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# TETRAGONAL CURVES, SCROLLS AND $K3$ SURFACES

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Abstract. In this paper we establish a theorem which determines the invariants of a general hyperplane section of a rational normal scroll of arbitrary dimension. We then construct a complete intersection surface on a four-dimensional scroll and prove it is regular with a trivial dualizing sheaf. We determine the invariants for which the surface is nonsingular, and hence a $K3$ surface. A general hyperplane section of this surface is a tetragonal curve; we use the first theorem to determine for which tetragonal invariants such a construction is possible. In particular we show that for every genus $g \geq 7$ there is a tetragonal curve of genus $g$ that is a hyperplane section of a $K3$ surface. Conversely, if the tetragonal invariants are not sufficiently balanced, then the complete intersection must be singular. Finally we determine for which additional sets of invariants this construction gives a tetragonal curve as a hyperplane section of a singular canonically trivial surface, and discuss the connection with other recent results on canonically trivial surfaces.

## 1. Introduction

In this paper we investigate the relationship among tetragonal curves, $K3$ surfaces, and rational normal scrolls. Fundamental to our study is the fact that any canonical tetragonal curve is a complete intersection of two divisors on a three-dimensional rational normal scroll. In a similar manner we construct a complete intersection surface on a four-dimensional scroll, and show that this surface, if nonsingular, is a $K3$ surface that contains a tetragonal curve as a general hyperplane section.

Accordingly, we begin by proving in Section 2 a theorem that determines the invariants of a general hyperplane section of a given $d$-dimensional scroll $S$. In Section 3 we recall general facts about tetragonal curves. Specifically, a canonical tetragonal curve is a complete intersection of two divisors on a three-dimensional scroll. Certain invariants of the curve are determined by the two divisors and by the invariants of the scroll. We give necessary conditions on these invariants for the tetragonal curve to be nonsingular. We then describe in Section 4 the construction of a complete intersection surface of two divisors on a four-dimensional scroll. We prove that this surface is regular and has a trivial dualizing sheaf; hence, if it is nonsingular, it will be a $K3$ surface. In Section 5 we relate the nonsingularity of this surface to the invariants of the four-dimensional scroll. We distinguish the surfaces having only isolated singularities from those which are singular along a curve.

We then address in Section 6 the following question: given a specific set of tetragonal invariants, does there exist a tetragonal curve with those invariants that

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is a general hyperplane section of a $K3$ surface as constructed above? We use the results of the previous section and the theorem of Section 2 to determine precisely for which sets of invariants the answer is affirmative. In particular, the answer is affirmative if the tetragonal invariants are balanced; we conclude in Corollary 6.2 that there exists a tetragonal curve of every genus $g \geq 7$ which lies on a $K3$ surface. For most other sets of invariants the construction yields only surfaces that are singular along a curve. However, there are a limited number of sets of invariants for which the answer to the question above is negative, but for which there does exist a tetragonal curve with those invariants that is a general hyperplane section of a regular surface with trivial dualizing sheaf and only isolated singularities. Such surfaces are examples of *canonically trivial* surfaces; we describe in Section 7 how the construction of such a surface can nonetheless yield information about whether there exists a tetragonal curve with specified invariants which lies on a $K3$ surface.

## 2. Hyperplane sections of scrolls

Define a $d$-dimensional rational normal scroll $S \subset \mathbf{P}^n$ with invariants according to the following standard construction (cf. [Sc] or [Ha, 8.26]). Let $f_1, \dots, f_d$ be non-negative integers with $f_i \geq f_{i+1}$ and $\sum f_i = n - d + 1$. Choose complementary subspaces $\Lambda_i$ of $\mathbf{P}^n$, each of dimension $f_i$, rational normal curves $C_i \subset \Lambda_i$, and isomorphisms $\varphi_i : \mathbf{P}^1 \to C_i$. Then

$$S = \bigcup_{\lambda \in \mathbf{P}^1} \overline{\varphi_1(\lambda), \dots, \varphi_d(\lambda)}$$

is a $d$-dimensional rational normal scroll, where $\overline{\varphi_1(\lambda), \dots, \varphi_d(\lambda)}$ is the linear span of the $\varphi_i(\lambda)$ in $\mathbf{P}^n$. The scroll $S$, which we also denote $S(f_1, ..., f_d)$, is determined up to projective equivalence by the numbers $f_i$, which are called the invariants of the scroll. Before we state the main theorem of this section we define related invariants that are central to the proof.

**Definition 2.1.** The *index of relative balance* of a $d$-dimensional scroll $S(f_1, ..., f_d)$, denoted $r(S)$, is given by

$$r(S) = \min \left\{ k \in \mathbf{N} : (d-k) f_k \leq \sum_{i=k}^{d} f_i \right\}.$$

**Definition 2.2.** If $S(f_1, ..., f_d)$ is a $d$-dimensional scroll and $1 \leq k \leq d-1$, define the integer

$$d_k = \left\lfloor \frac{\sum_{i=k}^{d} f_i}{d-k} \right\rfloor.$$

Notice that $r(S) \leq d-1$, since $f_{d-1} \leq f_{d-1} + f_d$. The integers $d_k$ are related to the index of relative balance in the following way.

**Proposition 2.3.** *If the index of relative balance $r(S) = r$, then* $\min\limits_{1 \leq k \leq d-1} \{d_k\} = d_r$.

*Proof.* We consider two cases. If $1 \leq k < r$, then from the definition of the index of relative balance $\sum_{i=r}^{d} f_i < (d-r) f_{r-1}$ , so that

$$\begin{aligned}(d-k)\sum_{i=r}^{d} f_i &< (d-r)\sum_{i=r}^{d} f_i + (r-k)(d-r)f_{r-1} \\ &\leq (d-r)\left(\sum_{i=r}^{d} f_i + \sum_{i=k}^{r-1} f_i\right) \\ &= (d-r)\sum_{i=k}^{d} f_i,\end{aligned}$$

and $d_r < d_k$.

If, on the other hand, $r < k < d$, then

$$\begin{aligned}(d-k)\sum_{i=r}^{d} f_i &\leq (d-k)(k-r)f_r + (d-k)\sum_{i=k}^{d} f_i \\ &= (k-r)\left[(d-r)f_r - (k-r)f_r\right] + (d-k)\sum_{i=k}^{d} f_i \\ &\leq (k-r)\left[\left(\sum_{i=r}^{d} f_i\right) - \left(\sum_{i=r}^{k-1} f_i\right)\right] + (d-k)\sum_{i=k}^{d} f_i \\ &= (d-r)\sum_{i=k}^{d} f_i,\end{aligned}$$

and again $d_r < d_k$. □

We can now state the main theorem of this section, which describes the invariants of a general hyperplane section of a $d$-dimensional scroll in terms of the index of relative balance.

**Theorem 2.4.** *Let $S = S(f_1, ..., f_d)$ be a $d$-dimensional rational normal scroll with index of relative balance $r(S) = r$. Then a general hyperplane section of $S$ is a $(d-1)$-dimensional scroll with invariants $e_i$ $(e_i \geq e_{i+1})$ satisfying*

$$\sum_{i=1}^{d-1} e_i = \sum_{i=1}^{d} f_i$$

*and the following conditions:*

1. $e_r \leq e_{d-1} + 1$ *and*
2. $e_i = f_i \quad \forall i, 1 \leq i \leq r-1.$

*Proof.* Since a general hyperplane section of $S$ is an irreducible non-degenerate variety of dimension $d-1$ in $\mathbf{P}^{n-1}$ with minimal degree $n-d+1$, it must be a rational normal scroll, say $S' = S'(e_1, \ldots, e_n)$ (cf. [Ha, 19.9]). The dual sequence of the resolution of $S$ over $\mathbf{P}^1$ is given by

$$0 \to \mathcal{O}_{\mathbf{P}^1} \to \bigoplus_{i=1}^{d} \mathcal{O}_{\mathbf{P}^1}(f_i) \to \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbf{P}^1}(e_i).$$

