TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 8, August 1997, Pages 3181–3234
S 0002-9947(97)01963-6

AN INFINITE DIMENSIONAL MORSE THEORY WITH APPLICATIONS

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Abstract. In this paper we construct an infinite dimensional (extraordinary) cohomology theory and a Morse theory corresponding to it. These theories have some special properties which make them useful in the study of critical points of strongly indefinite functionals (by strongly indefinite we mean a functional unbounded from below and from above on any subspace of finite codimension). Several applications are given to Hamiltonian systems, the one-dimensional wave equation (of vibrating string type) and systems of elliptic partial differential equations.

0. Introduction

Let E be a real Hilbert space with an inner product $\langle .\,,.\rangle$ and let Φ be a twice continuously differentiable functional. Denote the Fréchet derivative and the gradient of Φ at x by $\Phi'(x)$ and $\nabla\Phi(x)$ respectively, where as usual

$$\langle \nabla\Phi(x), y\rangle := \Phi'(x)y \quad \forall\, y \in E.$$

Recall that a point $x_0 \in E$ is said to be *critical* if $\Phi'(x_0) = 0$, or equivalently, if $\nabla\Phi(x_0) = 0$. The level $c \in \mathbf{R}$ will be called *regular* if $\Phi^{-1}(c)$ contains no critical points, and *critical* if $\nabla\Phi(x_0) = 0$ for some $x_0 \in \Phi^{-1}(c)$.

Let a, b, $a < b$, be two regular levels of Φ. Denote $M := \Phi^{-1}([a,b])$ and consider the restriction of Φ to M. In Morse theory one is interested in the local topological structure of the level sets of $\Phi|_M$ near a critical point and in the relation between this local structure and the topological structure of the set M. To be more specific, suppose that $x_0 \in M$ is an isolated critical point of Φ. Then one defines a sequence of *critical groups* of Φ at x_0 by setting

$$c_q(\Phi, x_0) := H_q(\Phi^c \cap U, \Phi^c \cap U - \{x_0\}), \qquad q = 0, 1, 2, ..., \tag{0.1}$$

where $c := \Phi(x_0)$, $\Phi^c := \{x \in E : \Phi(x) \le c\}$, H_q is the q-th singular homology group with coefficients in some field $\mathcal{F}$ and U is a neighbourhood of x_0. Define the *Morse index* of x_0 to be the maximal dimension of a subspace of E on which the quadratic form $\langle \Phi''(x_0)y, y\rangle$ is negative definite. One shows that if x_0 is a *nondegenerate* critical point, i.e., if $\Phi''(x_0) : E \to E$ is invertible, then $c_q(\Phi, x_0) = \mathcal{F}$

Received by the editors March 20, 1995.
1991 *Mathematics Subject Classification.* Primary 58E05; Secondary 34C25, 35J65, 35L05, 55N20, 58F05.
Key words and phrases. Filtration, cohomology, critical group, Morse inequalities, Morse index, degree theory, Hamiltonian system, wave equation, system of elliptic partial differential equations.
The first author was supported in part by the KBN Grant PB 513/2/91.
The second author was supported in part by the Swedish Natural Science Research Council.

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for $q =$ the Morse index of x_0, and $c_q(\Phi, x_0) = 0$ otherwise. So in this case the groups $c_q(\Phi, x_0)$ are uniquely determined by the Morse index. If x_0 is degenerate, no such simple relation exists. The global aspect of Morse theory is expressed by the *Morse inequalities* which relate the critical groups of all critical points of $\Phi|_M$ to the homology groups $H_q(\Phi^b, \Phi^a)$ (which are isomorphic with $H_q(M, \Phi^{-1}(a))$ by the excision property of homology).

In what follows we assume that the reader is somewhat familiar with Morse theory in Hilbert spaces. Necessary prerequisites (and much more!) may be found e.g. in Chang [10] and Mawhin and Willem [34]. Let us also mention that a different approach to Morse theory, based on the Conley index, has been developed by Benci (see [6] and the references there).

The purpose of this paper is to construct a Morse theory for strongly indefinite functionals. In order to explain why the ordinary theory fails in this case, let us consider the following very simple example: Let $E = E^+ \oplus E^-$ be an orthogonal decomposition of E into two infinite dimensional subspaces and suppose $\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2$, where $x^+ \in E^+$ and $x^- \in E^-$. Then 0 is the only critical point of Φ; it is nondegenerate and has the Morse index $+\infty$. Therefore $c_q(\Phi, 0) = 0$ for all q (this can also be easily computed directly from the definition of c_q). More generally, if $\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 + \psi(x)$, where $\nabla\psi$ is a *compact* mapping (i.e., it takes bounded sets to precompact ones), then the Morse index of any critical point must necessarily be $+\infty$. So in this case one cannot expect to obtain any useful information from the usual Morse theory.

In order to remedy this difficulty, the second author has introduced a different Morse theory in [41]. It was based on a suitably adapted version of an infinite dimensional cohomology theory due to Gęba and Granas [20, 21]. In the present paper we construct another infinite dimensional cohomology theory and a Morse theory associated with it. Let $(E_n)_{n=1}^\infty$ be a *filtration* of E, i.e., an increasing sequence of closed subspaces of E such that $E = \text{cl}(\bigcup_{n=1}^\infty E_n)$ (cl denotes the closure), and let $\mathcal{E} = \{E_n, d_n\}_{n=1}^\infty$, where $(d_n)_{n=1}^\infty$ is a sequence of nonnegative integers. Then for a pair (X, A) of closed sets in E, $A \subset X$, we define cohomology groups of (X, A) by setting

$$H^q_{\mathcal{E}}(X, A) := \{H^{q+d_n}(X \cap E_n, A \cap E_n)\}_{n=1}^\infty, \qquad q \in \mathbf{Z}. \tag{0.2}$$

Since we are only interested in the asymptotic behaviour of the sequences on the right-hand side above, we will consider $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$, where $\xi_n, \eta_n \in H^{q+d_n}(X \cap E_n, A \cap E_n)$, as equivalent (or representing the same element of $H^q_{\mathcal{E}}(X, A)$) if $\xi_n = \eta_n$ for almost all n. In applications E_n will be a direct summand of E_{n+1}, and we will have $\dim(E_{n+1} \ominus E_n) = k$ and $d_n = c + dn$, where $0 < d \le k$ and the constants c, d, k are independent of n.

In [41] the cohomology groups were obtained as the limit as $n \to \infty$ of the direct system $\{H^{q+n}(X \cap E_n, A \cap E_n), \Delta^q_n\}$, where $(E_n)_{n=1}^\infty$ is a filtration with $\dim(E_{n+1} \ominus E_n) = 1$ and $\Delta^q_n : H^{q+n}(X \cap E_n, A \cap E_n) \to H^{q+n+1}(X \cap E_{n+1}, A \cap E_{n+1})$ is a certain Mayer-Vietoris homomorphism. The objects we construct here are more unusual: each group $H^q_{\mathcal{E}}(X, A)$ is in fact a *sequence* of cohomology groups of the spaces $(X \cap E_n, A \cap E_n)$ approximating (X, A). Although this is a certain disadvantage compared to [41], our approach here has advantages which prevail. It is much more flexible and more elementary. In particular, the fact that we may have $\dim(E_{n+1} \ominus E_n) > 1$ and $d_{n+1} - d_n > 1$ will turn out to be very convenient in applications. E.g., for Hamiltonian systems in $\mathbf{R}^{2N}$ the natural choice of E_n and d_n

is such that $\dim(E_{n+1} \ominus E_n) = 4N$ and $d_n = N(1+2n)$ (E_n is in fact the space of n-th partial sums in the Fourier expansion of functions in E, so $\dim E_n = 2N(1+2n)$). By this choice it will be possible to avoid a tedious approximation procedure which was needed in [41, Section 6]. Furthermore, if the functional Φ is invariant with respect to an action of a group $\mathcal{G}$, then in most cases $E_{n+1} \ominus E_n$ cannot be one-dimensional if it is to be $\mathcal{G}$-invariant.

The cohomology theory (0.2) will be constructed in Section 1, and in Sections 2–5 we construct a corresponding Morse theory. We use an approach due to Gromoll and Meyer [10, 22] and combine it with some ideas from the Conley index theory [6, 11] (note that our cohomology is defined only on pairs of *closed* sets, so critical groups cannot be introduced by a formula similar to (0.1)). Let us point out that the functionals considered in [41] satisfied the Palais-Smale condition (PS) and were of the form $\Phi(x) = \frac{1}{2}\langle Lx, x\rangle + \psi(x)$, with L linear and $P_F \nabla\psi$ compact (P_F is the orthogonal projector onto a certain subspace of E). Here we assume that Φ satisfies the condition (PS)*, which is somewhat stronger than (PS) but makes any further assumptions on the form of Φ unnecessary. In Section 6 a degree theory is constructed. It is related to our Morse theory via formulas of Poincaré-Hopf type.

The remaining sections are devoted to applications. In Section 7 we consider the problem of existence of periodic solutions for a Hamiltonian system of differential equations

$$\dot{z} = JH_z(z, t)$$

with Hamiltonian H asymptotically quadratic at 0 and infinity. We extend several earlier results; see Remark 7.12 for more comments on previous work.

In Section 8 we are concerned with the asymptotically linear wave equation

$$u_{tt} - u_{xx} = f(x, t, u) \tag{0.3}$$

satisfying the boundary and the periodicity conditions

$$u(0, t) = u(\pi, t) = 0, \qquad u(x, t + 2\pi) = u(x, t). \tag{0.4}$$

We extend earlier results contained in [1, 27] (cf. Remark 8.7). Moreover, we work directly with the functional

$$\Phi(u) := \frac{1}{2}\int_\Omega (u_t^2 - u_x^2)\, dxdt + \int_\Omega F(x, t, u)\, dxdt$$

(where $\Omega = (0, \pi) \times (0, 2\pi)$ and F is the primitive of f) which is natural for this problem. This is in fact one of the main advantages of our approach. In [1] a finite dimensional reduction was performed, and for this purpose it was necessary to assume that the derivative f_u of f is bounded (here f need not be differentiable). The gradient $\nabla\psi$ of the non-quadratic part of Φ does not satisfy the previously mentioned compactness condition. Therefore the theory of [41] cannot be applied directly. Still, in [27] this theory was used, but only after introducing a different functional—which had all properties required in [41].

Section 9 is concerned with the system of elliptic partial differential equations

$$-\Delta v = F_u(x, u, v), \quad -\Delta u = F_v(x, u, v), \qquad u|_{\partial\Omega} = v|_{\partial\Omega} = 0$$

in a bounded domain $\Omega \subset \mathbf{R}^N$. Note that the functional

$$\Phi(u, v) := \int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega F(x, u, v)\, dx$$

corresponding to this problem is strongly indefinite.

Notation and terminology. The closure of a set X will be denoted by $\overline{X}$ or $\mathrm{cl}(X)$, the interior by $\mathrm{int}(X)$ and the boundary by ∂X. $B(p,r)$ is the open ball, $\overline{B}(p,r)$ the closed ball and $S(p,r)$ the sphere of radius r centered at p. Two sets A, B are said to be *bounded away* from each other if $d(A,B) := \inf\{\|x-y\| : x \in A,\ y \in B\} > 0$. An increasing sequence $(E_n)_{n=1}^{\infty}$ of closed subspaces of a Hilbert space E is called a *filtration* of E if $E = \mathrm{cl}(\bigcup_{n=1}^{\infty} E_n)$. A filtration together with a given sequence $(d_n)_{n=1}^{\infty}$ of nonnegative integers will be denoted by $\mathcal{E} = \{E_n, d_n\}_{n=1}^{\infty}$ (somewhat improperly, we will sometimes write $\mathcal{E} = (E_n)_{n=1}^{\infty}$ when the choice of the numbers d_n is immaterial). A continuous function is called a *mapping*, and a mapping $f : X \to Y$ (or $f : (X,A) \to (Y,B)$), where $X, Y \subset E$, is *filtration-preserving* if $f(X \cap E_n) \subset E_n$ for almost all n. Homotopies and group isomorphisms are denoted by $\simeq$ and $\cong$ respectively. For level sets and critical sets we use the customary notation

$$\Phi^c := \{x \in E : \Phi(x) \le c\},$$

$$K := \{x \in E : \nabla\Phi(x) = 0\} \quad \text{and} \quad K_c := K \cap \Phi^{-1}(c).$$

Occasionally we will write $K(\Phi)$ instead of K if we want to distinguish between critical sets of different functionals.

1. Cohomology of filtered spaces

Let X be a metric space and A a closed subset of X. In what follows we denote the Čech cohomology of the pair (X,A) with coefficients in a fixed field $\mathcal{F}$ by $H^*(X,A)$. It is well-known that the Čech cohomology satisfies all the Eilenberg-Steenrod axioms. It also has some additional properties which will be useful later.

Property 1.1. [39, Corollary 6.6.3] *Let (X,A) be a pair of closed subsets of a normed linear space E, $A \subset X$. Let Λ be the family of all pairs (U,V) of open subsets of E such that $X \subset U$ and $A \subset V \subset U$. Then Λ is an inverse system directed by inclusion, and*

$$H^*(X,A) = \varinjlim_{\Lambda} H^*(U,V). \tag{1.1}$$

Since open sets in a normed linear space are absolute neighbourhood retracts, the Čech and the singular cohomology of (U,V) coincide. It follows that for pairs (X,A) as above, Property 1.1 (with singular groups on the right-hand side of (1.1)) may be taken as a definition of the Čech cohomology. See also [17, Sec. VIII.6]. Note that in [39] Property 1.1 is shown to hold for the Alexander-Spanier cohomology. However, for paracompact Hausdorff spaces the Čech and the Alexander-Spanier theories are equivalent (by [39, Corollary 6.8.8] and the five lemma applied to the exact sequence of pairs).

Property 1.2. *(Strong excision) If A and B are closed subsets of a normed linear space E, then the inclusion $(A, A\cap B) \subset (A\cup B, B)$ induces an isomorphism (the so-called excision isomorphism)*

$$H^*(A, A\cap B) \overset{exc}{\cong} H^*(A\cup B, B).$$

The above property, in a more general form, may be found in [39, Theorem 6.6.5]. See also [17, VIII.6.15].

In order to introduce cohomology theory of filtered spaces we will need some preliminaries. Let $(\mathcal{G}_n)_{n=1}^{\infty}$ be a sequence of abelian groups. We define the asymptotic group $[(\mathcal{G}_n)_{n=1}^{\infty}]$ by the formula

$$[(\mathcal{G}_n)_{n=1}^{\infty}] := \prod_{n=1}^{\infty} \mathcal{G}_n \Big/ \bigoplus_{n=1}^{\infty} \mathcal{G}_n.$$

In other words, in the group

$$\prod_{n=1}^{\infty} \mathcal{G}_n = \{(\xi_n)_{n=1}^{\infty} : \xi_n \in \mathcal{G}_n\}$$

we introduce the equivalence relation $(\xi_n)_{n=1}^{\infty} \sim (\eta_n)_{n=1}^{\infty}$ if and only if $\xi_n = \eta_n$ for almost all $n \geq 1$, and set

$$[(\mathcal{G}_n)_{n=1}^{\infty}] = \prod_{n=1}^{\infty} \mathcal{G}_n \Big/ \sim .$$

If in particular $\mathcal{G}_n = \mathcal{G}$ for almost all n, we will write $[\mathcal{G}]$ instead of $[(\mathcal{G}_n)_{n=1}^{\infty}]$. Note that the above construction of asymptotic groups generalizes immediately to modules.

Assume now that E is a real Hilbert space and there is a filtration $(E_n)_{n=1}^{\infty}$ of E. Suppose that a sequence $(d_n)_{n=1}^{\infty}$ of nonnegative integers is given and let $\mathcal{E} = \{E_n, d_n\}_{n=1}^{\infty}$. If (X, A) is a closed pair of subsets of E, then for any integer q we define the q-th *$\mathcal{E}$-cohomology group of (X, A) with coefficients in $\mathcal{F}$* by the formula

$$H^q_{\mathcal{E}}(X, A) := \left[\left(H^{q+d_n}(X \cap E_n, A \cap E_n)\right)_{n=1}^{\infty}\right].$$

Since $\mathcal{F}$ is a field, $H^*_{\mathcal{E}}(X, A)$ is in fact a (graded) vector space over $\mathcal{F}$.

As admissible morphisms in the category of closed pairs in E we take all mappings $f : (X, A) \to (Y, B)$ which preserve the filtration. It is clear that each such f induces a homomorphism

$$f^* : H^*_{\mathcal{E}}(Y, B) \to H^*_{\mathcal{E}}(X, A)$$

given by the formula $f^* := [f_n^*]$, or more precisely, by

$$f^*[(\xi_n)_{n=1}^{\infty}] := [(f_n^*(\xi_n))_{n=1}^{\infty}],$$

where $f_n := f|_{(X \cap E_n, A \cap E_n)}$ and $\xi_n \in H^{*+d_n}(Y \cap E_n, B \cap E_n)$.

The coboundary homomorphism $\delta^* : H^*(A) \to H^{*+1}(X, A)$ is defined by setting $\delta^* := [\delta_n^*]$, where $\delta_n^* : H^{q+d_n}(A \cap E_n) \to H^{q+d_n+1}(X \cap E_n, A \cap E_n)$ is the usual coboundary homomorphism in the Čech theory.

A homotopy G between two admissible mappings $f, g : (X, A) \to (Y, B)$ will be called *admissible* or *filtration-preserving* if $G([0,1] \times (X \cap E_n)) \subset E_n$ for almost all n. It is easy to see that $H^*_{\mathcal{E}}$ is a cofunctor in our (extraordinary) cohomology theory of closed pairs in E and filtration-preserving mappings. More precisely, we have the following:

Proposition 1.3. *(i) (Contravariance of $H^*_{\mathcal{E}}$) If* id *is the identity mapping on (X, A), then* id* *is the identity on $H^*_{\mathcal{E}}(X, A)$, and if $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ are admissible, then $(g \circ f)^* = f^* \circ g^*$.*

(ii) (Naturality of δ^*) If $f : (X, A) \to (Y, B)$ is admissible, then $\delta^*(f|_A)^* = f^*\delta^*$.

(iii) (Exactness) For each pair (X,A), $A\subset X$, of closed subsets of E, let $i : A\subset X$ and $j : X\subset (X,A)$ be the inclusions. Then the cohomology sequence

$$\cdots\longrightarrow H^q_{\mathcal{E}}(X,A)\xrightarrow{j^*}H^q_{\mathcal{E}}(X)\xrightarrow{i^*}H^q_{\mathcal{E}}(A)\xrightarrow{\delta^*}H^{q+1}_{\mathcal{E}}(X,A)\longrightarrow\cdots$$

is exact.

(iv) (Strong excision) If A, B are closed subsets of E, then the inclusion $(A,A\cap B)\subset(A\cup B,B)$ induces the excision isomorphism

$$H^*_{\mathcal{E}}(A,A\cap B)\overset{exc}{\cong}H^*_{\mathcal{E}}(A\cup B,B).$$

(v) (Homotopy invariance) If $f,g : (X,A)\to(Y,B)$ are admissible and homotopic by an admissible homotopy, then $f^*=g^*$.

(vi) (Exact sequence of a triple) For each triple (X,A,B), where $B\subset A\subset X$ are closed subsets of E, let $i : (A,B)\subset(X,B)$ and $j : (X,B)\subset(X,A)$ be the inclusions. Then there exists a homomorphism $\delta^* : H^*_{\mathcal{E}}(A,B)\to H^{*+1}_{\mathcal{E}}(X,A)$ such that the cohomology sequence

$$\cdots\longrightarrow H^q_{\mathcal{E}}(X,A)\xrightarrow{j^*}H^q_{\mathcal{E}}(X,B)\xrightarrow{i^*}H^q_{\mathcal{E}}(A,B)\xrightarrow{\delta^*}H^{q+1}_{\mathcal{E}}(X,A)\longrightarrow\cdots$$

is exact.

The proofs follow immediately from the definitions and the corresponding properties of ordinary cohomology (see [17, 39] and Property 1.2). E.g., for a fixed n one has the exact sequence

$$\begin{aligned}\cdots\to H^{q+d_n}(X\cap E_n,A\cap E_n)&\overset{j_n^*}{\to}H^{q+d_n}(X\cap E_n,B\cap E_n)\\ &\overset{i_n^*}{\to}H^{q+d_n}(A\cap E_n,B\cap E_n)\overset{\delta_n^*}{\to}H^{q+d_n+1}(X\cap E_n,A\cap E_n)\to\cdots,\end{aligned}$$

and this gives (vi).

