}(w) \mid w \in \mathcal{B}\}$ is a basis of $Gr_{\mathcal{F}}(V)$ and, for $i \in I$, set V_i' to be the space generated by the set $\{w \in \mathcal{B} \mid \delta(w) \leq i\}$. Clearly $V_i' \subset V_i$, and we need to prove equality. Set i_0 to be the minimum element of I such that $V_{i_0}' \subsetneq V_{i_0}$ and fix $v \in V_{i_0} \setminus V_{i_0}'$. By the minimality of i_0 we have that $d_{\mathcal{F}}(v) = i_0$, so $s_{\mathcal{F}}(v) = \sum c_r s_{\mathcal{F}}(w_r)$ with $d_{\mathcal{F}}(w_r) = \delta(w_r) = i_0$. It follows that $v' = v - \sum c_r w_r \in V_{<i_0}$; thus, by the minimality of i_0, we have that $v' = \sum d_s w_s$ with $d_{\mathcal{F}}(w_s) < i_0$. This implies that $v = \sum c_r w_r + \sum d_s w_s \in V_{i_0}'$, which is absurd. $\qquad\square$

A consequence of Lemma 3.1 is the following fact:

Lemma 3.3. *Fix a filtration $\mathcal{F}$ on V and a basis $\mathcal{B}$ such that $\{s_{\mathcal{F}}(v) \mid v \in \mathcal{B}\}$ is a basis of $Gr_{\mathcal{F}}(V)$. Suppose further that a subset $\{v_j\}_{j \in J}$ of V is given having the property that for each $j \in J$ there is $w_j \in \mathcal{B}$ and a nonzero constant c_j such that $s_{\mathcal{F}}(v_j) = c_j s_{\mathcal{F}}(w_j)$.*

Then if the map $j \mapsto w_j$ is bijective, the set $\{v_j\}_{j \in J}$ is a basis of V.

Proof. Indeed $\{s_{\mathcal{F}}(v_j)\} = \{c_j s_{\mathcal{F}}(w_j)\}$. Since the map $j \mapsto w_j$ is bijective (and $c_j \neq 0$ for all j), it follows that $\{s_{\mathcal{F}}(v_j)\}$ is a basis of $Gr_{\mathcal{F}}(V)$. The result now follows from Lemma 3.1. □

The next result will be used in § 6. Suppose that there is a map $\phi : I \to \mathbb{N}$; we can define a new filtration (over $\mathbb{N}$) on V by setting, for $k \in \mathbb{N}$,

$$V_{(k)} = \sum_{\phi(i) \leq k} V_i.$$

Analogously we can define a new filtration on $Gr_{\mathcal{F}}(V)$ by setting

$$V_{[k]} = \sum_{\phi(i) \leq k} V^i.$$

Lemma 3.4. *Suppose that the map $\phi : I \to \mathbb{N}$ is order preserving and that $V_{(k)}$ is finite dimensional. Then $V_{[k]}$ is finite dimensional and*

$$\dim V_{(k)} = \dim V_{[k]}.$$

Proof. If $V_{(k)} = 0$ there is nothing to prove. If $V_{(k)} \neq 0$ then we define a sequence $i_0 < \cdots < i_h < \cdots$ in I as follows: set i_0 to be the minimal element of I; if i_h is given we set $i_{h+1} \in I$ to be the minimal $i \in I$ such that $i > i_h$, and $V^i \neq 0$. We observe that $V^{i_{h+1}} = V_{i_{h+1}}/V_{i_h}$ and, obviously, $V^{i_0} = V_{i_0}$. Indeed suppose that there is $i_h < i < i_{h+1}$ such that $V_{i_h} \subsetneq V_i$ and choose the minimal one, so that $V_{<i} \subset V_{i_h}$. By the definition of i_{h+1} we obtain that $V^i = 0$, so $V_i = V_{<i} \subset V_{i_h}$.

Since $\dim V_{i_h}/V_{i_{h-1}} = \dim V^{i_h}$, it is clear that $\dim V_{i_h} \geq h$, so there must be $r \in \mathbb{N}$ such that $\phi(i_r) \leq k$ and $\phi(i_h) > k$ for all $h > r$.

We claim that $V_{i_r} = V_{(k)}$. In fact, if $\phi(i) \leq k$, then $i < i_{r+1}$; thus $V_{(k)} \subset V_{<i_{r+1}} = V_{i_r}$. Our claim follows.

We now prove that

$$V_{[k]} = \bigoplus_{h=0}^{r} V^{i_h}.$$

This will conclude the proof. If $i \in I$ is such that $\phi(i) \leq k$ and $i \neq i_h$ for all $h = 1, \ldots, r$, then there is $0 \leq h \leq r$ such that $i_h < i < i_{h+1}$. The definition of i_{h+1} implies that $V^i = 0$. □

We now specialize to $\mathfrak{U}$. We will make extensive use of the filtrations on $\mathfrak{U}$ defined in [2]. In fact we will define two filtrations on $\mathfrak{U}$ and then relate them in Proposition 3.5.

If $\lambda = \sum a_i \alpha_i \in Q$ we let $|\lambda| = \sum_i |a_i|$; so, in particular, if $\alpha \in R^+$, then $|\alpha|$ is the height of the root. Suppose that $u = Y^J K_\lambda X^I$ is a standard monomial; then we define the total height of u to be

$$d_0(u) = \sum_{r=1}^{N} (i_r + j_r)|\beta_r|,$$

and if $\alpha \in R$ we let $o(\alpha, u)$ be the exponent of X_α in u. Then the total degree of u is the $(2N + 1)$-uple

$$d(u) = (o(-\beta_N, u), \ldots, o(-\beta_1, u), o(\beta_1, u), \ldots, o(\beta_N, u), d_0(u)).$$

We order the semigroup $\mathbb{N}^{2N+1}$ by the lexicographic order $<$ defined by

$$(i_1, \ldots, i_{2N+1}) < (j_1, \ldots, j_{2N+1})$$

if and only if there is $K \geq 0$ such that $i_{2N+1-k} = j_{2N+1-k}$ for $k < K$ and $i_{2N+1-K} < j_{2N+1-K}$.

Using the standard monomials as $\mathcal{B}$ and the total degree d as δ, we obtain a filtration $\mathcal{F}$ on $\mathfrak{U}$ as described after Lemma 3.1. It has been proven (see [2]) that the filtration $\mathcal{F}$ is an algebra filtration, meaning by this that $\mathfrak{U}_D \mathfrak{U}_{D'} \subset \mathfrak{U}_{D+D'}$ for all $D, D' \in \mathbb{N}^{2N+1}$.

In the rest of the paper we let $Gr(\mathfrak{U})$ denote $Gr_{\mathcal{F}}(\mathfrak{U})$, and, if $u \in \mathfrak{U}$, we set $d(u) = d_{\mathcal{F}}(u)$ and $\hat{u} / s_{\mathcal{F}}(u)$. Since the filtration $\mathcal{F}$ is an algebra filtration, then $Gr(\mathfrak{U})$ is an algebra itself; indeed it is the algebra over $\mathbb{C}(q)$ with generators $\hat{X}_\alpha$ ($\alpha \in R$) and $\hat{K}_\lambda$ ($\lambda \in Q$) subject to the relations described in [2], Proposition 10.1. Thus $Gr(\mathfrak{U})$ is a semi-commutative algebra (here we use the terminology of § 3.7 of [9]). It is finitely generated with generating subspace

(3.2) $$Gr(\mathfrak{U})_1 = \langle \hat{X}_\alpha, \hat{K}_{\pm\alpha_i} \mid \alpha \in R, i = 1, \ldots, n \rangle.$$

We can define another filtration $\mathcal{F}_0$ (over $\mathbb{N}$) of $\mathfrak{U}$, still using the standard monomials as $\mathcal{B}$ but choosing the total height d_0 as δ. This is also an algebra filtration, and if $u \in \mathfrak{U}$, we will use $d_0(u)$ for $d_{\mathcal{F}_0}(u)$ and $\overline{u}$ for $s_{\mathcal{F}_0}(u)$. We set also $\overline{\mathfrak{U}} = Gr_{\mathcal{F}_0}(\mathfrak{U})$.

If $\mathcal{B}$ is the basis of standard monomials for $\mathfrak{U}$, then, by Lemma 3.2, the set $\overline{\mathcal{B}} = \left\{ \overline{Y^R K_\lambda X^T} \right\}$ is a basis of $\overline{\mathfrak{U}}$. We take $\overline{\delta} : \overline{\mathcal{B}} \to \mathbb{N}^{2N+1}$ to be defined by $\overline{\delta}\left(\overline{Y^R K_\lambda X^T}\right) = d(Y^R K_\lambda X^T)$ and let $\overline{\mathcal{F}} = \mathcal{F}(\overline{\delta}, \overline{\mathcal{B}})$. This is a filtration on $\overline{\mathfrak{U}}$, and we denote by $\overline{d}$ its degree map. By Lemma 3.2, the set $\{ s_{\overline{\mathcal{F}}}\left(\overline{Y^R K_\lambda X^T}\right) \}$ is a basis of $Gr_{\overline{\mathcal{F}}}(\overline{\mathfrak{U}})$, so we can define a linear isomorphism $\Phi : Gr_{\overline{\mathcal{F}}}(\overline{\mathfrak{U}}) \to Gr(\mathfrak{U})$ by setting $\Phi(s_{\overline{\mathcal{F}}}\left(\overline{Y^R K_\lambda X^T}\right)) = (Y^R K_\lambda X^T)\hat{}$.

We claim that, if $u \in \mathfrak{U}$, then $\Phi(s_{\overline{\mathcal{F}}}(\overline{u})) = \hat{u}$. Let us check this: set $D = d(u)$ and $m = d_0(u)$ and write u in terms of the standard monomials,

$$u = \sum_{d(Y^R K_\lambda X^T)=D} h_{RT\lambda} Y^R K_\lambda X^T + \sum_{\substack{d(Y^R K_\lambda X^T)<D \\ d_0(Y^R K_\lambda X^T)=m}} k_{RT\lambda} Y^R K_\lambda X^T + u'$$

with $d_0(u') < m$. Clearly

$$\overline{u} = \sum_{d(Y^R K_\lambda X^T)=D} h_{RT\lambda} \overline{Y^R K_\lambda X^T} + \sum_{\substack{d(Y^R K_\lambda X^T)<D \\ d_0(Y^R K_\lambda X^T)=m}} k_{RT\lambda} \overline{Y^R K_\lambda X^T};$$

thus, by the definition of $\overline{\mathcal{F}}$, we have that $\overline{d}(\overline{u}) = D$. It follows that

$$s_{\overline{\mathcal{F}}}(\overline{u}) = \sum_{d(Y^R K_\lambda X^T)=D} h_{RT\lambda} s_{\overline{\mathcal{F}}}\left(\overline{Y^R K_\lambda X^T}\right);$$

thus

$$\Phi(s_{\overline{\mathcal{F}}}(\overline{u})) = \sum_{d(Y^R K_\lambda X^T)=D} h_{RT\lambda} (Y^R K_\lambda X^T)\hat{},$$

which is precisely $\hat{u}$. The above discussion aims at the following result:

Proposition 3.5. *Suppose that $\mathcal{C}$ is a basis of $\mathfrak{U}$ such that $\{\hat{v} \mid v \in \mathcal{C}\}$ is a basis of $Gr(\mathfrak{U})$. Define $\delta : \mathcal{C} \to \mathbb{N}$ by setting $\delta(v) = d_0(v)$ for all $v \in \mathcal{C}$. Then $\mathcal{F}_0 = \mathcal{F}(\delta, \mathcal{C})$.*

Proof. By Lemma 3.2, it is enough to check that $\{\overline{v} \mid v \in \mathcal{C}\}$ is a basis of $\overline{\mathfrak{U}}$. Since

$$\Phi(\{s_{\overline{\mathcal{F}}}(\overline{v}) \mid v \in \mathcal{C}\}) = \{\hat{v} \mid v \in \mathcal{C}\}$$

and since Φ is an isomorphism, it follows that $\{s_{\overline{\mathcal{F}}}(\overline{v}) \mid v \in \mathcal{C}\}$ is a basis of $Gr_{\overline{\mathcal{F}}}(\overline{\mathfrak{U}})$. Thus, by Lemma 3.1, $\{\overline{v} \mid v \in \mathcal{C}\}$ is a basis of $\overline{\mathfrak{U}}$. $\square$

The filtration on $\mathfrak{U}$ can be used to induce a filtration on modules. So for M any finitely generated left $\mathfrak{U}$-module and M_0 a finite dimensional generating subspace of M, then, as usual, we can define a filtration $\mathcal{F}_M$ on M by setting, for $D \in \mathbb{N}^{2N+1}$, $M_D = \mathfrak{U}_D \cdot M_0$. Since $\mathcal{F}$ is an algebra filtration, the space $Gr_{\mathcal{F}_M}(M)$ is a graded $Gr(\mathfrak{U})$-module that we denote by $Gr(M)$. If there is no chance for confusion we set $d(v) = d_{\mathcal{F}_M}(v)$ and $\hat{v} = s_{\mathcal{F}_M}(v)$ for all $v \in M$. In particular, if $\mathcal{L}$ is a left ideal in $\mathfrak{U}$, then we always choose $(\mathbb{C}(q) + \mathcal{L})/\mathcal{L}$ as a generating subspace of $\mathfrak{U}/\mathcal{L}$ and denote $v_{\mathcal{L}} = (1 + \mathcal{L})\widehat{}$.

4. THE QUANTIZATION OF $\mathfrak{k}$

Let $\mathfrak{k}$ denote the subalgebra of $\mathfrak{g}$ whose set of simple roots is

$$\Delta(\mathfrak{k}) = \{2e_1\} \cup \{e_i - e_{i+1} \mid i = 2, \ldots, n\} \cup \{2e_n\}.$$

In the classical setting $\mathfrak{k}$ is the complexification of a maximal compact subalgebra of $\mathfrak{sp}(n-1, 1)$. We now define a quantization of $\mathfrak{k}$ by giving a set of elements of $\mathfrak{U}$ that, when $q = 1$, reduces to a set of generators for $\mathfrak{k}$.

Set

$$(4.1) \qquad X_0 = q^{-1} X_{e_1 - e_2} X_{e_1 + e_2} - q X_{e_1 + e_2} X_{e_1 - e_2} \qquad Y_0 = \Omega(X_0)$$

and

$$(4.2) \qquad K = \{X_0, Y_0\} \cup \{K_i^{\pm 1} \mid i = 1, \ldots, n\} \cup \{E_i, F_i \mid i = 2, \ldots, n\}.$$

We let $\mathfrak{K}$ denote the subalgebra of $\mathfrak{U}$ generated by K.

We now describe the characterization of a quantization of $\mathfrak{k}$ by means of invariant vectors. The left ideal generated by $\mathfrak{k}$ in $U(\mathfrak{g})$ is the annihilator of the vectors that realize the trivial representation in the class one representations for $(\mathfrak{g}, \mathfrak{k})$. This is the characterization of $\mathfrak{k}$ that we want to quantize. Thus we need a characterization of the analogs of the $\mathfrak{k}$-spherical vectors.

Let $\mathfrak{m}$ denote the Levi component of the parabolic subalgebra corresponding to the simple roots in $\Delta(\mathfrak{k})$ that belong to Δ. Thus a set of simple roots for $\mathfrak{m}$ is

$$(4.3) \qquad \Delta(\mathfrak{m}) = \{e_i - e_{i+1} \mid i = 2, \ldots, n\} \cup \{2e_n\}.$$

We observe that

$$(4.4) \qquad \mathfrak{m} = \mathfrak{d} \oplus \mathfrak{m}_0$$

where $\mathfrak{d} = \mathbb{C}h_{2e_1}$ and $\mathfrak{m}_0 = [\mathfrak{m}, \mathfrak{m}]$.

Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{U}$ generated by $K_i^{\pm 1}$ $(i = 1, \ldots, n)$ and by E_i, F_i with $i \geq 2$. We let $\mathfrak{M}_0$ be the subalgebra generated by $K_i^{\pm 1}, E_i, F_i$ with $i \geq 2$. Note that $\mathfrak{K}$ is the subalgebra generated by X_0, Y_0, and $\mathfrak{M}$.

If $k \in \mathbb{N}$ we let

$$(4.5) \qquad \lambda_k = k(e_1 + e_2).$$

Clearly $\lambda_k \in P^+$. These are exactly the highest weights of the class one representations of $(\mathfrak{g}, \mathfrak{k})$. By a classical branching theorem ([6]) the $\mathfrak{k}$-invariant vectors in $F(\lambda_k)$ are exactly the vectors that are invariant for both $\mathfrak{m}$ and $\mathfrak{m}_0$. We now show

that the trivial representation of $\mathfrak{M}$ and $\mathfrak{M}_0$ occurs in $V(\lambda_k)$ exactly once, and then we will show that we must choose the $\mathfrak{M}$-spherical vectors in $V(\lambda_k)$ as analogs of the $\mathfrak{k}$-invariant vectors.

Set $P_0 = \sum_{i=2}^{n} \mathbb{Z} e_i$ to be the lattice of integral weights of $\mathfrak{m}_0$, P_0^+ the set of dominant weights, and Q_0 the root lattice. If $\lambda_0 \in P_0^+$ we let $F_0(\lambda_0)$ denote the irreducible finite dimensional $\mathfrak{m}_0$-module of highest weight λ_0. Let $V_0(\lambda_0)$ be the finite dimensional irreducible representation of $\mathfrak{M}_0$ of highest weight λ_0.

Clearly $\mathfrak{M}_0$ is isomorphic to $\mathfrak{U}_q(\mathfrak{sp}(n-1))$. By Lusztig's theorem (Theorem 2.1), the character of $V_0(\lambda_0)$ is equal to the character of $F_0(\lambda_0)$.

Let $\psi : P \to P_0$ denote the restriction of weights to $\sqrt{-1}\mathfrak{t}_0 \cap \mathfrak{m}_0$. We extend ψ to the characters by setting $\psi(e^\lambda) = e^{\psi(\lambda)}$. It is clear that, if $\lambda \in P^+$, then $\psi(\chi(F(\lambda)))$ is the character of $F(\lambda)$ as an $\mathfrak{m}_0$-module. If M is an admissible $\mathfrak{U}$-module, $v \in M_\mu$, and $K_\nu \in \mathfrak{U}^0 \cap \mathfrak{M}_0$, it is easy to check that $K_\nu v = q^{(\psi(\mu),\nu)} v$. It follows that the character of M as an $\mathfrak{M}_0$-module is equal to $\psi(\chi(M))$, hence the character of $V(\lambda)$ as an $\mathfrak{M}_0$-module is equal to the character of $F(\lambda)$ as an $\mathfrak{m}_0$-module. In particular we have that $V_0(\lambda_0)$ occurs in $V(\lambda)$ with the same multiplicity by which $F_0(\lambda_0)$ occurs in $F(\lambda)$.

To obtain the same result for $\mathfrak{M}$, we first introduce PBW-bases for $\mathfrak{M}_0$ and $\mathfrak{M}$. Let $R(\mathfrak{m})$ denote the set of roots of $\mathfrak{m}$. We have chosen the set $\Delta(\mathfrak{m})$ defined in (4.3) as a set of simple roots for $R(\mathfrak{m})$, and we let $R^+(\mathfrak{m})$ be the corresponding set of positive roots. In order to distinguish the root vectors that correspond to roots in $R(\mathfrak{m})$ we set $F_\alpha = X_{-\alpha}$ and $E_\alpha = X_\alpha$ whenever $\alpha \in R^+(\mathfrak{m})$, and, if $J \in \mathbb{N}^{N'}$, we set $E^J = \prod_{i=1}^{N'} E_{\gamma_i}^{j_i}$, where N' is the number of positive roots for $\mathfrak{M}$. Let F^J denote $\Omega(E^J)$. If $\lambda \in Q$ then we can form the monomials $F^J K_\lambda E^{J'}$; note that these monomials are particular standard monomials of $\mathfrak{U}$.