A hyperplane section of $S$ determines homogeneous polynomials $p_i(s,t)$ of degree $f_i$ over $\mathbf{P}^1$, $1 \leq i \leq d$. To determine the invariants $e_i$ for $S'$, we seek relations among the polynomials $p_i(s,t)$. More specifically, $\forall k, 1 \leq k \leq d-1$, we say there is a relation of type $k$ if

$$(s,t)^j p_k + (s,t)^{j+f_k-f_{k+1}} p_{k+1} + \cdots + (s,t)^{j+f_k-f_d} p_d = 0,$$

where $(s,t)^j$ denotes a homogeneous polynomial of degree $j$. Thus, the degree of a relation of type $k$ is $j + f_k$. □

**Proposition 2.5.** *The integer $d_r$ is a lower bound for the least possible degree for which there must be a relation among the polynomials $p_i(s,t)$.*

*Proof.* A relation of type $k$ has degree $j+f_k$, of which there are $j+f_k+1$ independent monomials in $s$ and $t$. In order to force a relation of type $k$, the condition on $j$ is

$$\begin{gathered}(j+1)+(j+f_k-f_{k+1}+1)+\cdots+(j+f_k-f_{k+1}+1)>j+f_k+1,\\ (d-k)(j+f_k+1)>\textstyle\sum_{i=k}^d f_i,\\ j+f_k\geq\left\lfloor\frac{\sum_{i=k}^d f_i}{d-k}\right\rfloor.\end{gathered}$$

The degree of the relation is $j+f_k\geq d_k$, which is in turn $\geq d_r$ by Proposition 2.3. ☐

We show next that the lower bound $d_r$ is sharp by exhibiting monomials $q_j(s,t)$ of degree $f_j$ such that the smallest degree of a relation among the monomials is $d_r$. First we define integers $n_j$ for all $j$ such that $r\leq j\leq d$ as follows:

$$n_j=\begin{cases} f_j & \text{if } j<r,\\ \left\lceil\frac{d-j}{d-r}\left(\sum_{i=r}^j f_i\right)-\frac{j-r}{d-r}\left(\sum_{i=j+1}^d f_i\right)\right\rceil & \text{if } r\leq j\leq d.\end{cases}$$

Notice that $n_r=f_r$, $n_d=0$, and $n_j\geq n_{j+1}$, because for $j$ such that $r\leq j\leq d-1$,

$$\begin{aligned}\frac{d-j}{d-r}\left(\sum_{i=r}^{j} f_i\right)-\frac{j-r}{d-r}\left(\sum_{i=j+1}^{d} f_i\right)-\frac{d-j-1}{d-r}\left(\sum_{i=r}^{j+1} f_i\right)+\frac{j-r+1}{d-r}\left(\sum_{i=j+2}^{d} f_i\right)\\ =\frac{\sum_{i=r}^d f_i}{d-r}-f_{j+1},\end{aligned}$$

which is non-negative since $(d-r)f_{j+1}\leq(d-r)f_r\leq\sum_{i=r}^d f_i$. We then define monomials $q_j$ by

$$q_j(s,t)=s^{n_j}t^{f_j-n_j}$$

and prove the following useful proposition.

**Proposition 2.6.** *If $\delta(f,g)$ denotes the degree of the least common multiple of two polynomials $f$ and $g$, then the monomials $q_j$ defined above satisfy*

$$d_r\leq\delta(q_j,q_{j+1})\leq d_r+1,$$

*for all $j$, $r\leq j\leq d-1$, with equality on the left if $j=r$.*

*Proof.* Since the integers $n_j$ are decreasing,

$$\begin{aligned}\delta(q_j,q_{j+1})&=n_j+f_{j+1}-n_{j+1}\\ &=\left\lceil\tfrac{d-j}{d-r}\left(\textstyle\sum_{i=r}^j f_i\right)\right\rceil-\left\lceil\tfrac{d-j-1}{d-r}\left(\textstyle\sum_{i=r}^j f_i\right)\right\rceil\\ &=\tfrac{\left(\sum_{i=r}^j f_i\right)-k}{d-r}+\left\lceil\tfrac{(d-j)k}{d-r}\right\rceil-\left\lceil\tfrac{(d-j-1)k}{d-r}\right\rceil,\end{aligned}$$

where $\sum_{i=r}^j f_i\equiv k\pmod{d-r}$, $0\leq k\leq d-r-1$. The first term in the last expession is $d_r$, and the difference of the last two is either 0 or 1, since

$$0\leq\frac{(d-j)k}{d-r}-\frac{(d-j-1)k}{d-r}=\frac{k}{d-r}<1.$$

Finally, in the case $j=r$, notice that $k-1<\frac{(d-r-1)k}{d-r}\leq k$, so that

$$\left\lceil\frac{(d-r-1)k}{d-r}\right\rceil=\lceil k\rceil=k,$$

and $\delta(q_r,q_{r+1})=d_r$. ☐

We now complete the proof of Theorem 2.4. From the previous two propositions we see that the smallest degree for which there must be a relation among the polynomials $p_i$ is $d_r$. In terms of the invariants of a general hyperplane section $S'(e_1 \dots e_{d-1})$, this means $e_{d-1} = d_r$. Further, since there are $d - r + 1$ terms in a relation of type $r$, there are $d - r$ non-redundant relations, all of degree $d_r$ or $d_r + 1$, by the previous proposition, among these terms. Consequently, the smallest $d - r$ invariants of $S'$ are given by $e_r \leq e_{d-1} + 1$, which determines the invariants completely if $r = 1$. If $r > 1$, then any additional non-redundant relations must involve a polynomial $p_k$, where $k < r$. From the definition of $r(S) = r$ we see that $f_{r-1} > d_r$, so the next smallest degree of a relation must be $f_{r-1}$. Thus $e_{r-1} = f_{r-1}$ and, in the same manner, $e_i = f_i \quad \forall\ i,\ 1 \leq i \leq r - 1$. □

## 3. Tetragonal curves

We recall several basic facts about tetragonal curves (cf. [Sc], [Br]). If $C$ is a tetragonal curve of genus $g$ canonically embedded in $\mathbf{P}^{g-1}$, then it is a complete intersection of divisors $Y_i', 1 \leq i \leq 2$, of class $2H' - b_iR'$ on a three-dimensional scroll $S'(e_1, e_2, e_3)$, where the hyperplane section $H'$ and fibre of the ruling $R'$ are generators of $Pic(S')$, and $2 + \sum b_i = \sum e_j = g - 3$. We define the divisors $Y_i'$ on $S'$ by quadratic forms

$$\psi_i' = \sum_{1 \leq j \leq k \leq 3} P_{ijk}\varphi_j\varphi_k,$$

where $\varphi_j \in H^0(\mathcal{O}_{S'}(H' - e_jR'))$ and $P_{ijk} \in \mathbf{C}[s,t]$ are homogeneous polynomials of degree

$$\deg(P_{ijk}) = e_j + e_k - b_i.$$

The following proposition describes necessary conditions for the nonsingularity of the curve $C$.

**Proposition 3.1** (cf. [Sc],[Br]). *If the tetragonal curve $C$ with invariants $b_i$ and $e_j$ described above is nonsingular, then*

1. $e_1 \leq \frac{g-1}{2}$,
2. *if* $e_3 = 0$, *then* $b_2 = 0$,
3. $b_1 \leq 2e_2$,
4. $b_2 \leq 2e_3$,
5. *if* $e_3 > 0$, *then* $b_1 \leq e_1 + e_3$ *and* $b_1 \leq \frac{2g-6}{3}$.