Note that $H^*_{\mathcal{E}}$ satisfies all the Eilenberg-Steenrod axioms for cohomology ((i)–(v) above) except the dimension axiom which is satisfied only in the trivial case $E_n=E$ and $d_n=0$ for almost all n. Note also that (iii) is a special case of (vi) (take $B=\emptyset$).

Instead of the dimension axiom we have the following basic example:

Example 1.4. Suppose that F is a closed subspace of E, $\dim(F\cap E_n)=k_n$ and $d:=\lim_{n\to\infty}(k_n-d_n)$ exists, $d\in\mathbf{Z}\cup\{\pm\infty\}$. Given $p\in F$ and $r>\|p\|$, let $D:=\overline{B}(p,r)\cap F$ and $S:=S(p,r)\cap F$. For each $n\ge 1$, $D\cap E_n$ is a closed ball with boundary $S\cap E_n$ and $\dim(D\cap E_n)=k_n$. So if $d=+\infty$ or $-\infty$, then for large n, $H^{q+d_n}(D\cap E_n,S\cap E_n)=0$, and $H^q_{\mathcal{E}}(D,S)=0$ for all $q\in\mathbf{Z}$. If $d\ne\pm\infty$, then $q+d_n=q+k_n-d$ for almost all n and

$$H^q_{\mathcal{E}}(D,S)=\begin{cases}[\mathcal{F}] & \text{for } q=d,\\ [0] & \text{otherwise.}\end{cases}$$

2. Critical groups

Let E be a real Hilbert space and $\mathcal{E}=(E_n)_{n=1}^\infty$ a filtration of E. Denote the orthogonal projector of E onto E_n by P_n. Observe that if $Z\subset E$ is a compact set, then $P_nx\to x$ as $n\to\infty$, uniformly for $x\in Z$.

Let $\Phi\in C^1(E,\mathbf{R})$. A sequence $(y_j)_{j=1}^\infty$ is said to be a *(PS)*-sequence* (with respect to $\mathcal{E}$) if

$$\Phi(y_j)\text{ is bounded, } y_j\in E_{n_j}\text{ for some } n_j,\ n_j\to\infty\text{ and } P_{n_j}\nabla\Phi(y_j)\to 0\ \text{ as } j\to\infty.$$

If each (PS)*-sequence has a convergent subsequence, then Φ is said to satisfy the *(PS)*-condition* (with respect to $\mathcal{E}$), and if this is true for each (PS)*-sequence contained in a closed set N, then Φ is said to satisfy the *(PS)*-condition on N*.

The (PS)*-condition (in a slightly different form) has been introduced by Bahri and Berestycki [4, 5] and Li and Liu [25]. Note that if Φ satisfies (PS)*, then each convergent subsequence of (y_j) tends to a critical point of Φ. Moreover, Φ satisfies the usual Palais-Smale condition (PS). Indeed, suppose (x_j) is a sequence such that $\Phi(x_j)$ is bounded and $\nabla\Phi(x_j) \to 0$. For each j there exists an $n_j \geq j$ such that setting $y_j := P_{n_j} x_j$, we obtain $|\Phi(x_j) - \Phi(y_j)| \leq 1$, $\|y_j - x_j\| \leq 1/j$ and $\|\nabla\Phi(x_j) - \nabla\Phi(y_j)\| \leq 1/j$. Hence $P_{n_j}\nabla\Phi(y_j) \to 0$, so (y_j), and therefore also (x_j), has a convergent subsequence.

In what follows we will usually assume that Φ satisfies (PS)* on the whole space E. However, let us remark that our results remain valid if (PS)* is satisfied only on a suitable closed subset of E.

Definition 2.1. Let $N \subset E - K$, where K is the critical set of Φ. A mapping $V : N \to E$ is called a *gradient-like vector field for Φ on N* if

- **(i):** V is locally Lipschitz continuous;
- **(ii):** $\|V(x)\| \leq 1$ for all $x \in N$;
- **(iii):** there is a function $\beta : N \to \mathbf{R}_+$ such that $\langle \nabla\Phi(x), V(x)\rangle \geq \beta(x)$ for all $x \in N$ and $\inf_{z\in Z}\beta(z) > 0$ for any set $Z \subset N$ which is bounded away from K and such that $\sup_{z\in Z}|\Phi(z)| < \infty$.

We say that a gradient-like vector field V for Φ on N is *related to $\mathcal{E}$* (or *$\mathcal{E}$-related*) if the mapping $V|_Z$ preserves the filtration $(E_n)_{n=1}^{\infty}$ on any set $Z \subset N$ which is bounded away from K and such that $\sup_Z |\Phi| < \infty$.

Lemma 2.2. *Let N be an open subset of E. If $\Phi \in C^1(E, \mathbf{R})$ satisfies the (PS)*-condition, then there exists an $\mathcal{E}$-related gradient-like vector field V for Φ on $N - K$.*

Proof. Let

$$N_k := \{x \in N : d(x, K) > \frac{1}{k},\ |\Phi(x)| < k\}.$$

Clearly, for each $k \geq 1$ the set N_k is open, $N_k \subset N_{k+1}$ and $\bigcup_{k=1}^{\infty} N_k = N - K$. Hence there is an integer $k_0 \geq 1$ such that $N_k \neq \emptyset$ for $k \geq k_0$.

For each $k \geq k_0$ and $n \geq 1$, let

$$\gamma_n(k) := \inf\{\|P_n\nabla\Phi(y)\| : y \in N_k \cap E_n\}$$

and

$$\gamma(k) := \frac{1}{2}\liminf_{n\to\infty}\gamma_n(k).$$

Evidently,

$$\gamma(k+1) \leq \gamma(k). \tag{2.1}$$

Moreover,

$$\gamma(k) > 0 \tag{2.2}$$

for any $k \geq k_0$. For if $\gamma(k) = 0$, then there is a sequence (y_j), $y_j \in N_k \cap E_{n_j}$, such that $n_j \to \infty$ and $P_{n_j}\nabla\Phi(y_j) \to 0$ as $j \to \infty$. Since $|\Phi(y_j)| < k$ for any j, it follows from (PS)* that after passing to a subsequence, $y_j \to y \in \overline{N}_k$. Therefore $y \notin K$. But $\nabla\Phi(y) = \lim_{j\to\infty} P_{n_j}\nabla\Phi(y_j) = 0$, a contradiction.

For any $x \in N - K$, let

$$m(x) := \min\{k \geq k_0 : x \in N_k\}$$

and define a function $\beta : N - K \to \mathbf{R}_+$ by the formula

$$\beta(x) := \frac{1}{2}\gamma(m(x)+1), \quad x \in N - K.$$

If $Z \subset N - K$, Z is bounded away from K and $\sup_Z |\Phi| < \infty$, then there is a $k \geq k_0$ such that $Z \subset N_k$. Hence by (2.1), (2.2),

$$\inf_{z\in Z} \beta(z) \geq \frac{1}{2}\gamma(k+1) > 0. \tag{2.3}$$

Let $x \in N - K$ and define

$$W_x := \{w \in S : \langle \nabla\Phi(x), w\rangle > \frac{1}{2}\gamma(m(x))\},$$

where $S := \{w \in E : \|w\| = 1\}$. Observe that W_x is open (in S) and nonempty (because $\|\nabla\Phi(x)\| = \lim_{n\to\infty} \|P_n\nabla\Phi(P_n x)\| \geq \liminf_{n\to\infty} \gamma_n(m(x)) = 2\gamma(m(x))$). Denote $S_n := S \cap E_n$. Since $\mathrm{cl}(\bigcup_{n=1}^{\infty} S_n) = S$, the number

$$n(x) := \min\{n \geq 1 : W_x \cap S_n \neq \emptyset\}$$

is well-defined. Let $w(x)$ be an arbitrary point of $W_x \cap S_{n(x)}$. Since $\nabla\Phi$ is continuous and $N_{m(x)}$ is open, there exists an open neighbourhood $U(x)$ of x such that

$$U(x) \subset N_{m(x)} - N_{m(x)-2}, \tag{2.4}$$

$$\langle \nabla\Phi(y), w(x)\rangle > \frac{1}{2}\gamma(m(x)) \quad \forall\, y \in U(x) \tag{2.5}$$

and

$$\|\nabla\Phi(y) - \nabla\Phi(x)\| < \frac{1}{2}\gamma(m(x)) \quad \forall\, y \in U(x). \tag{2.6}$$

By (2.4),

$$m(y) = m(x) \text{ or } m(y) = m(x) - 1 \quad \text{ whenever } y \in U(x). \tag{2.7}$$

The open covering $\{U(x)\}_{x\in N-K}$ admits a locally finite Lipschitz continuous partition of unity $\{\lambda_j\}_{j\in J}$ subordinate to it. For each $j \in J$ there is an $x_j \in N - K$ such that $\mathrm{supp}\,\lambda_j \subset U(x_j)$. Define $V : N - K \to E$ by the formula

$$V(y) := \sum_{j\in J} \lambda_j(y) w(x_j), \quad y \in N - K.$$

Clearly, V is locally Lipschitz continuous and $\|V(y)\| \leq 1$ for $y \in N - K$.

Let $y \in N - K$. If $\lambda_j(y) \neq 0$, then $y \in U(x_j)$, and in view of (2.7), $m(x_j) \leq m(y) + 1$. Therefore, by (2.5) and (2.1),

$$\langle \nabla\Phi(y), V(y)\rangle = \sum_{j\in J} \lambda_j(y)\langle \nabla\Phi(y), w(x_j)\rangle > \frac{1}{2}\gamma(m(y)+1) = \beta(y).$$

This together with (2.3) shows that V is a gradient-like vector field for Φ on $N-K$.

It remains to show that V is related to $\mathcal{E}$. Let $Z \subset N - K$ be bounded away from K and such that $\sup_Z |\Phi| < \infty$. Then $Z \subset N_m$ for some $m \geq k_0$. There exists an n_0 such that $\gamma_n(k) > \gamma(k)$ for all $n \geq n_0$ and $k_0 \leq k \leq m$. It follows that

$$\|P_n\nabla\Phi(y)\| > \gamma(k) \quad \text{ whenever } n \geq n_0,\ k_0 \leq k \leq m \text{ and } y \in N_k \cap E_n. \tag{2.8}$$

Let $n \geq n_0$ and $y \in Z \cap E_n$. If $\lambda_j(y) \neq 0$, then $y \in U(x_j)$, and by (2.7), $m(y) \leq m(x_j)$. Moreover, $m(y) \leq m$ since $Z \subset N_m$. Let $\overline{w} := \|P_n \nabla\Phi(y)\|^{-1} P_n \nabla\Phi(y) \in S_n$. By (2.8) (with $k = m(y)$) and (2.6),

$$\begin{aligned}\langle \nabla\Phi(x_j), \overline{w}\rangle = & \quad \langle \nabla\Phi(y), \overline{w}\rangle - \langle \nabla\Phi(y) - \nabla\Phi(x_j), \overline{w}\rangle \\ = & \quad \|P_n \nabla\Phi(y)\| - \langle \nabla\Phi(y) - \nabla\Phi(x_j), \overline{w}\rangle \\ > & \quad \gamma(m(y)) - \tfrac{1}{2}\gamma(m(x_j)) \geq \tfrac{1}{2}\gamma(m(x_j)).\end{aligned}$$

Hence $\overline{w} \in W_{x_j} \cap S_n$. It follows that $W_{x_j} \cap S_n \neq \emptyset$, and by the definition of $n(x)$, $n(x_j) \leq n$. Therefore $w(x_j) \in E_{n(x_j)} \subset E_n$ and $V(y) \in E_n$. □

Definition 2.3. Let A be an isolated compact subset of the critical set K of a functional $\Phi \in C^1(E, \mathbf{R})$. A pair (W, W^-) of closed subsets of E is said to be an *admissible pair for* Φ *and* A *with respect to* $\mathcal{E}$ if the following conditions are satisfied:

- **(i):** W is bounded away from $K - A$, $W^- \subset \partial W$ and $A \subset \mathrm{int}(W)$;
- **(ii):** $\Phi|_W$ is bounded;
- **(iii):** there are a neighbourhood N of W and an $\mathcal{E}$-related gradient-like vector field V for Φ on $N - A$;
- **(iv):** W^- is the union of finitely many (possibly intersecting) closed sets each of which lies on a C^1-manifold of codimension 1, V is transversal to each of these manifolds at points of W^-, the flow η of $-V$ can leave W only via W^-, and if $x \in W^-$, then $\eta(t, x) \notin W$ for any $t > 0$.

The gradient-like vector field V corresponding to (W, W^-) will be called an *admissible field*. In what follows we will usually omit the expressions "related to $\mathcal{E}$" and "with respect to $\mathcal{E}$".

Remark 2.4. Assume that a pair (W, W^-) of closed sets satisfies the conditions of Definition 2.3 except that the gradient-like field V in (iii) is defined only on a neighbourhood N of ∂W. If Φ satisfies (PS)*, then, using Lemma 2.2 and partition of unity, it is easy to construct a gradient-like field $\widetilde{V} : \widetilde{N} - A \to E$, where $\widetilde{N} = N \cup W$. Thus (W, W^-) is an admissible pair for Φ and A.

Proposition 2.5. *Assume that* $\Phi \in C^1(E, \mathbf{R})$ *satisfies (PS)*. Let* $a < b$, $W := \Phi^{-1}([a, b])$ *and* $W^- := \Phi^{-1}(a)$. *If* $A := K \cap \mathrm{int}(W)$ *and* W *is bounded away from* $K - A$, *then the pair* (W, W^-) *is admissible for* Φ *and* A.

Proof. Clearly, there exists an open neighbourhood N of W such that N is bounded away from $K - A$. In view of Lemma 2.2, there is an $\mathcal{E}$-related gradient-like vector field V for Φ on $N - A$. Since $\langle \nabla\Phi(x), V(x)\rangle > 0$ whenever $x \in W^-$, (W, W^-) is an admissible pair. □

Recall that $S(p, \delta) = \{x \in E : \|x - p\| = \delta\}$.

Proposition 2.6. *Suppose that* $\Phi \in C^1(E, \mathbf{R})$ *satisfies (PS)* and has an isolated critical point* p. *For each open neighbourhood* U *of* p *there exists an admissible pair* (W, W^-) *for* Φ *and* p *such that* $W \subset U$ *and* $\Phi|_{W^-} < c := \Phi(p)$. *Moreover, there is a* $\delta_1 > 0$ *such that* $\overline{B}(p, \delta_1) \subset \mathrm{int}(W)$ *and if* $x \in S(p, \delta_1) \cap \Phi^c$, *then* $\eta(t, x) \in W^-$ *for some* $t > 0$ *(*η *is the flow of* $-V$*).*

Proof. Choose $\delta > 0$ such that $0 < \delta < d(p, K - \{p\})$, $\overline{B}(p,\delta) \subset U$ and $\sup_{B(p,\delta)} |\Phi| < \infty$. Let $V : \overline{B}(p,\delta) - \{p\} \to E$ be a gradient-like vector field related to $\mathcal{E}$ and let

$$\alpha := \inf\{\beta(x) : \frac{\delta}{2} \le \|x-p\| \le \delta\}$$

($\beta(.)$ is the function in (iii) of Definition 2.1). Choose $\varepsilon > 0$ with

$$0 < \varepsilon < \frac{\alpha\delta}{4}. \tag{2.9}$$

Let $\delta_1, \delta_2 > 0$ be such that $\delta_2 < \delta_1/2 < \delta/4$ and

$$\overline{B}(p,\delta_1) \subset \{x \in E : |\Phi(x) - c| < \varepsilon\}.$$

Set $N := \overline{B}(p,\delta)$. Define now a locally Lipschitz continuous function $\omega : N \to [0,1]$ such that $\omega(x) = 0$ in a neighbourhood of p, $\omega(x) = 1$ for $\delta_2 \le \|x-p\| \le \delta$, and consider the initial value problem

$$\frac{d\sigma}{dt} = -\omega(\sigma)V(\sigma), \quad \sigma(0,x) = x \in N.$$

Having all this, define

$$W := \{\sigma(t,x) : t \ge 0,\ x \in \overline{B}(p,\delta_1),\ \Phi(\sigma(t,x)) \ge c - \varepsilon\}$$

and

$$W^- := W \cap \Phi^{-1}(c-\varepsilon).$$

We will show that $W \subset B(p,\delta)$. Assuming the contrary, there are $x \in \overline{B}(p,\delta_1)$ and $0 \le t_1 < t_2$ such that $\delta/2 < \|\sigma(t,x) - p\| < \delta$ for $t \in (t_1,t_2)$ and $\|\sigma(t_1,x)-p\| = \delta/2$, $\|\sigma(t_2,x) - p\| = \delta$. Hence

$$\frac{\delta}{2} \le \|\sigma(t_2,x) - \sigma(t_1,x)\| \le \int_{t_1}^{t_2} \|V(\sigma(s,x))\|\, ds \le t_2 - t_1. \tag{2.10}$$

Moreover, by Definition 2.1,

$$\begin{aligned}\Phi(\sigma(t_2,x)) - \Phi(\sigma(t_1,x)) &= \int_{t_1}^{t_2} \frac{d}{ds}\Phi(\sigma(s,x))\, ds \\ &= \int_{t_1}^{t_2} \langle \nabla\Phi(\sigma(s,x)), -V(\sigma(s,x))\rangle\, ds \le -\alpha(t_2 - t_1).\end{aligned}$$

Therefore, in view of (2.9), (2.10),

$$\Phi(\sigma(t_2,x)) \le \Phi(\sigma(t_1,x)) - \alpha(t_2-t_1) \le c + \varepsilon - \frac{\alpha\delta}{2} < c - \varepsilon,$$

a contradiction.

The set W is closed. For if $y_n := \sigma(t_n,x_n) \in W$, where $x_n \in \overline{B}(p,\delta_1)$, $t_n \ge 0$ and $y_n \to y \in \partial W$, then x_n may be chosen so that $\sigma(t,x_n) \notin B(p,\delta_2)$ for $0 \le t \le t_n$. Since $\Phi(\sigma(t_n,x_n)) \ge c - \varepsilon$, we obtain that the sequence (t_n) is bounded. Hence $t_n \to t \ge 0$ (possibly after passing to a subsequence), $x_n = \sigma(-t_n, y_n) \to \sigma(-t,y) =: x \in \overline{B}(p,\delta_1)$, and $y = \sigma(t,x) \in W$.

The set W^- is obviously closed, and it is a subset of the submanifold $B(p,\delta) \cap \Phi^{-1}(c-\varepsilon)$. Since the mapping $t \mapsto \sigma(t,x)$ is transversal to this manifold, (W, W^-) is an admissible pair (and $\Phi|_{W^-} < c$).

To prove the second conclusion, note that since $\inf\{\beta(x) : \delta_1/2 \le \|x-p\| \le \delta\} > 0$, there exists an $\varepsilon_0 > 0$ such that if $x \in S(p,\delta_1)$ and $\Phi(x) \le c$, then $\Phi(\sigma(t,x)) \le c - \varepsilon_0$ whenever $\|\sigma(t,x)\| = \delta_1/2$. Choosing δ_2 smaller if necessary, we obtain

$\Phi(x) > c - \varepsilon_0$ for each $x \in B(p, \delta_2)$. Therefore $\sigma(t, x)$ cannot enter $B(p, \delta_2)$. Since $\omega(x) = 1$ and $\beta(x)$ is bounded away from 0 as $\delta_2 \leq \|x - p\| \leq \delta$, $\Phi(\sigma(t_0, x)) = c - \varepsilon$ for some t_0, and $\eta(t_0, x) = \sigma(t_0, x) \in W^-$. □

From now on we assume that the sequence (d_n) has been given and that $\mathcal{E} = \{E_n, d_n\}$. Let p be an isolated critical point of a functional $\Phi \in C^1(E, \mathbf{R})$ satisfying the (PS)*-condition and let (W, W^-) be an admissible pair for Φ and p. We define the q-th *critical group* ($q \in \mathbf{Z}$) of Φ at p with respect to $\mathcal{E}$ by the formula

$$c^q_{\mathcal{E}}(\Phi, p) := H^q_{\mathcal{E}}(W, W^-).$$

Proposition 2.6 asserts the existence of an admissible pair (W, W^-). We will prove now that $c^q_{\mathcal{E}}(\Phi, p)$, $q \in \mathbf{Z}$, is well-defined, i.e., it does not depend on the particular choice of such a pair.