Proposition 4.1. *The monomials $F^J K_{\lambda_0} E^{J'}$ with $\lambda_0 \in Q_0$ form a basis for $\mathfrak{M}_0$. The standard monomials $F^J K_\lambda E^{J'}$ with $\lambda \in Q$ form a basis of $\mathfrak{M}$.*

Proof. Since $s^{(1)}$ (see (2.3)) fixes the roots of $\mathfrak{m}$ and since $s^{(2)} \ldots s^{(n-1)} s_n$ is the longest element of the Weyl group of $\mathfrak{m}$, the first result follows from the PBW theorem for $\mathfrak{M}_0$ and [2, § 9.5].

For $\mathfrak{M}$ one has to argue as follows. Let $\mathfrak{M}'$ denote the span of the standard monomials $F^R K_\lambda E^T$ with $\lambda \in Q$. Clearly $\mathfrak{M}' \subset \mathfrak{M}$ and $E_i, F_i \in \mathfrak{M}'$ if $i \geq 2$, while $K_i \in \mathfrak{M}'$ for all i. To conclude the proof it is enough to check that $\mathfrak{M}'$ is an algebra, so set $u = F^R K_\lambda E^T$ and $v = F^J K_\mu E^{J'}$. If we write $\lambda = \sum n_i \alpha_i$ and $\mu = \sum m_i \alpha_i$, then we set $\lambda_0 = \sum_{i>1} n_i \alpha_i$ and $\mu_0 = \sum_{i>1} m_i \alpha_i$. By the relations defining $\mathfrak{U}$ we know that there is $k \in \mathbb{Z}$ such that $uv = q^k K_1^{n_1+m_1} F^R K_{\lambda_0} E^T F^J K_{\mu_0} E^{J'}$. Since the monomials $F^K K_{\nu_0} E^L$ form a basis of $\mathfrak{M}_0$, we can write

$$uv = q^k K_1^{n_1+m_1} \left(\sum c_{KL\nu_0} F^K K_{\nu_0} E^L \right).$$

If we write $\nu = (n_1 + m_1)\alpha_1 + \nu_0$, then we can find constants $c'_{KL\nu}$ such that

$$uv = \sum c'_{KL\nu} F^K K_\nu E^L.$$

This concludes the proof. $\qquad\square$

If u is in the $\mathfrak{U}^0$-weight space $\mathfrak{U}_\mu$, we set $\lambda(u) = \mu$. Notice that $\lambda(X^R) = \sum r_i \beta_i$ and $\lambda(Y^R) = -\sum r_i \beta_i$. Notice also that, if $\lambda = \lambda(E^L)$, then $F^K K_\nu E^L =$

$q^{(\lambda,\nu)}F^K E^L K_\nu$. It follows easily that, if one defines

$$(4.6) \qquad\qquad N_{RT\lambda} = F^R E^T K_\lambda,$$

then the set $\{N_{RT\lambda}\}$ is a basis of $\mathfrak{M}$.

We now turn to considering the $\mathfrak{U}^0$-admissible finite dimensional representations of $\mathfrak{M}$. We note that we can extend the action of $\mathfrak{M}_0$ on $V_0(\lambda_0)$ to $\mathfrak{M}$ as follows:

Lemma 4.2. *Set $\mu_k = ke_1$. If $\mu \in P_0$ and $v \in V_0(\lambda_0)_\mu$, define*

$$\pi_k(F^R E^T K_\lambda)(v) = q^{(\lambda,\mu+\mu_k)}\pi(F^R E^T)(v),$$

where π denotes the action of $\mathfrak{M}_0$ on $V_0(\lambda_0)$.

Then $(\pi_k, V_0(\lambda_0))$ is a representation of $\mathfrak{M}$.

Proof. Since the monomials $F^R E^T K_\lambda$ form a basis of $\mathfrak{M}$, this action is certainly well defined. If $\lambda \in Q_0$ then $(\lambda, \mu_k) = 0$, so, if $u \in \mathfrak{M}_0$ and $v \in V_0(\lambda_0)$, then $\pi_k(u)(v) = \pi(u)(v)$. Moreover, if $\lambda \in Q$, $\mu \in P_0$, and $v \in V_0(\lambda_0)_\mu$, then $\pi_k(uK_\lambda)(v) = q^{(\lambda,\mu_k+\mu)}\pi(u)(v)$.

It remains to prove that

$$(4.7) \qquad \pi_k(F^R E^T K_\lambda)(\pi_k(F^H E^L K_\mu)(v)) = \pi_k(F^R E^T K_\lambda F^H E^L K_\mu)(v).$$

If $v \in V_0(\lambda_0)_\nu$ then $\pi(F^H E^L)(v) \in V_0(\lambda_0)_{\nu+\gamma}$, where $\gamma = \lambda(F^H E^L)$; thus

$$\begin{aligned}
\pi_k(F^R E^T K_\lambda)&(\pi_k((F^H E^L K_\mu)(v)) \\
&= q^{(\lambda,\nu+\gamma+\mu_k)}q^{(\mu,\nu+\mu_k)}\pi(F^R E^T)(\pi((F^H E^L)(v)).
\end{aligned}$$

On the other hand

$$\pi_k(F^R E^T K_\lambda F^H E^L K_\mu)(v) = q^{(\lambda,\gamma)}q^{(\lambda+\mu,\nu+\mu_k)}\pi(F^R E^T F^H E^L)(v).$$

Since $\pi(F^R E^T F^H E^L)(v) = \pi(F^R E^T)(\pi(F^H E^L)(v))$, we have (4.7). $\qquad\square$

We let $V(\lambda_0, k)$ denote the representation of $\mathfrak{M}$ we just defined.

Now let V be any $\mathfrak{U}^0$-admissible finite dimensional representation of $\mathfrak{M}$ and set

$$V_k = \{v \in V \mid K_{2e_1} \cdot v = q^{2k}v\}.$$

By the admissibility of V it is clear that $V = \bigoplus_k V_k$ and, since $[K_{2e_1}, \mathfrak{M}] = 0$, V_k is a $\mathfrak{U}^0$-admissible $\mathfrak{M}$-module. If $\lambda \in P$ and $(V_k)_\lambda \neq 0$, pick $v \in (V_k)_\lambda$ such that $v \neq 0$. If we write $\lambda = \sum n_i e_i$, then $K_{2e_1} \cdot v = q^{2n_1}v = q^{2k}v$; thus $n_1 = k$. If we consider V_k as an admissible $\mathfrak{M}_0$-module and set $\lambda_0 = \lambda - \mu_k = \sum_{i\geq 2} n_i e_i$, then it is clear that $(V_k)_\lambda \subset (V_k)_{\lambda_0}$ and that, if $(V_k)_{\lambda'} \cap (V_k)_{\lambda_0} \neq 0$ for some $\lambda' \in P$, then $\lambda' = \lambda_0 + \mu_k = \lambda$. Since $(V_k)_{\lambda_0} = \bigoplus_{\lambda'}(V_k)_{\lambda'} \cap (V_k)_{\lambda_0}$, it follows that $(V_k)_\lambda = (V_k)_{\lambda_0}$. Since V_k is $\mathfrak{U}^0 \cap \mathfrak{M}_0$-admissible, then, as a $\mathfrak{M}_0$-module, it is completely reducible and its constituents are of the type $V_0(\lambda_0)$. If $W \simeq V_0(\lambda_0)$ is such a constituent, then

$$W = \bigoplus_{\nu_0 \in P_0} W_{\nu_0} = \bigoplus W \cap (V_k)_{\nu_0} = \bigoplus W \cap (V_k)_{\nu_0+\mu_k};$$

thus, if $w \in W_{\nu_0}$ and $\lambda \in Q$, then $F^R E^T K_\lambda \cdot w = q^{(\lambda,\nu_0+\mu_k)}F^R E^T \cdot w \in W$. This implies that W is $\mathfrak{M}$-invariant and, as an $\mathfrak{M}$-module, $W \simeq V(\lambda_0, k)$.

This in turns proves the following: if V is a $\mathfrak{U}^0$-admissible finite dimensional $\mathfrak{M}$-module, then it is completely reducible and its constituents are of the type $V(\lambda_0, k)$.

Since $\chi(V(\lambda_0, k)) = e^{\mu_k}\chi(V_0(\lambda_0))$, then the character of $V(\lambda_0, k)$ is equal to the character of the irreducible $\mathfrak{m}$-module $F(\lambda_0, k) = \mathbb{C}_k \otimes F_0(\lambda_0)$, where $\mathbb{C}_k$ denotes the one dimensional representation of $\mathfrak{d}$ such that $h_{2e_1} \cdot v = 2kv$ for all $v \in \mathbb{C}_k$ (recall the decomposition (4.4)).

Since $V(\lambda)$ is an admissible $\mathfrak{M}$-module, we can write

$$\chi(V(\lambda)) = \sum_{k, \lambda_0} m_{k\lambda_0}\, \chi(V(\lambda_0, k))$$

$$= \sum_{k, \lambda_0} m_{k\lambda_0}\, \chi(F(\lambda_0, k)).$$

Since $\chi(V(\lambda)) = \chi(F(\lambda))$ we deduce from the linear independence of irreducible characters that the multiplicity of $V(\lambda_0, k)$ in $V(\lambda)$ is equal to the multiplicity of $F(\lambda_0, k)$ in $F(\lambda)$. We thus have proved:

Proposition 4.3. *The trivial representations of $\mathfrak{M}$ and of $\mathfrak{M}_0$ occur in $V(\lambda_k)$ exactly once.*

Proof. By the classical branching theorem ([6]) from $\mathfrak{sp}(n)$ to $\mathfrak{sp}(1) \times \mathfrak{sp}(n-1)$, we obtain that the multiplicities of the trivial representations of $\mathfrak{m}$ and $\mathfrak{m}_0$ in $F(\lambda_k)$ are both equal to one. $\qquad\square$

We are now ready to define a quantization of $U(\mathfrak{g})\mathfrak{k}$. Let $v_k \in V(\lambda_k)$ be such that $\mathbb{C}(q)v_k$ realizes the trivial representation of $\mathfrak{M}$ in $V(\lambda_k)$. Let ϵ be the homomorphism from $\mathfrak{U}$ to $\mathbb{C}(q)$ defined by

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = \epsilon(K_i^{-1}) = 1;$$

we recall that ϵ is the counit for a Hopf algebra structure of $\mathfrak{U}$. Let $\mathcal{A} = \mathrm{Ker}(\epsilon)$ denote the augmentation ideal of $\mathfrak{U}$.

Let $\mathcal{I}$ be the left ideal of $\mathfrak{U}$ defined by:

$$(4.8) \qquad\qquad \mathcal{I} = \{u \in \mathcal{A} \mid u \cdot v_k = 0 \ \forall\, k\}$$

We require that the representations $V(\lambda_k)$ be spherical for a quantization of $\mathfrak{k}$. We also require that $\mathfrak{M}$ should be contained in our quantization, so, by Proposition 4.3, the vectors v_k must be invariant for the quantization of $\mathfrak{k}$. Therefore we call $\mathcal{I}$ the quantization of $U(\mathfrak{g})\mathfrak{k}$.

We now relate $\mathcal{I}$ with the quantization $\mathfrak{K}$ of $\mathfrak{k}$ described after (4.2). We first show that $\mathfrak{K} \cap \mathcal{A} \subset \mathcal{I}$.

Lemma 4.4.

1. $[X_0, \mathfrak{M}_0] = [Y_0, \mathfrak{M}_0] = 0.$
2. $X_0, Y_0 \in \mathcal{I}.$
3. $\mathfrak{K} \cap \mathcal{A} \subset \mathcal{I}.$

Proof. First of all we observe that $X_0 \in \mathfrak{U}_{2e_1}$, therefore $[X_0, K_i] = 0$ if $i \geq 2$. Furthermore, from (A.2), (A.3), (A.5), (A.6), (A.8), (A.9), (A.11), (A.12) we obtain

that $[X_0, E_i] = [X_0, F_i] = 0$ for $i > 2$, and that, if $n > 2$,

$$\begin{aligned}
X_0 E_2 &= X_{e_1-e_2} E_2 X_{e_1+e_2} - q X_{e_1+e_2} X_{e_1-e_2} E_2 \\
&= q^{-1} E_2 X_{e_1-e_2} X_{e_1+e_2} - X_{e_1-e_3} X_{e_1+e_2} \\
&\quad - X_{e_1+e_2} E_2 X_{e_1-e_2} + q X_{e_1+e_2} X_{e_1-e_3} \\
&= q^{-1} E_2 X_{e_1-e_2} X_{e_1+e_2} - X_{e_1-e_3} X_{e_1+e_2} \\
&\quad - q E_2 X_{e_1+e_2} X_{e_1-e_2} + q X_{e_1+e_2} X_{e_1-e_3} \\
&= E_2 X - X_{e_1-e_3} X_{e_1+e_2} + q X_{e_1+e_2} X_{e_1-e_3}.
\end{aligned}$$

It follows that $[X_0, E_2] = -X_{e_1-e_3} X_{e_1+e_2} + q X_{e_1+e_2} X_{e_1-e_3}$. By [2, Theorem 9.3], we have that $-X_{e_1-e_3} X_{e_1+e_2} + q X_{e_1+e_2} X_{e_1-e_3} = 0$.

An analogous computation works for $[X_0, F_2]$:

$$\begin{aligned}
[X_0, F_2] &= q^{-1} X_{e_1-e_2} [X_{e_1+e_2}, F_2] - q [X_{e_1+e_2}, F_2] X_{e_1-e_2} \\
&= -q^{-2} X_{e_1-e_2} X_{e_1+e_3} K_{e_2-e_3} + X_{e_1+e_3} K_{e_2-e_3} X_{e_1-e_2} \\
&= -q^{-2} (X_{e_1-e_2} X_{e_1+e_3} - q X_{e_1+e_3} X_{e_1-e_2}) K_{e_2-e_3} = 0.
\end{aligned}$$

If $n = 2$ one obtains the same result using formulas (A.2), (A.3), (A.8), (A.9) instead.

Since $\Omega(\mathfrak{M}_0) = \mathfrak{M}_0$, it follows that

$$[Y_0, \mathfrak{M}_0] = [\Omega(X_0), \Omega(\mathfrak{M}_0)] = \Omega([\mathfrak{M}_0, X_0]) = 0,$$

and the proof of the first statement is complete.

By Proposition 4.3, we have that $X_0 \cdot v_k = c v_k$ for some $c \in \mathbb{C}(q)$. Since $K_{2e_1} \cdot v_k = v_k$, then

$$c v_k = K_{2e_1} X_0 \cdot v_k = q^4 c v_k;$$

thus $c = 0$. The result for Y_0 is obtained in the same way, thus concluding the proof of the second statement. Finally, the last statement follows from what we just proved by observing that $\mathfrak{M} \cap \mathcal{A} \subset \mathcal{I}$. $\square$

Set

(4.9) $$\mathcal{J} = \mathfrak{U}(\mathfrak{K} \cap \mathcal{A}).$$

The main result of the paper (Theorem 7.3) states that $\mathcal{I} = \mathcal{J}$. The proof of this result spans the next three sections. We now give an immediate corollary: if V is a finite dimensional $\mathfrak{U}$-module, a vector $v \in V$ is said to be $\mathfrak{K}$-spherical if $\mathfrak{K} \cap \mathcal{A} \cdot v = 0$.

Corollary 4.5. *Let $\mathcal{I}_K$ denote the annihilator of all the $\mathfrak{K}$-spherical vectors. Then*

$$\mathcal{I}_K = \mathfrak{U}(\mathfrak{K} \cap \mathcal{A}).$$

Proof. Clearly $\mathcal{I}_K \subset \mathcal{I}$ and $\mathcal{J} \subset \mathcal{I}_K$. $\square$

5. Bases for $\mathfrak{U}/\mathcal{J}$

Our aim is to prove that $\mathcal{I} = \mathcal{J}$. To this purpose we need to determine several bases of $\mathfrak{U}/\mathcal{J}$. More precisely, our argument is divided in three steps:

1. We prove that the Gelfand-Kirillov dimension of $\mathfrak{U}/\mathcal{I}$ is equal to the Gelfand-Kirillov dimension of $\mathfrak{U}/\mathcal{J}$ (see Theorem 6.4).
2. We show that certain elements of $\mathfrak{U}$ act injectively on $\mathfrak{U}/\mathcal{J}$ (see Corollary 5.8).

3. Using step (2) we show that, if there is an element $u \in \mathcal{I} \setminus \mathcal{J}$, then the Gelfand-Kirillov dimension of $\mathfrak{U}/\mathcal{J}$ would be strictly bigger than the Gelfand-Kirillov dimension of $\mathfrak{U}/\mathcal{I}$, and this contradicts step (1).

In order to carry out steps (1) and (3), we need to find a basis $\mathcal{B}^0$ of $\mathfrak{U}/\mathcal{J}$ such that the set $\{\hat{v} \mid v \in \mathcal{B}^0\}$ is a basis of the graded module $Gr(\mathfrak{U}/\mathcal{J})$ (here we are using the notation we set up at the end of § 3). This is the basis given in Proposition 5.6.

For step (2) we compute the action of some elements of $\mathfrak{U}$ on $\mathfrak{U}/\mathcal{J}$ by means of carefully chosen bases of $\mathfrak{U}/\mathcal{J}$. These are the bases described in Proposition 5.7.

We now start the construction of the basis $\mathcal{B}^0$ described above. We begin by finding a basis of $\mathfrak{U}$ other than the basis of standard monomials.

Set $M_{RT} = Y^R X^T$, with $Y^R X^T$ that satisfies the following property:

$$(5.1) \qquad o(\alpha, Y^R X^T) = 0 \quad \text{if } \alpha \text{ is a root of } \mathfrak{m},$$

where we recall that $o(\alpha, u)$ is the exponent of X_α in u. We let $\mathcal{B}'$ denote the set $\{M_{RT} N_{VW\lambda}\}$, where $N_{VW\lambda}$ is defined as in (4.6). An obvious consequence of Lemma 3.1 is the following:

Lemma 5.1. *The set $\mathcal{B}'$ is a basis of $\mathfrak{U}$.*

Proof. By the relations defining $Gr(\mathfrak{U})$ it is clear that there are nonzero constants $c_{RTVW\lambda} \in \mathbb{C}(q)$ such that

$$(M_{RT} N_{VW\lambda})\widehat{} = c_{RTVW\lambda} (F^V Y^R K_\lambda X^T E^W)\widehat{}.$$

By Lemmas 3.1 and 3.2, the result follows. $\qquad\square$

Next we consider the standard monomials $M_{RT}^0 = Y^R X^T$ that satisfy property (5.1) and the following property:

$$(5.2) \qquad \begin{aligned} &\text{if } o(e_1 - e_2, Y^R X^T) \neq 0 \text{ then } o(e_1 + e_2, Y^R X^T) = 0, \\ &\text{if } o(e_1 + e_2, Y^R X^T) \neq 0 \text{ then } o(e_1 - e_2, Y^R X^T) = 0, \\ &\text{if } o(-e_1 - e_2, Y^R X^T) \neq 0 \text{ then } o(-e_1 + e_2, Y^R X^T) = 0, \\ &\text{if } o(-e_1 + e_2, Y^R X^T) \neq 0 \text{ then } o(-e_1 - e_2, Y^R X^T) = 0, \end{aligned}$$

i.e. the root vectors $X_{e_1-e_2}$ and $X_{e_1+e_2}$ cannot occur together in X^T, and the root vectors $X_{-e_1+e_2}$ and $X_{-e_1-e_2}$ cannot occur together in Y^R. In the rest of this section the exponent 0 in a new definition of monomials will always indicate that the corresponding set will also satisfy the above conditions.