*Proof.*

1. The Riemann-Roch theorem shows $H^0(C, \omega_C(-n(g_1^4))) = 0$ for $n > \frac{g-1}{2}$.
2. If $e_3 > 0$ then the scroll $S$ is a cone over a two-dimensional scroll, say with vertex $V$. If $b_2 > 0$ as well, then $V \in Y_1' \cap Y_2'$, and $C$ is singular at $V$.
3. If $b_1 > 2e_2$, then $\psi_1' = \left(\sum_{k=1}^{3} P_{11k}\varphi_1\right)\varphi_k$ is reducible and so is $C$.
4. If $b_2 > 2e_3$ then $P_{133} = 0 = P_{233}$ and $e_3 > 0$ . Thus the rational curve on $S'$ given by $\varphi_1 = \varphi_2 = 0$ is a component of $C$, which is again reducible.
5. If $e_3 > 0$ and $b_1 > e_1 + e_3$ then

$$\psi_1' = P_{111}\varphi_1^2 + P_{112}\varphi_1\varphi_2 + P_{113}\varphi_2^2,$$

and $Y_1'$ is singular along the rational curve given by $\varphi_1 = \varphi_2 = 0$. If $b_2 < 2e_3$ and $[s_0, t_0]$ is a zero of $P_{233}$, then the singular point $[0, 0, 1] \in \pi^{-1}([s_0, t_0])$ is contained in $Y_1' \cap Y_2' = C$. If $b_2 = 2e_3$ and the constant $P_{233}$ is nonzero, then let $[s_0, t_0]$ be a zero of $P_{112}^2 - 4P_{111}P_{122}$, which has degree $b_2 + 4 > 0$. The fibre of $Y_1'$ over $[s_0, t_0]$ is then a double line which intersects the fibre of $Y_2'$ over $[s_0, t_0]$ in a singular point of $C$. The second inequality then follows from $b_1 \le e_1 + e_3$ and $b_1 \le 2e_2$. □

In addition, the invariants $b_i$ and $e_j$ distinguish which tetragonal curves are bi-elliptic (elliptic-hyperelliptic) and which lie on a Del Pezzo surface.

**Proposition 3.2** (cf. [Sc],[Br]). *Let $C$ be a nonsingular tetragonal curve with invariants $b_i$ and $e_j$.*

1. *$C$ is bi-elliptic if and only if $b_2 = e_3 = 0$.*
2. *$C$ lies on a Del Pezzo surface if and only if $b_2 = 0 < e_3$.*

*Proof.*

1. If $b_2 = e_3 = 0$, then $S'$ is a cone over a two-dimensional scroll and all fibres of $Y_1' \subset S'$ over $\mathbf{P}^1$ are singular conics. In this case the $g_4^1$ is given by a composition of $2:1$ maps $C \to E \to \mathbf{P}^1$, where the curve $E$ has geometric genus $p_a E = 1 + \frac{b_2}{2} = 1$ [Sc, 6.5]. Thus $C$ is a double cover of an elliptic curve. Conversely, if $C$ is bi-elliptic then $C$ is a quadric section of an elliptic cone $Y_1'$. An embedding of the elliptic curve in a two-dimensional scroll $S''$ gives an embedding of $Y_1'$ in a cone $S'$ over $S''$. Thus $Y_1' \subset S'$, where $S'$ is a three-dimensional scroll with $e_3 = 0$. The quadric surface $Y_2'$ has class $2H'$, so $b_2 = 0$ as well.
2. If $b_2 = 0$ then $C$ is contained in a surface of degree $g-1$ in $\mathbf{P}^{g-1}$: an elliptic cone or a Del Pezzo surface. If $e_3 > 0$ then the surface cannot be a cone, so $C$ must lie on a Del Pezzo surface. Conversely, if $C$ lies on a Del Pezzo surface then the $g_4^1$ is not unique, and Schreyer's minimal resolution of $\mathcal{O}_C$ shows that $b_2 = 0$ [Sc, 6.2]. Since the Del Pezzo surface cannot lie on a cone, $e_3 > 0$. □

## 4. $K3$ surfaces and 4-dimensional scrolls

In this section we construct a regular surface $X$ with trivial canonical divisor that lies on a four-dimensional scroll and contains a tetragonal curve as a hyperplane section. The surface $X$, if nonsingular, will then be a $K3$ surface.

Let $S(f_1, f_2, f_3, f_4)$ be a 4-dimensional scroll in $\mathbf{P}^n$, where $\sum_{i=1}^4 f_i = n - 3$, $f_i \ge f_{i+1}$, and $f_4 > 0$. As with three-dimensional scrolls, the Picard group of $S$ is generated by the class of a hyperplane $H$ and a ruling $R$ of the scroll. Let $X$ be the complete intersection of two effective divisors $Y_i$ on $S$ in the class of $2H - b_i R$, where $b_1 + b_2 = n - 5$. We show below that such a surface $X$ meets two necessary conditions of a $K3$ surface, although it need not be reduced or irreducible, much less nonsingular.

**Theorem 4.1.** *The complete intersection $X$ described above is a regular surface with trivial dualizing sheaf.*

*Proof.* On $S$ we have $\omega_S \simeq \mathcal{O}_S(-4H + (n-5)R)$, so

$$\omega_{Y_1} \simeq \omega_S(2H - b_1 R) \otimes \mathcal{O}_{Y_1} \simeq \mathcal{O}_{Y_1}(-2H + b_2 R)$$

and

$$\omega_X \simeq \omega_{Y_1}(2H - b_2R) \otimes \mathcal{O}_X \simeq \mathcal{O}_X.$$

To see that $H^1(\mathcal{O}_X) = 0$, consider the following exact sequences:

$$0 \to \mathcal{I}_X \to \mathcal{O}_S \to \mathcal{O}_X \to 0, \tag{1}$$

$$0 \to \mathcal{O}_S(-4H + (n-5)R) \to \bigoplus_{i=1}^{2} \mathcal{O}_S(-2H + b_iR) \to \mathcal{I}_X \to 0. \tag{2}$$

Sequence (1) shows that $H^1(\mathcal{O}_X)$ injects into $H^2(\mathcal{I}_X)$, since the scroll $S$ is projectively normal, and sequence (2) yields

$$\bigoplus_{i=1}^{2} H^2(\mathcal{O}_S(-2H + b_iR)) \to H^2(\mathcal{I}_X) \to H^3(\omega_S).$$

Now, the space $H^3(\omega_S)$ is dual to $H^1(\mathcal{O}_S) = 0$, and $H^2(\mathcal{O}_S(-2H+b_1R))$ is dual to $H^2(\mathcal{O}_S(-2H+b_2R))$ by Serre duality. The Leray spectral sequence shows that each of these last two spaces is zero. Thus $H^2(\mathcal{I}_X) = 0$, and consequently $H^1(\mathcal{O}_X) = 0$. □

Our interest will be in finding such surfaces that are nonsingular, i.e. $K3$ surfaces, or that have only isolated singularities. In those cases a general hyperplane section of the surface will be a nonsingular tetragonal curve. If a surface $X$ constructed as above is singular along a curve, however, then a general hyperplane section will be singular as well.

## 5. Singularities of surfaces on scrolls

In this section we describe the possible singularities of the complete intersection surface $X$ described in the previous section. We describe the number and type of singularities of $X$ according to the invariants $b_i$ and $f_j$.

**Theorem 5.1.** *Let $X$ be the complete intersection of two divisors $Y_i$ of class $2H - b_iR$, $1 \le i \le 2$, on the four-dimensional scroll $S(f_1, f_2, f_3, f_4)$, where $\sum f_j = 2 + \sum b_i$. If the surface $X$ is not singular along a curve, then the following conditions hold:*

1. $b_2 \le 2f_3$;
2. $b_2 \le f_2 + f_4$;
3. $b_1 \le 2f_2$;
4. $b_1 \le f_1 + f_3$;
5. $b_1 \le f_1 + f_4$ *or* $b_2 \le 2f_4$.

*Proof.* In each case we show that if the condition is not satisfied, then $X$ is singular along a curve. Define divisors $Y_i$ in the class of $2H - b_iR$ by the quadratic forms

$$\psi_i = \sum_{1 \le j \le k \le 4} P_{ijk}\varphi_j\varphi_k,$$

where $P_{ijk}(s,t)$ are homogeneous polynomials over $\mathbf{P}^1$ of degree $f_j + f_k - b_i$ . We let $[s_0, t_0] \in \mathbf{P}^1$ be arbitrary and show that if the condition is not satisfied, then $X$ is singular at a point $P \in \pi^{-1}([s_0, t_0])$.