Proposition 2.7. *Suppose that Φ satisfies (PS)* and (W_1, W_1^-), (W_2, W_2^-) are two admissible pairs for Φ and an isolated critical point p. Then $H^*_{\mathcal{E}}(W_1, W_1^-) \cong H^*_{\mathcal{E}}(W_2, W_2^-)$.*

Proof. To (W_i, W_i^-) there correspond a neighbourhood N_i of W_i and an admissible vector field V_i on $N_i - \{p\}$, $i = 1, 2$. According to Proposition 2.6, there is an admissible pair (W, W^-) for Φ and p such that $W \subset \operatorname{int}(W_1) \cap \operatorname{int}(W_2)$. It suffices to show that

$$H^*_{\mathcal{E}}(W_1, W_1^-) \cong H^*_{\mathcal{E}}(W, W^-).$$

Assuming that (W, W^-) is constructed as in Proposition 2.6, we easily obtain (using an appropriate partition of unity) a gradient-like vector field $\widetilde{V}$ which is admissible for both (W, W^-) and (W_1, W_1^-). Note in particular that since $W^- \subset \Phi^{-1}(c - \varepsilon)$, where $c := \Phi(p)$, the flow $\widetilde{\eta}$ of $-\widetilde{V}$ cannot re-enter W after leaving it.

Consider the initial value problem

$$\frac{d\sigma}{dt} = -\widetilde{\omega}(\sigma)\widetilde{V}(\sigma), \quad \sigma(0, x) = x \in W_1,$$

where $\widetilde{\omega} : N_1 \to [0, 1]$ is a locally Lipschitz continuous function such that $\widetilde{\omega} \equiv 0$ on $\overline{B}(p, \delta_0/2)$, $\widetilde{\omega} \equiv 1$ on $N_1 - B(p, \delta_0)$, and δ_0 is chosen so that $\overline{B}(p, \delta_0) \subset \operatorname{int}(W)$. Observe that whenever n is large enough, then

$$\sigma(t, x) \in W_1 \cap E_n \text{ provided } x \in W_1 \cap E_n \text{ and } \sigma(t, x) \in W_1 \tag{2.11}$$

(because the mapping $W_1 \ni x \mapsto \widetilde{\omega}(x)\widetilde{V}(x)$ preserves the filtration).

Since the mapping $W_1 \ni x \mapsto \widetilde{\omega}(x)\widetilde{V}(x)$ is locally Lipschitz continuous and bounded, it follows that for a given $x \in W_1$ either there is a unique $t = t(x) \in [0, \infty)$ such that $\sigma(t(x), x) \in W_1^-$ or $\sigma(t, x) \in W_1$ for all $t \geq 0$. In the latter case we set $t(x) = +\infty$. The implicit function theorem and the transversality condition on W_1^- imply that the function $x \mapsto t(x)$ is continuous on the set $\{x \in W_1 : t(x) < \infty\}$.

Let $A := \{\sigma(t, x) : t \geq 0,\ x \in W^-\} \cap W_1$ and let $\widetilde{W} := W \cup A$, $\widetilde{W}^- := \widetilde{W} \cap W_1^-$. Then $(\widetilde{W}, \widetilde{W}^-)$ is an admissible pair for Φ and p, and $\widetilde{V}$ is an admissible field. Since

(2.12)

$$\begin{aligned}\Phi(\sigma(t, x)) - \Phi(x) &= \int_0^t \widetilde{\omega}(\sigma(s, x))\langle \nabla\Phi(\sigma(s, x)), -\widetilde{V}(\sigma(s, x))\rangle\, ds \\ &\leq -\int_0^t \widetilde{\omega}(\sigma(s, x))\widetilde{\beta}(\sigma(s, x))\, ds,\end{aligned}$$

where $\widetilde{\beta}$ corresponds to $\widetilde{V}$ (in the sense of (iii) of Definition 2.1), and since $\widetilde{\omega} = 1$ and $\widetilde{\beta}$ is bounded away from 0 on $W_1 - \operatorname{int}(W)$, it follows that $t(x) < \infty$ whenever $x \in A$. Therefore the mapping

$$[0,1] \times A \ni (\lambda, x) \mapsto \sigma(\lambda t(x), x) \in A$$

is a strong deformation retraction of A onto $\widetilde{W}^-$. Since it preserves the filtration (cf. (2.11)), we have $H^*_{\mathcal{E}}(A, \widetilde{W}^-) = 0$. Now the exactness of the cohomology sequence of the triple $(\widetilde{W}, A, \widetilde{W}^-)$ and the excision property imply that

$$H^*_{\mathcal{E}}(\widetilde{W}, \widetilde{W}^-) \cong H^*_{\mathcal{E}}(\widetilde{W}, A) \overset{exc}{\cong} H^*_{\mathcal{E}}(W, W^-). \tag{2.13}$$

Let $W_0 = \widetilde{W} \cup W_1^-$. Then (W_0, W_1^-) is an admissible pair for Φ and p. Excising again, we see that

$$H^*_{\mathcal{E}}(\widetilde{W}, \widetilde{W}^-) \overset{exc}{\cong} H^*_{\mathcal{E}}(W_0, W_1^-). \tag{2.14}$$

We will show that $H^*_{\mathcal{E}}(W_0, W_1^-) \cong H^*_{\mathcal{E}}(W_1, W_1^-)$. There is a $T > 0$ such that for any $x \in W_1$ either $\sigma(t,x) \in \widetilde{W}$ for $T \le t \le t(x)$ or $T \ge t(x)$. Indeed, if $\sigma(t,x) \notin \widetilde{W}$, then by (2.12), $\Phi(\sigma(t,x)) \le \Phi(x) - t\beta$, where $\beta := \inf_{W_1 - \widetilde{W}} \widetilde{\beta}(x) > 0$. So we can choose $T = (\sup_{W_1} \Phi - \inf_{W_1} \Phi)/\beta$.

Consider the mapping $\xi : [0,T] \times W_1 \to W_1$ given by the formula

$$\xi(t,x) := \begin{cases} \sigma(t,x) & \text{if } 0 \le t < t(x), \\ \sigma(t(x),x) & \text{if } t(x) \le t \le T. \end{cases}$$

Since the function $x \mapsto t(x)$ is continuous on the set $\{x \in W_1 : t(x) < \infty\}$, we get that ξ is a filtration-preserving deformation of the pair (W_1, W_1^-) into (W_0, W_1^-) and $\xi([0,T] \times W_0) \subset W_0$, $\xi([0,T] \times W_1^-) \subset W_1^-$. It follows that the pairs (W_0, W_1^-) and (W_1, W_1^-) are homotopy equivalent by filtration-preserving homotopies. Indeed, if $i : (W_0, W_1^-) \to (W_1, W_1^-)$ is the inclusion and $\xi_T := \xi(T, .)$, then $\xi_T \circ i \simeq \mathrm{id}$ on (W_0, W_1^-) and $i \circ \xi_T \simeq \mathrm{id}$ on (W_1, W_1^-). Hence $H^*_{\mathcal{E}}(W_0, W_1^-) \cong H^*_{\mathcal{E}}(W_1, W_1^-)$, which together with (2.13), (2.14) completes the proof. □

The critical groups $c^*_{\mathcal{E}}(\Phi, p)$ have a certain continuity property which will be useful further on.

Proposition 2.8. *Suppose that Φ satisfies (PS)*, p is an isolated critical point of Φ and (W, W^-) is an admissible pair for Φ and p. There exists an $\varepsilon > 0$ such that if $\widetilde{\Phi} \in C^1(E, \mathbf{R})$ satisfies (PS)*, $\sup_W |\widetilde{\Phi}| < \infty$, $\sup_W \|\nabla\Phi(x) - \nabla\widetilde{\Phi}(x)\| < \varepsilon$, $\widetilde{\Phi}$ has only one critical point $\widetilde{p}$ in W and W is bounded away from $K(\widetilde{\Phi}) - \{\widetilde{p}\}$, then (W, W^-) is an admissible pair for $\widetilde{\Phi}$ and $\widetilde{p}$.*

Proof. Choose a neighbourhood N of W such that N is bounded away from $K(\Phi) - \{p\}$ and $\sup_N |\Phi| < \infty$. Let $V : N - \{p\} \to E$ be an admissible vector field for Φ and let $\overline{B}(p,\delta) \subset \operatorname{int}(W)$. Since the set $N - B(p,\delta)$ is bounded away from $K(\Phi)$, $\beta := \inf\{\beta(x) : x \in N - B(p,\delta)\}$ is positive (again, $\beta(\cdot)$ is the function in (iii) of Definition 2.1). Let $\varepsilon \in (0,\beta)$ be fixed. If $\widetilde{\Phi}$ satisfies our hypotheses, we may assume after shrinking N if necessary that $\sup_N |\widetilde{\Phi}| < \infty$ and $\sup_N \|\nabla\Phi(x) - \nabla\widetilde{\Phi}(x)\| < \varepsilon$. Evidently, for $x \in N - B(p,\delta)$,

$$\langle \nabla\widetilde{\Phi}(x), V(x)\rangle = \langle \nabla\Phi(x), V(x)\rangle + \langle \nabla\widetilde{\Phi}(x) - \nabla\Phi(x), V(x)\rangle \ge \beta(x) - \varepsilon \ge \beta - \varepsilon > 0.$$

So $\widetilde{p} \in B(p,\delta)$ and $N - B(p,\delta)$ is bounded away from $K(\widetilde{\Phi})$. Invoking Remark 2.4, we easily conclude the proof. □

Corollary 2.9. *Let $\{\Phi_\lambda\}_{\lambda\in[0,1]}$ be a family of C^1-functionals satisfying (PS)*, and suppose there is an open set U such that each Φ_λ has a unique critical point $p_\lambda \in U$, $\sup\{|\Phi_\lambda(x)| : x \in U,\ \lambda \in [0,1]\} < \infty$ and the mapping $\lambda \mapsto \nabla\Phi_\lambda$ is continuous, uniformly in $x \in U$ (i.e., $\sup_U \|\nabla\Phi_\mu(x) - \nabla\Phi_\lambda(x)\| \to 0$ as $\mu \to \lambda$). Then $c^*_{\mathcal{E}}(\Phi_\lambda, p_\lambda)$ is independent of $\lambda \in [0,1]$.*

Proof. Take any $\lambda \in [0,1]$ and a ball $B_\lambda \subset U$ around p_λ. There is an admissible pair (W_λ, W_λ^-), $W_\lambda \subset B_\lambda$, for Φ_λ and p_λ. By Proposition 2.8, there is an $\varepsilon > 0$ such that for each $\mu \in [0,1]$, $|\mu - \lambda| < \varepsilon$, (W_λ, W_λ^-) is an admissible pair for Φ_μ and p_μ (that $p_\mu \in B(p_\lambda,\delta) \subset W_\lambda$ is seen in the same way as in the proof of Proposition 2.8). Hence $c^*_{\mathcal{E}}(\Phi_\mu, p_\mu) = c^*_{\mathcal{E}}(\Phi_\lambda, p_\lambda)$. The conclusion follows from the connectedness of $[0,1]$. □

Recall our earlier observation that the results above remain valid if (PS)* is satisfied on a suitable closed subset of E. In particular, in Lemma 2.2 (PS)* is needed on $\overline{N}$, in Proposition 2.6 on a closed neighbourhood of p, in Proposition 2.7 on a closed neighbourhood of $W_1 \cup W_2$, and in Corollary 2.9 all Φ_λ should satisfy (PS)* on $\overline{U}$.

Corollary 2.10. *Suppose p is an isolated critical point of Φ and let $\Phi_p(x) := \Phi(x+p)$. If there exists a closed neighbourhood N of p such that Φ satisfies (PS)* on N and $\nabla\Phi$ is uniformly continuous on N, then $c^*_{\mathcal{E}}(\Phi, p) = c^*_{\mathcal{E}}(\Phi_p, 0)$.*

Proof. Let $p_n := P_n p$. We show first that Φ_{p-p_n} satisfies (PS)* on $\overline{B}(p,\delta)$ for n large and δ small enough. Suppose $x_k \in \overline{B}(p,\delta)\cap E_{n_k}$, $n_k \to \infty$ and $P_{n_k}\nabla\Phi_{p-p_n}(x_k) \to 0$. Since $\nabla\Phi$ is uniformly continuous on N, $\|\nabla\Phi(x_k+p-p_n) - \nabla\Phi(x_k+p_{n_k}-p_n)\| \to 0$, and therefore $P_{n_k}\nabla\Phi(x_k+p_{n_k}-p_n) \to 0$ as $k \to \infty$. Moreover, $x_k + p_{n_k} - p_n \in N \cap E_{n_k}$ for almost all k. Hence (x_k) has a convergent subsequence. Now let (W, W^-), $W \subset B(p,\delta)$, be an admissible pair for Φ and p. Choosing a larger n if necessary, it follows from Proposition 2.8 that (W, W^-) is also an admissible pair for Φ_{p-p_n} and p_n. Since the mapping $x \mapsto x - p_n$ is a filtration-preserving homeomorphism and $(W - p_n, W^- - p_n)$ (where $W - p_n := \{x - p_n : x \in W\}$) is an admissible pair for Φ_p and 0, $c^*_{\mathcal{E}}(\Phi, p) = H^*_{\mathcal{E}}(W, W^-) \cong H^*_{\mathcal{E}}(W - p_n, W^- - p_n) = c^*_{\mathcal{E}}(\Phi_p, 0)$. □

Remark 2.11. It is easy to see that if $A = K_c$, where c is an isolated critical value, then Propositions 2.6 and 2.7 remain valid for Φ and K_c.

Suppose now that the critical set $K = K(\Phi)$ is compact. A pair (W, W^-) of closed subsets of E will be called a *globally admissible pair for Φ and K with respect to $\mathcal{E}$* if (W, W^-) satisfies the conditions of Definition 2.3 with $A = K$ and $N = E$ (i.e., $K \subset \mathrm{int}(W)$ and the gradient-like vector field V is defined on $E - K$). The field V will be referred to as *globally admissible.* We also define the *critical groups* of the pair (Φ, K) by setting

$$c^q_{\mathcal{E}}(\Phi, K) := H^q_{\mathcal{E}}(W, W^-),$$

where (W, W^-) is a globally admissible pair.

Proposition 2.12. *Suppose that Φ satisfies (PS)* and the critical set K is compact. Then the critical groups $c^q_{\mathcal{E}}(\Phi, K)$ are well-defined.*

Proof. Let $(W, W^-) := (\Phi^{-1}([a,b]), \Phi^{-1}(a))$, where a and b are chosen in such a way that $K \subset \operatorname{int}(W)$, and let $V : E - K \to E$ be a gradient-like vector field related to $\mathcal{E}$. It is easily seen (cf. Proposition 2.5) that (W, W^-) is globally admissible.

Let (W_1, W_1^-) be another globally admissible pair with a corresponding globally admissible field $\widetilde{V}$. Choose a and b above so that $W_1 \subset W$. Since $\widetilde{V}$ is also globally admissible for (W, W^-), the argument of Proposition 2.7 shows that $H^*_{\mathcal{E}}(W, W^-) \cong H^*_{\mathcal{E}}(W_1, W_1^-)$ (the cutoff function $\widetilde{\omega}$ should be 0 in a neighbourhood of K and 1 outside a neighhbourhood U of K, where $\overline{U} \subset \operatorname{int}(W_1)$). □

Lemma 2.13. *Suppose that Φ satisfies (PS)* and the critical set K is compact. Then there exists a bounded globally admissible pair (W, W^-) for Φ and K.*

Proof. Choose a and b so that $a < \Phi(x) < b$ for $x \in K$. Let $V : E - K \to E$ be a gradient-like vector field related to $\mathcal{E}$ and let N, U be two neighbourhoods of K such that N is closed, U open and $N \subset U \subset \Phi^{-1}([a,b])$. Consider the initial value problem

$$\frac{d\sigma}{dt} = -\omega(\sigma)V(\sigma), \qquad \sigma(0,x) = x,$$

where $\omega : E \to [0,1]$ is locally Lipschitz continuous and $\omega(x) = 0$ for $x \in N$, $\omega(x) = 1$ for $x \notin U$. Define

$$W := \{\sigma(t,x) : t \geq 0,\ x \in \overline{U},\ \Phi(\sigma(t,x)) \geq a\}$$

and

$$W^- := W \cap \Phi^{-1}(a).$$

It is easy to see using the argument of Proposition 2.6 that (W, W^-) is a globally admissible pair and W, W^- are bounded sets. □

The critical groups $c^*_{\mathcal{E}}(\Phi, K)$ have a continuity property similar to the one known from the Conley index theory [6, 15]:

Proposition 2.14. *Let $\{\Phi_\lambda\}_{\lambda \in [0,1]}$ be a family of C^1-functionals satisfying (PS)*. Suppose that the mapping $\lambda \mapsto \nabla\Phi_\lambda$ is continuous, uniformly on bounded subsets of E, and there exist a bounded set N and a constant C such that $K(\Phi_\lambda) \subset N$ and $\sup_N |\Phi_\lambda| \leq C$ for all $\lambda \in [0,1]$. Then $c^*_{\mathcal{E}}(\Phi_\lambda, K(\Phi_\lambda))$ is independent of λ.*

Proof. Choose $\lambda \in [0,1]$ and a bounded globally admissible pair (W_λ, W_λ^-) for Φ_λ and $K(\Phi_\lambda)$. It is easily seen from the proof of Proposition 2.8 that (W_λ, W_λ^-) is a globally admissible pair for Φ_μ and $K(\Phi_\mu)$ whenever $|\lambda - \mu|$ is small enough. Note in particular that $\sup_{W_\lambda} |\Phi_\mu| < \infty$ (because W_λ is bounded and $\sup_{W_\lambda} |\Phi_\lambda| < \infty$) and $K(\Phi_\mu) \subset W_\lambda$ (because $K(\Phi_\mu) \subset N$ and

$$\|\nabla\Phi_\mu(x)\| \geq \|\nabla\Phi_\lambda(x)\| - \|\nabla\Phi_\mu(x) - \nabla\Phi_\lambda(x)\| > 0$$

for x in a neighbourhood of $N - \operatorname{int}(W_\lambda)$). Also, using partition of unity it is easy to construct a gradient-like vector field $V_\mu : E - K(\Phi_\mu) \to E$ for Φ_μ such that $V_\mu = V_\lambda$ in a neighbourhood of ∂W_λ. So the conclusion follows from the connectedness of $[0,1]$. □

Remark 2.15. Let M_0 be a C^2 Riemannian manifold and $(F_n)_{n=1}^\infty$ a filtration of a Hilbert space F. Denote the orthogonal projector of F onto F_n by Q_n and define $M := M_0 \times F$, $M_n := M_0 \times F_n$ and $P_n(x,y) := (x, Q_n y)$ for $(x,y) \in M_0 \times F$. It is easy to see by inspection that the results of this and the preceding section remain valid for M.

3. Morse inequalities

Denote $[\mathbf{Z}] := \prod_{n=1}^{\infty} \mathbf{Z} / \bigoplus_{n=1}^{\infty} \mathbf{Z}$ and $[\mathbf{Z}_+] := \{[(\xi_n)_{n=1}^{\infty}] \in [\mathbf{Z}] : \xi_n \geq 0$ for almost all $n\}$. Let (X, B) be a pair of closed subsets of E with the property that for each $q \in \mathbf{Z}$ there is an $n(q)$ such that

$$\dim H^{q+d_n}(X \cap E_n, B \cap E_n) < \infty$$

whenever $n \geq n(q)$. Then

$$\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, B) := \left[\left(\dim H^{q+d_n}(X \cap E_n, B \cap E_n)\right)_{n=1}^{\infty}\right] \tag{3.1}$$

is a well-defined element of $[\mathbf{Z}_+]$. The sequence on the right-hand side of (3.1) will often be constant for almost all n. In such a case we will write $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, B) = [d]$, d being the constant. We will say that the pair (X, B) is *of $\mathcal{E}$-finite type*, or *$\mathcal{E}$-finite* for short, if $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, B)$ is well-defined (in the above sense) and $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, B) = [0]$ for almost all $q \in \mathbf{Z}$.

Suppose that Φ satisfies (PS)* and (W, W^-) is an admissible pair for Φ and $A := \{p_1, ..., p_k\}$. We will say that p_j is *of $\mathcal{E}$-finite type*, or *$\mathcal{E}$-finite*, if some (and therefore every) admissible pair for Φ and p_j is $\mathcal{E}$-finite. If (W, W^-) and all p_j are $\mathcal{E}$-finite, then we define

$$M^q_{\mathcal{E}}(W, W^-) := \sum_{j=1}^{k} \dim_{\mathcal{E}} c^q_{\mathcal{E}}(\Phi, p_j), \qquad q \in \mathbf{Z}, \tag{3.2}$$

and

$$\beta^q_{\mathcal{E}}(W, W^-) := \dim_{\mathcal{E}} H^q_{\mathcal{E}}(W, W^-), \qquad q \in \mathbf{Z}. \tag{3.3}$$

Moreover, in such a case we may define the Morse and the Poincaré polynomials of (W, W^-) by setting

$$M_{\mathcal{E}}(t, W, W^-) := \sum_{q=-\infty}^{\infty} M^q_{\mathcal{E}}(W, W^-)\, t^q$$

and

$$P_{\mathcal{E}}(t, W, W^-) := \sum_{q=-\infty}^{\infty} \beta^q_{\mathcal{E}}(W, W^-)\, t^q.$$

Note that $M_{\mathcal{E}}$ and $P_{\mathcal{E}}$ are not polynomials in the usual sense (because some exponents q may be negative). Formally, $M_{\mathcal{E}}$ and $P_{\mathcal{E}}$ are elements of $[\mathbf{Z}][t, t^{-1}]$.