We now want to define a basis of $\mathfrak{U}$ that involves explicitly our special elements X_0 and Y_0 defined in (4.2). Set $M_{RTijVW\lambda}^0 = M_{RT}^0 Y_0^i X_0^j N_{VW\lambda}$, and let $\mathcal{C}^0$ denote the set of all the monomials $M_{RTijVW\lambda}^0$.

We wish to prove that $\mathcal{C}^0$ is a basis for $\mathfrak{U}$. For this we need to compute an expression for X_0 and Y_0 in terms of the standard monomials.

Lemma 5.2.

$$(5.3) \quad X_0 = (-q)^{n-1}(q+q^{-1})X_{2e_1} - (q-q^{-1})(\sum_{i=2}^{n}(-q)^{i-2}X_{e_1-e_i}X_{e_1+e_i}),$$

$$(5.3') \quad X_0 = (-q)^{-n+1}(q+q^{-1})X_{2e_1} - (q-q^{-1})(\sum_{i=2}^{n}(-q)^{-i+2}X_{e_1+e_i}X_{e_1-e_i}),$$

$$(5.4) \quad Y_0 = (-q)^{-n+1}(q+q^{-1})X_{-2e_1} + (q-q^{-1})(\sum_{i=2}^{n}(-q)^{-i+2}X_{-e_1-e_i}X_{-e_1+e_i}),$$

$$(5.4') \quad Y_0 = (-q)^{n-1}(q+q^{-1})X_{-2e_1} + (q-q^{-1})(\sum_{i=2}^{n}(-q)^{i-2}X_{-e_1+e_i}X_{-e_1-e_i}).$$

Proof. We prove by induction on $n-j$ that

$$
\begin{aligned}
(5.5) \quad & q^{-1}X_{e_1-e_j}X_{e_1+e_j} - qX_{e_1+e_j}X_{e_1-e_j} \\
& = -(q-q^{-1})\sum_{i=j}^{n}(-q)^{i-j}X_{e_1-e_i}X_{e_1+e_i} + (q+q^{-1})(-q)^{n-j+1}X_{2e_1}.
\end{aligned}
$$

If $j = n$ then, by (A.23),

$$
\begin{aligned}
q^{-1}X_{e_1-e_n}X_{e_1+e_n} - qX_{e_1+e_n}X_{e_1-e_n} \\
= -(q-q^{-1})X_{e_1-e_n}X_{e_1+e_n} - q[X_{e_1+e_n}, X_{e_1-e_n}] \\
= -(q-q^{-1})X_{e_1-e_n}X_{e_1+e_n} - q(q+q^{-1})X_{2e_1}.
\end{aligned}
$$

If $j < n$, then, by (A.25),

$$
\begin{aligned}
q^{-1}X_{e_1-e_j}X_{e_1+e_j} - qX_{e_1+e_j}X_{e_1-e_j} \\
= -(q-q^{-1})X_{e_1-e_j}X_{e_1+e_j} - q[X_{e_1+e_j}, X_{e_1-e_j}] \\
= -(q-q^{-1})X_{e_1-e_j}X_{e_1+e_j} \\
- q(q^{-1}X_{e_1-e_{j+1}}X_{e_1+e_{j+1}} - qX_{e_1+e_{j+1}}X_{e_1-e_{j+1}}).
\end{aligned}
$$

One easily verifies that (5.5) satisfies the recursive equation written above.

The other expression for X_0 given in (5.3') is obtained in the same way. One obtains (5.4) and (5.4') by applying Ω to (5.3) and (5.3'). $\square$

Lemma 5.3. *The set $\mathcal{C}^0$ is a basis for $\mathfrak{U}$.*

Proof. Let A denote the set of all $R \in \mathbb{N}^N$ such that X^R and Y^R satisfy properties (5.2) and (5.1). If $R = (r_1, \ldots, r_N) \in A$ and $i \in \mathbb{N}$, we set

$$R^i = (r_1 + i, r_2, \ldots, r_{2n-1} + i).$$

If $R \in \mathbb{N}^N$ and $i \le \min(r_1, r_{2n-1})$, we set

$$R_i = (r_1 - i, r_2, \ldots, r_{2n-1} - i, 0, \cdots, 0).$$

We begin by computing $\hat{M}^0_{RTijVW\lambda}$: by Lemma 5.2 we have

$$\hat{M}^0_{RTijVW\lambda} = (q-q^{-1})^{i+j}\hat{Y}^R\hat{X}^T(\hat{X}_{-e_1-e_2}\hat{X}_{-e_1+e_2})^i(\hat{X}_{e_1-e_2}\hat{X}_{e_1+e_2})^j\hat{F}^V\hat{E}^W\hat{K}_\lambda.$$

Thus, using the relations defining $Gr(\mathfrak{U})$, we find that there is a nonzero constant $c_{RTijVW\lambda}$ such that

$$(5.6) \qquad \hat{M}^0_{RTijVW\lambda} = c_{RTijVW\lambda}\hat{Y}^{(R^i,V)}\hat{K}_\lambda\hat{X}^{(T^j,W)}.$$

In view of Proposition 3.3, it is enough to check that the map

$$f : A \times A \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{N'} \times \mathbb{N}^{N'} \times Q \to \mathbb{N}^N \times Q \times \mathbb{N}^N$$

defined by

$$f : (R, T, i, j, V, W, \lambda) \mapsto ((R^i, V), \lambda, (T^i, W))$$

is bijective. But this is obvious since the inverse of f is

$$g : \mathbb{N}^N \times Q \times \mathbb{N}^N \to A \times A \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{N'} \times \mathbb{N}^{N'} \times Q,$$
$$g : (R, \lambda, T) \mapsto (R_i, T_i, i, j, V, W, \lambda),$$

where

$$i = \min(o(-e_1 - e_2, Y^R), o(-e_1 + e_2, Y^R)),$$
$$j = \min(o(e_1 - e_2, X^T), o(e_1 + e_2, X^T)),$$
$$V = (r_{2n}, \dots, r_N),$$
$$W = (t_{2n}, \dots, t_N).$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We now modify slightly our basis $\{M^0_{RTijVW\lambda}\}$ in order to get a basis of the ideal $\mathcal{A}$. If $\lambda \neq 0$ set $\tilde{N}_{VW\lambda} = F^V E^W (K_\lambda - 1)$, and if $\lambda = 0$ set $\tilde{N}_{VW\lambda} = N_{VW\lambda}$. We set

$$\tilde{M}^0_{RTijVW\lambda} = M^0_{RT} Y_0^i X_0^j \tilde{N}_{VW\lambda}.$$

It is rather easy to check that the set $\{\tilde{M}^0_{RTijVW\lambda}\}$ is a basis of $\mathfrak{U}$, and that $\tilde{M}^0_{RTijVW\lambda} \in \mathcal{A}$ unless $R, T, i, j, V, W, \lambda$ are all zero, in which case it is equal to 1. Since $\mathfrak{U} = \mathbb{C}(q) \oplus \mathcal{A}$, it follows that the set $\{\tilde{M}^0_{RTijVW\lambda}\} \setminus \{1\}$ is a basis of $\mathcal{A}$. Since, by Proposition 4.1, the monomials $N_{VW\lambda}$ are a basis for $\mathfrak{M}$, one can prove analogously that the set $\{\tilde{N}_{VW\lambda}\} \setminus \{1\}$ is a basis of $\mathcal{A} \cap \mathfrak{M}$.

We write $M^0_{RTij} = M^0_{RTij000} = \tilde{M}^0_{RTij000}$ for short, and denote by $\mathbf{P}^0$ the span of the elements M^0_{RTij}.

Lemma 5.4.

$$\mathfrak{U} = \mathbf{P}^0 \oplus \mathfrak{U}(\mathcal{A} \cap \mathfrak{M}).$$

Proof. We need only to prove that $\mathbf{P}^0 \cap (\mathfrak{U}(\mathcal{A} \cap \mathfrak{M})) = 0$.

Suppose that $\sum c_{RTij} M^0_{RTij} \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$; then

$$\sum c_{RTij} M^0_{RTij} = \sum u_{VW\lambda} \tilde{N}_{VW\lambda}$$

with $u_{VW\lambda} \in \mathfrak{U}$ and V, W, λ not all zero. We write $u_{VW\lambda} = \sum d^{VW\lambda}_{RTijHL\mu} M^0_{RTijHL\mu}$ and substituting we obtain that

$$\sum c_{RTij} M_{RTij} = \sum d^{VW\lambda}_{RTijHL\mu} M^0_{RTijHL\mu} \tilde{N}_{VW\lambda}$$
$$= \sum d^{VW\lambda}_{RTijHL\mu} M^0_{RTij} N_{HL\mu} \tilde{N}_{VW\lambda}.$$

Since $\mathcal{A} \cap \mathfrak{M}$ is an ideal of $\mathfrak{M}$, it follows that

$$N_{HL\mu} \tilde{N}_{VW\lambda} = \sum h^{HL\mu VW\lambda}_{AB\nu} \tilde{N}_{AB\nu}$$

with A, B, ν not all zero, and hence

$$\sum c_{RTij} M^0_{RTij} = \sum d^{VW\lambda}_{RTijHL\mu} h^{HL\mu VW\lambda}_{AB\nu} M^0_{RTij} \tilde{N}_{AB\nu}$$
$$= \sum d^{VW\lambda}_{RTijHL\mu} h^{HL\mu VW\lambda}_{AB\nu} \tilde{M}^0_{RTijAB\nu}$$

with A, B, ν not all zero.

This contradicts the linear independence of the $\tilde{M}^0_{RTijVW\lambda}$. $\square$

The following lemma is the crucial technical result of this section: we obtain it by a lengthy calculation that is described in Appendix B.

Lemma 5.5.

$$[X_0, Y_0] \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M}).$$

We finally arrive at the specific basis $\mathcal{B}^0$ for $\mathfrak{U}/\mathcal{J}$ described at the beginning of the section. Set $B^0_{RT} = M^0_{RT} + \mathcal{J} \in \mathfrak{U}/\mathcal{J}$. Let $\mathcal{B}^0$ be the set of the B^0_{RT}.

Proposition 5.6. *The set $\mathcal{B}^0$ is a basis of $\mathfrak{U}/\mathcal{J}$.*

Proof. We need only to prove that the B^0_{RT} are linearly independent, so suppose that $\sum c_{RT} B^0_{RT} = 0$ or, equivalently, that

$$\sum c_{RT} M^0_{RT00} \in \mathcal{J};$$

then we can write

$$\sum c_{RT} M^0_{RT00} = u_1 X_0 + u_2 Y_0 + u_3$$

with $u_1, u_2 \in \mathfrak{U}$ and $u_3 \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$.

Write $u_s = \sum d^s_{RTijVW\lambda} \tilde{M}^0_{RTijVW\lambda}$ $(s = 1, 2)$; we notice that, if $\lambda \neq 0$,

$$\tilde{M}^0_{RTijVW\lambda} X_0 = M^0_{RTij} \tilde{N}_{VW\lambda} X_0$$
$$= q^{(2e_1,\lambda)} M^0_{RTij} X_0 \tilde{N}_{VW\lambda} + (q^{(2e_1,\lambda)} - 1) M^0_{RTij} X_0 \tilde{N}_{VW0}$$

so, replacing u_s by

$$u'_s = \sum d^s_{RTij000} \tilde{M}^0_{RTij000} + \sum_{\lambda \neq 0} (q^{(-1)^{r-1}(2e_1,\lambda)} - 1) d^s_{RTij00\lambda} \tilde{M}^0_{RTij000}$$

for $s = 1, 2$, and u_3 by $u_3 + (u_1 - u'_1) X_0 + (u_2 - u'_2) Y_0$, we can assume that $u_s \in \mathbf{P}^0$ if $s = 1, 2$.

Assuming this, we obtain that

$$\tag{5.7} \sum c_{RT} M^0_{RT00} = \sum d_{RTij} M^0_{RTij} X_0 + \sum f_{RTij} M^0_{RTij} Y_0 + u_3.$$

Recall that $M^0_{RTij} = M^0_{RT} Y^i_0 X^j_0$, so, if $j \neq 0$ then

$$M^0_{RTij} Y_0 = M^0_{RT} Y^i_0 X^j_0 Y_0 = M^0_{RT} Y^i_0 X^{j-1}_0 Y_0 X_0 + M^0_{RT} Y^i_0 X^{j-1}_0 [X_0, Y_0].$$

Write $M^0_{RT} Y^i_0 X^{j-1}_0 Y_0$ in terms of the $\tilde{M}^0_{HLklVW\lambda}$, i.e.

$$M^0_{RT} Y^i_0 X^{j-1}_0 Y_0 = \sum h_{HLklVW\lambda} \tilde{M}^0_{HLklVW\lambda};$$

then, reasoning as for $u_1 X_0$ above, we can write

$$M^0_{RT} Y^i_0 X^{j-1}_0 Y_0 X_0 = \sum h'_{HLkl} M^0_{HLkl} X_0 + u'$$

with $u' \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$. By Lemma 5.5, we obtain that

$$M^0_{RTij}Y_0 = \sum h'_{HLkl}M^0_{HLkl}X_0 + u'' = \sum h'_{HLkl}M^0_{HLk(l+1)} + u''$$

with $u'' \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$. Substituting in (5.7), we find that

$$\begin{aligned}
\sum c_{RT}M^0_{RT00} &= \sum d_{RTij}M^0_{RTij}X_0 + \sum f_{RTij}M^0_{RTij}Y_0 + u_3 \\
&= \sum d_{RTij}M^0_{RTi(j+1)} + \sum f_{RTi0}M^0_{RTi0}Y_0 \\
&\quad + \sum_{j \neq 0} f_{RTij}M^0_{RTij}Y_0 + u_3 \\
&= \sum d_{RTij}M^0_{RTi(j+1)} + \sum f_{RTi0}M^0_{RT(i+1)0} \\
&\quad + \sum_{j \neq 0} g_{RTij}M^0_{RTij} + u'''
\end{aligned}$$

with $u''' \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$. By Lemma 5.4, it follows that

$$\sum c_{RT}M^0_{RT00} = \sum d_{RTij}M^0_{RTi(j+1)} + \sum f_{RTi0}M^0_{RT(i+1)0} + \sum_{j \neq 0} k_{RTij}M^0_{RTij},$$

and this contradicts the linear independence of the M^0_{RTij}. $\qquad\square$

We now start the construction of four new bases of $\mathfrak{U}/\mathcal{J}$ that we will use in computing the action of certain elements of $\mathfrak{U}$ on $\mathfrak{U}/\mathcal{J}$. The first basis is defined as follows: we denote by M^1_{RT} the standard monomials $Y^R X^T$ that satisfy property (5.1) and the following property:

(5.8) $$o(-2e_1, Y^R X^T) = o(2e_1, Y^R X^T) = 0.$$

This means that the root vector X_{2e_1} does not occur in X^T and the root vector X_{-2e_1} does not occur in Y^R.

We will show that the set $\{M^1_{RT} + \mathcal{J}\}$ is a basis of $\mathfrak{U}/\mathcal{J}$. This is the first of the four basis that we need. The other three bases are essentially the same except that we write the root vectors that occur in the monomials in a different order.

More precisely, we denote $^R X = X^{r_N}_{\beta_N} \cdots X^{r_1}_{\beta_1}$ and $^R Y = \Omega(^R X)$. Set $M^2_{RT} = {}^R Y X^T$, $M^3_{RT} = X^T Y^R$, and $M^4_{RT} = {}^T X Y^R$. We also assume that R, T are such that $Y^R X^T$ satisfies properties (5.1) and (5.8). We can (at last) state

Proposition 5.7. *If $r = 1, 2, 3, 4$, set $\mathcal{B}^r = \{M^r_{RT} + \mathcal{J}\}$.*
Then the sets $\mathcal{B}^r$ are bases of $\mathfrak{U}/\mathcal{J}$.

An obvious consequence of Proposition 5.7 is the following result.

Corollary 5.8. *Set $X_1 = X_{-e_1-e_2}$, $X_2 = X_{-e_1+e_2}$, $X_3 = X_{e_1-e_2}$, and $X_4 = X_{e_1+e_2}$.*
Then X_r acts injectively on $\mathfrak{U}/\mathcal{J}$ ($r = 1, 2, 3, 4$).

Proof. It is enough to compute the action of X_r using the basis $\mathcal{B}^r$. $\qquad\square$

The rest of this section is devoted to the proof of Proposition 5.7. We will give full technical details only for the proof that $\mathcal{B}^1$ is a basis for $\mathfrak{U}/\mathcal{J}$, the other cases being fairly similar.

We set $M^1_{RTijVW\lambda} = M^1_{RT}Y^i_0 X^j_0 N_{VW\lambda}$ and let $\mathcal{C}^1$ denote the set of the monomials $M^1_{RTijVW\lambda}$. The procedure for showing that $\mathcal{B}^1$ is a basis of $\mathfrak{U}/\mathcal{J}$ is the same

as in the construction of the basis $\mathcal{B}^0$ above: we first show that $\mathcal{C}^1$ is a basis for $\mathfrak{U}$ and then apply Lemma 5.5 as in the proof of Proposition 5.6.

We cannot argue exactly as for the $M^0_{RTijVW\lambda}$ because the set $\{\hat{M}^1_{RTijVW\lambda}\}$ is not a basis for $Gr(\mathfrak{U})$; thus we need to introduce a different filtration.

Using the basis $\mathcal{B}'$ introduced in Lemma 5.1, we define a map $\delta_1 : \mathcal{B}' \to \mathbb{N}^2$ by setting

$$\delta_1(M_{RT}N_{VW\lambda}) = (d_0(M_{RT}N_{VW\lambda}), o(2e_1, Y^R X^T) + o(-2e_1, Y^R X^T)).$$

We order $\mathbb{N}^2$ lexicographically; thus, as described in § 3, the map δ_1 defines a filtration on $\mathfrak{U}$ that we denote by $\mathcal{F}_1$. We let d_1 denote the corresponding degree map. We notice that this filtration is not an algebra filtration; it is however true that

$$(5.9) \qquad d_1(Y^R X^T N_{VW\lambda}) = d_1(Y^R) + d_1(X^T) + d_1(N_{VW\lambda}).$$

Indeed, since there is a constant C such that

$$(Y^R X^T N_{VW\lambda})\widehat{} = C(F^V Y^R K_\lambda X^T E^V)\widehat{},$$

it follows from the definition of d_0 that

$$d_0(Y^R X^T N_{VW\lambda}) = d_0(F^V) + d_0(Y^R) + d_0(X^T) + d_0(E^W)$$
$$= d_0(Y^R) + d_0(X^T) + d_0(N_{VW\lambda}).$$

By the definition of d_1 we find that

$$d_1(Y^R X^T N_{VW\lambda}) = (d_0(Y^R X^T N_{VW\lambda}), o(-2e_1, Y^R X^T) + o(2e_1, Y^R X^T))$$
$$= (d_0(Y^R) + d_0(X^T) + d_0(N_{VW\lambda}), o(-2e_1, Y^R) + o(2e_1, X^T))$$
$$= (d_0(Y^R), o(-2e_1, Y^R)) + (d_0(X^T), o(2e_1, X^T)) + (d_0(N_{VW\lambda}), 0)$$
$$= d_1(Y^R) + d_1(X^T) + d_1(N_{VW\lambda}),$$

which is exactly (5.9).

Lemma 5.9. *Suppose that $R \in \mathbb{N}^N$ is such that X^R and Y^R satisfy property* (5.1), *and set $R^i = (r_1, \ldots, r_n + i, \ldots, r_{2n-1}, 0, \ldots, 0)$.*
Then there are nonzero constants C and D such that

1.
$$Y^R Y^i_0 = (C)^i Y^{R^i} + \sum c_T Y^T$$

   with $d_1(Y^T) < d_1(Y_{R^i})$.