1. If $b_2 > 2f_3$, then $P_{ijk} = 0$ for $3 \leq j \leq k$. Let $[\varphi_3, \varphi_4] \in \mathbf{P}^1$ be a zero of the determinant of the matrix

$$A = \begin{bmatrix} P_{113}(s_0,t_0)\varphi_3 + P_{114}(s_0,t_0)\varphi_4 & P_{213}(s_0,t_0)\varphi_3 + P_{214}(s_0,t_0)\varphi_4 \\ P_{123}(s_0,t_0)\varphi_3 + P_{124}(s_0,t_0)\varphi_4 & P_{223}(s_0,t_0)\varphi_3 + P_{224}(s_0,t_0)\varphi_4 \end{bmatrix}$$

   and let $[a, b] \in \mathbf{P}^1$ be a non-zero element of the kernel of $A$. If we set $\widehat{\psi}_2 = a\psi_1 + b\psi_2$, then

$$\begin{bmatrix} \widehat{P}_{213}(s_0,t_0) & \widehat{P}_{214}(s_0,t_0) \\ \widehat{P}_{223}(s_0,t_0) & \widehat{P}_{224}(s_0,t_0) \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \varphi_4 \end{bmatrix} = A \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

   and $\widehat{Y}_2$ is singular at $P = [0, 0, \varphi_3, \varphi_4] \in \pi^{-1}([s_0, t_0])$. Since $P_{ijk} = 0$ for $3 \leq j \leq k$, then $X$ is singular at $P$ as well.
2. If $b_2 > f_2 + f_4$, then $P_{ij4} = 0$ for $j \geq 2$. We let

$$\widehat{\psi}_2 = -P_{214}(s_0,t_0)\psi_1 + P_{114}(s_0,t_0)\psi_2;$$

   then $\widehat{P}_{2j4}(s_0, t_0) = 0$ for all $j$, so that $\widehat{Y}_2$ and $X$ are singular at $P = [0, 0, 0, 1] \in \pi^{-1}([s_0, t_0])$.
3. If $b_1 > 2f_2$, then $P_{i2k} = 0$ for $k \geq 2$. Let $[\varphi_2, \varphi_3, \varphi_4] \in \mathbf{P}^2$ be a point of intersection of the plane conic with equation

$$\sum_{2 \leq j \leq k \leq 4} P_{2jk}(s_0,t_0)\varphi_j\varphi_k = 0$$

   and the line with equation

$$\sum_{k=2}^{4} P_{11k}(s_0,t_0)\varphi_k = 0.$$

   Then $Y_1$ is singular at the point $P = [0, \varphi_2, \varphi_3, \varphi_4] \in \pi^{-1}([s_0, t_0])$ and $P \in Y_2$, so $X$ is singular at $P$ as well.
4. If $b_1 > f_1 + f_3$, then $P_{1jk} = 0$ for all $k \geq 3$. Let $[\varphi_3, \varphi_4] \in \mathbf{P}^1$ be a zero of

$$\sum_{3 \leq j \leq k \leq 4} P_{2jk}(s_0,t_0)\varphi_j\varphi_k = 0;$$

   then $Y_1$ and $X$ are singular at $P = [0, 0, \varphi_3, \varphi_4] \in \pi^{-1}([s_0, t_0])$.
5. If $b_1 > f_1 + f_3$ and $b_2 > 2f_4$, then $P_{11k} = 0$ for all $k$ and $P_{224} = 0$. Thus $Y_1$ and $X$ are singular at $P = [0, 0, 0, 1] \in \pi^{-1}([s_0, t_0])$. □

This imposes severe restrictions on the possible values of the invariants $b_i$ and $f_j$, as the following corollary illustrates.

**Corollary 5.2.** *If $X$ is not singular along a curve, then*

1. $b_1 \leq b_2 + 4$, *with equality only if* $b_1 \equiv 0 \pmod 2$*;*
2. $f_1 \leq f_2 + 4$, *with equality if and only if* $(f_1, f_2, f_3, f_4) = (f_4+8, f_4+4, f_4+2, f_4)$ *and* $(b_1, b_2) = (2f_4 + 4, 2f_4)$*;*
3. $f_2 \leq f_3 + 2$*;*
4. $f_3 \leq f_4 + 2$*;*
5. $4f_3 \geq \left(\sum_1^4 f_i\right) - 6$, *with equality only if* $b_1 = b_2 + 4$.

*Proof.* We repeatedly apply the inequalities derived in the previous theorem.

1. If $b_1 \le f_1 + f_4$, then
$$b_1 = b_2 + 4 - 2(f_1 + f_4 - b_1) - (2f_3 - b_2) - (2f_2 - b_1) \le b_2 + 4.$$
If equality holds, then $b_1 = 2f_2$. If, on the other hand, $b_1 > f_1 + f_4$, then $b_2 \le 2f_4$ and
$$b_1 = b_2 + 4 - (2f_4 - b_2) - (2f_2 - b_1) - 2(f_1 + f_3 - b_1) \le b_2 + 4.$$
If equality holds in this case, then $b_1 = 2f_2$.
2. If $b_1 \le f_1 + f_4$, then
$$f_1 = f_2 + 4 - (f_1 + f_4 - b_1) - (2f_3 - b_2) - (2f_2 - b_1) - (f_2 + f_4 - b_2) \le f_2 + 4.$$
If equality holds, then the first three non-negative terms in parentheses must be zero, and $b_1 = b_2 + 4$, as above. Since the last three terms are zero, it follows that $f_2 = f_3 + 2 = f_4 + 4$. If, on the other hand, $b_1 \ge f_1 + f_4 + 1$, then $b_2 \le 2f_4$ and
$$f_1 = f_2 + 1 - (b_1 - f_1 - f_4 - 1) - (2f_4 - b_2) - (2f_2 - b_1) - (f_1 + f_3 - b_1) \\ \le f_2 + 1.$$
3. If $b_1 \le f_1 + f_4$, then
$$f_2 = f_3 + 2 - (f_1 + f_4 - b_1) - (2f_3 - b_2) \le f_3 + 2.$$
If, on the other hand, $b_1 \ge f_1 + f_4 + 1$, then $b_2 \le 2f_4$ and
$$f_2 = f_3 + 1 - (b_1 - f_1 - f_4 - 1) - (2f_4 - b_2) - 2(f_1 + f_3 - b_1) \le f_3 + 1.$$
4. If $b_1 \le f_1 + f_4$, then
$$f_3 = f_4 + 2 - (f_1 + f_4 - b_1) - (f_2 + f_4 - b_2) \le f_4 + 2.$$
If, on the other hand, $b_1 \ge f_1 + f_4 + 1$, then $b_2 \le 2f_4$ and
$$f_3 \le f_2 = f_4 + 2 - (2f_4 - b_2) - (f_1 + f_3 - b_1) \le f_4 + 2.$$
5. Since $b_1 \le b_2 + 4$, we have
$$4f_3 - (b_1 + b_2 + 2) + 6 = 2(2f_3 - b_2) + (b_2 - b_1 + 4) \ge 0,$$
with equality only if $b_1 = b_2 + 4$. □

This narrows considerably the list of invariants for those surfaces which are not singular along a curve. The number of such surfaces depends on the congruence of $\sum f_j$ mod 4; we set $\sum f_j = 4k + m$, where $0 \le m \le 3$. Tables 1–4 in the appendix give a complete list of invariants $b_i$ and $f_j$ for which the general surface $X$ is not singular along a curve. In the sequel we determine which of these surfaces have isolated singularities, and which are nonsingular.

**Theorem 5.3.** *If $X$ is a surface constructed as above, then $X$ is nonsingular if its invariants satisfy one of the following conditions:*

1. $b_1 \le f_3 + f_4$ *; or*
2. $b_1 = b_2 + 2,\ f_1 = f_2 = f_3 + 1 = f_4 + 1$ *; or*
3. $b_1 = b_2 + 4,\ f_1 = f_2 + 4 = f_3 + 6 = f_4 + 8$ .