Theorem 3.1. *(Morse inequalities) Suppose that $\Phi \in C^1(E, \mathbf{R})$ satisfies (PS)* and (W, W^-) is an admissible pair for Φ and $A := \{p_1, ..., p_k\}$. If all p_j are $\mathcal{E}$-finite, then the pair (W, W^-) is $\mathcal{E}$-finite and there is a polynomial $Q(t) = \sum_{q=-\infty}^{\infty} a_q t^q$ such that $a_q \in [\mathbf{Z}_+]$ for all q and*

$$M_{\mathcal{E}}(t, W, W^-) = P_{\mathcal{E}}(t, W, W^-) + (1 + t)Q(t).$$

Note that an equivalent way of expressing the Morse inequalities is

$$\sum_{j=-\infty}^{q} (-1)^{q-j} M^j_{\mathcal{E}}(W, W^-) \geq \sum_{j=-\infty}^{q} (-1)^{q-j} \beta^j_{\mathcal{E}}(W, W^-), \qquad q \in \mathbf{Z}.$$

First we prove the following special case of Theorem 3.1:

Lemma 3.2. *Under the hypotheses of Theorem 3.1, suppose that* $\Phi(p_1) = \Phi(p_2) = ... = \Phi(p_k)$. *Then*

$$M^q_{\mathcal{E}}(W, W^-) = \beta^q_{\mathcal{E}}(W, W^-) \qquad \textit{for all } q \in \mathbf{Z}.$$

Proof. Let (W_j, W_j^-) be an admissible pair for Φ and p_j, $1 \le j \le k$. We may assume that the W_j's are pairwise disjoint. Then $(\bigcup_{j=1}^k W_j, \bigcup_{j=1}^k W_j^-)$ is an admissible pair for Φ and A. Using Remark 2.11, we obtain

$$H^q_{\mathcal{E}}(W, W^-) \cong H^q_{\mathcal{E}}\Big(\bigcup_{j=1}^k W_j, \bigcup_{j=1}^k W_j^-\Big) \cong \bigoplus_{j=1}^k H^q_{\mathcal{E}}(W_j, W_j^-) = \bigoplus_{j=1}^k c^q_{\mathcal{E}}(\Phi, p_j).$$

So the conclusion follows from the definitions (3.1)–(3.3). □

Proof of Theorem 3.1. Our argument follows closely [34] and [41].

Let $X \supset Y \supset Z$ be closed subsets of E. In the exact sequence of the triple (X, Y, Z) (cf. (vi) of Proposition 1.3) denote the range of a mapping by R and $\dim_{\mathcal{E}} H^q_{\mathcal{E}}(.)$ by $\beta^q_{\mathcal{E}}(.)$. Assume that the pairs (X, Y), (X, Z) and (Y, Z) are $\mathcal{E}$-finite. It follows (using the exactness) that

$$\begin{aligned} \beta^q_{\mathcal{E}}(X, Z) &= \dim_{\mathcal{E}} R(j^q) + \dim_{\mathcal{E}} R(i^q), \\ \beta^q_{\mathcal{E}}(Y, Z) &= \dim_{\mathcal{E}} R(i^q) + \dim_{\mathcal{E}} R(\delta^q), \\ \beta^q_{\mathcal{E}}(X, Y) &= \dim_{\mathcal{E}} R(\delta^{q-1}) + \dim_{\mathcal{E}} R(j^q). \end{aligned}$$

Hence

$$\beta^q_{\mathcal{E}}(X, Y) + \beta^q_{\mathcal{E}}(Y, Z) = \beta^q_{\mathcal{E}}(X, Z) + \dim_{\mathcal{E}} R(\delta^{q-1}) + \dim_{\mathcal{E}} R(\delta^q). \tag{3.4}$$

Denote the Poincaré polynomial of (X, Y) by $P_{\mathcal{E}}(t, X, Y)$, and set

$$Q(t, X, Y, Z) := \sum_{q=-\infty}^{\infty} \dim_{\mathcal{E}} R(\delta^q)\, t^q. \tag{3.5}$$

Since $\dim_{\mathcal{E}} R(\delta^q) = [0]$ for almost all q, it follows by multiplying (3.4) by t^q and summing over q that

$$P_{\mathcal{E}}(t, X, Y) + P_{\mathcal{E}}(t, Y, Z) = P_{\mathcal{E}}(t, X, Z) + (1 + t)Q(t, X, Y, Z). \tag{3.6}$$

Let $c_1 < c_2 < ... < c_m$ be the critical values of $\Phi|_W$. Chooose numbers d_i such that $d_0 := \inf_W \Phi$, $d_m := \sup_W \Phi$ and

$$d_0 < c_1 < d_1 < c_2 < ... < d_{m-1} < c_m < d_m.$$

Define

$$W_i := (W \cap \Phi^{d_i}) \cup W^-, \qquad i = 0, 1, ..., m,$$

and, for all $i = 1, \dots, m$,

$$\widetilde{W}_i := \{x \in W_i : \Phi(x) \ge d_{i-1}\}, \quad \widetilde{W}_i^- := \{x \in W_{i-1} : \Phi(x) \ge d_{i-1}\}.$$

Note that $W_m = W$, $W_0 = W^-$ and

$$H^*_{\mathcal{E}}(W_i, W_{i-1}) \overset{exc}{\cong} H^*_{\mathcal{E}}(\widetilde{W}_i, \widetilde{W}_i^-), \qquad i = 1, ..., m. \tag{3.7}$$

Since $\langle \nabla\Phi(x), V(x)\rangle > 0$ whenever $x \in W \cap \Phi^{-1}(d_i)$ (cf. (iii) of Definition 2.1), it is easy to see that $(\widetilde{W}_i, \widetilde{W}_i^-)$ is an admissible pair for Φ and the critical points p_j satisfying $\Phi(p_j) = c_i$. Hence

$$M^q_{\mathcal{E}}(W, W^-) = \sum_{i=1}^{m} M^q_{\mathcal{E}}(\widetilde{W}_i, \widetilde{W}_i^-) = \sum_{i=1}^{m} \beta^q_{\mathcal{E}}(\widetilde{W}_i, \widetilde{W}_i^-) \tag{3.8}$$

according to Lemma 3.2.

By (3.8), each pair $(\widetilde{W}_i, \widetilde{W}_i^-)$ is $\mathcal{E}$-finite, and by (3.7), the same is true for (W_i, W_{i-1}). Exactness of the cohomology sequence of the triple (W_i, W_{i-1}, W_{i-2}) implies that also the pair (W_i, W_{i-2}) is $\mathcal{E}$-finite. So the $\mathcal{E}$-finiteness of $(W, W^-) = (W_m, W_0)$ follows by induction.

Substituting $X = W_m \equiv W$, $Y = W_i$ and $Z = W_{i-1}$ in (3.6), we obtain

$$P_{\mathcal{E}}(t, W, W_i) + P_{\mathcal{E}}(t, W_i, W_{i-1}) = P_{\mathcal{E}}(t, W, W_{i-1}) + (1+t)Q(t, W, W_i, W_{i-1}).$$

Adding these equalities gives

$$\sum_{i=1}^{m} P_{\mathcal{E}}(t, W_i, W_{i-1}) = P_{\mathcal{E}}(t, W, W^-) + (1+t)Q(t), \tag{3.9}$$

where $Q(t)$ has coefficients $a_q \in [\mathbf{Z}_+]$ and $a_q = [0]$ for almost all q (cf. (3.5)). Finally, multiplying (3.8) by t^q, summing over q and employing (3.7), (3.9) and the definitions, we obtain

$$\begin{aligned} M_{\mathcal{E}}(t, W, W^-) &= \sum_{i=1}^{m} P_{\mathcal{E}}(t, \widetilde{W}_i, \widetilde{W}_i^-) \\ &= \sum_{i=1}^{m} P_{\mathcal{E}}(t, W_i, W_{i-1}) = P_{\mathcal{E}}(t, W, W^-) + (1+t)Q(t). \end{aligned}$$

□

If Φ satisfies (PS)*, $a < b$, $\Phi^{-1}(a)$, $\Phi^{-1}(b)$ are bounded away from the critical set K and $K \cap \Phi^{-1}(a, b)$ is finite, then $(W, W^-) := (\Phi^{-1}([a, b]), \Phi^{-1}(a))$ is an admissible pair, as follows from Proposition 2.5. Since

$$H^*_{\mathcal{E}}(W, W^-) \overset{exc}{\cong} H^*_{\mathcal{E}}(\Phi^b, \Phi^a),$$

we may denote the Morse and Poincaré polynomials of (W, W^-) by $M_{\mathcal{E}}(t, \Phi^b, \Phi^a)$ and $P_{\mathcal{E}}(t, \Phi^b, \Phi^a)$ in this case.

Corollary 3.3. *Suppose that $\Phi \in C^1(E, \mathbf{R})$ satisfies (PS)*, $\Phi^{-1}(a)$, $\Phi^{-1}(b)$ (where $a < b$) are bounded away from K and the set $A := K \cap \Phi^{-1}(a, b)$ is finite. If all points of A are $\mathcal{E}$-finite, then*

$$M_{\mathcal{E}}(t, \Phi^b, \Phi^a) = P_{\mathcal{E}}(t, \Phi^b, \Phi^a) + (1+t)Q(t),$$

where $Q(t)$ is as in Theorem 3.1.

Remark 3.4. Note for further reference that if $(E_n)_{n=1}^{\infty}$ is the trivial filtration of E (i.e., $E_n = E$ for all n) and $d_n = 0$ for all n, then essentially our theory is equivalent to the usual Morse theory. In particular, in this case our notion of admissible pair is a variant of the notion of Gromoll-Meyer pair as defined in [10].

4. Auxiliary results on linear mappings

In order to study the local behaviour of a functional Φ near an isolated critical point p one usually assumes that Φ satisfies (PS), the second Fréchet derivative of Φ exists, at least at p, and $L := \Phi''(p)$ is a Fredholm operator [10, 34, 41]. Then L is necessarily self-adjoint, of index 0 and

$$E = R(L) \oplus N(L),$$

where $R(L)$ and $N(L)$ are the range and the null space of L. Note that if L is Fredholm, it is proper on bounded sets, i.e., if C is compact, then the intersection of $L^{-1}(C)$ with any closed ball is compact. In other words, if (x_j) is a bounded sequence such that $Lx_j \to y$, then (x_j) possesses a convergent subsequence.

If Φ satisfies (PS)* instead of (PS), it seems natural to replace the condition $Lx_j \to y$ by $x_j \in E_{n_j}$ and $P_{n_j} Lx_j \to y$, i.e., to assume that L is A-proper. More precisely, let E be a real Hilbert space with a given filtration $\mathcal{E} = (E_n)_{n=1}^{\infty}$. Recall that a mapping $f : D \to E$, D a closed subset of E, is said to be *A-proper* (with respect to $\mathcal{E}$) if each bounded sequence $(x_j)_{j=1}^{\infty} \subset D$ such that $x_j \in D \cap E_{n_j}$ for some n_j, $n_j \to \infty$ and $P_{n_j} f(x_j) \to y \in E$ as $j \to \infty$, has a convergent subsequence. Clearly, if $x_{j_k} \to x$ as $k \to \infty$, then $f(x_{j_k}) \to f(x)$ and $f(x) = y$. It is easily seen that if B is a compact mapping and $f(x) = x + B(x)$, then f is A-proper. More generally, f is A-proper if $f(x) = Ax + B(x)$, where A is a bounded linear Fredholm operator of index 0, $A(E_n) \subset E_n$ for all n and B is compact. A survey of A-proper mappings may be found e.g. in Petryshyn [35, 36]. The definition given there is more general than ours. On the other hand, in [35, 36] it is assumed that $\dim E_n < \infty$, which is not necessarily the case here.

In [36, Theorem II.3.1] it is shown that if L is an A-proper bounded linear operator, then L is Fredholm of index ≥ 0. For the sake of completeness and because we do not assume that $\dim E_n < \infty$, we give a proof for a self-adjoint L.

Denote the space of bounded linear operators from E to F by $\mathcal{L}(E, F)$.

Proposition 4.1. *If $L \in \mathcal{L}(E, E)$ is A-proper and self-adjoint, then L is a Fredholm operator of index* 0.

Proof. In order to prove that $\dim N(L) < \infty$, assume the contrary. Then there exists a sequence $(x_j) \subset N(L)$ such that $\|x_j\| = 1$ and $\|x_i - x_j\| \geq 1$ if $i \neq j$. For each j there is a $z_j \in E_{n_j}$, $n_j \geq j$, such that $\|z_j - x_j\| \leq 1/j$. Then $\|z_i - z_j\| \geq 1/2$ for all i, j sufficiently large, $i \neq j$. On the other hand,

$$\|P_{n_j} Lz_j\| = \|P_{n_j} L(z_j - x_j)\| \leq \frac{1}{j}\|L\| \to 0,$$

so (z_j) has a convergent subsequence, a contradiction.

Since L is self-adjoint,

$$E = \overline{R(L)} \oplus N(L).$$

To show that $R(L)$ is closed, let $y_j \to y$, where $y_j \in R(L)$. Then $y_j = Lx_j$ for some x_j and we may assume that $x_j \in \overline{R(L)}$. Again, there is a $z_j \in E_{n_j}$, $n_j \geq j$, such that $\|z_j - x_j\| \leq 1/j$. Hence

$$P_{n_j} Lz_j = P_{n_j} Lx_j + P_{n_j} L(z_j - x_j) = P_{n_j} y_j + P_{n_j} L(z_j - x_j) \to y. \tag{4.1}$$

Therefore, if (z_j) is bounded, $z_j \to z$ after passing to a subsequence, and $Lz = y$. So $y \in R(L)$. If (z_j) is unbounded, we may assume $\|z_j\| \to \infty$. Let $w_j := z_j/\|z_j\|$.

It follows from (4.1) that $P_{n_j}Lw_j \to 0$, so after passing to a subsequence, $w_j \to w$ and $w \in N(L)$, $\|w\| = 1$. On the other hand, since $z_j - x_j \to 0$,

$$\lim_{j\to\infty} \frac{x_j}{\|x_j\|} = \lim_{j\to\infty} w_j = w.$$

Hence $w \in \overline{R(L)}$. This contradiction completes the proof of the closedness of $R(L)$.

We have shown that L is a Fredholm operator. Since it is self-adjoint, its index must be 0. □

Assume from now on that $L \in \mathcal{L}(E,E)$ is a self-adjoint Fredholm operator (of index 0) and $\mathcal{E}$ is a given filtration. As usual, P_n is the orthogonal projector of E onto E_n.

Lemma 4.2. *(i) There exists an n_0 such that if $n \geq n_0$, then $P_n|_{N(L)} : N(L) \to P_nN(L)$ is a linear isomorphism and $\|z\| \leq 2\|P_nz\|$ for all $z \in N(L)$.*

(ii) $E_n = (R(L) \cap E_n) \oplus P_nN(L)$, and the spaces $R(L) \cap E_n$ and $P_nN(L)$ are orthogonal.

Proof. (i) This is obvious because $\dim N(L) < \infty$ and $P_n \to I$ uniformly on compact sets (I is the identity operator).

(ii) Let $x \in E_n$ and suppose that x is orthogonal to $P_nN(L)$. Then

$$0 = \langle x, P_nz\rangle = \langle x, z\rangle \quad \forall\, z \in N(L).$$

It follows that $x \in R(L)$, so the orthogonal complement of $P_nN(L)$ in E_n is $R(L) \cap E_n$. □

Let Q_n be the orthogonal projector of $R(L)$ onto $R(L) \cap E_n$. Since $P_n|_{R(L)}$ and Q_n map $R(L)$ into E, $P_n - Q_n$ may be considered as an element of $\mathcal{L}(R(L), E)$.

Proposition 4.3. *$P_n - Q_n \to 0$ in $\mathcal{L}(R(L), E)$ as $n \to \infty$.*

Proof. Let $x \in R(L)$. Since $P_ny = Q_ny = y$ whenever $y \in R(L) \cap E_n$,

$$\langle P_nx - Q_nx, y\rangle = 0 \quad \forall\, y \in R(L) \cap E_n.$$

Therefore, in view of (ii) of Lemma 4.2, $P_nx - Q_nx \in P_nN(L)$. So $P_nx - Q_nx = P_nz_n$ for some $z_n \in N(L)$. By (i) of Lemma 4.2,

$$\|z_n\| \leq 2\|P_nz_n\| = 2\|(P_n - Q_n)x\| \leq 4\|x\| \tag{4.2}$$

for almost all n. Since $x \in R(L)$, $Q_nx \in R(L) \cap E_n$ and $z_n \in N(L)$,

(4.3)
$$\begin{aligned}\|(P_n - Q_n)x\|^2 &= \langle P_nz_n, (P_n - Q_n)x\rangle \\ &= \langle P_nz_n, x\rangle = \langle P_nz_n - z_n, x\rangle \leq \|(P_n - I)z_n\|\|x\|.\end{aligned}$$

Since $P_n \to I$ uniformly on bounded subsets of $N(L)$, it follows from (4.2) and (4.3) that $(P_n - Q_n)x \to 0$ as $n \to \infty$, uniformly in $x \in R(L)$, $\|x\| \leq 1$. Hence the conclusion. □

Corollary 4.4. *The sequence $(R(L) \cap E_n)_{n=1}^{\infty}$ is a filtration of $R(L)$. More precisely, for each $x \in R(L)$, $Q_nx \to x$ as $n \to \infty$.*

Proof. Immediate from Proposition 4.3 since $P_nx \to x$ as $n \to \infty$. □

Theorem 4.5. *Let $L \in \mathcal{L}(E, E)$ be a self-adjoint operator. Then the following conditions are equivalent:*

(i) L is A-proper;

(ii) L is a Fredholm operator of index 0 and there exist $c > 0$ and $n_0 \geq 1$ such that if $n \geq n_0$, then $\|P_n Lx\| \geq c\|x\|$ for all $x \in R(L) \cap E_n$.

Proof. (i) $\Rightarrow$ (ii) By Proposition 4.1, L is Fredholm of index 0. Suppose that for any $j \geq 1$ there exist $n_j \geq j$ and $x_{n_j} \in R(L) \cap E_{n_j}$ such that

$$\|P_{n_j} L x_{n_j}\| < \frac{1}{j} \|x_{n_j}\|.$$

Then $P_{n_j} L y_{n_j} \to 0$, where $y_{n_j} := x_{n_j} / \|x_{n_j}\|$. The A-properness of L implies that, after passing to a subsequence, $y_{n_j} \to y \in R(L)$ and $\|y\| = 1$. But $Ly = 0$, i.e., $y \in N(L)$. This contradiction shows that (ii) is satisfied.

(ii) $\Rightarrow$ (i) Although this implication will not be used, we prove it for the sake of completeness.

Suppose that L is Fredholm of index 0. Let $(x_j) \subset E$ be a bounded sequence such that $x_j \in E_{n_j}$, $n_j \to \infty$ and $P_{n_j} L x_j \to y$ as $j \to \infty$. Passing to a subsequence, $x_j \to x$ weakly and $P_{n_j} L x_j \to Lx$ weakly. Therefore $y = Lx$ and $y \in R(L)$. According to (ii) of Lemma 4.2, $x_j = u_j + P_{n_j} z_j$, where $u_j \in R(L) \cap E_{n_j}$ and $z_j \in N(L)$. Passing to a subsequence again, $z_j \to z$ and

$$P_{n_j} L u_j = P_{n_j} L x_j - P_{n_j} L P_{n_j} z_j \to y + Lz = y.$$

There is a $u \in R(L)$ such that $Lu = y$. Since $u_j - Q_{n_j} u \in R(L) \cap E_{n_j}$, it follows from the inequality in (ii) and Corollary 4.4 that

$$c\|u_j - Q_{n_j} u\| \leq \|P_{n_j} L(u_j - Q_{n_j} u)\| = \|P_{n_j} L u_j - P_{n_j} L Q_{n_j} u\| \to \|y - Lu\| = 0.$$

Hence $u_j \to u$, and therefore $x_j = u_j + P_{n_j} z_j \to u + z$. This shows that L is A-proper. □

5. Computation of critical groups

Suppose $(E_n)_{n=1}^{\infty}$ is a filtration of a real Hilbert space E and $\mathcal{E} = \{E_n, d_n\}_{n=1}^{\infty}$.