2.
$$X^R X^i_0 = (D)^i X^{R^i} + \sum d_T X^T$$

   with $d_1(X_T) < d_1(X^{R^i})$.

Proof. The proof is simply a careful computation: the details are given in Appendix B ☐

Lemma 5.10. *There is a nonzero constant C_{ij} such that*

$$s_{\mathcal{F}_1}(M^1_{RTijVW\lambda}) = C_{ij} s_{\mathcal{F}_1}(Y^{R^i} X^{T^j} N_{VW\lambda}).$$

Proof. We write explicitly $M^1_{RT} = Y^R X^T$; then, by [2], Remark 10.1 (c),

$$M^1_{RTijVW\lambda} = Y^R X^T Y^i_0 X^j_0 N_{VW\lambda}$$
$$= Y^R Y^i_0 X^T X^j_0 N_{VW\lambda} + u'$$

with $d_0(u') < d_0(M^1_{RTijVW\lambda})$. Thus from the definition of $\mathcal{F}_1$ we conclude that $d_1(u') < d_1(M^1_{RTijVW\lambda})$. It follows that

$$s_{\mathcal{F}_1}(M^1_{RTijVW\lambda}) = s_{\mathcal{F}_1}(Y^R Y^i_0 X^T X^j_0 N_{VW\lambda}).$$

By Lemma 5.9, we can write

$$Y^R Y^i_0 X^T X^j_0 N_{VW\lambda} = (C)^i(D)^j Y^{R^i} X^{T^j} N_{VW\lambda} + \sum c_A d_B Y^A X^B N_{VW\lambda}$$

with $d_1(Y^A) + d_1(X^B) < d_1(Y^{R^i}) + d_1(X^{T^j})$.

Using (5.9) we find that $d_1(Y^A X^B N_{VW\lambda}) < d_1(Y^{R^i} X^{T^j} N_{VW\lambda})$; thus, setting $C_{ij} = (C)^i(D)^j$, we find that

$$s_{\mathcal{F}_1}(M^1_{RTijVW\lambda}) = C_{ij} s_{\mathcal{F}_1}(Y^{R^i} X^{T^j} N_{VW\lambda}),$$

as we wished to prove. $\qquad\square$

Lemma 5.11. *The set $\mathcal{C}^1$ is a basis for $\mathfrak{U}$.*

Proof. Because of Proposition 3.3 and Lemma 5.10, it is enough to check that the map $(R, T, i, j, V, W, \lambda) \mapsto Y^{R^i} X^{T^j} N_{VW\lambda}$ is bijective. This is obvious since its inverse is the map $Y^R X^T N_{VW\lambda} \mapsto (R_0, T_0, i, j, V, W, \lambda)$, where $i = r_n$, $j = t_n$, $R_0 = (r_1, \ldots, r_{n-1}, 0, r_{n+1}, \ldots, r_N)$, and $T_0 = (t_1, \ldots, t_{n-1}, 0, t_{n+1}, \ldots, t_N)$. $\qquad\square$

From now on we can follow the same steps as in the proof that $\mathcal{B}^0$ is a basis: we write $M^1_{RTij} = M^1_{RTij000}$ for short and let $\mathbf{P}^1$ denote the span of the elements M^1_{RTij}. Exactly as in Lemma 5.4 we can check that

$$(5.10) \qquad\qquad \mathfrak{U} = \mathbf{P}^1 \oplus \mathfrak{U}(\mathcal{A} \cap \mathfrak{M}).$$

The proof that $\mathcal{B}^1$ is a basis of $\mathfrak{U}/\mathcal{J}$ is identical to the proof of Proposition 5.6 if we just replace the M^0_{RTij} by the M^1_{RTij} and use (5.10) instead of Lemma 5.4. $\quad\square$

6. GELFAND-KIRILLOV DIMENSION

We refer to [10] and [5] for definitions and generalities on Gelfand-Kirillov dimension. If M is a finitely generated module for a finitely generated algebra A, then we let $GK(M)$ denote its Gelfand-Kirillov dimension. We set

$$p = \dim \mathfrak{g} - \dim \mathfrak{k} = 4(n-1).$$

Lemma 6.1.

$$GK(\mathfrak{U}/\mathcal{I}) \geq p.$$

Proof. Recall from §4 that $\lambda_k = k(e_1 + e_2)$ and v_k denotes a nonzero vector in $V(\lambda_k)$ on which $\mathfrak{M}$ acts trivially. If $k \in \mathbb{N}$, we set $V(k) = \bigoplus_{r \leq k} V(\lambda_r)$ and $v(k) = \sum_{r \leq k} v_r$. Let $\mathfrak{U}_1$ be the space generated (over $\mathbb{C}(q)$) by $\overline{X}_\alpha$ $(\alpha \in R)$ and $K^{\pm}_i$. Clearly $\mathfrak{U}_1$ generates $\mathfrak{U}$ (as an algebra), and we set $\mathfrak{U}^k = (\mathfrak{U}_1)^k$. Let $(\mathfrak{U}/\mathcal{I})^k = \mathfrak{U}^k \cdot (1 + \mathcal{I})$. By the definition of Gelfand-Kirillov dimension,

$$GK(\mathfrak{U}/\mathcal{I}) = \limsup(\log_k(\dim(\mathfrak{U}/\mathcal{I})^k)).$$

We will show that there is a positive constant $C \in \mathbb{N}$ such that $\dim(\mathfrak{U}/\mathcal{I})^{Ck} \geq \dim V(k)$. Set $C' = |e_1 + e_2|$ and $C = 3C'$. We define a map $\phi_k : (\mathfrak{U}/\mathcal{I})^{Ck} \to V(k)$ by setting $\phi_k(u + \mathcal{I}) = u \cdot v(k)$. Since $\mathcal{I} \cdot v(k) = 0$, it is clear that the map is well defined. We claim that ϕ_k is surjective. We will show this by induction on k.

If $k = 0$ there is nothing to prove, so let us assume that $k > 0$. We observe that there is $u \in \mathfrak{U}^+$ such that $u \cdot v_k = v_{\lambda_k}$, where v_{λ_k} denotes a highest vector for $V(\lambda_k)$. We can clearly assume that $u \in \mathfrak{U}_{\lambda_k}$, hence $d_0(u) = |\lambda_k| = C'k$, and this implies that $u \in \mathfrak{U}^{C'k}$. In fact u is a sum of monomials of the type $X^R = \prod_i X_{\beta_i}^{r_i}$ with $\sum r_i \beta_i = \lambda_k$. Hence

$$|R| = \sum r_i \leq \sum r_i |\beta_i| = |\lambda_k|.$$

Therefore $u \in \mathfrak{U}^{C'k}$.

Fix $w \in V(\lambda_k)$ and suppose that $w \in V(\lambda_k)_\mu$. We know that there is $u' \in \mathfrak{U}^-$ such that $u' \cdot v_{\lambda_k} = w$; and, reasoning as above, we can assume that $u' \in \mathfrak{U}_{\mu - \lambda_k}$. Hence u' is a sum of monomials Y^R with $|R| \leq |\lambda_k - \mu|$. Let $\bar\lambda_k = -\lambda_k$ denote the lowest weight of $V(\lambda_k)$. Since $\mu - \bar\lambda_k$ is a sum of positive roots,

$$|\lambda_k - \mu| \leq |\lambda_k - \mu + \mu - \bar\lambda_k| = |\lambda_k - \bar\lambda_k| = |2\lambda_k|,$$

and it follows that $|R| \leq |2\lambda_k| = 2C'k$, i.e. $u' \in \mathfrak{U}^{2C'k}$. Since $V(\lambda_k)$ is admissible, the outcome of the above discussion is that for any $w \in V(\lambda_k)$ there is $u' \in \mathfrak{U}^{2C'k}$ such that $u'u \cdot v_k = w$.

If $r < k$, the weight $\lambda_r - \lambda_k$ is not a sum of positive roots; hence $u \cdot v_r = 0$ if $r < k$.

Fix $v \in V(k)$, and write $v = w_0 + w$ with $w_0 \in V(k-1)$ and $w \in V(\lambda_k)$. By the induction hypothesis we know that there is $u_0 \in \mathfrak{U}^{C(k-1)}$ such that $\phi_{k-1}(u_0 + \mathcal{I}) = u_0 \cdot v(k-1) = w_0$. We have proven above that there are $u_1, u_2 \in \mathfrak{U}^{2C'k}$ such that $u_1 u \cdot v_k = u_0 \cdot v_k$ and $u_2 u \cdot v_k = w$. Then

$$v = w_0 + w = u_0 \cdot v(k) - u_1 u \cdot v(k) + u_2 u \cdot v(k)$$

and $u_0 - u_1 u + u_2 u \in \mathfrak{U}^{Ck}$. We have therefore proven that ϕ_k is surjective.

It follows that $\dim(\mathfrak{U}/\mathcal{I})^{Ck} \geq \dim V(k)$. By a result of Wallach ([13]; see also Proposition 5.2 of [12] and the remark thereafter), it is known that $\dim V(k) = k^p(A + C_k)$, where A is a positive constant and C_k tends to zero as k tends to infinity. Hence

$$GK(\mathfrak{U}/\mathcal{I}) = \limsup \log_k(\dim(\mathfrak{U}/\mathcal{I})^k) \geq \limsup \log_{Ck}(\dim(\mathfrak{U}/\mathcal{I})^{Ck})$$
$$\geq \limsup \log_{Ck}(k^p(A + C_k)) = p,$$

and the proof is complete. $\square$

We now prove a result that slightly improves the results of [9] on the Gelfand-Kirillov dimension of quantized algebras.

Lemma 6.2. *If M is a finitely generated $\mathfrak{U}$-module, then*

$$GK(M) = GK(Gr(M)).$$

Proof. We wish to use a suitable generalization of Theorem 1.3 of [11].

Let $k \in \mathbb{N}$ and set $D_1 = (0, \dots, 0, 1) \in \mathbb{N}^{2N+1}$. We define a new filtration $\mathcal{F}^k$ over $\mathbb{N}^{2N+1}$ by choosing

$$\{X_\alpha \mid \alpha \in R\} \cup \{K_i^{\pm 1} \mid i = 1, \dots, n\}$$

as a set of generators for $\mathfrak{U}$ and defining $d_{\mathcal{F}^k}(X_\alpha) = kd(X_\alpha)$ and $d_{\mathcal{F}^k}(K_i^{\pm 1}) = D_1$. We also define a filtration $\hat{\mathcal{F}}$ on $Gr(\mathfrak{U})$ by setting $d_{\hat{\mathcal{F}}}(\hat{X}_\alpha) = d(X_\alpha)$ and $d_{\hat{\mathcal{F}}}(\hat{K}_i^{\pm 1}) = D_1$.

As shown in Theorem 2.17 of [11], in order to prove our result, it is enough to prove that $GK(\mathfrak{U}/\mathcal{L}) = GK(Gr(\mathfrak{U}/\mathcal{L}))$ for any left ideal $\mathcal{L}$ of $\mathfrak{U}$. We observe that one can generalize both Theorem 2.9 and Lemma 2.14 of [11] to our setting, so we can conclude that there is $\bar{k}$ such that

$$GK(Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})) \leq GK(Gr_{\hat{\mathcal{F}}}(Gr(\mathfrak{U}/\mathcal{L}))).$$

To prove the lemma it is enough to show that

(6.1) $$GK(Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})) = GK(\mathfrak{U}/\mathcal{L})$$

and

(6.2) $$GK(Gr_{\hat{\mathcal{F}}}(Gr(\mathfrak{U}/\mathcal{L}))) = GK(Gr(\mathfrak{U}/\mathcal{L})).$$

Let $\phi : \mathbb{N}^{2N+1} \to \mathbb{N}$ be the map defined by

$$\phi(d_1, \ldots, d_{2N+1}) = d_{2N+1}.$$

Notice that ϕ is an order preserving homomorphism. We can define a filtration over $\mathbb{N}$ on $\mathfrak{U}$ by setting $\mathfrak{U}_{(m)} = \sum_{\phi(D) \leq m} \mathcal{F}_D^{\bar{k}}(\mathfrak{U})$. Since ϕ is a homomorphism, this filtration is an algebra filtration. We also set $(\mathfrak{U}/\mathcal{L})_{(m)} = (\mathfrak{U}_{(m)} + \mathcal{L})/\mathcal{L}$. We observe that $\dim(U/\mathcal{L})_{(m)}$ is finite for all m. It follows from Proposition 8.6.5 of [10] that

$$GK(\mathfrak{U}/\mathcal{L}) = \limsup(\log_m(\dim(\mathfrak{U}/\mathcal{L})_{(m)}).$$

Analogously we can define a filtration over $\mathbb{N}$ on $Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U})$ and on $Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})$ by setting

$$Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U})_{[m]} = \bigoplus_{\phi(D) \leq m} \mathcal{F}_D^{\bar{k}}(\mathfrak{U})/\mathcal{F}_{<D}^{\bar{k}}(\mathfrak{U})$$

and

$$Gr_{F^{\bar{k}}}(\mathfrak{U}/\mathcal{L})_{[m]} = (Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U})_{[m]} + Gr_{\mathcal{F}^{\bar{k}}}(\mathcal{L}))/Gr_{\mathcal{F}^{\bar{k}}}(\mathcal{L}).$$

Lemma 3.4 implies that the dimension of $Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})_{[m]}$ is finite; thus

$$GK(Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})) = \limsup(\log_m(\dim Gr_{\mathcal{F}^{\bar{k}}}(\mathfrak{U}/\mathcal{L})_{[m]}).$$

Applying Lemma 3.4 again, (6.1) follows immediately. The same argument applied to $Gr(\mathfrak{U}/\mathcal{L})$ gives (6.2). $\qquad\square$

The results in § 5 coupled with Lemma 6.2 have as an easy consequence the fact that the Gelfand-Kirillov dimension of $\mathfrak{U}/\mathcal{I}$ is exactly equal to p. Indeed, we prove

Lemma 6.3.

$$GK(\mathfrak{U}/\mathcal{J}) \leq p.$$

Proof. By Lemma 6.2, $GK(\mathfrak{U}/\mathcal{J}) = GK(Gr(\mathfrak{U}/\mathcal{J}))$. Since $Gr(\mathfrak{U})$ is a semicommutative algebra, the Gelfand-Kirillov dimension can be computed as if $Gr(\mathfrak{U})$ were a commutative algebra: if $R \in \mathbb{N}^N$ we set $|R| = \sum r_i$ and define

$$Gr(\mathfrak{U})_m = \{\hat{Y}^R \hat{K}_\lambda \hat{X}^T \mid |R| + |\lambda| + |T| \leq m\}.$$

Let $v_{\mathcal{J}} = (1 + \mathcal{J})\hat{}$ and set $Gr(\mathfrak{U}/\mathcal{J})_m = Gr(\mathfrak{U})_m \cdot v_{\mathcal{J}}$. Then

$$GK(Gr(\mathfrak{U}/\mathcal{J})) = \limsup \log_m(\dim Gr(\mathfrak{U}/\mathcal{J})_m).$$

It is clear that the elements $\hat{M}^0_{RTijVW\lambda} \cdot v_{\mathcal{J}}$ generate $Gr(\mathfrak{U}/\mathcal{J})$ and that

$$\hat{M}^0_{RTijVW\lambda} \cdot v_{\mathcal{J}} = \hat{M}^0_{RTijVW0} \cdot v_{\mathcal{J}}.$$

Furthermore, if i, j, V, W are not all zero, then $\hat{M}^0_{RTijVW0} \cdot v_{\mathcal{J}} = 0$. Thus $Gr(\mathfrak{U}/\mathcal{J})_m$ is generated by $\{\hat{Y}^R\hat{X}^T \cdot w_{\mathcal{J}}\}$ with Y^RX^T that satisfy (5.1), (5.2) and such that $|R| + |T| \leq m$; it follows that $\dim Gr(\mathfrak{U}/\mathcal{J})_m$ is bounded above by a polynomial in m of degree p, and hence $GK(\mathfrak{U}/\mathcal{J}) \leq p$. $\square$

An immediate result is the following:

Theorem 6.4.

$$GK(\mathfrak{U}/\mathcal{J}) = GK(\mathfrak{U}/\mathcal{I}) = p.$$

Proof. Since $\mathcal{J} \subset \mathcal{I}$, then $GK(\mathfrak{U}/\mathcal{J}) \geq GK(\mathfrak{U}/\mathcal{I})$. The result now follows from Lemmas 6.1 and 6.3. $\square$

7. Proof of the main result

We have just proven that $GK(\mathfrak{U}/\mathcal{I}) = GK(\mathfrak{U}/\mathcal{J})$. Thus, by Lemma 6.2,

$$GK(Gr(\mathfrak{U}/\mathcal{I})) = GK(Gr(\mathfrak{U}/\mathcal{J})).$$

This is not enough to conclude right away that $\mathcal{I} = \mathcal{J}$ as it would be in the classical case. The problem here is that the annihilator of the generator of $Gr(\mathfrak{U}/\mathcal{J})$ in $Gr(\mathfrak{U})$ is not completely prime. The subject of this section is the work needed to obtain the desired equality.

Set $v_{\mathcal{I}} = (1 + \mathcal{I})\hat{\ } \in Gr(\mathfrak{U}/\mathcal{I})$; since $\mathcal{J} \subset \mathcal{I}$ it is clear that $Gr(\mathfrak{U}/\mathcal{I})$ is generated, as a vector space, by $\hat{\mathcal{B}}^0 = \{\hat{M}^0_{RT} \cdot v_{\mathcal{I}}\}$.

Let A denote the set of all M^0_{RT}. We divide A into four subsets A_s (s=1,2,3,4) as follows

1. $A_1 = \{Y^RX^T \mid o(e_1 - e_2, Y^RX^T) = 0 \text{ and } o(-e_1 - e_2, Y^RX^T) = 0\}$,
2. $A_2 = \{Y^RX^T \mid o(e_1 + e_2, Y^RX^T) = 0 \text{ and } o(-e_1 - e_2, Y^RX^T) = 0\}$,
3. $A_3 = \{Y^RX^T \mid o(e_1 - e_2, Y^RX^T) = 0 \text{ and } o(-e_1 + e_2, Y^RX^T) = 0\}$,
4. $A_4 = \{Y^RX^T \mid o(e_1 + e_2, Y^RX^T) = 0 \text{ and } o(-e_1 + e_2, Y^RX^T) = 0\}$.

Let $\hat{A}_s$ denote the space generated by $\hat{M}^0_{RT} \cdot v_{\mathcal{I}}$ with $M^0_{RT} \in A_s$ and, if $i \in \mathbb{N}$, let $(\hat{A}_s)_i$ be the subspace of $\hat{A}_s$ generated by the $\hat{M}^0_{RT} \cdot v_{\mathcal{I}}$ such that $|R| + |T| \leq i$. Set also $p_s(i) = \dim(A_s)_i$. Arguing as in Lemma 6.3, it is clear that $Gr(\mathfrak{U})_i \cdot v_{\mathcal{I}} = \sum_{s=1}^4 (\hat{A}_s)_i$.

If there are $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_{2n-1}, 0, \ldots, 0)$ and $\bar{T} = (\bar{t}_1, \ldots, \bar{t}_{2n-1}, 0, \ldots, 0)$ such that $Y^{\bar{R}}X^{\bar{T}} \in A_s$ and $\hat{Y}^{\bar{R}}\hat{X}^{\bar{T}} \cdot v_{\mathcal{I}} = 0$, then for all R, T such that $r_i \geq \bar{r}_i$ and $t_i \geq \bar{t}_i$ for all i, we have that $\hat{Y}^R\hat{X}^T \cdot v_{\mathcal{I}} = 0$. It follows that $(\hat{A}_s)_i$ is generated by all $\hat{Y}^R\hat{X}^T \cdot v_{\mathcal{I}}$ such that $Y^RX^T \in A_s$, $|R| + |T| \leq i$, and there is j such that either $r_j < \bar{r}_j$ or $t_j < \bar{t}_j$. Counting the number of these generators, we find that $p_s(i)$ is bounded above by a polynomial of degree at most $p - 1$.