*Proof.*

1. In this case the degree of $P_{ijk}(s,t)$ is non-negative unless $j = k = 4$. The fibres of $X = Y_1 \cap Y_2$ over $\mathbf{P}^1$ will be nonsingular except where the determinant $\det(D) = \det(\lambda_1\Sigma_1 + \lambda_2\Sigma_2) = 0$, where $\Sigma_i$ is the symmetric matrix
$$\Sigma_i = \begin{bmatrix} 2P_{i11} & P_{i12} & P_{i13} & P_{i14} \\ P_{i12} & 2P_{i22} & P_{i23} & P_{i24} \\ P_{i13} & P_{i23} & 2P_{i33} & P_{i34} \\ P_{i14} & P_{i24} & P_{i34} & * \end{bmatrix}.$$
For any given $[s,t] \in \mathbf{P}^1$ there are four values of $[\lambda_1, \lambda_2] \in \mathbf{P}^1$ for which $\det(D) = 0$, and hence four values of $[\varphi_1, \varphi_2, \varphi_3, \varphi_4] \in \mathbf{P}^3$ in the kernel of $D$. Since the divisors $Y_i$ are general, we may assume that none of these four values satisfies
$$\sum_{1\le j\le k\le 4} P_{1jk}(s,t)\varphi_j\varphi_k = 0,$$
and hence the fibre of $X$ is nonsingular. Since every fibre of $X$ is nonsingular, $X$ is nonsingular as well.
2. In this case $P_{1jk} = 0$ and $\deg(P_{2jk}) = 1$ for $j, k \ge 3$, and $\deg(P_{1jk}) = 0$ for $1 \le j \le 2$ and $3 \le j \le 4$. For any given $[s,t] \in \mathbf{P}^1$, let $D = \lambda_1\Sigma_1 + \lambda_2\Sigma_2$ as above. If $\det(D) = 0$, then $\lambda_2 \ne 0$ since
$$\det \begin{bmatrix} P_{113} & P_{114} \\ P_{123} & P_{124} \end{bmatrix} \ne 0,$$
so we assume $\lambda_2 = 1$. To each zero $\lambda_1$ of $\det(\lambda_1\Sigma_1 + \Sigma_2)$ there corresponds an element $[\varphi_1, \varphi_2, \varphi_3, \varphi_4] \in \mathbf{P}^3$ in the kernel of $D$. If $\varphi_1 = \varphi_2 = 0$, then $[s,t]$ is one of the two zeros of $(4P_{233}P_{234} - P_{234}^2)$ and
$$\begin{bmatrix} \lambda_1 P_{113} + P_{213}(s,t) & \lambda_1 P_{114} + P_{214}(s,t) \\ \lambda_1 P_{123} + P_{223}(s,t) & \lambda_1 P_{124} + P_{224}(s,t) \end{bmatrix} \begin{bmatrix} -P_{234}(s,t) \\ 2P_{233}(s,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which is not true for the general surface $\dot{X}$. Thus we assume that $\varphi_1$ and $\varphi_2$ are not both zero and, by generality,
$$\sum_{1\le j\le k\le 4} P_{1jk}(s,t)\varphi_j\varphi_k \ne 0.$$
Consequently, any element of the kernel of $D$ is not in the fibre of $Y_1$ or $X$, and $X$ is nonsingular.
3. In this case $P_{1jk} = 0$ if $j \ge 2$ and $k \ge 3$, $P_{234} = P_{244} = 0$, and $P_{114}$, $P_{122}$, $P_{224}$ and $P_{233}$ are all constants which we may assume are nonzero by generality. For any given $[s,t] \in \mathbf{P}^1$, let $D = \lambda_1\Sigma_1 + \lambda_2\Sigma_2$ as above. The only element $[\varphi_1, \varphi_2, \varphi_3, \varphi_4] \in \mathbf{P}^3$ in the kernel of $\Sigma_1$ is $[0, 0, -P_{114}, P_{113}(s,t)]$, which is not in $Y_2$ since
$$\sum_{1\le j\le k\le 4} P_{2jk}(s,t)\varphi_j\varphi_k = P_{233}P_{114}^2 \ne 0.$$
If $\lambda_2 \ne 0$ there are three values of $[\lambda_1, \lambda_2] \in \mathbf{P}^1$ for which $\det(D) = 0$, and hence three values of $[\varphi_1, \varphi_2, \varphi_3, \varphi_4] \in \mathbf{P}^3$ in the kernel of $D$. For each of these, $\varphi_1 \ne 0$ since $P_{224}P_{233} \ne 0$, and, by generality once again,
$$\sum_{1\le j\le k\le 4} P_{1jk}(s,t)\varphi_j\varphi_k \ne 0,$$
and $X$ is nonsingular. □

The first part of this theorem applies to the surfaces with the most balanced invariants; we use it to prove in the next section the existence of a tetragonal curve of any genus $g \geq 7$ which lies on a $K3$ surface.

We turn our attention next to those surfaces which have isolated singularities.

**Proposition 5.4.** *The surface $X$ is singular in each of the following cases:*

1. $f_1 + f_4 < b_1 < 2f_2$;
2. $f_2 + f_3 < b_1 < 2f_2$;
3. $f_1 + f_4 < b_1$ *and* $b_2 < 2f_4$;
4. $f_2 + f_3 < b_1 \leq f_1 + f_4$ *and* $b_2 < 2f_3$;
5. $f_3 \leq f_4 + 1$ *and* $b_2 > f_3 + f_4$;
6. $f_2 + f_4 < b_1 < f_1 + f_4$ *and* $2f_4 < b_2 \leq f_3 + f_4$;
7. $f_2 + f_3 < b_1 < f_1 + f_3$ *and* $b_2 \leq f_3 + f_4$.

*Proof.* In all cases we assume $X$ is not singular along a curve and use the inequalities of Corollary 5.2.

1. In this case $P_{1j4} = 0\ \forall j$, $\deg(P_{113}) = 0$ and $\deg(P_{122}) = 1$. Let $[s_0, t_0]$ be the zero of
$$\begin{vmatrix} 2P_{111}(s,t) & P_{112}(s,t) & P_{113}(s,t) \\ P_{112}(s,t) & 2P_{122}(s,t) & P_{123}(s,t) \\ P_{113}(s,t) & P_{123}(s,t) & 0 \end{vmatrix},$$
and let $[\varphi_1, \varphi_2, \varphi_3] \in \mathbf{C}^3$ be a nonzero element of the kernel of the resulting singular matrix. Define $\varphi_4$ to be one of the two roots of the quadratic equation
$$P_{244}\varphi_4^2 + \sum_{j=1}^{3} P_{2j4}(s_0, t_0)\varphi_j\varphi_4 + \sum_{1\leq j\leq k\leq 3} P_{2jk}(s_0, t_0)\varphi_j\varphi_k = 0.$$
Then $Y_1$ is singular at the point $P = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]$ in the fibre over $[s_0, t_0]$ and $P \in Y_2$, so $X$ is singular at $P$ as well.
2. In this case $P_{1jk} = 0\ \forall j \geq 2$ and $\forall k \geq 3$, and $\deg(P_{122}) = 1$. Let $[s_0, t_0]$ be the zero of $P_{122}(s, t)$ and let $[\varphi_2, \varphi_3, \varphi_4]$ be one of the two points of intersection in $\mathbf{P}^2$ of the plane conic with equation
$$\sum_{2\leq j\leq k\leq 4} P_{2jk}(s_0, t_0)\varphi_j\varphi_k = 0$$
and the line with equation
$$\sum_{k=2}^{4} P_{11k}(s_0, t_0)\varphi_k = 0.$$
Then $Y_1$ is singular at the point $P = [0, \varphi_2, \varphi_3, \varphi_4]$ in the fibre over $[s_0, t_0]$ and $P \in Y_2$, so $X$ is singular at $P$ as well.
3. In this case $P_{1j4} = 0\ \forall j$ and $\deg(P_{244}) = 1$. Let $[s_0, t_0]$ be the zero of $P_{244}(s, t)$. Then $Y_1$ is singular at the point $P = [0, 0, 0, 1]$ in the fibre over $[s_0, t_0]$ and $P \in Y_2$, so $X$ is singular at $P$ as well.
4. In this case $P_{1jk} = 0\ \forall j \geq 2$ and $\forall k \geq 3$, $\deg(P_{114}) = 0$, and $\deg(P_{233}) = 1$. Let $[s_0, t_0]$ be a zero of
$$P_{233}(s,t)P_{114}^2 - P_{234}(s,t)P_{113}(s,t)P_{114} + P_{244}(s,t)P_{113}^2(s,t).$$
Then $Y_1$ is singular at the point $P = [0, 0, -P_{114}, P_{113}(s_0, t_0)]$ in the fibre over $[s_0, t_0]$ and $P \in Y_2$, so $X$ is singular at $P$ as well.