For an arbitrary self-adjoint operator $L \in \mathcal{L}(E, E)$, denote the Morse index of L (or more precisely, of the quadratic form $x \mapsto \langle Lx, x\rangle$) by $M^-(L)$. Suppose L is a Fredholm operator, and recall that

$$Q_n : R(L) \to R(L) \cap E_n$$

is the orthogonal projector of $R(L)$ onto $R(L) \cap E_n$. Define the *$\mathcal{E}$-Morse index* $M^-_{\mathcal{E}}(L)$ of L by the formula

$$M^-_{\mathcal{E}}(L) := \lim_{n\to\infty} (M^-(Q_n L|_{R(L)\cap E_n}) - d_n). \tag{5.1}$$

The above limit exists in many important cases, as will be shown later. However, in general it does not, so $M^-_{\mathcal{E}}(L)$ is not always well-defined.

As usual, $M^0(L)$ will denote the nullity of L, i.e.,

$$M^0(L) := \dim N(L).$$

Observe that $M^-(Q_n L|_{R(L)\cap E_n})$ is the Morse index of the quadratic form $R(L) \cap E_n \ni x \mapsto \langle Lx, x\rangle$. It might seem more natural to consider the form $E_n \ni x \mapsto \langle Lx, x\rangle$ and therefore define the $\mathcal{E}$-Morse index by

$$\widetilde{M^-_{\mathcal{E}}}(L) := \lim_{n\to\infty} (M^-(P_n L|_{E_n}) - d_n). \tag{5.2}$$

However, in view of Lemma 4.2, $E_n = (R(L) \cap E_n) \oplus P_n N(L)$, and since the quadratic form $P_n N(L) \ni x \mapsto \langle Lx, x \rangle$ tends to 0 as $n \to \infty$ (in the sense that $|\langle Lx, x\rangle| \le \varepsilon_n \|x\|^2$, where $\varepsilon_n \to 0$), its contribution to the Morse index as $n \to \infty$ should be neglected. This justifies the definition (5.1). See also Theorem 5.4.

Remark 5.1. If $N(L) \subset E_n$, then $E_n = (R(L) \cap E_n) \oplus N(L)$, and the quadratic forms $(R(L) \cap E_n) \ni x \mapsto \langle Lx, x\rangle$ and $E_n \ni x \mapsto \langle Lx, x \rangle$ have the same Morse index. Therefore if $N(L) \subset E_{n_0}$ for some n_0, then $M^-_{\mathcal{E}}(L)$ is well-defined if and only if $\widetilde{M^-_{\mathcal{E}}}(L)$ is and $M^-_{\mathcal{E}}(L) = \widetilde{M^-_{\mathcal{E}}}(L)$.

Proposition 5.2. *Suppose $A \in \mathcal{L}(E, E)$ is a self-adjoint Fredholm operator of index 0 such that $A(E_n) \subset E_n$ for almost all n and $B \in \mathcal{L}(E, E)$ is a self-adjoint compact operator. Then $A + B$ is A-proper. If $M^-(A|_{E_n}) = d_n + k$ for almost all n and some $k \in \mathbf{Z}$, then $M^-_{\mathcal{E}}(A + B)$ is well-defined and finite.*

Proof. Set $L := A + B$. It is easy to verify that L is A-proper.

Let us introduce some auxiliary notation. $Q : E \to R(L)$ is the orthogonal projector,

$$F_n := (R(L) \cap E_{n+1}) \cap (R(L) \cap E_n)^{\perp}, \qquad G_n := E_{n+1} \cap E_n^{\perp}$$

and

$$U_n := P_{n+1} - P_n : E \to G_n.$$

According to Lemma 4.2, $E_n = (R(L) \cap E_n) \oplus P_n N(L)$ and $E_{n+1} = (R(L) \cap E_{n+1}) \oplus P_{n+1} N(L)$. Therefore

$$E_{n+1} = (R(L) \cap E_n) \oplus F_n \oplus P_{n+1} N(L) = (R(L) \cap E_n) \oplus P_n N(L) \oplus G_n$$

and the sums are orthogonal. So each $x \in R(L) \cap E_{n+1}$ may be represented as

$$x = y + z = y + P_n \zeta + w, \tag{5.3}$$

where $y \in R(L) \cap E_n$, $z \in F_n$, $\zeta \in N(L)$ and $w \in G_n$.

It follows from Proposition 4.3 and Theorem 4.5 that there exist $\varepsilon > 0$ and $m_0 \ge 1$ such that

$$\|Q_n Lx\| \ge 2\varepsilon \|x\| \quad \text{whenever } x \in R(L) \cap E_n \quad \text{and } n \ge m_0. \tag{5.4}$$

For $n \ge m_0$ and $x \in R(L) \cap E_{n+1}$, define

$$T_n x := Q_{n+1} Lx - (Q_n L Q_n x + U_n A U_n x).$$

Using the decomposition (5.3) of x and observing that $\langle Ay, w\rangle = 0$ (because $A(E_n) \subset E_n$), we obtain

$$\begin{aligned} \langle T_n x, x\rangle &= \langle L(y+z), y+z\rangle - \langle Ly, y\rangle - \langle Aw, w\rangle \\ &= 2\langle L(y+w), P_n\zeta\rangle + \langle LP_n\zeta, P_n\zeta\rangle + 2\langle By, w\rangle + \langle Bw, w\rangle. \end{aligned}$$

Since B is compact, $w \in G_n \subset E_n^{\perp}$ and $LP_n\zeta \to 0$ uniformly in $\zeta \in N(L) \cap \overline{B}(0,1)$, it follows that $\|T_n\| = \sup\{|\langle T_n x, x\rangle| : x \in R(L) \cap E_{n+1} \cap \overline{B}(0,1)\} \le \varepsilon$ if m_0 is large enough. Hence by (5.4),

$$\|Q_{n+1} Lx - \lambda T_n x\| \ge \|Q_{n+1} Lx\| - \lambda \|T_n x\| \ge \varepsilon \|x\|$$

for each $\lambda \in [0, 1]$ and $x \in R(L) \cap E_{n+1}$. Therefore, for $n \ge m_0$,

$$\begin{aligned} M^-(Q_{n+1} L|_{R(L)\cap E_{n+1}}) &= M^-((Q_n L Q_n + U_n A U_n)|_{R(L)\cap E_{n+1}}) \\ &= M^-(Q_n L|_{R(L)\cap E_n}) + M^-(A|_{G_n}). \end{aligned} \tag{5.5}$$

The second equality follows because $R(L)\cap E_n$ is orthogonal to G_n. Finally, since $A(E_n)\subset E_n$,

$$d_{n+1}-d_n = M^-(A|_{E_{n+1}}) - M^-(A|_{E_n}) = M^-(A|_{G_n}).$$

This and (5.5) imply that $M^-(Q_nL|_{R(L)\cap E_n})-d_n$ is constant for almost all n. Moreover, it is finite because $M^-(Q_nQA|_{R(L)\cap E_n})\le d_n+k$, $\dim N(A)<\infty$ and B is compact. □

Theorem 5.3. *Suppose that $\Phi\in C^1(E,\mathbf{R})$, p is an isolated critical point of Φ and*

$$\Phi(x)=\Phi(p)+\frac{1}{2}\langle L(x-p),x-p\rangle+\varphi(x), \tag{5.6}$$

where L is an invertible A-proper operator and $\nabla\varphi(x)=o(\|x-p\|)$ as $x\to p$. If $M^-_{\mathcal{E}}(L)$ is well-defined and finite, then $c^q_{\mathcal{E}}(\Phi,p)=[\mathcal{F}]$ for $q=M^-_{\mathcal{E}}(L)$ and $[0]$ otherwise. If $M^-_{\mathcal{E}}(L)=+\infty$ or $-\infty$, then $c^q_{\mathcal{E}}(\Phi,p)=[0]$ for all q.

Proof. Note first that since $R(L)=E$, we have $P_n=Q_n$ and $M^-_{\mathcal{E}}(L)=\widetilde{M}^-_{\mathcal{E}}(L)$. Consider the family of functionals

$$\Phi_\lambda(x):=\Phi(p)+\frac{1}{2}\langle L(x-p),x-p\rangle+(1-\lambda)\varphi(x),\qquad \lambda\in[0,1],\ x\in E.$$

By Theorem 4.5, there are $c>0$ and $n_0\ge 1$ such that for $n\ge n_0$ and $x\in E_n$, $\|P_nLx\|\ge c\|x\|$. Take $\delta>0$ such that $\|\nabla\varphi(x)\|\le\frac{c}{2}\|x-p\|$ for $x\in B(p,\delta)$. We claim that each Φ_λ satisfies (PS)* on $\overline{B}(p,\delta)$. Indeed, assume $x_k\in\overline{B}(p,\delta)\cap E_{n_k}$, $n_k\to\infty$ and $P_{n_k}\nabla\Phi_\lambda(x_k)\to 0$. Let $p_k:=P_{n_k}p\in E_{n_k}$. Since $\nabla\Phi_\lambda(x_k)=L(x_k-p_k)+L(p_k-p)+(1-\lambda)\nabla\varphi(x_k)$,

$$\begin{aligned}\|P_{n_k}\nabla\Phi_\lambda(x_k)\| &\ge \|P_{n_k}L(x_k-p_k)\|-\|L(p_k-p)\|-\|\nabla\varphi(x_k)\|\\ &\ge c\|x_k-p_k\|-\|L(p_k-p)\|-\frac{c}{2}\|x_k-p\|\end{aligned}$$

for $n_k\ge n_0$. Hence $x_k\to p$ as $k\to\infty$.

In view of Corollary 2.9,

$$c^*_{\mathcal{E}}(\Phi_1,p)=c^*_{\mathcal{E}}(\Phi,p).$$

Since $\Phi_1(x)=\Phi(p)+\frac{1}{2}\langle L(x-p),x-p\rangle$ clearly satisfies the hypotheses of Corollary 2.10, we may assume that $p=0$ (and of course that $\Phi_1(0)=0$).

Let $W:=\Phi_1^{-1}([-1,1])$ and $W^-:=\Phi_1^{-1}(-1)$. Since Φ_1 satisfies (PS)* on E, (W,W^-) is an admissible pair for Φ_1 and 0 (see Proposition 2.5). Moreover, if $n\ge n_0$, then $\Phi_1|_{E_n}$ is a nondegenerate quadratic form and $(W\cap E_n,W^-\cap E_n)$ is an admissible pair for $\Phi_1|_{E_n}$ and 0 (with respect to the trivial filtration of E_n, cf. Remark 3.4).

It remains to compute $H^*(W\cap E_n,W^-\cap E_n)$. Choose an equivalent inner product in E_n such that $\Phi_1(x)=\frac{1}{2}\|x^+\|^2-\frac{1}{2}\|x^-\|^2$, where $x=x^++x^-\in E_n^+\oplus E_n^-=E_n$. For $x\in\Phi_1^1\cap E_n$, let $t(x)$ be the smallest $t\ge 0$ for which $x^++(1-t)x^-\in\Phi_1^{-1}([-1,1])$. Then the mapping $(\lambda,x)\mapsto x^++(1-\lambda t(x))x^-$, $\lambda\in[0,1]$, is a strong deformation retraction of $(\Phi_1^1\cap E_n,\Phi_1^{-1}\cap E_n)\equiv(\{x\in E_n:\Phi_1(x)\le 1\},\{x\in E_n:\Phi_1(x)\le -1\})$ onto $(W\cap E_n,W^-\cap E_n)$. Similarly, the mapping $(\lambda,x)\mapsto(1-\lambda)x^++x^-$, $\lambda\in[0,1]$, is a strong deformation retraction of $(\Phi_1^1\cap E_n,\Phi_1^{-1}\cap E_n)$ onto $(\Phi_1^1\cap E_n^-,\Phi_1^{-1}\cap E_n^-)=(E_n^-,E_n^- - B(0,1))$. Hence $(W\cap E_n,W^-\cap E_n)$ is homotopy equivalent to $(B,\partial B)$, where B is the closed unit ball in E_n^-, and therefore $H^*(W\cap E_n,W^-\cap E_n)\cong H^*(B,\partial B)$. Since $\dim E_n^-=M^-(P_nL|_{E_n})$, we obtain $H^{q+d_n}(W \cap E_n, W^- \cap E_n) \cong \mathcal{F}$ if $q = M^-(P_n L|_{E_n}) - d_n$ and 0 otherwise. Now the conclusion follows from the definitions of $H^*_{\mathcal{E}}$ and $M^-_{\mathcal{E}}(L)$. □

Suppose $\Phi \in C^2(U, \mathbf{R})$, where U is a neighbourhood of a critical point p. Then Φ admits the representation (5.6). Assume that the operator L is Fredholm, let $x = p + z + y$, where $z \in N(L)$, $y \in R(L)$, and denote the orthogonal projector onto $R(L)$ by Q. Then

$$\nabla\Phi(p+z+y) = Ly + \nabla\varphi(p+z+y),$$
$$\nabla\Phi(p) = 0 \quad \text{and} \quad \Phi''(p) = L.$$

Since $L|_{R(L)}$ is invertible, it follows from the implicit function theorem that there exist $\delta > 0$ and a C^1-function $y = \alpha(z) : B(0,\delta) \cap N(L) \to R(L)$ such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and

$$Q\nabla\Phi(p + z + \alpha(z)) \equiv 0. \tag{5.7}$$

Define

(5.8)
$$\widetilde{\varphi}(z) := \Phi(p + z + \alpha(z)) - \Phi(p) = \frac{1}{2}\big\langle L\alpha(z), \alpha(z)\big\rangle + \varphi(p + z + \alpha(z)).$$

Suppose 0 is an isolated critical point of $\widetilde{\varphi}$ and let $c^q(\widetilde{\varphi}, 0) := H^q(\widetilde{W}, \widetilde{W}^-)$, where $(\widetilde{W}, \widetilde{W}^-)$ is an admissible pair for $\widetilde{\varphi}$ and 0 in $N(L)$ (with respect to the trivial filtration of $N(L)$).

Theorem 5.4. *Suppose U is a neighbourhood of an isolated critical point p of $\Phi \in C^2(U, \mathbf{R})$ and the operator L (cf. (5.6)) is A-proper. If $M^-_{\mathcal{E}}(L)$ is well-defined and finite, then $c^q_{\mathcal{E}}(\Phi, p) \cong [c^{q - M^-_{\mathcal{E}}(L)}(\widetilde{\varphi}, 0)]$ for all q ($\widetilde{\varphi}$ is given by the formula (5.8)). If $M^-_{\mathcal{E}}(L) = +\infty$ or $-\infty$, then $c^q_{\mathcal{E}}(\Phi, p) = [0]$ for all q.*

Proof. Consider a family of functionals

$$\Phi_\lambda(p+z+y) := \Phi(p) + \frac{1}{2}\langle Ly, y\rangle + \frac{1}{2}\lambda(2-\lambda)\langle L\alpha(z), \alpha(z)\rangle$$
$$+\lambda\varphi(p + z + \alpha(z)) + (1-\lambda)\varphi(p + z + y + \lambda\alpha(z)),$$

where $z + y \in (N(L) \oplus R(L)) \cap B(0,\delta)$ and $0 \le \lambda \le 1$ (this family has been introduced by Dancer in [15]). Observe that $\Phi_0 = \Phi$,

$$\Phi_1(p+z+y) = \Phi(p) + \frac{1}{2}\langle Ly, y\rangle + \widetilde{\varphi}(z)$$

and

$$\nabla_y\Phi_\lambda(p+z+y) = Ly + (1-\lambda)Q\nabla\varphi(p+z+y+\lambda\alpha(z)), \tag{5.9}$$

(5.10)
$$\begin{aligned}\nabla_z&\Phi_\lambda(p+z+y)\\ &= \lambda(2-\lambda)\langle L\alpha(z), \alpha'(z)\,\cdot\rangle + \lambda\langle\nabla\varphi(p+z+\alpha(z)), \cdot + \alpha'(z)\,\cdot\rangle\\ &\quad + (1-\lambda)\langle\nabla\varphi(p+z+y+\lambda\alpha(z)), \cdot + \lambda\alpha'(z)\,\cdot\rangle.\end{aligned}$$

We will show that the family $\{\Phi_\lambda\}$ satisfies the hypotheses of Corollary 2.9 on a suitably small ball $B(p,r)$, where $0 < r < \delta$ and $K(\Phi_\lambda) \cap B(p,r) = \{p\}$.

First we verify that each Φ_λ satisfies (PS)* on $\overline{B}(p,r)$. Suppose $x_k = p+z_k+y_k \in \overline{B}(p,r)\cap E_{n_k}$, $n_k\to\infty$ and $P_{n_k}\nabla\Phi_\lambda(x_k)\to 0$. Then, in view of (5.9),

$$P_{n_k}\nabla_y\Phi_\lambda(p+z_k+y_k) = P_{n_k}Ly_k+(1-\lambda)P_{n_k}Q\nabla\varphi(p+z_k+y_k+\lambda\alpha(z_k)) =: w_k \to 0.$$

Setting $v_k := y_k-(1-\lambda)\alpha(z_k)$, we get

$$(1-\lambda)P_{n_k}L\alpha(z_k)+P_{n_k}Lv_k+(1-\lambda)P_{n_k}Q\nabla\varphi(p+z_k+\alpha(z_k)+v_k) = w_k.$$

Since $L\alpha(z_k)+Q\nabla\varphi(p+z_k+\alpha(z_k)) = 0$ (cf. (5.7)),

(5.11)

$$P_{n_k}Lv_k+(1-\lambda)P_{n_k}Q\Big(\nabla\varphi(p+z_k+\alpha(z_k)+v_k)-\nabla\varphi(p+z_k+\alpha(z_k))\Big) = w_k.$$

Since $v_k = x_k-p-z_k-(1-\lambda)\alpha(z_k)$ and $x_k\in E_{n_k}$,

$$P_{n_k}v_k-v_k = (I-P_{n_k})(p+z_k+(1-\lambda)\alpha(z_k)).$$

So $P_{n_k}v_k-v_k\to 0$ (because $N(L)$ is finite dimensional) and $Q_{n_k}v_k-v_k\to 0$ according to Proposition 4.3. It follows from Theorem 4.5 that

(5.12)

$$\begin{aligned}\|P_{n_k}Lv_k\| &\geq \|P_{n_k}LQ_{n_k}v_k\|-\|P_{n_k}L(Q_{n_k}v_k-v_k)\|\\ &\geq c\|Q_{n_k}v_k\|-\|L\|\|Q_{n_k}v_k-v_k\| \geq c\|v_k\|-(\|L\|+c)\|Q_{n_k}v_k-v_k\|\end{aligned}$$

for almost all k. Since $\varphi\in C^2(U,\mathbf{R})$ and $\varphi''(p)=0$, then taking the radius r of the ball $B(p,r)$ smaller if necessary, we obtain

$$\|\nabla\varphi(p+z_k+\alpha(z_k)+v_k)-\nabla\varphi(p+z_k+\alpha(z_k))\| \leq \frac{c}{2}\|v_k\|.$$

Combining this with (5.11) and (5.12) gives

$$c\|v_k\|-(\|L\|+c)\|Q_{n_k}v_k-v_k\| \leq \frac{c}{2}\|v_k\|+\|w_k\|.$$

Hence $v_k\to 0$ as $k\to\infty$. Passing to a subsequence, $z_k\to\overline{z}$ and $x_k = p+z_k+y_k\to p+\overline{z}+(1-\lambda)\alpha(\overline{z})$. This completes the proof of (PS)*.

Suppose $p+z+y\in B(p,r)$ and $\nabla\Phi_\lambda(p+z+y)=0$. Since $Q\Phi_\lambda''(p)|_{R(L)} = L|_{R(L)}$ (cf. (5.9)), it follows from the implicit function theorem that (5.9) has a unique solution $y = y(z,\lambda)$ provided r is small enough (r independent of λ). A direct verification using (5.7) shows that $y = (1-\lambda)\alpha(z)$. Inserting this in (5.10) we obtain

$$\begin{aligned}&\nabla_z\Phi_\lambda(p+z+(1-\lambda)\alpha(z))\\ &\quad = \lambda(2-\lambda)\langle\nabla\Phi(p+z+\alpha(z)),\alpha'(z)\,\cdot\rangle+(I-Q)\nabla\varphi(p+z+\alpha(z))\\ &\quad = (I-Q)\nabla\varphi(p+z+\alpha(z))\end{aligned}$$

because $\alpha'(z)\cdot\in R(L)$ and (5.7) is satisfied. Recall that $K(\Phi)\cap B(p,r) = \{p\}$. Since $\nabla\Phi(p+z+y)=0$ if and only if $y=\alpha(z)$ and $(I-Q)\nabla\varphi(p+z+\alpha(z))=0$, we must have $z=0$ and $y=(1-\lambda)\alpha(0)=0$. So $K(\Phi_\lambda)\cap B(p,r)=\{p\}$.