We will show that, if $\mathcal{I} \neq \mathcal{J}$, then for each $s = 1, 2, 3, 4$ there are $\bar{R}$ and $\bar{T}$ (that depend on s) such that $Y^{\bar{R}}X^{\bar{T}} \in A_s$ and $\hat{Y}^{\bar{R}}\hat{X}^{\bar{T}} \cdot v_{\mathcal{I}} = 0$. Since

$$\dim(Gr(\mathfrak{U}))_i \cdot v_{\mathcal{I}} \leq \sum_{s=1}^4 p_s(i),$$

it follows that $\dim Gr(\mathfrak{U})_i$ is bounded above by a polynomial of degree at most $p - 1$; hence $GK(Gr(\mathfrak{U}/\mathcal{I})) < p$ and this gives a contradiction.

The rest of this section is devoted to carrying out this program. First of all we set up some notation: if $i = 1$ we set $i' = 2$ and if $i = 2$ we set $i' = 1$; if $i = 3$ we set $i' = 4$ and if $i = 4$, we set $i' = 3$. Set also $\epsilon_1 = -e_1 - e_2$, $\epsilon_2 = -e_1 + e_2$, $\epsilon_3 = e_1 - e_2$ and $\epsilon_4 = e_1 + e_2$. We start with a simple lemma:

Lemma 7.1. *Let X_i $(i = 1, 2, 3, 4)$ be as defined in Corollary 5.8 and suppose that Y^R and X^T satisfy property (5.2).*

1. *If $i = 1, 2$ then*

$$X_i Y^R = \sum c_H Y^H + c_0^R Y^{R_0} Y_0$$

with Y^H that satisfy property (5.2), $o(\epsilon_{i'}, Y^H) \le o(\epsilon_{i'}, Y^R)$, and $d_0(Y^H) = d_0(Y^R) + d_0(X_i)$.
    Furthermore, $o(\epsilon_{i'}, Y^H) = o(\epsilon_{i'}, Y^R)$ if and only if $o(\epsilon_{i'}, Y^R) = 0$.
2. *If $i = 3, 4$ then*

$$X_i X^T = \sum d_H X^H + d_0^T X^{T_0} X_0$$

with X^H that satisfy property (5.2), $o(\epsilon_{i'}, X^H) \le o(\epsilon_{i'}, X^T)$, and $d_0(X^H) = d_0(X^R) + d_0(X_i)$.
    Furthermore, $o(\epsilon_{i'}, X^H) = o(\epsilon_{i'}, X^T)$ if and only if $o(\epsilon_{i'}, X^T) = 0$.

Proof. We write explicitly

$$Y^R = X_{-e_1-e_2}^{r_{2n-1}} Y^{R'} X_{-e_1+e_2}^{r_1},$$

where $R' = (0, r_2, \ldots, r_{2n-2}, 0, \ldots, 0)$. If $i = 1$ and $o(\epsilon_{i'}, Y^R) = r_1 = 0$ then $X_1 Y^R = X_{-e_1-e_2}^{r_{2n-1}+1} Y^{R'}$ and there is nothing to prove (here $c_0^R = 0$). If $r_1 \ne 0$, then, since Y^R satisfies property (5.2), $r_{2n-1} = 0$, and hence, by [2], setting $c(q) = q^{-(r_2 + \cdots + 2r_n + \cdots + r_{2n-2})}$,

$$
\begin{aligned}
X_1 Y^R &= X_{-e_1-e_2} Y^{R'} X_{-e_1+e_2}^{r_1} \\
&= c(q) Y^{R'} X_{-e_1-e_2} X_{-e_1+e_2}^{r_1} \\
&= \sum_{i=1}^{r_1-1} c(q) Y^{R'} X_{-e_1+e_2}^{i-1} [X_{-e_1-e_2}, X_{-e_1+e_2}] X_{-e_1+e_2}^{r_1-i} \\
&\quad + c(q) Y^{R'} X_{-e_1+e_2}^{r_1-1} X_{-e_1-e_2} X_{-e_1+e_2}.
\end{aligned}
$$

(7.1)

By Lemma 5.2

$$
\begin{aligned}
X_{-e_1-e_2} X_{-e_1+e_2} &= \frac{1}{q - q^{-1}} (Y_0 - (-q)^{-n+1}(q + q^{-1}) X_{-2e_1}) \\
&\quad - \sum_{i=3}^{n} (-q)^{-i+2} X_{-e_1-e_i} X_{-e_1+e_i}.
\end{aligned}
$$

Furthermore, by (A.25),

$$[X_{-e_1-e_2}, X_{-e_1+e_2}] = q^{-1} X_{-e_1+e_3} X_{-e_1-e_3} - q X_{-e_1-e_3} X_{-e_1+e_3},$$

so, substituting in (7.1), we find that

$$
\begin{aligned}
X_1 Y^R = & \sum_{i=1}^{r_1-1} c(q)q^{-1}Y^{R'}X_{-e_1+e_2}^{i-1}X_{-e_1+e_3}X_{-e_1-e_3}X_{-e_1+e_2}^{r_1-i} \\
& - \sum_{i=1}^{r_1-1} c(q)q Y^{R'}X_{-e_1+e_2}^{i-1}X_{-e_1-e_3}X_{-e_1+e_3}X_{-e_1+e_2}^{r_1-i} \\
& - \frac{(-q)^{-n+1}(q+q^{-1})}{q-q^{-1}}c(q)Y^{R'}X_{-e_1+e_2}^{r_1-1}X_{-2e_1} \\
& - \sum_{i>2}(-q)^{-i+2}c(q)Y^{R'}X_{-e_1+e_2}^{r_1-1}X_{-e_1-e_i}X_{-e_1+e_i} \\
& + \frac{c(q)}{q-q^{-1}}Y^{R'}X_{-e_1+e_2}^{r_1-1}Y_0.
\end{aligned}
$$

Since $X_{-e_1+e_2}X_{-e_1\pm e_i} = qX_{-e_1\pm e_i}X_{-e_1+e_2}$, it follows that

$$
\begin{aligned}
(7.2) \quad X_1 Y^R = & \sum_{i=1}^{r_1-1} c(q)q^{2i-3}Y^{R'}X_{-e_1+e_3}X_{-e_1-e_3}X_{-e_1+e_2}^{r_1-1} \\
& - \sum_{i=1}^{r_1-1} c(q)q^{2i-1}Y^{R'}X_{-e_1-e_3}X_{-e_1+e_3}X_{-e_1+e_2}^{r_1-1} \\
& + \frac{-(-q)^{-n+1}(q+q^{-1})}{q-q^{-1}}c(q)q^{2r_1-2}Y^{R'}X_{-2e_1}X_{-e_1+e_2}^{r_1-1} \\
& - \sum_{i>2}(-q)^{-i+2}c(q)q^{2r_1-2}Y^{R'}X_{-e_1-e_i}X_{-e_1+e_i}X_{-e_1+e_2}^{r_1-1} \\
& + \frac{c(q)}{q-q^{-1}}Y^{R'}X_{-e_1+e_2}^{r_1-1}Y_0.
\end{aligned}
$$

We observe that, if $i > 2$, then $Y^{R'}X_{-e_1\pm e_i}$ and $Y^{R'}X_{-2e_1}$ are linear combinations of the type $\sum c'_{H'}Y^{H'}$ with $o(-e_1 \pm e_2, Y^{H'}) = 0$. This depends on our choice of the convex ordering on R^+ and [2, Theorem 9.3]. Using this fact in (7.2), it follows that

$$
X_1 Y^R = \sum c'_{H'}Y^{H'}X_{-e_1+e_2}^{r_1-1} + \frac{c(q)}{q-q^{-1}}Y^{R'}X_{-e_1+e_2}^{r_1-1}Y_0;
$$

so, setting $c_H = c'_{H'}$, $Y^H = Y^{H'}X_{-e_1+e_2}^{r_1-1}$, $c_0^R = \frac{c(q)}{q-q^{-1}}$, and $Y^{R_0} = Y^{R'}X_{-e_1+e_2}^{r_1-1}$, we have our result. It remains only to prove that $d_0(Y^H) = d_0(Y^R) + d_0(X_i)$. By eventually collecting terms we can assume that the Y^H are distinct, thus linearly independent in $\mathfrak{U}$. Set $\lambda = \lambda(X_1 Y^R)$, and note that $Y^{R_0}Y_0 \in \mathfrak{U}_\lambda$, thus $\sum c_H Y^H \in \mathfrak{U}_\lambda$. By the linear independence of the Y^H, it follows that $Y^H \in \mathfrak{U}_\lambda$ for all H; but then $d_0(Y^H) = |\lambda| = d_0(X_1 Y^R) = d_0(X_1) + d_0(Y^R)$.

The other cases are similar. $\qquad\square$

If we write $u \in \mathfrak{U}$ as a linear combination of the $\tilde{M}_{RTijVW\lambda}^0$, i.e.,

$$
u = \sum k_{RTijVW\lambda}\tilde{M}_{RTijVW\lambda}^0,
$$

then

$$
(7.3) \qquad d_0(u) = \max\{d_0(\tilde{M}_{RTijVW\lambda}^0) \mid k_{RTijVW\lambda} \neq 0\}.
$$

In fact it is easy to check that

$$d(M^0_{RTijVW\lambda}) = d(M^0_{RTijVW0});$$

hence $(\tilde{M}^0_{RTijVW\lambda})\widehat{} = \hat{M}^0_{RTijVW\lambda} - \hat{M}^0_{RTijVW0}$ if $\lambda \neq 0$, while $(\tilde{M}^0_{RTijVW0})\widehat{} = \hat{M}^0_{RTijVW0}$. Furthermore we know from the proof of Lemma 5.3 that the $\hat{M}^0_{RTijVW\lambda}$ form a basis of $Gr(\mathfrak{U})$. It follows easily that the set $\{(\tilde{M}^0_{RTijVW\lambda})\widehat{}\}$ is a basis of $Gr(\mathfrak{U})$. Applying Proposition 3.5, we obtain (7.3).

Now fix $v \in \mathfrak{U}/\mathcal{J}$ and write it in terms of the basis $\mathcal{B}^0$. Then $v = \sum h_{RT} B^0_{RT} = \sum h_{RT} Y^R X^T + \mathcal{J}$. Set $\ell(v) = d_0(\sum h_{RT} Y^R X^T)$,

$$v_1 = \sum_{d_0(Y^R X^T)=\ell(v)} h_{RT} Y^R X^T + \mathcal{J},$$

and $v_2 = v - v_1$.

Note that, if $u \in \mathfrak{U}$, then $\ell(u + \mathcal{J}) \leq d_0(u)$. Indeed, if we write $u + \mathcal{J} = \sum h_{RT} Y^R X^T + \mathcal{J}$, then $u = \sum h_{RT} Y^R X^T + u'$ with $u' \in \mathcal{J}$. Write u' as a linear combination of the $\tilde{M}^0_{RTijVW\lambda}$:

$$u' = \sum k_{RTijVW\lambda} \tilde{M}^0_{RTijVW\lambda}$$

with i, j, V, W or λ not all zero. This implies that

$$u = \sum h_{RT} \tilde{M}^0_{RT00000} + \sum k_{RTijVW\lambda} \tilde{M}^0_{RTijVW\lambda};$$

thus our assertion follows from (7.3).

This implies that if $u \in \mathfrak{U}$, then $\ell(u \cdot v) \leq d_0(u) + \ell(v)$. In fact,

$$\ell(u \cdot v) \leq d_0(u(\sum h_{RT} Y^R X^T)) \leq d_0(u) + d_0(\sum h_{RT} Y^R X^T) = d_0(u) + \ell(v).$$

If $i = 1, 2, 3, 4$, set

$$o_i(v) = \max\{o(\epsilon_i, Y^R X^T) \mid h_{RT} \neq 0 \text{ and } d_0(Y^R X^T) = \ell(v)\}.$$

Lemma 7.2. *If $v \in \mathfrak{U}/\mathcal{J}$, then $o_{i'}(X_i \cdot v) \leq o_{i'}(v)$, and $o_{i'}(X_i \cdot v) = o_{i'}(v)$ if and only if $o_{i'}(v) = 0$. Furthermore, if $j \neq i, i'$, then $o_j(X_i \cdot v) = o_j(v)$.*

Proof. Suppose $i = 1, 2$. Clearly $X_i \cdot v = X_i \cdot v_1 + X_i \cdot v_2$ and

$$\ell(X_i \cdot v_2) \leq d_0(X_i) + \ell(v_2) < d_0(X_i) + \ell(v_1).$$

Write $v_1 = \sum_T p_T(Y) X^T + \mathcal{J}$, where $p_T(Y) = \sum_R h_{RT} Y^R$. By Lemma 7.1,

$$X_i p_T(Y) = \sum_H c_{HT} Y^H + \sum_R h_{RT} c_0^R Y^{R_0} Y_0$$

with $o(\epsilon_{i'}, Y^H) \leq o_{i'}(v)$ and $d_0(\sum_H c_{HT} Y^H) = d_0(p_T(Y)) + d_0(X_i)$.

Moreover, $o(\epsilon_{i'}, Y^H) = o_{i'}(v)$ if and only if $o_{i'}(v) = 0$. This implies that

$$X_i \cdot v_1 = \sum_T X_i p_T(Y) X^T + \mathcal{J}$$

$$= \sum_T (\sum_H c_{HT} Y^H) X^T + \sum_T (\sum_R h_{RT} c_0^R Y^{R_0}) Y_0 X^T + \mathcal{J}$$

$$= \sum_T (\sum_H c_{HT} Y^H) X^T + \sum_T (\sum_R h_{RT} c_0^R Y^{R_0})[Y_0, X^T] + \mathcal{J}.$$

Now, by [2], $d_0(Y^{R_0}[Y_0, X^T]) < d_0(Y^{R_0} Y_0 X^T)$, and, by Lemma 7.1,

$$d_0(Y^{R_0} Y_0 X^T) \leq \ell(v) + d_0(X_i),$$

so

$$X_i \cdot v_1 = \sum_{HT} c_{HT} Y^H X^T + \mathcal{J} + v'$$

with $\ell(v') < \ell(v) + d_0(X_i)$. Since $d_0(Y^H X^T) = \ell(v) + d_0(X_i)$, we are done if we show that $\sum_{HT} c_{HT} Y^H X^T + \mathcal{J} \neq 0$. By the linear independence of the B_{RT}^0, we have that $\sum_{HT} c_{HT} Y^H X^T + \mathcal{J} = 0$ if and only if the c_{HT} are all zero; but this implies that $X_i p_T(Y) = \sum_R h_{RT} c_0^R Y^{R_0} Y_0$, and hence $X_i \cdot (p_T(Y) + \mathcal{J}) = 0$ for all T. By Corollary 5.8, this says that $p_T(Y) + \mathcal{J} = 0$ for all T, and this is impossible unless $p_T(Y) = 0$ for all T. In such a case we would have that $v_1 = 0$, which is impossible. □

It is now clear how to proceed: one can use the various X_i to move from one $\hat{A}_s$ to another.

Theorem 7.3. *The annihilator of all the vectors v_k is equal to $\mathfrak{U}(\mathfrak{K} \cap \mathcal{A})$, i.e.,*

$$\mathcal{I} = \mathcal{J}.$$

Proof. If $\mathcal{I} \neq \mathcal{J}$ and $u \in \mathcal{I} \setminus \mathcal{J}$, then $v = u + \mathcal{J} \neq 0$ in $\mathfrak{U}/\mathcal{J}$, so $v = \sum h_{RT} B_{RT}^0 = \sum h_{RT} Y^R X^T + \mathcal{J}$. By picking the term $Y^{\bar{R}} X^{\bar{T}}$ of top total degree, we have that $\hat{Y}^{\bar{R}} \hat{X}^{\bar{T}} \cdot v_{\mathcal{I}} = 0$. Set $i = 1, 2$ and $j = 3, 4$. Because of Lemma 7.2, by applying X_i and X_j to v enough times we can assume that $o_{i'}(v) = o_{j'}(v) = 0$. Thus for each $s = 1, 2, 3, 4$ there is $Y^R X^T \in A_s$ such that $\hat{Y}^R \hat{X}^T \cdot v_{\mathcal{I}} = 0$. □

8. APPLICATIONS

We first prove a result involving the so-called coideal property: we say that a subspace S of $\mathfrak{U}$ has the coideal property if

$$\Delta(S) \subset S \otimes \mathfrak{U} + \mathfrak{U} \otimes S.$$

In the classical case any subspace of $\mathfrak{g}$ has the coideal property.

Our result is negative: not only does $\mathfrak{U}(\mathfrak{K} \cap \mathcal{A})$ not have the coideal property but we also show that $\mathfrak{U}(\mathfrak{M} \cap \mathcal{A})$ is the maximal left ideal that annihilates our vectors v_k and that has the coideal property. This contrasts sharply with the classical case. The precise statement is even stronger:

Theorem 8.1. *Suppose that $S \subset \mathcal{A}$ is a space having the coideal property such that $S v_1 = 0$.*
Then $S \subset \mathfrak{U}(\mathfrak{M} \cap \mathcal{A})$.

The proof of the theorem is done in several steps: we begin with a computation.

Lemma 8.2.

$$X_0(v_1 \otimes v_1) \neq 0.$$

Proof. Using the formulas in Proposition A.1 and the recursion formula given at the end of the proof of Lemma 5.2, we find that

$$X_0(v_1 \otimes v_1) = (1 - q^2) \sum_{i=2}^n (-q)^{2n-i-1} X_{e_1 - e_i} v_1 \otimes X_{e_1 + e_i} v_1$$

$$- (1 - q^2) \sum_{i=2}^n (-q)^{i-3} X_{e_1 + e_i} v_1 \otimes X_{e_1 - e_i} v_1.$$

If $X_{e_1-e_2}v_1 = 0$, since v_1 is $\mathfrak{M}$-spherical, we would have that $\mathfrak{U}^+v_1 = 0$, which is impossible. If $X_{e_1+e_2}v_1 = 0$, then, using the PBW basis of $\mathfrak{U}^+$, it would follow that the weight $e_1 + e_2$ cannot occur in $\mathfrak{U}^+v_1$; but this is absurd since $v_1 \in V(e_1 + e_2)$. Hence $X_0(v_1 \otimes v_1) \neq 0$. □

Using the branching theorem of [6], we find that the trivial representation of $\mathfrak{M}$ occurs in $V(\lambda)$ if and only if $\lambda = he_1 + ke_2$ with $h - k$ even and positive. Also the multiplicity of the trivial representation of $\mathfrak{M}$ in $V(\lambda)$ is at most one. We fix a $\mathfrak{M}$-spherical vector $v_{(h,k)}$ in $V(he_1 + ke_2)$. Note that we can and do choose $v_k = v_{(k,k)}$.

Lemma 8.3. *Suppose that S is a subspace of $\mathfrak{U}$ that has the coideal property.*
   If $Sv_1 = 0$ then $Sv_{(2,0)} = 0$.

Proof. If we decompose the $\mathfrak{U}$-module $M = V(\lambda_1) \otimes V(\lambda_1)$ (recall that $\lambda_1 = e_1 + e_2$), we find that the representations occurring in M that contain the trivial representation of $\mathfrak{M}$ are only $V(0)$, $V(e_1 + e_2)$, $V(2e_1 + 2e_2)$, and $V(2e_1)$. Since $v_1 \otimes v_1$ is obviously $\mathfrak{M}$-invariant, we can write

$$v_1 \otimes v_1 = c_1v_{(0,0)} + c_2v_{(1,1)} + c_3v_{(2,2)} + c_4v_{(2,0)}.$$

By Lemma 8.2 it follows that $c_4 \neq 0$. Since S has the coideal property, we know that $S(v_1 \otimes v_1) = 0$. Our claim follows. □

We now prove that the annihilator of all the vectors $v_{(h,k)}$ is $\mathfrak{U}(\mathfrak{M} \cap \mathcal{A})$.