5. In this case $P_{ij4} = 0$ for $j \geq 3$, and
$$\deg \begin{vmatrix} P_{114}(s,t) & P_{214}(s,t) \\ P_{124}(s,t) & P_{224}(s,t) \end{vmatrix} = 1.$$
If $[s_0, t_0]$ is the zero of this determinant, then $X$ is singular at $[0,0,0,1]$ in the fibre over $[s_0, t_0]$.
6. Here $P_{1j4} = 0$ for $j \geq 2$, $P_{244} = 0$, and $\deg(P_{114}) = 1$, so $X$ is singular at $[0,0,0,1]$ in the fibre over the zero of $P_{114}(s,t)$.
7. In this case $P_{ijk} = 0$ for $j \geq 2$ and $k \geq 3$, $f_3 = f_4$, $\deg(P_{113}) = 1$ and $\deg(P_{2jk}) = 0$ for $j, k \geq 3$. Let $[\varphi_3, \varphi_4]$ be a zero of
$$P_{233}\varphi_3^2 + P_{234}\varphi_3\varphi_4 + P_{244}\varphi_4^2;$$
and let $[s_0, t_0]$ be the zero of $P_{113}(s,t)\varphi_3 + P_{114}(s,t)\varphi_4$. Then $X$ is singular at $[0,0,\varphi_3,\varphi_4]$ in the fibre over $[s_0, t_0]$. □

**Corollary 5.5.** *If $X$ is a surface with invariants satisfying $b_1 = b_2 + 3$, then $X$ is singular. If the invariants satisfy the conditions of Theorem 5.1 as well, then $X$ has only isolated singularities.*

*Proof.* Let $\sum f_j = 4k+m$, where $0 \leq m \leq 3$. We have $(b_1, b_2) = (2k+1, 2k-2)$ for $m = 1$, and there are four possible combinations of the invariants $f_j$ for which $X$ is not singular along a curve (cf. Table 2). If $(f_1, f_2, f_3, f_4) = (k+1, k+1, k, k-1)$, then part 1 of the previous proposition applies; whereas if $(f_1, f_2, f_3, f_4)$ is equal to $(k+2, k+1, k-1, k-1)$, $(k+3, k+1, k-1, k-2)$ or $(k+4, k+1, k-1, k-3)$, then 2 applies. Part 5 also applies if $(f_1, f_2, f_3, f_4) = (k+3, k+1, k-1, k-2)$. In that case there are three isolated singularities; in the other cases there are two. The fact that these are the only singularities follows from the generality of the divisors $Y_i$ on $S$.

If $m = 3$, then $(b_1, b_2) = (2k+2, 2k-1)$, and once again there are four possible combinations of the invariants $f_j$ for which $X$ is not singular along a curve (cf. Table 4). If $(f_1, f_2, f_3, f_4) = (k+1, k+1, k+1, k)$, then Part 3 applies, and if $(f_1, f_2, f_3, f_4) = (k+2, k+1, k, k)$, $(k+3, k+1, k, k-1)$ or $(k+4, k+1, k, k-2)$, then Part 4 applies. In each of these four cases the surface has one isolated singularity. □

We use this corollary in Section 7 to show the existence of tetragonal curves that are hyperplane sections, not of $K3$ surfaces, but of regular surfaces with trivial dualizing sheaf and isolated singularities.

## 6. Tetragonal curves on $K3$ surfaces

Let $C$ be a nonsingular tetragonal curve of genus $g$, canonically embedded in $\mathbf{P}^{g-1}$, as in Section 3. For any such curve we can construct a surface $X$ as in Section 4 which contains, as a general hyperplane section, a tetragonal curve with the same invariants as $C$. We show below that the surface $X$ will be nonsingular, and hence a $K3$ surface, if these invariants are relatively balanced. Slightly less-balanced invariants can give rise to a surface with only isolated singularities, whereas more unbalanced invariants dictate that the surface $X$ must be singular along a curve, and hence that a general hyperplane section curve will be singular as well.

**Theorem 6.1.** *Let $(b_1, b_2, e_1, e_2, e_3)$ be a collection of integers satisfying $b_1 \geq b_2 > 0$, $e_j \geq e_{j+1}$, and $2 + \sum b_i = \sum e_j$. If $e_1 \leq e_3 + 1$ and either $b_1 \leq b_2 + 2$ or*

$b_1 = b_2 + 4 \equiv 0$ (*mod* 2)*, then there exists a tetragonal curve of genus* $g = 3 + \sum e_j$ *with invariants* $(b_1, b_2, e_1, e_2, e_3)$ *that is a hyperplane section of a* $K3$ *surface.*

*Proof.* If $b_1 \leq b_2 + 2$, let $S(f_1, f_2, f_3, f_4)$ be a four-dimensional scroll with $\sum f_i = \sum e_j$, $f_i \geq f_{i+1}$, and $f_1 \leq f_4 + 1$. Define divisors $Y_i$ on $S$ in the class of $2H - b_iR$, and let $X = Y_1 \cap Y_2$. If $b_1 \leq b_2 + 1$ or $b_1 = b_2 + 2 \equiv 0 \pmod 2$, then $b_1 \leq f_3 + f_4$ and $X$ is nonsingular by the first part of Theorem 5.3. If $b_1 = b_2 + 2 \equiv 1 \pmod 2$, then $X$ is nonsingular by the second part of the same theorem. If, on the other hand, $b_1 = b_2 + 4 \equiv 0 \pmod 2$, then define $S$ by $f_1 = f_2 + 4 = f_3 + 6 = f_4 = \frac{b_1}{2} + 4$. Then $X$ is nonsingular by the third part of Theorem 5.3.

In all cases $X$ is a regular surface with trivial dualizing sheaf by Theorem 4.1, and hence a $K3$ surface. A general hyperplane section of this surface is a canonical tetragonal curve of genus $g$ defined by the complete intersection of two divisors $Y_i'$ of class $2H' - b_iR'$ on a three-dimensional scroll $S'(e_1, e_2, e_3)$, with invariants satisfying $e_1 \leq e_3 + 1$ by Theorem 2.4. □

This immediately gives the following corollary, where the bound on the genus follows from the condition that $b_2 > 0$.

**Corollary 6.2.** *For every* $g \geq 7$*, there exists a tetragonal curve of genus* $g$ *which lies on a* $K3$ *surface.*

At the opposite end of the spectrum, such a construction is not possible for tetragonal curves with unbalanced invariants. Recall that bi-elliptic curves have the most unbalanced invariants of all tetragonal curves, with $b_2 = e_3 = 0$. A theorem of Reid states that bi-elliptic curves of sufficiently high genus ($g \geq 11$) cannot lie on a $K3$ surface [Re]. In the following theorem we use Theorem 2.4 to give a partial converse toTheorem 6.1. We show that if a tetragonal curve of sufficiently high genus is a general hyperplane section of a $K3$ surface constructed as in Section 4, then its tetragonal invariants must satisfy the same conditions of Theorem 6.1.