Since $\nabla\varphi$ is locally Lipschitz continuous at p, it is easy to see from (5.9), (5.10) that if r is small enough, then $|\Phi_\lambda(x)|$ is bounded by a constant independent of $x\in B(p,r)$ and $\lambda\in[0,1]$, and the mapping $\lambda\mapsto\nabla\Phi_\lambda$ is continuous, uniformly in $x\in B(p,r)$. Now all the hypotheses of Corollary 2.9 are verified; hence

$$c_{\mathcal{E}}^*(\Phi,p)\cong c_{\mathcal{E}}^*(\Phi_1,p).$$

Moreover, by Corollary 2.10, we may assume that $p=0$ (and $\Phi(p)=0$).

Let $\chi(y) := \frac{1}{2}\langle Ly, y\rangle$. Then $\Phi_1(z+y) = \chi(y) + \widetilde{\varphi}(z)$. Since $(R(L)\cap E_n)_{n=1}^{\infty}$ is a filtration of $R(L)$ (cf. Corollary 4.4), there exists an admissible pair (W_1, W_1^-) for χ and 0 (in $R(L)$) such that W_1 is bounded. Denote the corresponding admissible field by $V_1(y)$. Let $(\widetilde{W}, \widetilde{W}^-)$ be an admissible pair for $\widetilde{\varphi}$ and 0, $\widetilde{W} \subset B(0,\delta)\cap N(L)$, and let $\widetilde{V}(z)$ be a corresponding admissible field. Choose m_0 so that $P_{m_0}|_{N(L)} : N(L) \to P_{m_0}N(L)$ is a linear isomorphism, and define

$$W_2 := P_{m_0}\widetilde{W}, \qquad W_2^- := P_{m_0}\widetilde{W}^-$$

and

$$(W, W^-) := \big(W_1 + W_2,\ (W_1^- + W_2) \cup (W_1 + W_2^-)\big).$$

Note that $W_1 \cap W_2 = \{0\}$ since $W_1 \subset R(L)$ and $W_2 \subset P_{m_0}N(L)$. We claim that (W, W^-) is an admissible pair for Φ_1 and 0. For each $x \in E$ we have the unique decompositions $x = z + y = P_{m_0}\zeta + \xi$, where $z, \zeta \in N(L)$ and $y, \xi \in R(L)$. Let

$$V_0(x) := \omega(\|\xi\|)V_1(\xi) + \omega(\|\zeta\|)P_{m_0}\widetilde{V}(\zeta) \quad \text{and} \quad V(x) := \frac{V_0(x)}{1 + \|V_0(x)\|},$$

where $\omega : \mathbf{R} \to [0,1]$ is a Lipschitz continuous function such that $\omega(s) = 0$ for $s \le \varepsilon/2$, $\omega(s) = 1$ for $s \ge \varepsilon$ and $\varepsilon > 0$ is so small that $\overline{B}(0,\varepsilon)\cap R(L) \subset \operatorname{int}(W_1)$, $\overline{B}(0,\varepsilon)\cap N(L) \subset \operatorname{int}(\widetilde{W})$. According to Remark 2.4, it suffices to show that V is an admissible field in a neighbourhood of ∂W. Note that without the cutoff function ω, V would not be defined on the subspaces $\xi = 0$ and $\zeta = 0$ (because respectively V_1 and $\widetilde{V}$ are not). Let η_1, $\widetilde{\eta}$ and η be the flows of $-V_1$, $-\widetilde{V}$ and $-V$. Then $\eta(t,x) = \eta_1(t_1,\xi) + P_{m_0}\widetilde{\eta}(\widetilde{t},\zeta)$, and it is easy to see that condition (iv) of Definition 2.3 is satisfied. Since $\xi \in R(L)\cap E_n$ whenever $x \in E_n$ and $n \ge m_0$, V is related to $\mathcal{E}$. It remains to show that V is gradient-like in some neighbourhood N of ∂W. Since W is bounded, we may assume that N is bounded. Then $\|\xi - y\| = \|P_{m_0}\zeta - z\| \to 0$ uniformly in $x \in N$ as $m_0 \to \infty$, and therefore

$$\begin{aligned}\langle V_0(x), Ly + \nabla\widetilde{\varphi}(z)\rangle &= \omega(\|\xi\|)\langle V_1(\xi), Ly\rangle + \omega(\|\zeta\|)\langle \widetilde{V}(\zeta), \nabla\widetilde{\varphi}(z)\rangle \\ &\quad + \omega(\|\zeta\|)\langle (P_{m_0} - I)\widetilde{V}(\zeta), Ly + \nabla\widetilde{\varphi}(z)\rangle \\ &\ge \omega(\|\xi\|)\beta_1(\xi) + \omega(\|\zeta\|)\widetilde{\beta}(\zeta) - \varepsilon_{m_0},\end{aligned}$$

where $\varepsilon_{m_0} \to 0$ as $m_0 \to \infty$ and β_1, $\widetilde{\beta}$ are as in Definition 2.1. We may assume the neighbourhood N has been chosen in such a way that $x \notin N$ if $\|\xi\| < \varepsilon$ and $\|\zeta\| < \varepsilon$. Taking m_0 large enough, we see that $\langle V(x), Ly + \nabla\widetilde{\varphi}(z)\rangle$ is positive and bounded away from 0 on N. Hence (W, W^-) is an admissible pair.

If $n \ge m_0$, then $(W_1 + W_2)\cap E_n = (W_1 \cap E_n) + W_2$. We need to compute the cohomology of

$$\Big((W_1\cap E_n) + W_2,\ ((W_1^- \cap E_n) + W_2) \cup ((W_1 \cap E_n) + W_2^-)\Big).$$

Topologically this pair is equivalent to

$$(W_1 \cap E_n, W_1^- \cap E_n) \times (W_2, W_2^-),$$

where we have used the customary notation

$$(A, A_0) \times (B, B_0) = (A \times B,\ (A \times B_0) \cup (A_0 \times B)).$$

Let B be a closed ball of dimension $m_n := M^-(Q_n L|_{R(L)\cap E_n})$. For almost all n, $(W_1 \cap E_n, W_1^- \cap E_n)$ is homotopy equivalent to $(B, \partial B)$ (cf. the proof of Theorem

5.3). It follows from the Künneth formula [17, Proposition VI.12.16], [39, Theorem 5.6.1] that

$$\begin{aligned} H^{q+d_n}((W_1 \cap E_n, W_1^- \cap E_n) \times (W_2, W_2^-)) &\cong H^{q+d_n}((B, \partial B) \times (W_2, W_2^-)) \\ &\cong [H^*(B, \partial B) \otimes H^*(W_2, W_2^-)]^{q+d_n} \cong H^{q+d_n-m_n}(W_2, W_2^-) \end{aligned}$$

(note that the hypotheses in the Künneth formula are satisfied because the Čech cohomology has the strong excision property and $(B, \partial B)$ is a pair of ANRs). If $M_{\mathcal{E}}^-(L)$ is finite, then $q + d_n - m_n = q - M_{\mathcal{E}}^-(L)$ for almost all n, and the first conclusion follows. If $M_{\mathcal{E}}^-(L) = +\infty$ or $-\infty$, then respectively $q + d_n - m_n < 0$ or $q + d_n - m_n > \dim N(L)$ for almost all n, so the right-hand side above is 0 for such n. This completes the proof. □

Corollary 5.5. *Suppose that Φ satisfies the hypotheses of Theorem 5.4 and $M_{\mathcal{E}}^-(L)$ is finite. If $\varphi(x) \geq \varphi(p) = 0$ for all $x \in U$ (φ is given by the formula (5.6)), then $c_{\mathcal{E}}^q(\Phi, p) = [\mathcal{F}]$ for $q = M_{\mathcal{E}}^-(L)$ and $[0]$ otherwise. If $\varphi(x) \leq 0$ for all $x \in U$, then $c_{\mathcal{E}}^q(\Phi, p) = [\mathcal{F}]$ for $q = M_{\mathcal{E}}^-(L) + M^0(L)$ and $[0]$ otherwise.*

Proof. Let $R(L) = E^+ \oplus E^-$ be the decomposition corresponding to the positive and the negative part of the spectrum of L. There exists a constant $c > 0$ such that $\langle Ly, y\rangle \geq c\|y\|^2$ for $y \in E^+$ and $\langle Ly, y\rangle \leq -c\|y\|^2$ for $y \in E^-$. According to (5.8) (with $\alpha(z) = \alpha^+(z) + \alpha^-(z) \in E^+ \oplus E^-$),

$$\begin{aligned} \widetilde{\varphi}(z) &= \frac{1}{2}\langle L\alpha^+(z), \alpha^+(z)\rangle + \varphi(p + z + \alpha^+(z)) \\ &+ \quad \frac{1}{2}\langle L\alpha^-(z), \alpha^-(z)\rangle + \big(\varphi(p + z + \alpha(z)) - \varphi(p + z + \alpha^+(z))\big) \end{aligned}$$

for all $z \in B(0, \delta) \cap N(L)$. We claim that $\widetilde{\varphi}$ has a local minimum at 0 if $\varphi(x) \geq 0$ for all $x \in U$. Since

$$\widetilde{\varphi}(z) \geq \frac{1}{2}\langle L\alpha^-(z), \alpha^-(z)\rangle + \big(\varphi(p + z + \alpha(z)) - \varphi(p + z + \alpha^+(z))\big),$$

it suffices to show that the right-hand side above is nonnegative.

Let $\sigma : [0, 1] \to E$ be given by $\sigma(t) := p + z + \alpha^+(z) + t\alpha^-(z)$. Then

$$\begin{aligned} \varphi(p + z + \alpha(z)) - \varphi(p + z + \alpha^+(z)) &= \int_0^1 \frac{d}{dt}\varphi(\sigma(t))\, dt = \int_0^1 \langle \nabla\varphi(\sigma(t)), \alpha^-(z)\rangle\, dt \\ &= \int_0^1 \langle \nabla\varphi(\sigma(t)) - \nabla\varphi(p + z + \alpha(z)), \alpha^-(z)\rangle\, dt + \langle \nabla\varphi(p + z + \alpha(z)), \alpha^-(z)\rangle \\ &= \int_0^1 \langle \nabla\varphi(\sigma(t)) - \nabla\varphi(p + z + \alpha(z)), \alpha^-(z)\rangle\, dt - \langle L\alpha^-(z), \alpha^-(z)\rangle, \end{aligned}$$

where the last equality follows from (5.7). Moreover, since $\varphi''(p) = 0$, we may assume after choosing a smaller δ if necessary that

$$\|\nabla\varphi(\sigma(t)) - \nabla\varphi(p + z + \alpha(z))\| \leq c\|\sigma(t) - p - z - \alpha(z)\| = (1 - t)c\|\alpha^-(z)\|$$

for all $z \in B(0, \delta) \cap N(L)$. Hence

$$\begin{aligned} \widetilde{\varphi}(z) &\geq \frac{1}{2}\langle L\alpha^-(z), \alpha^-(z)\rangle + \big(\varphi(p + z + \alpha(z)) - \varphi(p + z + \alpha^+(z))\big) \\ &\geq -\frac{1}{2}\langle L\alpha^-(z), \alpha^-(z)\rangle - \int_0^1 (1 - t)c\|\alpha^-(z)\|^2\, dt \geq 0, \end{aligned}$$

which proves the claim. So $c^q(\widetilde{\varphi},0)=\mathcal{F}$ if $q=0$ and 0 otherwise (see [15] or [34, Corollary 8.4]). It follows from Theorem 5.4 that $c^q_{\mathcal{E}}(\Phi,p)\cong[c^{q-M^-_{\mathcal{E}}(L)}(\widetilde{\varphi},0)]=[\mathcal{F}]$ for $q=M^-_{\mathcal{E}}(L)$ and $[0]$ otherwise.

If $\varphi(x)\le 0$ for all $x\in U$, a similar argument shows that $\widetilde{\varphi}$ has a local maximum at 0. Hence employing Theorem 5.4 and [15, 34] we obtain $c^q_{\mathcal{E}}(\Phi,p)\cong[c^{q-M^-_{\mathcal{E}}(L)}(\widetilde{\varphi},0)]=[\mathcal{F}]$ for $q=M^-_{\mathcal{E}}(L)+M^0(L)$ and $[0]$ otherwise. □

A functional Φ is said to satisfy the *local linking condition at* 0 if there exist a decomposition $E=Y\oplus Z$ and a constant $\rho>0$ such that

$$\Phi(y)\le c:=\Phi(0)\quad \text{for each } y\in B(0,\rho)\cap Y$$

and

$$\Phi(z)\ge c\quad \text{for each } z\in B(0,\rho)\cap Z.$$

This is a variant of a condition introduced in [25].

Clearly, the functional Φ in Corollary 5.5 satisfies the local linking condition at p. If $\Phi\notin C^2(E,\mathbf{R})$, the argument of this corollary can no longer be applied. However, we still have the following weaker result:

Theorem 5.6. *Suppose that $\Phi\in C^1(E,\mathbf{R})$ satisfies (PS)* and the local linking condition at* 0 *(with Y and Z as above). If* 0 *is an isolated critical point of Φ, $E_n=(Y\cap E_n)\oplus(Z\cap E_n)$ and $\dim(Y\cap E_n)=q_0+d_n$ for almost all n, then $c^{q_0}_{\mathcal{E}}(\Phi,0)\ne[0]$.*

Proof. We may assume that $\Phi(0)=0$, 0 is the only critical point of Φ in $B(0,\rho)$ and $\sup_{B(0,\rho)}|\Phi|<\infty$. Let $\delta\in(0,\rho)$ and let (W,W^-), $W\subset B(0,\delta)$, be an admissible pair having the additional properties given in Proposition 2.6. Define

$$A:=\{\eta(t,x)\in W: t\ge 0,\ x\in S(0,\delta_1)\cap Y\}.$$

Since $\Phi\le 0$ on $S(0,\delta_1)\cap Y$, for each $x\in A$ there is a unique $t(x)$ such that $\eta(t(x),x)\in W^-$. According to (iv) of Definition 2.3, $t(x)$ depends continuously on x. Hence the mapping

$$\alpha(\lambda,x):=\begin{cases}\eta(\lambda t(x),x) & \text{if } x\in A,\ 0\le\lambda\le 1,\\ x & \text{if } x\in W^-,\ 0\le\lambda\le 1,\end{cases}$$

is a filtration-preserving strong deformation retraction of $A\cup W^-$ onto W^-. So it follows from the exact sequence of the triple $(W,A\cup W^-,W^-)$ that

$$H^*_{\mathcal{E}}(W,W^-)\cong H^*_{\mathcal{E}}(W,A\cup W^-).$$

For $z\in Z$, let $\delta_1(z):=\min\{\delta_1,d(z,A\cup W^-)\}$ and

$$D:=\{y+z\in Y\oplus Z:\|y\|<\delta_1(z)\}.$$

Since $\delta_1(z)>0\ \forall z\in Z$ and $\Phi|_{W^-}<0$, $(A\cup W^-)\cap Z=\emptyset$. Hence D is an open set, $Z\subset D$ and $(A\cup W^-)\cap D=\emptyset$. Denote $F_\delta:=(\overline{B}(0,\delta)\cap Y)\oplus(\overline{B}(0,\delta)\cap Z)$ and let $i:(\overline{B}(0,\delta_1)\cap Y,S(0,\delta_1)\cap Y)\to(W,A\cup W^-)$, $j:(W,A\cup W^-)\to(F_\delta,F_\delta-D)$ be the inclusion mappings. Then we have

(5.13)
$$H^*_{\mathcal{E}}(F_\delta,F_\delta-D)\xrightarrow{j^*}H^*_{\mathcal{E}}(W,A\cup W^-)\xrightarrow{i^*}H^*_{\mathcal{E}}(\overline{B}(0,\delta_1)\cap Y,S(0,\delta_1)\cap Y),$$

where i^*, j^* are the induced homomorphisms. It is easy to see that the mapping

$$\gamma(\lambda, y+z) := \begin{cases} \dfrac{2\lambda\delta_1 y}{\max\{\|y\|, \delta_1(z)\}} + (1-2\lambda)y + z, & 0 \le \lambda \le \frac{1}{2}, \\ \dfrac{\delta_1 y}{\max\{\|y\|, \delta_1(z)\}} + (2-2\lambda)z, & \frac{1}{2} \le \lambda \le 1, \end{cases}$$

is a deformation of $(F_\delta, F_\delta - D)$ onto $(\overline{B}(0,\delta_1) \cap Y, S(0,\delta_1) \cap Y)$. It preserves the filtration since $y, z \in E_n$ whenever $y + z \in E_n$. Moreover, the restriction of γ to $[0,1] \times (\overline{B}(0,\delta_1) \cap Y, S(0,\delta_1) \cap Y)$ is a homotopy between $\gamma(1,.) \circ (ji)$ and the identity on $(\overline{B}(0,\delta_1) \cap Y, S(0,\delta_1) \cap Y)$. Similarly, γ is a homotopy between $(ji) \circ \gamma(1,.)$ and the identity on $(F_\delta, F_\delta - D)$. So the inclusion mapping ji is a homotopy equivalence by filtration-preserving homotopies, and it follows that i^*j^* in (5.13) is an isomorphism. In particular, $H^{q_0}_{\mathcal{E}}(W, W^-) \cong H^{q_0}_{\mathcal{E}}(W, A \cup W^-)$ is nontrivial because $H^{q_0}_{\mathcal{E}}(\overline{B}(0,\delta_1) \cap Y, S(0,\delta_1) \cap Y) = [\mathcal{F}]$ according to Example 1.4. □

6. Relation to degree theory

Assume that $U \subset E$ is an open neighbourhood of an isolated critical point p of $\Phi \in C^2(U, \mathbf{R})$, $\nabla\Phi(x) = x - T(x)$ and T is compact. Define the Leray-Schauder index of $\nabla\Phi$ at p as

$$\mathrm{ind}(\nabla\Phi, p) := \deg(I - T, B(p,\rho), 0),$$

where $B(p,\rho) \subset U$ and on the right-hand side we have the Leray-Schauder degree of $I - T$ with respect to $B(p,\rho)$ and 0 (see e.g. [31]). According to [10, Theorem II.3.2], cf. also [34, Theorem 8.5],

$$\mathrm{ind}(\nabla\Phi, p) = \sum_{q=0}^{\infty} (-1)^q \dim c_q(\Phi, p) \tag{6.1}$$

(c_q are the critical groups defined in (0.1)). It follows by inspection of the proof in [10] that (6.1) remains valid with $c_q(\Phi, p)$ replaced by the critical Čech cohomology groups $c^q(\Phi, p)$ of Φ at p (with respect to the trivial filtration of E; cf. Remark 3.4). Recall that the Euler characteristic for a pair (X, B) of finite type is defined by

$$\chi(X, B) = \sum_{q=0}^{\infty} (-1)^q \dim H^q(X, B) \equiv P(-1, X, B),$$

where $P(t, X, B)$ is the Poincaré polynomial of (X, B).

Let (W, W^-) be a bounded admissible pair for Φ and p (again, with respect to the trivial filtration of E). Since $c^*(\Phi, p) = H^*(W, W^-)$, (6.1) may be reformulated as

$$\deg(\nabla\Phi, \mathrm{int}(W), 0) = \chi(W, W^-), \tag{6.2}$$

and this relation remains valid also if (W, W^-) is admissible for Φ and a set $A \subset K$, cf. [10, Theorem II.3.3]. Formula (6.2) may be seen as a generalization of the Poincaré-Hopf theorem.

Recall from [31] that a mapping $T : X \to Y$ is said to be a *k-set contraction* if for each bounded set $B \subset X$, $\alpha(T(B)) \le k\alpha(B)$, where α is the (Kuratowski) measure of noncompactness. T is called a *strict set contraction* if it is a k-set contraction with $k < 1$. For mappings f of the form $f(x) = x - T(x)$, where T is a strict set contraction, there exists a degree theory (see [31, Section 6.2]) which coincides with the Leray-Schauder degree if T is compact (compact mappings are 0-set contractions). One can verify that formula (6.2) remains valid if $\nabla\Phi(x) = x - T(x)$ and T is a strict set contraction (this will be done in the course of the proof of Theorem 6.1).

In this section we will look for a possible generalization of formula (6.2) to the case of strongly indefinite functionals.