Lemma 8.4. *Set $\mathcal{L} = \{u \in \mathfrak{U} \mid uv_{(h,k)} = 0 \text{ for all } h, k\}$. Then*
$$\mathcal{L} = \mathfrak{U}(\mathfrak{M} \cap \mathcal{A}).$$

Proof. By the same argument we gave in Lemma 6.1, we have that

$$GK(\mathfrak{U}/\mathcal{L}) \geq \dim \mathfrak{g} - \dim \mathfrak{m}.$$

Set $v_{\mathcal{L}} = (1 + \mathcal{L})\hat{\ }$. Arguing as in Lemma 6.3, we can check that $Gr(\mathfrak{U}/\mathcal{L})$ is generated by the set $\{\hat{Y}^R\hat{X}^Tv_{\mathcal{L}}\}$ with the monomials Y^RX^T that satisfy property 5.1.

If there is $u \in \mathcal{L} \setminus \mathfrak{U}(\mathfrak{M} \cap \mathcal{A})$, then, as in the proof of Theorem 7.3, there is a monomial $Y^{\bar{R}}X^{\bar{T}}$ that satisfies property 5.1 such that $\hat{Y}^{\bar{R}}\hat{X}^{\bar{T}}v_{\mathcal{L}} = 0$; but this implies that the Gelfand-Kirillov dimension of $Gr(\mathfrak{U}/\mathcal{L})$ is less than $\dim \mathfrak{g} - \dim \mathfrak{m}$. In light of Lemma 6.2 this is absurd; hence $\mathcal{L} = \mathfrak{U}(\mathfrak{M} \cap \mathcal{A})$. □

We are now ready to give the

Proof of Theorem 8.1. Pick $u \in S$. Because of Lemma 8.4 it is enough to check that $uv_{(h,k)} = 0$ for all h, k. Since $u \in \mathcal{A}$, it follows that $uv_{(0,0)} = 0$. By our hypothesis $uv_{(1,1)} = 0$; hence, by Lemma 8.3, $uv_{(2,0)} = 0$.

Now fix $h, k \in \mathbb{N}$ with $h - k = 2a$, and $a \geq 0$. Since S has the coideal property, it is clear that

$$u((\otimes^a v_{(2,0)}) \otimes (\otimes^k v_{(1,1)})) = 0.$$

We observe that

$$(\otimes^a v_{(2,0)}) \otimes (\otimes^k v_{(1,1)}) = C(q)v_{(h,k)} + \text{other terms},$$

so to conclude we need only to check that $C(q) \neq 0$.

We do this by specialization at $q = 1$ (see [8, § 4]). We can choose our vector $v_{(h,k)}$ in such a way that its specialization $\bar{v}_{(h,k)}$ at $q = 1$ is a nonzero $\mathfrak{m}$-invariant vector in $F(he_1 + ke_2)$. Since specialization respects tensor product, it follows that

$$(\otimes^a \bar{v}_{(2,0)}) \otimes (\otimes^k \bar{v}_{(1,1)}) = C(1)\bar{v}_{(h,k)} + \text{other terms.}$$

If we check that $C(1) \neq 0$ we are done. For doing this, it is enough to check that

$$\bar{X}_{2e_1}^a \bar{X}_{e_1+e_2}^k ((\otimes^a \bar{v}_{(2,0)}) \otimes (\otimes^k \bar{v}_{(1,1)})) \neq 0$$

(here $\bar{X}_{2e_1}$ and $\bar{X}_{e_1+e_2}$ denote the root vectors for $\mathfrak{m}$). But it is easy to compute, using weight considerations, that

$$\bar{X}_{2e_1}^a \bar{X}_{e_1+e_2}^k ((\otimes^a \bar{v}_{(2,0)}) \otimes (\otimes^k \bar{v}_{(1,1)})) = a!k!(\otimes^a \bar{X}_{2e_1} \bar{v}_{(2,0)}) \otimes (\otimes^k \bar{X}_{e_1+e_2} \bar{v}_{(1,1)}).$$

Our result now follows because, by realizing $F(2e_1)$ and $F(e_1 + e_2)$ inside $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ with the standard action of $\mathfrak{sp}(n)$, one can check directly that $\bar{X}_{e_1+e_2}\bar{v}_{(1,1)} \neq 0$ and $\bar{X}_{2e_1}\bar{v}_{(2,0)} \neq 0$. $\qquad\square$

The next result involves the analog of the algebra of invariant differential operators on a symmetric space: let $\mathfrak{K}$ denote the subalgebra of $\mathfrak{U}$ generated by X_0, Y_0 and $\mathfrak{M}$. Let $C_{\mathfrak{U}}(\mathfrak{K})$ denote the centralizer of $\mathfrak{K}$ in $\mathfrak{U}$ and set

$$D_q(\mathfrak{U}, \mathfrak{K}) = C_{\mathfrak{U}}(\mathfrak{K})/C_{\mathfrak{U}}(\mathfrak{K}) \cap \mathfrak{U}(\mathfrak{K} \cap \mathcal{A}).$$

Theorem 8.5. *The set $D_q(\mathfrak{U}, \mathfrak{K})$ is a commutative algebra.*

Proof. Since $C_{\mathfrak{U}}(\mathfrak{K}) \cap \mathfrak{U}(\mathfrak{K} \cap \mathcal{A})$ is obviously a bilateral ideal in $C_{\mathfrak{U}}(\mathfrak{K})$, it follows immediately that $D_q(\mathfrak{U}, \mathfrak{K})$ is an algebra.

If $X, Y \in C_{\mathfrak{U}}(\mathfrak{K})$, then, by multiplicity one, $[X, Y]v_k = 0$ for all k. Hence, by Theorem 7.3, we have that $[X, Y] \in \mathcal{J} = \mathfrak{U}(\mathfrak{K} \cap \mathcal{A})$. $\qquad\square$

Appendix A. Relations between root vectors

In order to clean up our computations, we need one further bit of notation: for $\alpha \in Q$ and $u \in \mathfrak{U}$ we denote by $\mathrm{ad}_\alpha(u)$ the linear map on $\mathfrak{U}$ defined by

$$\mathrm{ad}_\alpha(u)(v) = uv - Ad(K_\alpha)(v)u.$$

Recall (see [2]) that $\mathrm{ad}_\alpha(u)$ is a twisted derivation; that is,

(A.1) $$\mathrm{ad}_\alpha(u)(vw) = \mathrm{ad}_\alpha(u)(v)w + Ad(K_\alpha)(v)\,\mathrm{ad}_\alpha(u)(w).$$

The following notations will also be useful:

$$T^{(i)} = T_i T_{i+1} \ldots T_n T_{n-1} \ldots T_i,$$
$$T^{!i} = T_1 \ldots T_i,$$
$$T^{i!} = T_i \ldots T_1.$$

Proposition A.1. *Relations between positive roots and simple positive roots:*

(A.2) $\qquad \mathrm{ad}_{e_i-e_j}(X_{e_i-e_j})(X_{2e_n}) = -\delta_{jn}X_{e_i+e_n}, \qquad i < j \le n,$

(A.3) $\qquad \mathrm{ad}_{e_i+e_j}(X_{e_i+e_j})(X_{2e_n}) = 0, \qquad\qquad\quad i < j \le n,$

(A.4) $\qquad \mathrm{ad}_{2e_i}(X_{2e_i})(X_{2e_n}) = -\frac{1-q^2}{q+q^{-1}}X^2_{e_i+e_n}, \qquad i < n,$

(A.5) $\qquad \mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{j+1}}) = -\delta_{jh}X_{e_i-e_{j+1}}, \quad i < h,\ i < j < n,$

(A.6) $\qquad \mathrm{ad}_{e_i+e_h}(X_{e_i+e_h})(X_{e_j-e_{j+1}}) = -\delta_{j+1\,h}X_{e_i+e_j}, \quad i < h,\ i < j < n,$

(A.7) $\qquad \mathrm{ad}_{2e_i}(X_{2e_i})(X_{e_j-e_{j+1}}) = 0, \qquad\qquad\quad i < j < n.$

Proof. We will only give the details of the proof of (A.4) (which is the most difficult one). All the others can be derived similarly and are left to the reader.

We want to show that

$$X_{2e_i}X_{2e_n} - X_{2e_n}X_{2e_i} = -\tfrac{1-q^2}{q+q^{-1}}X^2_{e_i+e_n}.$$

First of all we observe that we can assume that $i = 1$. In fact, by Proposition 4.1, the computation for $i > 1$ is the same as for $i = 1$ in the quantized algebra corresponding to $\mathfrak{sp}(n-i+1, \mathbb{C})$. So we are left to verify that

$$X_{2e_1}X_{2e_n} - X_{2e_n}X_{2e_1} = -\tfrac{1-q^2}{q+q^{-1}}X^2_{e_1+e_n}.$$

By the definition of the root vectors X_α given in § 2, we have that $X_{2e_1} = T^{!n-1}(E_n)$. Hence

$$\begin{aligned} X_{2e_1}X_{2e_n} - X_{2e_n}X_{2e_1} &= T^{!n-1}(E_n)T^{(1)}(E_n) - T^{(1)}(E_n)T^{!n-1}(E_n) \\ &= T^{!n-2}(T_{n-1}(E_n)T^{(n-1)}T^{n-2!}(E_n) \\ &\quad - T^{(n-1)}T^{n-2!}(E_n)T_{n-1}(E_n)). \end{aligned}$$

By [2], § 9.5,

$$T^{(n-1)}T^{n-2!}(E_n) = E_n,$$

so

$$X_{2e_1}X_{2e_n} - X_{2e_n}X_{2e_1} = T^{!n-2}(T_{n-1}(E_n)E_n - E_nT_{n-1}(E_n))$$

and, by the relations between root vectors given for $\mathfrak{g}$ of rank 2 in [8], § 5.2,

$$T_{n-1}(E_n)E_n - E_nT_{n-1}(E_n) = -\tfrac{1-q^2}{q+q^{-1}}T_{n-1}T_n(E^2_{n-1}).$$

Therefore

$$\begin{aligned} X_{2e_1}X_{2e_n} - X_{2e_n}X_{2e_1} &= T^{!n-2}(-\tfrac{1-q^2}{q+q^{-1}}T_{n-1}T_n(E^2_{n-1})) \\ &= -\tfrac{1-q^2}{q+q^{-1}}(T^{!n}(E_{n-1}))^2 \\ &= -\tfrac{1-q^2}{q+q^{-1}}X^2_{e_1+e_n}, \end{aligned}$$

since $X_{e_1+e_n} = T^{!n}(E_{n-1})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition A.2. *Relations between positive roots and simple negative roots*:

(A.8) $[X_{e_i - e_j}, X_{-2e_n}] = 0,$ $i < j \le n,$

(A.9) $[X_{e_i + e_j}, X_{-2e_n}] = -\delta_{jn} \dfrac{X_{e_i - e_n} K_{2e_n}}{q^2},$ $i < j \le n,$

(A.10) $[X_{2e_i}, X_{-2e_n}] = \dfrac{(1 - q^{-2}) X_{e_i - e_n}^2 K_{2e_n}}{q^2 (q + q^{-1})},$ $i < n,$

(A.11) $[X_{e_i - e_h}, X_{-e_j + e_{j+1}}] = -\delta_{h\,j+1} \dfrac{X_{e_i - e_j} K_{e_j - e_{j+1}}}{q},$ $i < j < n,\ i < h,$

(A.12) $[X_{e_i + e_h}, X_{-e_j + e_{j+1}}] = -\delta_{hj} \dfrac{X_{e_i + e_{j+1}} K_{e_j - e_{j+1}}}{q},$ $i < j < n,\ i < h,$

(A.13) $[X_{2e_i}, X_{-e_j + e_{j+1}}] = 0,$ $i < j < n.$

Proof. As for Proposition A.1 we will give only an example, namely we will prove (A.11).

As before we can assume that $i = 1$. If $h = j + 1$ then

$$
\begin{aligned}
[X_{e_1 - e_{j+1}}, X_{-e_j + e_{j+1}}] &= [T^{!j-1}(E_j), T^{(1)}(F_j)] \\
&= T^{!j-2}([T_{j-1}(E_j), T^{(j-1)}T^{j-2!}(F_j)]) \\
&= T^{!j-2}([-E_{j-1}E_j + q^{-1}E_j E_{j-1}, F_j]) \\
&= T^{!j-2}(-E_{j-1}[E_j, F_j] + q^{-1}[E_j, F_j]E_{j-1}) \\
&= T^{!j-2}\Big(-E_{j-1}\Big(\frac{K_{e_j - e_{j+1}} - K_{-e_j + e_{j+1}}}{q - q^{-1}}\Big) \\
&\quad + q^{-1}\Big(\frac{K_{e_j - e_{j+1}} - K_{-e_j + e_{j+1}}}{q - q^{-1}}\Big)E_{j-1}\Big) \\
&= T^{!j-2}\Big(-E_{j-1}\Big(\frac{K_{e_j - e_{j+1}} - K_{-e_j + e_{j+1}}}{q - q^{-1}}\Big) \\
&\quad + q^{-1}E_{j-1}\Big(\frac{q^{-1}K_{e_j - e_{j+1}} - qK_{-e_j + e_{j+1}}}{q - q^{-1}}\Big)\Big) \\
&= \frac{T^{!j-2}(-E_{j-1}(1 - q^{-2})K_{e_j - e_{j+1}})}{q - q^{-1}} \\
&= -\frac{X_{e_1 - e_j} K_{e_j - e_{j+1}}}{q}.
\end{aligned}
$$

If $h > j + 1$, then

$$
[X_{e_1 - e_h}, X_{-e_j + e_{j+1}}] = [T^{!h-2}(E_{h-1}), T^{(1)}(F_j)],
$$

so

$$
\begin{aligned}
[X_{e_1 - e_h}, X_{-e_j + e_{j+1}}] &= T^{!h-2}([E_{h-1}, T^{(h-1)}T^{h-2!}(F_j)]) \\
&= T^{!h-2}([E_{h-1}, T^{(h-1)}(F_{j-1})]) \\
&= T^{!h-2}([E_{h-1}, F_{j-1}]) = 0.
\end{aligned}
$$

If $h \leq j$,

$$[X_{e_1-e_h}, X_{-e_j+e_{j+1}}] = [T^{!h-2}(E_{h-1}), T^{(1)}(F_j)]$$
$$= T^{!h-2}([E_{h-1}, T^{(h-1)}T^{h-2!}(F_j)])$$
$$= T^{!h-2}([E_{h-1}, F_j]) = 0.$$

The other formulas are obtained similarly. $\square$

Proposition A.3. *Relations between generic positive roots*:

(A.14) $\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_r}) = -\delta_{jh}X_{e_i-e_r}$, $i < h \leq j < r$,

(A.15) $\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j+e_r}) = -\delta_{jh}X_{e_i+e_r}$, $i < h \leq j < r$,

(A.16) $\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j+e_r}) = (q - q^{-1})X_{e_i+e_r}X_{e_j-e_h}$
$$= (q - q^{-1})X_{e_j-e_h}X_{e_i+e_r}, \quad i < j < h < r.$$

Proof. The proof of (A.14) is by induction on $r - j$. If $r - j = 1$ then (A.14) becomes (A.5). If $r - j > 1$ then, by (A.5) again,

$$\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_r}) = -\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(\mathrm{ad}_{e_j-e_{r-1}}(X_{e_j-e_{r-1}})(X_{e_{r-1}-e_r}))$$
$$= -\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{r-1}}X_{e_{r-1}-e_r} - q^{-1}X_{e_{r-1}-e_r}X_{e_j-e_{r-1}})$$
$$= -\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{r-1}})X_{e_{r-1}-e_r}$$
$$- Ad(K_{e_i-e_h})(X_{e_j-e_{r-1}})\,\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_{r-1}-e_r})$$
$$+ q^{-1}\,\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_{r-1}-e_r})X_{e_j-e_{r-1}}$$
$$+ q^{-1}Ad(K_{e_i-e_h})(X_{e_{r-1}-e_r})\,\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{r-1}})$$
$$= -\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{r-1}})X_{e_{r-1}-e_r}$$
$$+ q^{-1}Ad(K_{e_i-e_h})(X_{e_{r-1}-e_r})\,\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_{r-1}}),$$

so, by applying the induction hypothesis, we find that

$$\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(X_{e_j-e_r}) = \delta_{jh}X_{e_i-e_{r-1}}X_{e_{r-1}-e_r} - \delta_{jh}q^{-1}X_{e_{r-1}-e_r}X_{e_i-e_{r-1}}$$
$$= -\delta_{jh}X_{e_i-e_r}.$$

The proof of (A.15) is very similar to the proof of (A.14) above: we prove it by induction on $n - r$. For $r = n$ we use (A.2) to write

$$X_{e_j+e_n} = -X_{e_j-e_n}X_{2e_n} + q^{-2}X_{2e_n}X_{e_j-e_n}$$

and then apply (A.2) and (A.14) to check that

$$\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(-X_{e_j-e_n}X_{2e_n} + q^{-2}X_{2e_n}X_{e_j-e_n})$$

is equal to $-\delta_{jh}X_{e_i+e_n}$. The inductive step is similar, except that we use (A.6) to write

$$X_{e_j+e_r} = -X_{e_j+e_{r+1}}X_{e_r-e_{r+1}} + q^{-1}X_{e_r-e_{r+1}}X_{e_j+e_{r+1}}$$

and then we use (A.5) and the induction hypothesis to compute

$$\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(-X_{e_j+e_{r+1}}X_{e_r-e_{r+1}} + q^{-1}X_{e_r-e_{r+1}}X_{e_j+e_{r+1}}).$$

Finally, the proof of (A.16) is done by induction on $h - j$. We use (A.15) to write

$$X_{e_j+e_r} = -X_{e_j-e_{j+1}}X_{e_{j+1}+e_r} + q^{-1}X_{e_{j+1}+e_r}X_{e_j-e_{j+1}}.$$

To compute

$$\mathrm{ad}_{e_i-e_h}(X_{e_i-e_h})(-X_{e_j-e_{j+1}}X_{e_{j+1}+e_r} + q^{-1}X_{e_{j+1}+e_r}X_{e_j-e_{j+1}})$$

we use (A.5), (A.6), (A.15) when $j = h - 1$, and (A.5), (A.6), (A.14), and the induction hypothesis when $h - j > 1$. $\qquad\square$

Proposition A.4. *Relations between generic positive and negative roots*:

(A.17) $\qquad [X_{e_i-e_h}, X_{-e_i+e_j}] = qX_{-e_h+e_j}K_{e_i-e_h}, \qquad i < h < j,$

(A.18) $\qquad [X_{e_i-e_h}, X_{-e_i-e_j}] = qX_{-e_h-e_j}K_{e_i-e_h}, \qquad i < h < j,$

(A.19) $\qquad [X_{e_i+e_h}, X_{-e_j+e_r}] = 0, \qquad\qquad\qquad i < j < h, \ j < r,$

(A.20) $\qquad [X_{e_i+e_n}, X_{-e_j-e_n}] = -q^{-1}X_{e_i-e_j}K_{e_j+e_n}, \quad i < j < n, \ j < r,$

(A.21) $\qquad [X_{e_i+e_h}, X_{-e_j-e_h}] = -q^{-1}X_{e_i-e_j}K_{e_j+e_h}, \quad i < j < h.$

Proof. The proof of (A.17) is by induction on $j - h$. Applying Ω to (A.5), one finds that

$$X_{-e_i+e_j} = -X_{-e_{j-1}+e_j}X_{-e_i+e_{j-1}} + qX_{-e_i+e_{j-1}}X_{-e_{j-1}+e_j}.$$

We observe also that

(A.22) $$[X_{e_i-e_h}, X_{-e_i+e_h}] = \frac{K_{e_i-e_h} - K_{-e_i+e_h}}{q - q^{-1}};$$

so, by computing

$$[X_{e_i-e_h}, -X_{-e_{j-1}+e_j}X_{-e_i+e_{j-1}} + qX_{-e_i+e_{j-1}}X_{-e_{j-1}+e_j}],$$

using (A.11) and (A.22) when $j = h + 1$, the induction hypothesis, (A.5), and (A.11) if $j > h + 1$, one derives (A.17).