**Theorem 6.3.** *Let* $(b_1, b_2, e_1, e_2, e_3)$ *be a collection of integers satisfying* $b_1 \geq b_2 > 0$, $e_j \geq e_{j+1}$*, and* $2 + \sum b_i = \sum e_j > 42$*. Let* $S$ *be any four-dimensional scroll with general hyperplane section of type* $S'(e_1, e_2, e_3)$*, and* $X = Y_1 \cap Y_2$ *any* nonsingular *complete intersection surface, where* $Y_i$ *is a divisor of class* $2H - b_iR$ *on* $S$*. If* $C$ *is a general hyperplane section of* $X$*, then it is a tetragonal curve of genus* $g > 45$ *with invariants* $(b_1, b_2, e_1, e_2, e_3)$ *satisfying* $e_1 \leq e_3 + 1$ *and either* $b_1 \leq b_2 + 2$ *or* $b_1 = b_2 + 4 \equiv 0$ (*mod* 2).

*Proof.* The conditions on $b_1$ and $b_2$ follow immediately from the nonsingularity of $X$ and Corollaries 5.2 and 5.5. Since $\sum_1^4 f_i = \sum_1^3 e_i > 42$, the inequalities of Corollary 5.2 show that $f_3 \geq 10$,

$$(f_1 - f_2) + (f_1 - f_3) \leq 10 \leq f_3 \leq f_4 + 2,$$

and $3f_1 \leq \left(\sum_1^4 f_i\right) + 2$. If $3f_1 \leq \left(\sum_1^4 f_i\right)$, then $r(S) = 1$, and $e_1 \leq e_3 + 1$ follows directly from Theorem 2.4. If $3f_1 > \left(\sum_1^4 f_i\right)$ then $r(S) = 2$, since

$$f_2 \leq f_3 + 2 < f_3 + 8 \leq f_3 + f_4,$$

so $e_1 = f_1$ and $e_2 \le e_3 + 1$, again by Theorem 2.4. Finally,

$$2e_1 = 2f_1 \le \left(\sum_{i=2}^{4} f_i\right) + 2 = e_2 + e_3 + 2 \le 2e_3 + 3,$$

and again $e_1 \le e_3 + 1$. □

Notice that the lower bound on the genus is sharp. If $X$ is a general complete intersection surface with invariants $(f_1, f_2, f_3, f_4) = (15, 11, 9, 7)$ and $(b_1, b_2) = (22, 18)$, then it is nonsingular by the third part of Theorem 6.1. A general hyperplane section of $X$ is a tetragonal curve of genus $g = 45$ on a three-dimensional scroll $S'(15, 14, 13)$. There are a number of other tetragonal curves of smaller genus with $e_1 > e_3 + 1$ which may be constructed in this manner as a general hyperplane section of a $K3$ surface, all with $e_1 \le e_3 + 5$.

## 7. Canonically trivial surfaces

In the previous section we focused our attention on tetragonal invariants $(b_1, b_2, e_1, e_2, e_3)$ for which we could produce a tetragonal curve as a general hyperplane section of a $K3$ surface $X$. We now relax the nonsingularity condition on $X$ and make the following definition.

**Definition 7.1.** A surface is *canonically trivial* if it is normal, regular, and has a trivial dualizing sheaf.

Normality implies that a canonically trivial surface can have only isolated singularities; while a nonsingular canonically trivial surface is clearly $K3$. We can use Corollary 5.5 to give examples of tetragonal invariants for which the construction of Section 4 yields a tetragonal curve that is a hyperplane section of a canonically trivial surface.

**Theorem 7.2.** *Let $(b_1, b_2, e_1, e_2, e_3)$ be a collection of integers satisfying $b_1 = b_2 + 3 > 3$, $e_j \ge e_{j+1}$, and $2 + \sum b_i = \sum e_j$. If $e_1 \le e_3 + 1$, then there exists a tetragonal curve of genus $g = 3 + \sum e_j$ with invariants $(b_1, b_2, e_1, e_2, e_3)$ that is a hyperplane section of a singular canonically trivial surface.*

*Proof.* If $\sum e_j \equiv 1 \pmod 4$, then let $f_1 = f_2 = f_3 + 1 = f_4 + 2$ and let $X$ be the complete intersection of divisors $Y_1$ and $Y_2$ of class $2H - b_iR$ on $S(f_1, f_2, f_3, f_4)$. Then $f_1 = \frac{b_2}{2} + 2 \ge 3$ and $3f_1 \le 4f_1 - 3 = \sum f_i$, so a general hyperplane section of $S$ is a scroll $S'(e_1, e_2, e_3)$, where $e_1 \le e_3 + 1$ follows from Theorem 2.4. By Corollary 5.5, $X$ has only two isolated singularities. If $\sum e_j \equiv 3 \pmod 4$, then let $f_1 = f_2 = f_3 = f_4 + 1$, and let $S$ and $X$ be the corresponding varieties. Then $f_1 = \frac{b_1}{2} \ge 2$ and $3f_1 \le 4f_1 - 2 < \sum f_i$ so a general hyperplane section of $S$ is a scroll $S'(e_1, e_2, e_3)$, where $e_1 \le e_3 + 1$ follows from Theorem 2.4. By Corollary 5.5, $X$ has only one isolated singularity. In both cases $X$ is normal, since it is a complete intersection of divisors on a rational normal scroll. A general hyperplane section of $X$ is a nonsingular tetragonal curve with the desired invariants satisfying $b_1 = b_2 + 3$ and $e_1 \le e_3 + 1$. □

The following theorem shows that if a tetragonal curve of sufficiently high genus is a general hyperplane section of a canonically trivial surface constructed as above and $b_1 = b_2 + 3$, then the invariants $e_j$ must be as balanced as possible.

**Theorem 7.3.** *Let* $(b_1, b_2, e_1, e_2, e_3)$ *be a collection of integers satisfying* $b_1 = b_2+3$, $e_j \geq e_{j+1}$, *and* $2+\sum b_i = \sum e_j$. *Let* $S(f_1, f_2, f_3, f_4)$ *be any four-dimensional scroll with general hyperplane section of type* $S'(e_1, e_2, e_3)$ *and* $X = Y_1 \cap Y_2$ *any* canonically trivial *complete intersection surface, where* $Y_i$ *is a divisor of class* $2H - b_iR$ *on* $S$. *Then* $X$ *is singular, and if* $\sum e_j > 33$ *then* $e_1 \leq e_3 + 1$.

*Proof.* The fact that $X$ is singular follows from Corollary 5.5, but it cannot be singular along a curve since it is canonically trivial. If $\sum e_j > 33$, then the inequalities of Corollary 5.2 and $b_1 = b_2 + 3$ show that $f_3 \geq 8$,

$$(f_1 - f_2) + (f_1 - f_3) \leq 8 \leq f_3 \leq f_4 + 2,$$

and $3f_1 \leq \left(\sum_1^4 f_i\right) + 2$. If $3f_1 \leq \left(\sum_1^4 f_i\right)$, then $r(S) = 1$, and $e_1 \leq e_3 + 1$ follows directly from Theorem 2.4. If $3f_1 > \left(\sum_1^4 f_i\right)$ then $r(S) = 2$, since

$$f_2 \leq f_3 + 2 < f_3 + 6 \leq f_3 + f_4,$$

so $e_1 = f_1$ and $e_2 \leq e_3 + 1$, again by Theorem 2.4. Finally,

$$2e_1 = 2f_1 \leq \left(\sum_{i=2}^{4} f_i\right) + 2 = e_2 + e_3 + 2 \leq 2e_3 + 3,$$

and again $e_1 \leq e_3 + 1$. □

Recent work of J. Wahl ([Wa]) shows that the maximum dimension of a family of canonically trivial surfaces containing a curve is $corank(\Phi_K) - 1$, where $\Phi_K$ is the Gaussian-Wahl map for the curve. If, for example, a curve lies on a $d$-dimensional family of singular canonically trivial surfaces, and $corank(\Phi_K) = d + 1$, then the curve cannot lie on a $K3$ surface. From Corollary 5.2 we see that for any single combination of invariants $b_i$ there are at most eight inequivalent scrolls for which the surface $X$ is canonically trivial (cf. Tables 1–4 as well). Meanwhile, we have shown in [Br] that $corank(\Phi_K) = 9$ for the general tetragonal curve of genus $g \geq 7$. This suggests two possible avenues of further exploration.