Let $(E_n)_{n=1}^{\infty}$ be a filtration of E, $(d_n)_{n=1}^{\infty}$ a sequence of nonnegative integers and $\mathcal{E} := \{E_n, d_n\}_{n=1}^{\infty}$. Suppose that $\Phi \in C^1(E, \mathbf{R})$, U is an open bounded subset of E and

$$\Phi \text{ satisfies } (PS)^* \text{ on } \overline{U}, \tag{6.3}$$

$$\Phi|_U \text{ is bounded}, \tag{6.4}$$

$$\Phi \text{ has no critical points on } \partial U, \tag{6.5}$$

$$P_n\nabla\Phi(x) = x - T_n(x) \text{ for all } n \ge 1 \text{ and } x \in \overline{U \cap E_n}, \text{ where } T_n : \overline{U \cap E_n} \to E_n \text{ is a strict set contraction.} \tag{6.6}$$

Clearly, (6.6) is satisfied if all E_n are finite dimensional or all T_n are compact. Denote $U_n := U \cap E_n$ and $\Phi_n := \Phi|_{E_n}$. It is easy to see that U_n is nonempty for almost all n, $\overline{U}_n \subset \overline{U} \cap E_n$, $\partial U_n \subset \partial U \cap E_n$ and $\nabla\Phi_n = P_n\nabla\Phi|_{E_n}$. We are going to define a generalized topological degree of $\nabla\Phi$ with respect to U and 0. Observe that for large n, say $n \ge n_0$, $0 \notin \nabla\Phi_n(\partial U_n)$. Indeed, otherwise for each j there are $n_j \ge j$ and $y_j \in \partial U_{n_j}$ such that $\nabla\Phi_{n_j}(y_j) = 0$. By (PS)* (recall $\Phi|_U$ is bounded), (y_j) has a subsequence converging to some $y \in K \cap \partial U$, a contradiction to (6.5). Therefore

$$s_n := \deg(\nabla\Phi_n, U_n, 0)$$

is well-defined for $n \ge n_0$. For $n < n_0$ we put $s_n := 0$. Now we define the *$\mathcal{E}$-degree* of $\nabla\Phi$ with respect to U and 0 by the formula

$$\mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, U, 0) := \left[\left((-1)^{d_n}s_n\right)_{n=1}^{\infty}\right].$$

Note that $\mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, U, 0) \in [\mathbf{Z}]$. The above definition is modelled on the definition of the generalized degree for A-proper mappings given in [24] (in [24] there are no terms $(-1)^{d_n}$).

It is easy to see that $\mathrm{Deg}_{\mathcal{E}}$ satisfies the usual properties of topological degree except that $\mathrm{Deg}_{\mathcal{E}}(I, U, 0) = [((-1)^{d_n})_{n=1}^{\infty}]$ (instead of being equal to $[(1)_{n=1}^{\infty}]$) if $0 \in U$. Admissible homotopies are the ones which preserve the filtration and map $\partial U_n \times [0, 1]$ into $E_n - \{0\}$ for almost all n. By the excision property, it is possible to define the *$\mathcal{E}$-index* of $\nabla\Phi$ at an isolated critical point $p \in U$ by setting

$$\mathrm{Ind}_{\mathcal{E}}(\nabla\Phi, p) := \mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, B(p, \rho), 0),$$

where ρ is such that $B(p, \rho) \subset U$ and $\overline{B}(p, \rho) \cap K = \{p\}$.

Define the *$\mathcal{E}$-Euler characteristic* of an $\mathcal{E}$-finite pair (X, B) of closed subsets of E by

$$\chi_{\mathcal{E}}(X, B) := P_{\mathcal{E}}(-1, X, B) \equiv \sum_{q=-\infty}^{\infty} (-1)^q \dim_{\mathcal{E}} H^q_{\mathcal{E}}(X, B).$$

The pair (X, B) will be called *strongly $\mathcal{E}$-finite* if there is a $k \ge 1$ such that $\dim H^{q+d_n}(X \cap E_n, B \cap E_n)$ is finite for all $q \in \mathbf{Z}$ and $n \ge k$, and zero for all $|q| \ge k$ and $n \ge k$. Similarly, an isolated critical point p of a functional Φ which satisfies (PS)* will be called *strongly $\mathcal{E}$-finite* if so is some (and therefore every) admissible pair for Φ and p. Note that strongly $\mathcal{E}$-finite implies $\mathcal{E}$-finite (but not conversely, as will be seen from Remark 6.3 below). The reason for introducing the notion of strong $\mathcal{E}$-finiteness is that then

$$\chi_{\mathcal{E}}(X,B) = \left[\left((-1)^{d_n}\chi(X\cap E_n, B\cap E_n)\right)_{n=1}^{\infty}\right], \tag{6.7}$$

and in general this equality is not true for $\mathcal{E}$-finite pairs (again, Remark 6.3 will provide an example). To show (6.7), observe that if $n \geq k$, then

$$\begin{aligned}(-1)^{d_n}\chi(X\cap E_n, B\cap E_n) &= (-1)^{d_n}\sum_{q=0}^{\infty}(-1)^q \dim H^q(X\cap E_n, B\cap E_n)\\ &= \sum_{q=-\infty}^{\infty}(-1)^q \dim H^{q+d_n}(X\cap E_n, B\cap E_n).\end{aligned}$$

Since all terms on the right-hand side are zero for $|q| \geq k$, the conclusion follows from the definition of $\chi_{\mathcal{E}}$.

Now we state the main result of this section.

Theorem 6.1. *Suppose that $\Phi \in C^2(E,\mathbf{R})$ satisfies (PS)*, (W,W^-) is a bounded admissible pair for Φ and A, and $\nabla\Phi$ has the form (6.6) on W. Then*

(i) $\mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, \mathrm{int}(W), 0) = \left[\left((-1)^{d_n}\chi(W\cap E_n, W^-\cap E_n)\right)_{n=1}^{\infty}\right]$*;*

(ii) If (W,W^-) is strongly $\mathcal{E}$-finite, then $\mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, \mathrm{int}(W), 0) = \chi_{\mathcal{E}}(W,W^-)$*;*

(iii) (Poincaré-Hopf formula) If $A = \{p_1, ..., p_k\}$ and all p_j are strongly $\mathcal{E}$-finite, then

$$\sum_{j=1}^{k}\mathrm{Ind}_{\mathcal{E}}(\nabla\Phi, p_j) = \chi_{\mathcal{E}}(W,W^-).$$

Proof. (i) Since (6.3)–(6.6) are satisfied (with $U = \mathrm{int}(W)$), $\mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, \mathrm{int}(W), 0)$ is well-defined. It is easy to see that for almost all n, $(W_n, W_n^-) := (W\cap E_n, W^-\cap E_n)$ is an admissible pair for Φ_n and $A_n := W\cap K(\Phi_n)$ (with respect to the trivial filtration of E_n). Indeed, if V is an admissible field for (W,W^-) and N is a sufficiently small neighbourhood of ∂W, then V maps $N\cap E_n$ into E_n for all large n. According to Remark 2.4, this suffices for (W_n, W_n^-) to be an admissible pair. Since $\nabla\Phi_n|_{W_n}$ satisfies (6.6), it is a proper mapping; cf. [31, Corollary 6.2.2]. Therefore Φ_n satisfies (PS) on W_n. In particular, the set A_n is compact. Now it follows essentially from [10, Theorem II.3.3] that

$$\deg(\nabla\Phi_n, \mathrm{int}(W)\cap E_n, 0) = \chi(W_n, W_n^-), \tag{6.8}$$

and (i) is satisfied in view of the definition of $\mathrm{Deg}_{\mathcal{E}}$.

For the reader's convenience and since the hypotheses in [10] are somewhat different from ours, we give a proof of (6.8). Since A_n is compact, we may find a $\delta > 0$ and a function $\varepsilon \in C^2(E_n, [0,1])$ such that $\varepsilon(x) = 1$ if $x \in G := \{x \in E_n : d(x, A_n) \leq \delta\}$ and $\varepsilon(x) = 0$ if $d(x, \partial W_n) \leq \delta$. Moreover, we may assume that $\|\nabla\varepsilon(x)\| \leq m$ and $\|\varepsilon''(x)\| \leq m$ for some constant m and all x. Let $M := \sup\{\|x\| : x \in W_n\}$ and

$$\beta := \inf_{x\in W_n - G}\|\nabla\Phi_n(x)\|.$$

Clearly, $\beta > 0$. It follows from a version of the Sard-Smale theorem for k-set contractions [44] that we can find an arbitrarily small $x_0 \in E_n$ such that $(\nabla\Phi_n)^{-1}(x_0)\cap$

$W_n = \{p_1, ..., p_s\} \subset G$ and $\Phi_n''(p_j)$ is invertible for all j. Let

$$\psi_n(x) := \Phi_n(x) + \varepsilon(x)\langle x, x_0\rangle.$$

For $x \in W_n - G$ we have

$$\|\nabla\psi_n(x)\| \geq \|\nabla\Phi_n(x)\| - \|x_0\| - \|\nabla\varepsilon(x)\|\|x\|\|x_0\| \geq \frac{\beta}{2}$$

provided $\|x_0\|$ is small enough. Hence $K(\psi_n) \cap W_n = \{p_1, ..., p_s\}$. Since $\psi_n(x) = \Phi_n(x)$ if $d(x, \partial W_n) \leq \delta$, (W_n, W_n^-) is an admissible pair for ψ_n and $\{p_1, ..., p_s\}$. Now observe that for $x \in W_n$,

$$\nabla\psi_n(x) = x - T_n(x) + \langle x, x_0\rangle\nabla\varepsilon(x) + \varepsilon(x)x_0,$$

where T_n is a k-set contraction. Since $x \mapsto \langle x, x_0\rangle\nabla\varepsilon(x) + \varepsilon(x)x_0$ is a Lipschitz continuous mapping with Lipschitz constant $(M+2)m\|x_0\|$ which will be less than $1-k$ if we choose a sufficiently small x_0, we obtain that $x \mapsto T_n(x) - \langle x, x_0\rangle\nabla\varepsilon(x) - \varepsilon(x)x_0$ is a strict set contraction [31, Theorems 6.1.8 and 6.1.9]. Hence $\deg(\nabla\psi_n, U_n, 0)$, where $U_n := \mathrm{int}(W) \cap E_n$, is well-defined. Since $\psi_n = \Phi_n$ on ∂U_n,

$$\deg(\nabla\Phi_n, U_n, 0) = \deg(\nabla\psi_n, U_n, 0). \tag{6.9}$$

We will apply Theorem 3.1 to ψ_n, (W_n, W_n^-) and $\{p_1, ..., p_s\}$. Since each p_j is nondegenerate,

$$\mathrm{ind}(\nabla\psi_n, p_j) = (-1)^{k_j} = \sum_{q=0}^{\infty}(-1)^q \dim c^q(\psi_n, p_j),$$

where k_j is the Morse index of $\psi_n''(p_j)$. For the first equality, see [31, Theorem 8.1.1 and Remark on p. 122]. The second equality is an easy consequence of Theorem 5.3 (recall that c^q are the critical groups with respect to the trivial filtration of E_n). By the additivity property of degree,

$$\begin{aligned}\deg(\nabla\psi_n, U_n, 0) = \sum_{j=1}^{s}\mathrm{ind}(\nabla\psi_n, p_j) &= \sum_{q=0}^{\infty}(-1)^q \sum_{j=1}^{s}\dim c^q(\psi_n, p_j)\\ &= M(-1, W_n, W_n^-)\end{aligned}$$

(M is the Morse polynomial). By Theorem 3.1, $M(-1, W_n, W_n^-) = P(-1, W_n, W_n^-) \equiv \chi(W_n, W_n^-)$, and (6.8) follows from (6.9).

(ii) This is a direct consequence of (6.7) and (i).

(iii) Let (W_j, W_j^-) be an admissible pair for Φ and p_j. We may assume that $W_j \subset W$ for all j and $W_i \cap W_j = \emptyset$ if $i \neq j$. By the definition of $M_{\mathcal{E}}$,

$$M_{\mathcal{E}}(t, W, W^-) = \sum_{j=1}^{k} M_{\mathcal{E}}(t, W_j, W_j^-).$$

Hence, according to (ii) and Theorem 3.1,

$$\sum_{j=1}^{k}\mathrm{Ind}_{\mathcal{E}}(\nabla\Phi, p_j) = M_{\mathcal{E}}(-1, W, W^-) = P_{\mathcal{E}}(-1, W, W^-) \equiv \chi_{\mathcal{E}}(W, W^-).$$

Alternatively, one can show (cf. the proof of Theorem 3.1) that (W, W^-) is strongly $\mathcal{E}$-finite if all p_j are, and then the conclusion follows from (ii) and the additivity property of degree. $\square$

Remark 6.2. If all E_n are finite dimensional, then (6.6) is trivially satisfied and it suffices to assume that $\Phi \in C^1(E, \mathbf{R})$. Indeed, $\Phi_n|_{W_n}$ can be approximated in the C^1-topology by a C^2-function $\widetilde{\Phi}_n$ such that the degree on U_n is unchanged and (W_n, W_n^-) is admissible for $\widetilde{\Phi}_n$ and $K(\widetilde{\Phi}_n) \cap W_n$.

Remark 6.3. If (W, W^-) is $\mathcal{E}$-finite, but not strongly, then (ii) and (iii) of Theorem 6.1 need not hold. To see this, let Φ and p be as in Theorem 5.3, with $m_n := M^-(P_n L|_{E_n}) < \infty$ and $M^-_{\mathcal{E}}(L) = +\infty$ or $-\infty$. If (W, W^-) is an admissible pair for Φ and p, then $H^q_{\mathcal{E}}(W, W^-) = [0]$ for all q. So $\chi_{\mathcal{E}}(W, W^-) = [0]$, while

$$\mathrm{Ind}_{\mathcal{E}}(\nabla\Phi, p) = \mathrm{Deg}_{\mathcal{E}}(\nabla\Phi, \mathrm{int}(W), 0) = \left[\left((-1)^{m_n - d_n}\right)_{n=1}^{\infty}\right] \neq [0].$$

7. Hamiltonian systems

Let

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

be the standard symplectic matrix and consider the Hamiltonian system of differential equations

$$\dot{z} = JH_z(z, t), \tag{7.1}$$

where $H \in C^1(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic in t. In this section we study the existence of 2π-periodic solutions of (7.1). Let $E := H^{1/2}(S^1, \mathbf{R}^{2N})$ be the Sobolev space of 2π-periodic $\mathbf{R}^{2N}$-valued functions

$$z(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt, \qquad a_0, a_k, b_k \in \mathbf{R}^{2N},$$

such that $\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty$. Then E is a Hilbert space with inner product $\langle . , . \rangle$ defined by

$$\langle z, z' \rangle := 2\pi a_0 \cdot a_0' + \pi \sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k').$$

Assume that H_z is asymptotically linear at infinity. Then in particular

$$|H_z(z, t)| \le C(1 + |z|)$$

for some $C > 0$ and all (z, t). It is well-known [38] that under this condition $z(t)$ is a 2π-periodic solution of (7.1) if and only if it is a critical point of the functional

$$\Phi(z) := \frac{1}{2}\int_0^{2\pi} (-J\dot{z} \cdot z)\, dt - \int_0^{2\pi} H(z, t)\, dt = \frac{1}{2}\langle \widetilde{L}z, z \rangle - \psi(z). \tag{7.2}$$

Moreover, $\Phi \in C^1(E, \mathbf{R})$ and $\nabla\psi$ is a compact mapping. Sometimes we will make a stronger assumption that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ and

$$\|H_{zz}(z, t)\| \le C'(1 + |z|^s) \tag{7.3}$$

for some $C' > 0$, $s \in (0, \infty)$ and all (z, t). Then $\Phi \in C^2(E, \mathbf{R})$ [38].

Let

$$F_0 := \mathbf{R}^{2N}, \qquad F_k := \{a_k \cos kt + b_k \sin kt : a_k, b_k \in \mathbf{R}^{2N}\}, \quad \text{where } k \ge 1,$$

and

$$E_n := F_0 \oplus \bigoplus_{k=1}^{n} F_k \equiv \{z \in E : z(t) = a_0 + \sum_{k=1}^{n} a_k \cos kt + b_k \sin kt\}.$$

Then $(E_n)_{n=1}^{\infty}$ is a filtration of E. Set

$$d_n := N(1+2n) \quad \text{and} \quad \mathcal{E} := \{E_n, d_n\}_{n=1}^{\infty}.$$

Consider the linear Hamiltonian system

$$\dot{z} = JAz,$$

where A is a symmetric $2N \times 2N$ matrix (with constant entries a_{ij}). Let $B : E \to E$ be the linear operator defined by

$$\langle Bz, w \rangle := \int_0^{2\pi} Az \cdot w \, dt.$$

Then B is compact, and it is easy to see that

$$Bz = Aa_0 + \sum_{k=1}^{\infty} \frac{1}{k}(Aa_k \cos kt + Ab_k \sin kt). \tag{7.4}$$

Following [26], cf. also [1, 2], we now proceed to define the index and the nullity of A. According to (7.2) and (7.4),

$$(\widetilde{L} - B)(a \cos kt + b \sin kt) = (-Jb - \frac{1}{k}Aa) \cos kt + (Ja - \frac{1}{k}Ab) \sin kt.$$

Hence $\widetilde{L} - B$ maps F_k into itself and $(\widetilde{L} - B)|_{F_k}$, $k \geq 1$, corresponds to a linear operator on $\mathbf{R}^{4N}$ given by the matrix

$$T_k(A) := \begin{pmatrix} -\frac{1}{k}A & -J \\ J & -\frac{1}{k}A \end{pmatrix}.$$

Since T_k is symmetric, it has only real eigenvalues. For k large enough, $M^-(T_k(A)) = M^+(T_k(A)) = 2N$ and $M^0(T_k(A)) = 0$, cf. [2, Section 2]. As usual, M^- and M^0 are the Morse index and the nullity of a corresponding quadratic form, and $M^+(T_k(A)) := M^-(-T_k(A))$. Hence the numbers

$$i^-(A) := M^-(-A) - N + \sum_{k=1}^{\infty}(M^-(T_k(A)) - 2N),$$
$$i^+(A) := M^+(-A) - N + \sum_{k=1}^{\infty}(M^+(T_k(A)) - 2N),$$

and

$$i^0(A) := M^0(-A) + \sum_{k=1}^{\infty} M^0(T_k(A))$$

are well-defined and finite. Moreover, $i^-(A) + i^+(A) + i^0(A) = 0$. Note that the numbers $i^{\pm}(A)$ differ by N from those introduced in [26].

Let $L := \widetilde{L} - B$ and $M_{\mathcal{E}}^+(L) := M_{\mathcal{E}}^-(-L)$.

Proposition 7.1. *$i^0(A) = M^0(L)$, $i^-(A) = M_{\mathcal{E}}^-(L) = \widetilde{M}_{\mathcal{E}}^-(L)$ and $i^+(A) = M_{\mathcal{E}}^+(L) = \widetilde{M}_{\mathcal{E}}^+(L)$.*

Proof. Clearly, $M^0(L) \equiv \dim N(L) = i^0(A)$. Since L maps each F_k, $k \geq 0$, into itself, $N(L) \subset E_n$ for some n and $P_n L|_{E_n} = L|_{E_n}$. Therefore $M^-_{\mathcal{E}}(L) = \widetilde{M}^-_{\mathcal{E}}(L)$ according to Remark 5.1 and

$$M^-(P_n L|_{E_n}) - d_n = M^-(-A) - N + \sum_{k=1}^{n}(M^-(T_k(A)) - 2N).$$

Invoking (5.2) we see that $i^-(A) = M^-_{\mathcal{E}}(L)$. Similarly, $i^+(A) = M^+_{\mathcal{E}}(L)$. □

Suppose now that $A(t)$ is a symmetric $2N \times 2N$-matrix with continuous 2π-periodic entries $a_{ij}(t)$. Then $i^-(A)$ and $i^+(A)$ are no longer defined. Since the operator B given by the formula

$$\langle Bz, w\rangle := \int_0^{2\pi} A(t)z \cdot w\, dt$$

is compact, it follows from Proposition 5.2 that $L := \widetilde{L} - B$ is A-proper and $M^-_{\mathcal{E}}(L)$ is well-defined and finite. Furthermore, $M^0(L)$ is the number of linearly independent 2π-periodic solutions of the linear system

$$\dot{z} = JA(t)z,$$

and therefore $0 \leq M^0(L) \leq 2N$. Denote

$$j^-(A) := M^-_{\mathcal{E}}(L), \quad j^+(A) := M^+_{\mathcal{E}}(L) \quad \text{and} \quad j^0(A) := M^0(L).$$

Since $M^-(Q_n L|_{R(L)\cap E_n}) + M^+(Q_n L|_{R(L)\cap E_n}) + M^0(L) = \dim E_n = 2d_n$ for almost all n (cf. Lemma 4.2 and the beginning of Section 5), $j^-(A) + j^+(A) + j^0(A) = 0$. Although we will only be concerned with the numbers $j^{\pm}(A)$ and $j^0(A)$, the following remark is in order:

Remark 7.2. To each matrix $A(t)$ as above there corresponds a unique solution $\gamma(t)$ of the initial value problem $\dot{\gamma} = JA(t)\gamma$, $\gamma(0) = I$ (the fundamental solution), and $\gamma(t)$ is a path in the space of symplectic matrices. If $j^0(A) = 0$, it is possible to introduce an equivalence relation for these paths and show that there exists a constant matrix A_1 such that the corresponding fundamental solution $\gamma_1(t)$ is equivalent to $\gamma(t)$. Now one can define the Maslov index of γ by setting $j(\gamma) := i^-(A_1)$. See [13, 33] for more details. To be more precise, the definition of $j(\gamma)$ in [13, 33] differs from ours. However, it follows from [33, Theorems 2.1, 3.1], cf. also [10, Theorems IV.1.1, IV.1.2] and [13, Theorem 1 and Lemma 2.4], that $j^-(A) = j(\gamma) = j(\gamma_1) = j^-(A_1)$. Since $i^-(A_1) = j^-(A_1)$ according to Proposition 7.1, the two definitions of $j(\gamma)$ are equivalent. If $j^0(A) \neq 0$, one can still define a Maslov-type index $(j(\gamma), n(\gamma))$ as has been shown by Long [32]. Moreover, $j(\gamma) = j^-(A)$ and $n(\gamma) = j^0(A)$ [32, Theorem 6].