For (A.18) we use induction on $n - j$. The proof is completely analogous to the previous one except that we use (A.2) to write

$$X_{-e_i-e_n} = -X_{-2e_n}X_{-e_i+e_n} + q^2X_{-e_i+e_n}X_{-2e_n}$$

and (A.6) to write

$$X_{-e_i-e_j} = -X_{-e_j+e_{j+1}}X_{-e_i-e_{j+1}} + qX_{-e_i-e_j}X_{-e_j+e_{j+1}}.$$

We then use (A.17) and (A.2) to prove (A.18) when $j = n$, and (A.5) and the induction hypothesis to prove it when $j < n$.

The induction for (A.19) is on $r - j$. If $j + 1 = r$ then (A.19) reduces to (A.12). If $r - j > 1$ then we apply Ω to (A.5) to obtain that

$$X_{-e_j+e_r} = -X_{-e_{r-1}+e_r}X_{-e_j+e_{r-1}} + qX_{-e_j+e_{r-1}}X_{-e_{r-1}+e_r}.$$

Our formula now follows from (A.12) and the induction hypothesis.

The relation (A.20) is obtained by induction on $n - j$. For the base of the induction one uses (A.2) to write

$$X_{-e_{n-1}-e_n} = -X_{-2e_n}X_{-e_{n-1}+e_n} + q^2X_{-e_{n-1}+e_n}X_{-2e_n},$$

and then one uses (A.9), (A.12), and (A.11). The inductive step is obtained by starting with (A.14) to write

$$X_{-e_j-e_n} = -X_{-e_{j+1}-e_n}X_{-e_j+e_{j+1}} + qX_{-e_j+e_{j+1}}X_{-e_{j+1}+e_n},$$

then using the induction hypothesis, (A.12), and (A.14).

Finally, (A.21) is obtained in a completely analogous fashion by induction on $n - h$. The base of the induction is (A.20), while the inductive step is obtained

starting with (A.6) to give a different expression for $X_{-e_j-e_h}$ and then applying (A.12), (A.19), and the induction hypothesis to compute the bracket. $\square$

Proposition A.5. *Relations between orthogonal but not strongly orthogonal roots*:

$$\text{(A.23)} \qquad [X_{e_i+e_n}, X_{e_i-e_n}] = (q + q^{-1})X_{2e_i}, \qquad i < n,$$

$$\text{(A.24)} \qquad [X_{e_i-e_n}, X_{-e_i-e_n}] = q^2(q + q^{-1})X_{-2e_n}K_{e_i-e_n}, \qquad i < n,$$

$$\text{(A.25)} \qquad \begin{aligned} [X_{e_i+e_j}, X_{e_i-e_j}] &= q^{-1}X_{e_i-e_{j+1}}X_{e_i+e_{j+1}} \\ &\quad - qX_{e_i+e_{j+1}}X_{e_i-e_{j+1}}, \qquad i < j < n, \end{aligned}$$

$$\text{(A.26)} \qquad \begin{aligned} [X_{e_i-e_j}, X_{-e_i-e_j}] &= -qX_{-e_j+e_{j+1}}X_{-e_j-e_{j+1}}K_{e_i-e_j}+, \\ &\quad + q^3 X_{-e_j-e_{j+1}}X_{-e_j+e_{j+1}}K_{e_i-e_j}, \qquad i < j < n. \end{aligned}$$

Proof. Without loss of generality we may assume that $i = 1$. To prove (A.23) we observe that, by [2], § 9.3,

$$\text{(A.27)} \qquad [X_{e_1+e_n}, X_{e_1-e_n}] = cX_{2e_1}$$

for some c. On the other hand, by Proposition A.1,

$$\begin{aligned} X_{e_1+e_n}X_{e_1-e_n}X_{2e_n} &= X_{e_1+e_n}(q^{-2}X_{2e_n}X_{e_1-e_n} - X_{e_1+e_n}) \\ &= q^{-2}X_{e_1+e_n}X_{2e_n}X_{e_1-e_n} - X_{e_1+e_n}^2 \\ &= q^{-2}q^2 X_{2e_n}X_{e_1+e_n}X_{e_1-e_n} - X_{e_1+e_n}^2 \\ &= X_{2e_n}X_{e_1+e_n}X_{e_1-e_n} - X_{e_1+e_n}^2, \end{aligned}$$

$$\begin{aligned} X_{e_1-e_n}X_{e_1+e_n}X_{2e_n} &= X_{e_1-e_n}(q^2 X_{2e_n}X_{e_1+e_n}) \\ &= q^2(q^{-2}X_{2e_n}X_{e_1-e_n} - X_{e_1+e_n})X_{e_1+e_n} \\ &= X_{2e_n}X_{e_1-e_n}X_{e_1+e_n} - q^2 X_{e_1+e_n}^2, \end{aligned}$$

and

$$X_{2e_1}X_{2e_n} = X_{2e_n}X_{2e_1} - \frac{1-q^2}{q+q^{-1}}X_{e_1+e_n}^2.$$

It follows that

$$[X_{e_1+e_n}, X_{e_1-e_n}]X_{2e_n} = X_{2e_n}[X_{e_1+e_n}, X_{e_1-e_n}] - X_{e_1+e_n}^2 + q^2 X_{e_1+e_n}^2.$$

Using (A.27), we obtain that

$$cX_{2e_1}X_{2e_n} = cX_{2e_n}X_{2e_1} - X_{e_1+e_n}^2 + q^2 X_{e_1+e_n}^2.$$

Hence

$$cX_{2e_n}X_{2e_1} - c\frac{1-q^2}{q+q^{-1}}X_{e_1+e_n}^2 = cX_{2e_n}X_{2e_1} - X_{e_1+e_n}^2 + q^2 X_{e_1+e_n}^2.$$

It follows that

$$c\frac{1-q^2}{q+q^{-1}}X_{e_1+e_n}^2 = X_{e_1+e_n}^2 - q^2 X_{e_1+e_n}^2.$$

This implies that $c(1 - q^2)/(q + q^{-1}) = 1 - q^2$ and therefore $c = (q + q^{-1})$.

We now turn to (A.25). Indeed, by (A.11),

$$X_{e_1-e_i} = -q[X_{e_1-e_{i+1}}, X_{-e_i+e_{i+1}}]K_{-e_i+e_{i+1}};$$

therefore

$$X_{e_1+e_i}X_{e_1-e_i} = X_{e_1+e_i}(-q[X_{e_1-e_{i+1}}, X_{-e_i+e_{i+1}}]K_{-e_i+e_{i+1}})$$
$$= -qX_{e_1+e_i}X_{e_1-e_{i+1}}X_{-e_i+e_{i+1}}K_{-e_i+e_{i+1}}$$
$$+ qX_{e_1+e_i}X_{-e_i+e_{i+1}}X_{e_1-e_{i+1}}K_{-e_i+e_{i+1}},$$

so, using (A.12) and [2], § 9.3,

$$X_{e_1+e_i}X_{e_1-e_i} = -X_{e_1-e_{i+1}}X_{e_1+e_i}X_{-e_i+e_{i+1}}K_{-e_i+e_{i+1}}$$
$$+ qX_{-e_i+e_{i+1}}X_{e_1+e_i}X_{e_1-e_{i+1}}K_{-e_i+e_{i+1}}$$
$$- X_{e_1+e_{i+1}}K_{e_i-e_{i+1}}X_{e_1-e_{i+1}}K_{-e_i+e_{i+1}}$$
$$= -X_{e_1-e_{i+1}}X_{-e_i+e_{i+1}}X_{e_1+e_i}K_{-e_i+e_{i+1}} + q^{-1}X_{e_1-e_{i+1}}X_{e_1+e_{i+1}}$$
$$+ X_{-e_i+e_{i+1}}X_{e_1-e_{i+1}}X_{e_1+e_i}K_{-e_i+e_{i+1}} - qX_{e_1+e_{i+1}}X_{e_1-e_{i+1}}$$
$$= -qX_{e_1-e_{i+1}}X_{-e_i+e_{i+1}}K_{-e_i+e_{i+1}}X_{e_1+e_i} + q^{-1}X_{e_1-e_{i+1}}X_{e_1+e_{i+1}}$$
$$+ qX_{-e_i+e_{i+1}}X_{e_1-e_{i+1}}K_{-e_i+e_{i+1}}X_{e_1+e_i} - qX_{e_1+e_{i+1}}X_{e_1-e_{i+1}}$$
$$= -q[X_{e_1-e_{i+1}}, X_{-e_i+e_{i+1}}]K_{-e_i+e_{i+1}}X_{e_1+e_i}$$
$$+ q^{-1}X_{e_1-e_{i+1}}X_{e_1+e_{i+1}} - qX_{e_1+e_{i+1}}X_{e_1-e_{i+1}}.$$

By (A.11) again, we have that

$$X_{e_1+e_i}X_{e_1-e_i} = X_{e_1-e_i}X_{e_1+e_i} + q^{-1}X_{e_1-e_{i+1}}X_{e_1+e_{i+1}} - qX_{e_1+e_{i+1}}X_{e_1-e_{i+1}},$$

as we wished to prove.

Formulas (A.24) and (A.26) are easier: applying Ω to (A.2), we write

$$X_{-e_i-e_n} = -X_{-2e_n}X_{-e_i+e_n} + q^2X_{-e_i+e_n}X_{-2e_n}.$$

Hence, using (A.8),

$$[X_{e_i-e_n}, X_{-e_i-e_n}] = -X_{-2e_n}[X_{e_i-e_n}, X_{-e_i+e_n}] + q^2[X_{e_i-e_n}, X_{-e_i+e_n}]X_{-2e_n}$$
$$= -X_{-2e_n}\left(\frac{K_{e_i-e_n} - K_{-e_i+e_n}}{q - q^{-1}}\right) + q^2\left(\frac{K_{e_i-e_n} - K_{-e_i+e_n}}{q - q^{-1}}\right)X_{-2e_n}$$
$$= q^2(q + q^{-1})X_{-2e_n}K_{e_i-e_n}.$$

This proves (A.24).

For (A.26) we write, using (A.6),

$$X_{-e_i-e_j} = -X_{-e_j+e_{j+1}}X_{-e_i-e_{j+1}} + qX_{-e_i-e_{j+1}}X_{-e_j+e_{j+1}},$$

and then we use (A.11) together with (A.18) to compute

$$[X_{e_i-e_j}, -X_{-e_j+e_{j+1}}X_{-e_i-e_{j+1}} + qX_{-e_i-e_{j+1}}X_{-e_j+e_{j+1}}]$$

and obtain the desired result. $\square$

In the next proposition we write $u \equiv v$ to mean $u \equiv v \mod \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$.

Proposition A.6. *Relations* $\mod \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$:

(A.28) $[X_{e_1+e_2}, X_{-e_2+e_j}] \equiv (-q)^{-j+2}X_{e_1+e_j}, \qquad j > 2,$

$$(A.29) \qquad [X_{e_2 - e_j}, X_{-e_1 - e_2}] \equiv -(-q)^{-j+3} X_{-e_1 - e_j}, \quad j > 2,$$

$$(A.30) \qquad [X_{e_1 + e_2}, X_{-e_2 - e_j}] \equiv -(-q)^{j-2n} X_{e_1 - e_j}, \quad j > 2,$$

$$(A.31) \qquad [X_{e_2 + e_j}, X_{-e_1 - e_2}] \equiv (-q)^{j+1-2n} X_{-e_1 + e_j}, \quad j > 2.$$

Proof. The proof of (A.28) is by induction on j. If $j = 3$ then (A.28) reduces to (A.12); if $j > 3$ then we write, using (A.5),

$$X_{-e_2 + e_j} = -X_{-e_{j-1} + e_j} X_{-e_2 + e_{j-1}} + q X_{-e_2 + e_{j-1}} X_{-e_{j-1} + e_j},$$

and so, using (A.12) and the fact that $X_{-e_{j-1} + e_j} \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$,

$$[X_{e_1 + e_2}, X_{-e_2 + e_j}] \equiv [X_{e_1 + e_2}, -X_{-e_{j-1} + e_j} X_{-e_2 + e_{j-1}} + q X_{-e_2 + e_{j-1}} X_{-e_{j-1} + e_j}]$$
$$\equiv -X_{-e_{j-1} + e_j} [X_{e_1 + e_2}, X_{-e_2 + e_{j-1}}].$$

Thus we can apply the induction hypothesis and find that

$$[X_{e_1 + e_2}, X_{-e_2 + e_j}] \equiv -(-q)^{-j+3} X_{-e_{j-1} + e_j} X_{e_1 + e_{j-1}}$$
$$\equiv -(-q)^{-j+3} [X_{-e_{j-1} + e_j}, X_{e_1 + e_{j-1}}]$$
$$\equiv (-q)^{-j+2} X_{e_1 + e_j},$$

where in the last equation we have used (A.12).

The proof of (A.29) is completely analogous.

The induction for (A.30) is on $n - j$: if $j = n$ then, by (A.2),

$$X_{-e_2 - e_n} = -X_{-2e_n} X_{-e_2 + e_n} + q^2 X_{-e_2 + e_n} X_{-2e_n},$$

and this implies that

$$[X_{e_1 + e_2}, X_{-e_2 + e_n}] \equiv [X_{e_1 + e_2}, -X_{-2e_n} X_{-e_2 + e_n} + q^2 X_{-e_2 + e_n} X_{-2e_n}]$$
$$\equiv -X_{-2e_n} [X_{e_1 + e_2}, X_{-e_2 + e_n}]$$
$$\equiv -(-q)^{-n+2} X_{-2e_n} X_{e_1 + e_n}$$
$$\equiv -(-q)^{-n+2} [X_{-2e_n}, X_{e_1 + e_n}]$$
$$\equiv -(-q)^{-n} X_{e_1 - e_n}.$$

Here we have used (A.9), (A.28), and the fact that $X_{-2e_n} \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$. The inductive step for (A.30) and formula (A.31) are completely analogous. $\qquad \square$

Proposition A.7. *Big relations*:

$$\text{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_j+e_i}) = \sum_{k>i}(-q)^{k-i-1}(q-q^{-1})X_{e_1-e_k}X_{e_j+e_k}$$

$$(A.32)\qquad \begin{aligned} &- \sum_{k=j+1}^{i-1}(-q)^{2n-i-k+1}(q-q^{-1})X_{e_1+e_k}X_{e_j-e_k} \\ &+ (q-q^{-1})(q^{-1}+q^{2n-2i+1})X_{e_1+e_i}X_{e_j-e_i} \\ &- \sum_{k\geq i+1}(-q)^{2n-i-k+1}(q-q^{-1})X_{e_1+e_k}X_{e_j-e_k} \\ &+ (-q)^{2n-j-i+1}X_{e_1+e_j}, \qquad 1<j<i, \end{aligned}$$

$$\text{ad}_{e_j+e_i}(X_{-e_j-e_i})(X_{-e_1+e_i}) = \sum_{k>i}(-q)^{k-i}(q-q^{-1})X_{-e_1+e_k}X_{-e_j-e_k}$$

$$(A.33)\qquad \begin{aligned} &- \sum_{k=j+1}^{i-1}(-q)^{2n-i-k+2}(q-q^{-1})X_{-e_1-e_k}X_{-e_j+e_k} \\ &- (q-q^{-1})(1+q^{2n-2i+2})X_{-e_1-e_i}X_{-e_j+e_i} \\ &- \sum_{k\geq i+1}(-q)^{2n-i-k+2}(q-q^{-1})X_{-e_1-e_k}X_{-e_j+e_k} \\ &+ (-q)^{2n-j-i+1}X_{-e_1-e_j}, \qquad 1<j<i. \end{aligned}$$

Proof. We will prove only (A.32), the proof of (A.33) being identical.

If $i=n$ then (A.32) specializes to

$$(A.34)\qquad \begin{aligned} \text{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_j+e_n}) = &-\sum_{k=j+1}^{n-1}(-q)^{n-k+1}(q-q^{-1})X_{e_1+e_k}X_{e_j-e_k} \\ &+ (q-q^{-1})(q^{-1}+q)X_{e_1+e_n}X_{e_j-e_n} \\ &+ (-q)^{n-j+1}X_{e_1+e_j}. \end{aligned}$$

We start by proving (A.34) by induction on $n-j$. If $j=n-1$ then, by (A.2),

$$(A.35)\qquad X_{e_{n-1}+e_n} = -X_{e_{n-1}-e_n}X_{2e_n} + q^{-2}X_{2e_n}X_{e_{n-1}-e_n},$$

so, by applying $\text{ad}_{e_1-e_n}(X_{e_1-e_n})$ to (A.35) and using (A.1), (A.2), and (A.5), we find that

$$\text{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_{n-1}+e_n}) = qX_{e_{n-1}-e_n}X_{e_1+e_n} - q^{-2}X_{e_1+e_n}X_{e_{n-1}-e_n}.$$

Using (A.6), we now find that

$$\text{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_{n-1}+e_n}) = (q^2-q^{-2})X_{e_1+e_n}X_{e_{n-1}-e_n} + q^2X_{e_1+e_{n-1}},$$

which is (A.34) for $j=n-1$.

If $j<n-1$, then we use (A.15) and write

$$X_{e_j+e_n} = -X_{e_j-e_{j+1}}X_{e_{j+1}+e_n} + q^{-1}X_{e_{j+1}+e_n}X_{e_j-e_{j+1}}$$

and apply $\mathrm{ad}_{e_1-e_n}(X_{e_1-e_n})$ to this last expression. By (A.1) and (A.5), we find that

$$\mathrm{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_j+e_n}) = -X_{e_j-e_{j+1}}\,\mathrm{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_{j+1}+e_n})$$
$$+ q^{-1}\,\mathrm{ad}_{e_1-e_n}(X_{e_1-e_n})(X_{e_{j+1}+e_n})X_{e_j-e_{j+1}}$$

so, using the induction hypothesis, (A.6), and (A.14), one can easily check (A.34).