If $C$ is a tetragonal curve with invariants satisfying $e_1 \leq e_3 + 1$, $b_1 = b_2 + 2$, and $\sum e_j \equiv 0 \pmod 4$, then there are exactly eight non-equivalent four-dimensional scrolls containing a canonically trivial surface whose general hyperplane section is a tetragonal curve with the same invariants as $C$ (cf. Table 1). Of these eight canonically trivial surfaces, three have isolated singularities and the rest are $K3$ surfaces. In [Br] we show that for the general tetragonal curve with these invariants $corank(\Phi_K) = 9$ as well. If it could be shown that the eight canonically trivial surfaces are independent components in the space of canonically trivial surfaces containing $C$, then Wahl's result would imply that if such a curve lies on a $K3$ surface, it must be a surface constructed as in Section 4.

In the case of a tetragonal curve $C$ with invariants satisfying $e_1 \leq e_3 + 1$ and $b_1 = b_2+3$, we have seen that there are exactly four non-equivalent four-dimensional scrolls containing a canonically trivial surface whose general hyperplane section is a tetragonal curve with the same invariants as $C$, and all four canonically trivial surfaces are singular. In [Br] we show that for the general tetragonal curve with these invariants $corank(\Phi_K) = 9$ as well. If these surfaces generate an eight-dimensional family of singular canonically trivial surfaces containing $C$, then we could conclude, again using Wahl's result, that the curve does not lie on a $K3$ surface.

## Appendix A. Tables for $X$ with at most isolated singularities

The tables in this appendix list the invariants for all possible four-dimensional scrolls which have at most isolated singularities. We set $\sum f_i = 4k+m$, $0 \leq m \leq 3$, and list the invariants separately for each of the four values of $m$.

Table 1. $X$ with $m = 0$

| $\sum f_j = 4k$ | $(b_1, b_2)$ | $(f_1, f_2, f_3, f_4)$ | **Singularities** |
|---|---|---|---|
| $k \geq 1$ | $(2k-1, 2k-1)$ | $(k, k, k, k)$ | none |
| $k \geq 2$ | $(2k-1, 2k-1)$ | $(k+1, k, k, k-1)$ | none |
| $k \geq 3$ | $(2k-1, 2k-1)$ | $(k+1, k+1, k, k-2)$ | none |
| $k \geq 2$ | $(2k, 2k-2)$ | $(k, k, k, k)$ | none |
| $k \geq 2$ | $(2k, 2k-2)$ | $(k+1, k, k, k-1)$ | none |
| $k \geq 2$ | $(2k, 2k-2)$ | $(k+1, k+1, k-1, k-1)$ | none |
| $k \geq 2$ | $(2k, 2k-2)$ | $(k+2, k, k-1, k-1)$ | 2 |
| $k \geq 3$ | $(2k, 2k-2)$ | $(k+2, k, k, k-2)$ | none |
| $k \geq 3$ | $(2k, 2k-2)$ | $(k+2, k+1, k-1, k-2)$ | 1 |
| $k \geq 3$ | $(2k, 2k-2)$ | $(k+3, k, k-1, k-2)$ | 1 |
| $k \geq 4$ | $(2k, 2k-2)$ | $(k+3, k+1, k-1, k-3)$ | none |

Table 2. $X$ with $m = 1$

| $\sum f_j = 4k+1$ | $(b_1, b_2)$ | $(f_1, f_2, f_3, f_4)$ | **Singularities** |
|---|---|---|---|
| $k \geq 1$ | $(2k, 2k-1)$ | $(k+1, k, k, k)$ | none |
| $k \geq 2$ | $(2k, 2k-1)$ | $(k+1, k+1, k, k-1)$ | none |
| $k \geq 2$ | $(2k, 2k-1)$ | $(k+2, k, k, k-1)$ | 1 |
| $k \geq 3$ | $(2k, 2k-1)$ | $(k+2, k+1, k, k-2)$ | none |
| $k \geq 2$ | $(2k+1, 2k-2)$ | $(k+1, k+1, k, k-1)$ | 2 |
| $k \geq 2$ | $(2k+1, 2k-2)$ | $(k+2, k+1, k-1, k-1)$ | 2 |
| $k \geq 3$ | $(2k+1, 2k-2)$ | $(k+3, k+1, k-1, k-2)$ | 3 |
| $k \geq 4$ | $(2k+1, 2k-2)$ | $(k+4, k+1, k-1, k-3)$ | 2 |

Table 3. $X$ with $m = 2$

| $\sum f_j = 4k+2$ | $(b_1, b_2)$ | $(f_1, f_2, f_3, f_4)$ | **Singularities** |
|---|---|---|---|
| $k \geq 1$ | $(2k, 2k)$ | $(k+1, k+1, k, k)$ | none |
| $k \geq 1$ | $(2k, 2k)$ | $(k+2, k, k, k)$ | none |
| $k \geq 2$ | $(2k, 2k)$ | $(k+1, k+1, k+1, k-1)$ | none |
| $k \geq 2$ | $(2k, 2k)$ | $(k+2, k+1, k, k-1)$ | 1 |
| $k \geq 3$ | $(2k, 2k)$ | $(k+2, k+2, k, k-2)$ | none |
| $k \geq 1$ | $(2k+1, 2k-1)$ | $(k+1, k+1, k, k)$ | none |
| $k \geq 2$ | $(2k+1, 2k-1)$ | $(k+2, k+1, k, k-1)$ | none |
| $k \geq 3$ | $(2k+1, 2k-1)$ | $(k+3, k+1, k, k-2)$ | none |
| $k \geq 2$ | $(2k+2, 2k-2)$ | $(k+1, k+1, k+1, k-1)$ | none |
| $k \geq 2$ | $(2k+2, 2k-2)$ | $(k+2, k+1, k, k-1)$ | none |
| $k \geq 2$ | $(2k+2, 2k-2)$ | $(k+3, k+1, k-1, k-1)$ | none |
| $k \geq 3$ | $(2k+2, 2k-2)$ | $(k+4, k+1, k-1, k-2)$ | 1 |
| $k \geq 4$ | $(2k+2, 2k-2)$ | $(k+5, k+1, k-1, k-3)$ | none |

TABLE 4. $X$ with $m = 3$

| $\sum f_j = 4k+3$ | $(b_1, b_2)$ | $(f_1, f_2, f_3, f_4)$ | **Singularities** |
|---|---|---|---|
| $k \geq 1$ | $(2k+1, 2k)$ | $(k+1, k+1, k+1, k)$ | none |
| $k \geq 1$ | $(2k+1, 2k)$ | $(k+2, k+1, k, k)$ | none |
| $k \geq 2$ | $(2k+1, 2k)$ | $(k+2, k+1, k+1, k-1)$ | none |
| $k \geq 2$ | $(2k+1, 2k)$ | $(k+2, k+2, k, k-1)$ | 1 |
| $k \geq 2$ | $(2k+1, 2k)$ | $(k+3, k+1, k, k-1)$ | 1 |
| $k \geq 3$ | $(2k+1, 2k)$ | $(k+3, k+2, k, k-2)$ | none |
| $k \geq 1$ | $(2k+2, 2k-1)$ | $(k+1, k+1, k+1, k)$ | 1 |
| $k \geq 1$ | $(2k+2, 2k-1)$ | $(k+2, k+1, k, k)$ | 1 |
| $k \geq 2$ | $(2k+2, 2k-1)$ | $(k+3, k+1, k, k-1)$ | 1 |
| $k \geq 3$ | $(2k+2, 2k-1)$ | $(k+4, k+1, k, k-2)$ | 1 |

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