Let us prove (A.32) by induction on $n-i$. Clearly (A.34) is the base of the induction. For the inductive step we can write, using (A.6),

$$X_{e_j+e_i} = -X_{e_j+e_{i+1}}X_{e_i-e_{i+1}} + q^{-1}X_{e_i-e_{i+1}}X_{e_j+e_{i+1}};$$

applying $\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})$ to this expression, we find using (A.1) that

$$\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_j+e_i}) = -\,\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_j+e_{i+1}})X_{e_i-e_{i+1}}$$
$$- X_{e_j+e_{i+1}}\,\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_i-e_{i+1}})$$
$$+ q^{-1}\,\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_i-e_{i+1}})X_{e_j+e_{i+1}}$$
$$+ q^{-2}X_{e_i-e_{i+1}}\,\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_j+e_{i+1}}).$$

Using (A.5), (A.6), and (A.16), we can rewrite the above expression as follows:

$$\mathrm{ad}_{e_1-e_i}(X_{e_1-e_i})(X_{e_j+e_i}) = -q\,\mathrm{ad}_{e_1-e_{i+1}}(X_{e_1-e_{i+1}})(X_{e_j+e_{i+1}})$$
$$+ (q-q^{-1})(X_{e_1+e_{i+1}}X_{e_j-e_{i+1}} + q^{-1}X_{e_1+e_i}X_{e_j-e_i} + X_{e_1-e_{i+1}}X_{e_j+e_{i+1}}).$$

One easily verifies that (A.32) satisfies the recursion formula above. $\qquad\square$

APPENDIX B. COMPUTATIONAL RESULTS

Proof of Lemma 5.5. We write $u \equiv v$ to mean $u - v \in \mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$. In general, if $u_i \in \mathfrak{U}$ for $i = 1, 2, 3, 4$, then

$$[u_1u_2, u_3u_4] = u_3[u_1, u_4]u_2 + u_3u_1[u_2, u_4] + [u_1, u_3]u_2u_4 + u_1[u_2, u_3]u_4.$$

We note that $X_{e_1+e_2}$ and $X_{-e_1-e_2}$ both commute with $[X_{e_1-e_2}, X_{-e_1+e_2}]$, while $X_{e_1-e_2}$ and $X_{-e_1+e_2}$ commute with $[X_{e_1+e_2}, X_{-e_1-e_2}]$. By using this fact and the above relation one immediately obtains that

$$[X_0, Y_0] \equiv -q^{-2}X_{-e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}$$
$$- q^{-2}X_{e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{-e_1-e_2} - q^2X_{e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{-e_1+e_2}$$
$$- q^2X_{-e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{e_1-e_2} + [X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2}$$
$$+ [X_{e_1+e_2}, X_{-e_1+e_2}]X_{e_1-e_2}X_{-e_1-e_2} + X_{-e_1-e_2}X_{e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]$$
$$+ X_{-e_1+e_2}X_{e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}].$$

Since by (A.26) $[X_{e_1-e_2}, X_{-e_1-e_2}]$ and $[X_{e_1+e_2}, X_{-e_1+e_2}]$ belong to $\mathfrak{U}(\mathcal{A} \cap \mathfrak{M})$, we get rid of the last two terms in the above expression.

For the remaining terms we will show that

(B.1) $\quad X_{-e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2} \equiv -X_{e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{-e_1-e_2},$

(B.2) $\quad X_{e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{-e_1+e_2} \equiv -X_{-e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{e_1-e_2},$

(B.3) $\quad [X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2} \equiv -[X_{e_1+e_2}, X_{-e_1+e_2}]X_{e_1-e_2}X_{-e_1-e_2};$

this will conclude the proof.

If $n = 2$ then one can verify the above congruences directly, using (A.2), (A.9), and (A.24). If $n > 3$, we begin by calculating the left hand side of (B.1): by (A.26) we have that

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2} = -qX_{-e_2+e_3}X_{-e_2-e_3}K_{e_1-e_2}X_{e_1+e_2}$$
$$+ q^3 X_{-e_2-e_3}X_{-e_2+e_3}K_{e_1-e_2}X_{e_1+e_2}$$
$$\equiv -qX_{-e_2+e_3}[X_{-e_2-e_3}, X_{e_1+e_2}]$$
$$+ q^3 X_{-e_2-e_3}[X_{-e_2+e_3}, X_{e_1+e_2}],$$

so, by (A.28) and (A.30),

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2} \equiv q^{4-2n}X_{-e_2+e_3}X_{e_1-e_3} + q^2 X_{-e_2-e_3}X_{e_1+e_3}$$
$$\equiv q^{4-2n}[X_{-e_2+e_3}, X_{e_1-e_3}] + q^2 [X_{-e_2-e_3}, X_{e_1+e_3}]$$
$$\equiv (q^{3-2n} + q)X_{e_1-e_2},$$

where the last congruence follows by (A.11) and (A.21). Therefore

$$X_{-e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2} \equiv (q^{3-2n} + q)X_{-e_1+e_2}X_{e_1-e_2}.$$

Following the same sort of computation but using (A.29) and (A.31) instead of (A.28) and (A.30), we obtain that

$$X_{e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{-e_1-e_2} \equiv -(q^{3-2n} + q)X_{e_1-e_2}X_{-e_1+e_2},$$

thus proving (B.1).

We now compute the left hand side of (B.2): by (A.5), (A.15), and (A.26), one obtains that

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{-e_1+e_2} \equiv -q^{-1}X_{-e_2+e_3}X_{-e_2-e_3}X_{-e_1+e_2}$$
$$+ qX_{-e_2-e_3}X_{-e_2+e_3}X_{-e_1+e_2}$$
$$\equiv q^{-1}X_{-e_2+e_3}X_{-e_1-e_3} - qX_{-e_2-e_3}X_{-e_1+e_3}$$
$$\equiv -(q^{-1} + q^{2n-3})X_{-e_1-e_2}$$

(here we used (A.6) and (A.33) in the last step). Therefore

$$X_{e_1+e_2}[X_{e_1-e_2}, X_{-e_1-e_2}]X_{-e_1+e_2} \equiv -(q^{-1} + q^{2n-3})X_{e_1+e_2}X_{-e_1-e_2}.$$

Doing the same calculation for the right hand side of (B.2) but using (A.32) instead of (A.33), we find that

$$X_{-e_1-e_2}[X_{e_1+e_2}, X_{-e_1+e_2}]X_{e_1-e_2} \equiv (q^{-1} + q^{2n-3})X_{-e_1-e_2}X_{e_1+e_2}.$$

Thus (B.2) is verified.

We now turn to (B.3): because of (A.16) we know that $[X_{e_1+e_2}, X_{-e_1+e_2}] \in \mathcal{A} \cap \mathfrak{M}$, so we can write

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2} \equiv [X_{e_1-e_2}, X_{-e_1-e_2}]X_{-e_1+e_2}X_{e_1+e_2}.$$

Thus, by a straightforward calculation that makes use of (A.5), (A.6), (A.12), and (A.15) to bring the terms in $\mathcal{A} \cap \mathfrak{M}$ on the right, we obtain

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2} \equiv -X_{-e_2+e_3}X_{-e_1+e_2}X_{-e_2-e_3}X_{e_1+e_2}$$
$$- qX_{-e_2-e_3}X_{-e_1+e_3}X_{e_1+e_2} + q^2 X_{-e_1+e_2}X_{-e_2-e_3}X_{e_1+e_3}$$
$$- (q - q^{-1})X_{-e_1-e_3}X_{e_1+e_3} - q^{-1}X_{-e_1-e_2}X_{e_1+e_2}.$$

Using again (A.5), (A.11), together with (A.21), and (A.30), one obtains

$$[X_{e_1-e_2}, X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2} \equiv -(q-q^{-1})X_{-e_1-e_3}X_{e_1+e_3}$$
$$+ (q+q^{-2n+3})X_{-e_1+e_2}X_{e_1-e_2} - q^{-1}X_{-e_1-e_2}X_{e_1+e_2}$$
$$- q^{-2n+3}X_{-e_1+e_3}X_{e_1-e_3} - qX_{-e_2-e_3}X_{-e_1+e_3}X_{e_1+e_2}.$$

We now work on the term $X_{-e_2-e_3}X_{-e_1+e_3}X_{e_1+e_2}$. Using (A.33), we have that

$$X_{-e_2-e_3}X_{-e_1+e_3} = qX_{-e_1+e_3}X_{-e_2-e_3} + \sum_{k>3}(-q)^{k-3}(q-q^{-1})X_{-e_1+e_k}X_{-e_2-e_k}$$
$$- (q-q^{-1})(1+q^{2n-4})X_{-e_1-e_3}X_{-e_2+e_3}$$
$$- \sum_{k\geq 4}(-q)^{2n-k-1}(q-q^{-1})X_{-e_1-e_k}X_{-e_2+e_k}$$
$$+ q^{2n-4}X_{-e_1-e_2}.$$

Therefore, by (A.28) and (A.30),

$$-qX_{-e_2-e_3}X_{-e_1+e_3}X_{e_1+e_2} \equiv q^{5-2n}X_{-e_1+e_3}X_{e_1-e_3} - q^{2n-3}X_{-e_1-e_2}X_{e_1+e_2}$$
$$+ (q-q^{-1})(1+q^{2n-4})X_{-e_1-e_3}X_{e_1+e_3}$$
$$+ \sum_{k>3}(-q)^{-2n+2k-2}(q-q^{-1})X_{-e_1+e_k}X_{e_1-e_k}$$
$$+ \sum_{k\geq 4}(-q)^{2n-2k+2}(q-q^{-1})X_{-e_1-e_k}X_{e_1+e_k}.$$

Collecting terms, we find that

$$[X_{e_1-e_2}X_{-e_1-e_2}]X_{e_1+e_2}X_{-e_1+e_2} \equiv (q+q^{-2n+3})X_{-e_1+e_2}X_{e_1-e_2}$$
$$- (q^{-1}+q^{2n-3})X_{-e_1-e_2}X_{e_1+e_2}$$
$$+ (q^{5-2n}-q^{-2n+3})X_{-e_1+e_3}X_{e_1-e_3}$$
$$+ (q^{2n-3}-q^{2n-5})X_{-e_1-e_3}X_{e_1+e_3}$$
$$+ \sum_{k>3}q^{-2n+2k-2}(q-q^{-1})X_{-e_1+e_k}X_{e_1-e_k}$$
$$+ \sum_{k\geq 4}q^{2n-2k+2}(q-q^{-1})X_{-e_1-e_k}X_{e_1+e_k}.$$

For the right hand side of (B.3) the same calculations using (A.29), (A.31), and (A.32) instead of (A.28), (A.30), and (A.33) yield that

$$[X_{e_1+e_2}X_{-e_1+e_2}]X_{e_1-e_2}X_{-e_1-e_2} \equiv -(q+q^{-2n+3})X_{e_1-e_2}X_{-e_1+e_2}$$
$$+ (q^{-1}+q^{2n-3})X_{e_1+e_2}X_{-e_1-e_2}$$
$$- (q^{5-2n}-q^{-2n+3})X_{e_1-e_3}X_{-e_1+e_3}$$
$$- (q^{2n-3}-q^{2n-5})X_{e_1+e_3}X_{-e_1-e_3}$$
$$- \sum_{k>3}q^{-2n+2k-2}(q-q^{-1})X_{e_1-e_k}X_{-e_1+e_k}$$
$$- \sum_{k\geq 4}q^{2n-2k+2}(q-q^{-1})X_{e_1+e_k}X_{-e_1-e_k}.$$

thus (B.3) is verified. We conclude that $[X_0, Y_0] \equiv 0$, which is what we wanted to prove. $\qquad\square$

Proof of Lemma 5.9. For the proof of the lemma we will need to use the following formulas that can be derived easily from [2], Theorem 9.3: if $i < j$ then

$$
\begin{aligned}
X_{-e_1+e_i} X_{-e_1+e_j} &= q X_{-e_1+e_j} X_{-e_1+e_i}, \\
X_{-e_1+e_i} X_{-e_1-e_j} &= q X_{-e_1-e_j} X_{-e_1+e_i}, \\
X_{-e_1-e_i} X_{-e_1+e_j} &= q^{-1} X_{-e_1+e_j} X_{-e_1-e_i}, \\
X_{-e_1-e_i} X_{-e_1-e_j} &= q^{-1} X_{-e_1-e_j} X_{-e_1-e_i},
\end{aligned}
$$
(B.4)

and, furthermore,

$$
\begin{aligned}
X_{-2e_1} X_{-e_1+e_i} &= q^{-2} X_{-e_1+e_i} X_{-2e_1}, \\
X_{-2e_1} X_{-e_1-e_i} &= q^2 X_{-e_1-e_i} X_{-2e_1}.
\end{aligned}
$$
(B.5)

Let us now prove (1). It is clear that, if $d_1(Y^T) < d_1(Y^R)$, then $d_1(Y^{T^i}) < d_1(Y^{R^i})$; hence an obvious induction on i reduces the problem to checking that there is a constant C such that

$$
Y^R Y_0 = C Y^{R^1} + \sum c_T Y^T
$$
(B.6)

with $d_1(Y^T) < d_1(Y^{R^1})$.

Suppose first that $Y^R = X_{-e_1+e_n}^{r_{n-1}} \dots X_{-e_1+e_2}^{r_1}$. We will prove by induction on $n - i$ that there is a nonzero constant $C_i(R)$ such that

$$
Y^R X_{-e_1-e_i} X_{-e_1+e_i} = C_i(R) X_{-2e_1} Y^R + u
$$

with $d_1(u) < d_1(X_{-2e_1} Y^R)$.

If $i = n$, we write $Y^R = X_{-e_1+e_n}^{r_{n-1}} Y^{R'}$, where $Y^{R'} = X_{-e_1+e_{n-1}}^{r_{n-2}} \dots X_{-e_1+e_2}^{r_1}$; we also write $q^{2|R'|} = q^{2r_1} \dots q^{2r_{n-2}}$. By (A.23), (B.4), and (B.5) we compute that

$$
\begin{aligned}
Y^R X_{-e_1-e_n} X_{-e_1+e_n} &= q^{2|R'|} X_{-e_1+e_n}^{r_{n-1}} X_{-e_1-e_n} X_{-e_1+e_n} Y^{R'} \\
&= q^{2|R'|} \sum_{h=1}^{r_{n-1}} X_{-e_1+e_n}^{r_{n-1}-h} [X_{-e_1+e_n}, X_{-e_1-e_n}] X_{-e_1+e_n}^h Y^{R'} \\
&\quad + q^{2|R'|} X_{-e_1-e_n} X_{-e_1+e_n}^{r_{n-1}+1} Y^{R'} \\
&= (q + q^{-1}) q^{2|R'|} \sum_{h=1}^{r_{n-1}} q^{2(r_{n-1}-h)} X_{-2e_1} X_{-e_1+e_n}^{r_{n-1}} Y^{R'} \\
&\quad + q^{2|R'|} X_{-e_1-e_n} X_{-e_1+e_n}^{r_{n-1}+1} Y^{R'} \\
&= (q + q^{-1}) q^{2|R'|} \frac{1 - q^{2r_{n-1}}}{1 - q^2} X_{-2e_1} Y^R \\
&\quad + q^{2|R'|} X_{-e_1-e_n} X_{-e_1+e_n}^{r_{n-1}+1} \dots X_{-e_1+e_2}^{r_1},
\end{aligned}
$$

so

$$
C_n(R) = (q + q^{-1}) q^{2r_1} \dots q^{2r_{n-2}} \frac{1 - q^{2r_{n-1}}}{1 - q^2}.
$$
(B.7)

Suppose now that $i < n$, and write $Y^R = Y^{R'} X_{-e_1+e_i}^{r_{i-1}} Y^{R''}$, where

$$Y^{R'} = X_{-e_1+e_n}^{r_{n-1}} \ldots X_{-e_1+e_{i+1}}^{r_i},$$

$$Y^{R''} = X_{-e_1+e_{i-1}}^{r_{i-2}} \ldots X_{-e_1+e_2}^{r_1}.$$

Set $q^{2|R''|} = q^{2r_1} \ldots q^{2r_{i-2}}$ and $q^{2|R'|} = q^{2r_i} \ldots q^{2r_{n-1}}$.

In this case a similar computation gives that

$$Y^R X_{-e_1-e_i} X_{-e_1+e_i} = q^{2|R''|} \sum_{h=1}^{r_{i-1}} Y^{R'} X_{-e_1+e_i}^{r_{i-1}-h} [X_{-e_1+e_i}, X_{-e_1-e_i}] X_{-e_1+e_i}^h Y^{R''}$$

$$+ q^{2|R''|} q^{r_i} \ldots q^{r_{n-1}} X_{-e_1-e_i} Y^{R'} X_{-e_1+e_i}^{r_{i-1}+1} Y^{R''}.$$

Using (A.23) and (A.25), it is easy to check that

$$[X_{-e_1+e_i}, X_{-e_1-e_i}] = (q+q^{-1})(-q)^{i-n} X_{-2e_1}$$

$$+ (q-q^{-1}) \sum_{j>i} (-q)^{i-j+1} X_{-e_1-e_j} X_{-e_1+e_j};$$

so, substituting above, we find that

$$Y^R X_{-e_1-e_i} X_{-e_1+e_i} = (q+q^{-1})(-q)^{i-n} q^{2|R''|} \sum_{h=1}^{r_{i-1}} Y^{R'} X_{-e_1+e_i}^{r_{i-1}-h} X_{-2e_1} X_{-e_1+e_i}^h Y^{R''}$$

$$+ (q-q^{-1}) q^{2|R''|} \sum_{j>i} (-q)^{i-j+1} \sum_{h=1}^{r_{i-1}} Y^{R'} X_{-e_1+e_i}^{r_{i-1}-h} X_{-e_1-e_j} X_{-e_1+e_j} X_{-e_1+e_i}^h Y^{R''} + u$$

with $d_1(u) < d_1(Y^R X_{-e_1-e_i} X_{-e_1+e_i})$. Using (B.4) and (B.5), one obtains that

$$Y^R X_{-e_1-e_i} X_{-e_1+e_i} = (q+q^{-1})(-q)^{i-n} q^{2|R'|} q^{2|R''|} \frac{1-q^{2r_{i-1}}}{1-q^2} X_{-2e_1} Y^R$$

$$+ (q-q^{-1}) \sum_{j>i} (-q)^{i-j+1} \frac{q^{-2}-q^{-2r_{i-1}-2}}{1-q^{-2}} Y^R X_{-e_1-e_j} X_{-e_1+e_j} + u.$$

Thus, applying the induction hypothesis, we find that

(B.8)
$$C_i(R) = (q+q^{-1})(-q)^{i-n} q^{2|R'|} q^{2|R''|} \frac{1-q^{2r_{i-1}}}{1-q^2} - (1-q^{-2r_{i-1}}) \sum_{j>i} (-q)^{i-j} C_j(R).$$

Using (B.7) as initial condition for the recursion equation (B.8), we find that

(B.9) $$C_i(R) = (q+q^{-1})(-q)^{i-n} q^{2r_1} \ldots q^{2r_{i-2}} \frac{1-q^{2r_{i-1}}}{1-q^2}.$$

If we write Y_0 as in (5.4), we find that

$$Y^R Y_0 = (-q)^{-n+1}(q+q^{-1}) Y^R X_{-2e_1} + (q-q^{-1})(\sum_{i=2}^{n} (-q)^{-i+2} Y^R X_{-e_1-e_i} X_{-e_1+e_i}$$

$$= (-q)^{-n+1}(q+q^{-1}) q^{2|R|} X_{-2e_1} Y^R$$

$$+ (q-q^{-1}) \sum_{i=2}^{n} (-q)^{-i+2} C_i(R) X_{-2e_1} Y^R + u'$$

W. BALDONI AND P. MÖSENEDER FRAJRIA

with $d_r(u') < d_r(X_{-2e_1} Y^R)$; so, by (B.9),
$$Y^R Y_0 = (q + q^{-1})(-q)^{-n+1} X_{-2e_1} Y^R + u'$$
with $d_r(u') < d_r(X_{-2e_1} Y^R)$.

More generally, if
$$Y^R = X_{-e_1-e_2}^{r_{2n-1}} \dots X_{-e_1-e_n}^{r_{n+1}} X_{-2e_1}^{r_n} X_{-e_1+e_n}^{r_{n-1}} \dots X_{-e_1+e_2}^{r_1},$$
then
$$Y^R Y_0 = (q + q^{-1})(-q)^{-n+1} Y^{R^1} + X_{-e_1-e_2}^{r_{2n-1}} \dots X_{-e_1-e_n}^{r_{n+1}} X_{-2e_1}^{r_n} u'.$$
Using (B.4) and (B.5), it is easy to check that
$$d_r(X_{-e_1-e_2}^{r_{2n-1}} \dots X_{-e_1-e_n}^{r_{n+1}} X_{-2e_1}^{r_n} u') < d_r(Y^{R^1}),$$
and this proves (B.6), with $C = (q + q^{-1})(-q)^{-n+1}$.

The analogous formulas for (2) of Lemma 5.9 can be obtained similarly using (5.3) instead of (5.4). □

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