In what follows we assume that there exist two symmetric $2N \times 2N$ matrices $A(t)$ and $A_0(t)$ with 2π-periodic entries such that

$$H(z,t) = \frac{1}{2}A(t)z \cdot z + G(z,t), \quad \text{where } G_z(z,t) = o(|z|) \text{ uniformly in } t \text{ as } |z| \to \infty \tag{7.5}$$

and

$$H(z,t) = \frac{1}{2}A_0(t)z \cdot z + G_0(z,t), \quad \text{where } (G_0)_z(z,t) = o(|z|) \text{ uniformly in } t \text{ as } |z| \to 0. \tag{7.6}$$

We will use the notation

(7.7)
$$\Phi(z) = \frac{1}{2}\int_0^{2\pi} (-J\dot z - A(t)z)\cdot z\,dt - \int_0^{2\pi} G(z,t)\,dt =: \frac{1}{2}\langle Lz, z\rangle - \varphi(z)$$
$$= \frac{1}{2}\int_0^{2\pi} (-J\dot z - A_0(t)z)\cdot z\,dt - \int_0^{2\pi} G_0(z,t)\,dt =: \frac{1}{2}\langle L_0 z, z\rangle - \varphi_0(z).$$

It is well-known (cf. e.g. [26] or [38]) that $\nabla\varphi(z) = o(\|z\|)$ as $\|z\| \to \infty$ and $\nabla\varphi_0(z) = o(\|z\|)$ as $z \to 0$. Indeed, for each $\varepsilon > 0$ there is a $C(\varepsilon)$ such that $|G_z(z,t)| \le \varepsilon|z| + C(\varepsilon)$. Hence

$$|\langle\nabla\varphi(z), y\rangle| \le \int_0^{2\pi} (\varepsilon|z| + C(\varepsilon))|y|\,dt \le (\varepsilon\|z\| + C'(\varepsilon))\|y\| \tag{7.8}$$

for all $y \in E$. Taking the supremum over $\|y\| \le 1$, dividing by $\|z\|$ and letting $\|z\| \to \infty$, we see that $\nabla\varphi(z) = o(\|z\|)$. Similarly, for each $\varepsilon > 0$ there is a $C(\varepsilon)$ such that $|(G_0)_z(z,t)| \le \varepsilon|z| + C(\varepsilon)|z|^2$. Hence

$$|\langle\nabla\varphi_0(z), y\rangle| \le (\varepsilon\|z\| + C'(\varepsilon)\|z\|^2)\|y\|$$

and $\nabla\varphi_0(z) = o(\|z\|)$ as $z \to 0$.

Lemma 7.3. *Suppose that H satisfies (7.5). Then Φ satisfies (PS)* (with respect to any filtration) in each of the following two cases:*

(i) $j^0(A) = 0$;

(ii) G_z is bounded and $G(z,t) \to \infty$ (or $G(z,t) \to -\infty$) uniformly in t as $|z| \to \infty$.

Moreover, under these hypotheses $\Phi|_{E_n}$ satisfies (PS) for each n.

Proof. (i) Let (z_j) be a (PS)*-sequence. Then $P_{n_j}\nabla\Phi(z_j) = P_{n_j}Lz_j - P_{n_j}\nabla\varphi(z_j) \to 0$. Since $\nabla\varphi(z) = o(\|z\|)$ as $\|z\| \to \infty$ and $\|P_{n_j}Lz_j\| \ge c\|z_j\|$ according to Theorem 4.5, (z_j) is bounded and it follows from the compactness of $\nabla\varphi$ that (z_j) has a convergent subsequence.

(ii) Assume $G(z,t) \to \infty$ (the other case is similar). Let (z_j) be a (PS)*-sequence and let $z_j = y_j + w_j \in R(L) \oplus N(L)$. Since $P_{n_j}Ly_j - P_{n_j}\nabla\varphi(z_j) \to 0$ and $\|P_{n_j}Ly_j\| \ge c\|y_j\|$, the sequence (y_j) is bounded. Hence $\varphi(z_j)$ is bounded (because $\Phi(z_j)$ is). By the mean value theorem,

$$|\varphi(y_j + w_j) - \varphi(w_j)| \le \sup_{z\in E}\|\nabla\varphi(z)\|\|y_j\|.$$

So $\varphi(w_j)$ is bounded as well. On the other hand, $\varphi(w_j) = \int_0^{2\pi} G(w_j(t), t)\,dt \to \infty$ if $\|w_j\| \to \infty$ (recall that $N(L)$ is finite dimensional). Hence (w_j) has a convergent subsequence, and the same is true for (y_j) because $\nabla\varphi$ is compact.

Since E_n is finite dimensional, it is clear that $\Phi|_{E_n}$ satisfies (PS). □

Theorem 7.4. *Suppose that H satisfies (7.5) and (7.6). If $j^0(A) = j^0(A_0) = 0$ and $j^-(A) \ne j^-(A_0)$, then (7.1) has a nontrivial 2π-periodic solution.*

Proof. It follows immediately from (7.6) that (7.1) has the trivial solution $z = 0$. Let $\Phi_\lambda(z) := \frac{1}{2}\langle Lz, z\rangle - (1-\lambda)\varphi(z)$, $0 \le \lambda \le 1$. It follows from Lemma 7.3 that all Φ_λ satisfy (PS)*. Since L is invertible and $\nabla\varphi(z) = o(\|z\|)$ as $\|z\| \to \infty$, there is a bounded set N such that $K(\Phi_\lambda) \subset N$ and $\sup_N|\Phi_\lambda| \le C$ for some $C > 0$ and all $\lambda \in [0,1]$. So according to Propositions 2.12 and 2.14, the critical groups $c^*_{\mathcal{E}}(\Phi, K(\Phi))$ are well-defined and $c^*_{\mathcal{E}}(\Phi, K(\Phi)) = c^*_{\mathcal{E}}(\Phi_1, K(\Phi_1)) = c^*_{\mathcal{E}}(\Phi_1, 0)$.

By Theorem 5.3, $c^q_{\mathcal{E}}(\Phi_1, 0) = [\mathcal{F}]$ if $q = j^-(A)$ and $c^q_{\mathcal{E}}(\Phi_1, 0) = [0]$ otherwise. Since $\nabla\varphi_0(z) = o(\|z\|)$ as $z \to 0$, we obtain—invoking Theorem 5.3 again—that $c^q_{\mathcal{E}}(\Phi, 0) \neq [0]$ if and only if $q = j^-(A_0)$. So $c^*_{\mathcal{E}}(\Phi, K) \neq c^*_{\mathcal{E}}(\Phi, 0)$, and Φ must have a critical point $z \neq 0$. □

Theorem 7.5. *Suppose that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies (7.3), (7.5) and (7.6). If G_z is bounded, then (7.1) has a nontrivial 2π-periodic solution in each of the following two cases:*

(i) $G(z,t) \to -\infty$ uniformly in t as $|z| \to \infty$ and

$$j^-(A) \notin [j^-(A_0),\ j^-(A_0) + j^0(A_0)];$$

(ii) $G(z,t) \to \infty$ uniformly in t as $|z| \to \infty$ and

$$j^+(A) \notin [j^+(A_0),\ j^+(A_0) + j^0(A_0)].$$

Proof. (i) Introduce a new filtration $\mathcal{E}' := \{E'_n, d_n\}_{n=1}^\infty$, where $E'_n := (R(L) \cap E_n) \oplus N(L)$ and $d_n = N(1 + 2n)$ as before. According to Lemma 7.3, Φ satisfies (PS)* with respect to $\mathcal{E}'$. Since $R(L) \cap E'_n = R(L) \cap E_n$, $M^-_{\mathcal{E}'}(L) = M^-_{\mathcal{E}}(L) \equiv j^-(A)$. It is easy to see that L and L_0 (cf. (7.7)) are A-proper with respect to $\mathcal{E}'$ (because they are with respect to $\mathcal{E}$). Furthermore, $E_n = (R(L) \cap E_n) \oplus P_n N(L)$ by Lemma 4.2, $E'_n = (R(L) \cap E_n) \oplus N(L)$ and

$$\|P_n z - z\| \leq \varepsilon_n \|z\| \quad \text{for all } z \in N(L), \tag{7.9}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Let $Q'_{0,n}$ be the orthogonal projector of $R(L_0)$ onto $R(L_0) \cap E'_n$. It follows from Theorem 4.5 and Proposition 4.3 that there is a $c > 0$ such that $\|Q'_{0,n} L_0 z\| \geq c\|z\|$ for almost all n and all $z \in R(L_0) \cap E'_n$. Using this and (7.9) we see that the quadratic form $z \mapsto \langle L_0 z, z\rangle$ is nondegenerate and has the same Morse index on $R(L_0) \cap E_n$ and on $R(L_0) \cap E'_n$, provided n is sufficiently large. So $M^-_{\mathcal{E}'}(L_0) = M^-_{\mathcal{E}}(L_0) \equiv j^-(A_0)$.

In Lemma 7.6 below we will show that if Φ has finitely many critical points, then (7.5), (i) and the boundedness of G_z imply that

$$c^q_{\mathcal{E}'}(\Phi, K) = \begin{cases} [\mathcal{F}] & \text{if } q = j^-(A), \\ [0] & \text{otherwise.} \end{cases} \tag{7.10}$$

On the other hand, if 0 is an isolated critical point of Φ, it follows from Theorem 5.4 that

$$c^q_{\mathcal{E}'}(\Phi, 0) = [c^{q - j^-(A_0)}(\widetilde{\varphi}_0, 0)],$$

where $\widetilde{\varphi}_0$ is defined on a subset of $N(L_0)$. So the right-hand side above can be nonzero only if $0 \leq q - j^-(A_0) \leq j^0(A_0)$. Since $j^-(A) \notin [j^-(A_0),\ j^-(A_0) + j^0(A_0)]$, $c^{j^-(A)}_{\mathcal{E}'}(\Phi, 0) = [0] \neq c^{j^-(A)}_{\mathcal{E}'}(\Phi, K)$. Hence (7.1) must have a nonzero solution.

(ii) This follows by the same argument applied to $-\Phi$. (An alternative proof may be obtained by working with Φ, using (ii) of Lemma 7.6 and the fact that $j^-(A) + j^+(A) + j^0(A) = 0$.) □

Observe that if the matrix A is t-independent, then $N(L) \subset E_{n_0}$ for some n_0 and $E_n = E'_n$ for almost all n. So in this case we can use the filtration $\mathcal{E}$.

Lemma 7.6. *Suppose that $\Phi \in C^1(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies (7.5), G_z is bounded and the critical set $K = K(\Phi)$ is finite.*

(i) If $G(z,t) \to -\infty$ uniformly in t as $|z| \to \infty$, then $c^*_{\mathcal{E}'}(\Phi, K)$ is given by (7.10).

(ii) If $G(z,t) \to \infty$ uniformly in t as $|z| \to \infty$, then

$$c^q_{\mathcal{E}'}(\Phi, K) = \begin{cases} [\mathcal{F}] & \text{if } q = j^-(A) + j^0(A), \\ [0] & \text{otherwise.} \end{cases}$$

Proof. (i) Let Q_n be the orthogonal projector of $R(L)$ onto $R(L) \cap E_n$. Find $c > 0$ and n_0 such that $\|Q_n L z\| \geq c\|z\|$ for all $z \in R(L) \cap E_n$ and $n \geq n_0$. Let $E'_n \equiv (R(L) \cap E) \oplus N(L) = E^+_n \oplus E^-_n \oplus N(L)$ be the decomposition (into L-invariant subspaces) corresponding to the positive, the negative and the zero part of the quadratic form $z \mapsto \langle Lz, z\rangle$ on E'_n. If $z = z^+ + z^- + z^0 \in E^+_n \oplus E^-_n \oplus N(L)$, then $\langle Lz^+, z^+\rangle \geq c\|z^+\|^2$ and $\langle Lz^-, z^-\rangle \leq -c\|z^-\|^2$. Therefore

$$\langle \nabla\Phi(z), z^-\rangle = \langle Lz^-, z^-\rangle - \langle \nabla\varphi(z), z^-\rangle \leq -c\|z^-\|^2 + C_0\|z^-\|,$$

where $C_0 := \sup_E \|\nabla\varphi(z)\|$. Hence there is an $R > 0$ such that setting

$$U := \{z \in E'_n : \|z^-\| < R\},$$

we obtain

$$\langle \nabla\Phi(z), z^-\rangle < 0 \quad \text{for all } z \in E'_n - U \text{ and } n \geq n_0. \tag{7.11}$$

In particular, $\Phi|_{E'_n}$ has no critical points in $E'_n - U$. For $z \in \overline{U}$ we have

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\langle Lz^+, z^+\rangle + \frac{1}{2}\langle Lz^-, z^-\rangle - \varphi(z) \\ &\geq \frac{1}{2}c\|z^+\|^2 - \frac{1}{2}\|L\|R^2 - \varphi(z^0) - (\varphi(z) - \varphi(z^0)) \\ &\geq \frac{1}{2}c\|z^+\|^2 - \frac{1}{2}\|L\|R^2 - \varphi(z^0) - C_0(R + \|z^+\|). \end{aligned} \tag{7.12}$$

Since $\varphi(z^0) \to -\infty$ as $\|z^0\| \to \infty$, it follows that

$$\Phi(z) \to \infty \quad \text{as } \|z^+ + z^0\| \to \infty,$$

and the convergence is uniform with respect to the choice of $n \geq n_0$ and $z^- \in E^-_n \cap \overline{B}(0,R)$. Hence we can find $0 < a < b$ and $R_0 > 0$ such that $K \subset \{z \in E : |\Phi(z)| < a\}$, and for each $n \geq n_0$,

$$\Phi^{-a} \cap E'_n \subset E'_n - U \quad \text{and} \quad \Phi^a \cap \overline{U} \subset D := \{z \in \overline{U} : \|z^+ + z^0\| \leq R_0\} \subset \Phi^b \cap \overline{U}.$$

It is easy to see that there exists a strong deformation retraction γ of E'_n onto $D \cup \partial U$. Using (7.11) we may construct a pseudogradient vector field V for Φ on E'_n in such a way that $\langle V(z), z^-\rangle < 0$ whenever $\|z^-\| \geq R$. It follows from (PS)* that if n is large enough, then $\Phi|_{E'_n}$ has no critical values in $[a,b]$. So $K(\Phi|_{E'_n}) \subset U - \Phi^{-1}([a,b])$. Since $\Phi|_{E'_n}$ satisfies (PS), the flow of $-V$ induces a strong deformation retraction η of $(E'_n - U) \cup D$ onto $\Phi^a \cap E'_n$. Indeed, if $z \in E'_n - U$, then $\eta(\lambda, z) \in E'_n - U$ (because $\langle V(z), z^-\rangle < 0$) and if $z \in D$, then $\eta(\lambda, z) \in \Phi^b \cap U$ (because $D \subset \Phi^b \cap \overline{U}$). So η may be constructed in such a way that it is a deformation onto $\Phi^a \cap E'_n$. Now we see that the mapping $\eta * \gamma$ given by

$$(\eta * \gamma)(\lambda, z) := \begin{cases} \gamma(2\lambda, z) & \text{for } 0 \leq \lambda \leq \frac{1}{2}, \\ \eta(2\lambda - 1, \gamma(1, z)) & \text{for } \frac{1}{2} \leq \lambda \leq 1, \end{cases}$$

is a strong deformation retraction of E'_n onto $\Phi^a \cap E'_n$. Since $\Phi^{-a} \cap E'_n \subset E'_n - U$ and $\langle V(z), z^-\rangle < 0$ as $\|z^-\| \geq R$, one can use the flow of $-V$ again in order to construct a strong deformation retraction of $E'_n - U$ onto $\Phi^{-a} \cap E'_n$. Therefore

$$H^q(\Phi^a \cap E'_n, \Phi^{-a} \cap E'_n) \cong H^q(E'_n, E'_n - U) = \begin{cases} \mathcal{F} & \text{if } q = j^-(A) + d_n, \\ 0 & \text{otherwise,} \end{cases}$$

provided n is large enough. Since $H^*_{\mathcal{E}'}(\Phi^a, \Phi^{-a}) \cong H^*_{\mathcal{E}'}(\Phi^{-1}([-a,a]), \Phi^{-1}(-a))$ (by excision) and the second pair is admissible for Φ and K, the conclusion follows.

(ii) Here we have

$$\langle \nabla\Phi(z), z^+ \rangle \geq c\|z^+\|^2 - C_0\|z^+\|,$$

and we may choose R such that if

$$M := \{z \in E'_n : \|z^+\| \leq R\},$$

then

$$\langle \nabla\Phi(z), z^+ \rangle > 0 \quad \text{for all } z \in E'_n - \operatorname{int}(M) \text{ and } n \geq n_0.$$

As in (7.12), we obtain

$$\Phi(z) \leq \frac{1}{2}\|L\|R^2 - \frac{1}{2}c\|z^-\|^2 - \varphi(z_0) + C_0(R + \|z^-\|)$$

for all $z \in M$. Therefore

$$\Phi(z) \to -\infty \quad \text{ as } \|z^- + z^0\| \to \infty,$$

and the convergence is uniform with respect to the choice of $n \geq n_0$ and $z^+ \in E_n^+ \cap \overline{B}(0,R)$. Hence we can find $0 < a < b$ and $0 < R_2 < R_1$ such that $K \subset \{z \in E : |\Phi(z)| < a\}$, $M \subset \Phi^a \cap E'_n$ and

$$\begin{aligned} D_1 := \{z \in M : \|z^- + z^0\| \geq R_1\} &\subset \Phi^{-b} \cap M \\ \subset D_2 := \{z \in M : \|z^- + z^0\| \geq R_2\} &\subset \Phi^{-a} \cap M. \end{aligned}$$

Obviously, there exists a strong deformation retraction γ of D_2 onto D_1. By (PS)*, we may assume that $K(\Phi|_{E'_n}) \subset M - \Phi^{-1}([-b,-a])$. Using the flow of $-V$, where V is a pseudogradient vector field on E'_n satisfying $\langle V(z), z^+ \rangle > 0$ for $\|z^+\| \geq R$, it is easy to construct a strong deformation retraction η of $\Phi^{-a} \cap M$ onto $\Phi^{-b} \cap M$. Hence $\gamma * \eta$ is a strong deformation retraction of $\Phi^{-a} \cap M$ onto D_1. Using the flow of $-V$ once again, we also obtain a strong deformation retraction of $\Phi^a \cap E'_n$ onto $(\Phi^{-a} \cap E'_n) \cup M$. Hence

$$\begin{aligned} H^*(\Phi^a \cap E'_n, \Phi^{-a} \cap E'_n) &\cong H^*((\Phi^{-a} \cap E'_n) \cup M, \Phi^{-a} \cap E'_n) \\ &\overset{exc}{\cong} H^*(M, \Phi^{-a} \cap M) \cong H^*(M, D_1). \end{aligned}$$

Since for all n large enough,

$$H^q(M, D_1) = \begin{cases} \mathcal{F} & \text{if } q = j^-(A) + j^0(A) + d_n, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the conclusion. □

It is clear that if $j^0(A_0) = 0$, then it suffices to assume that $H \in C^1$ in Theorem 7.5, and if $j^0(A) = 0$, G_z need not be bounded and G need not tend to infinity.

Suppose now that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies (7.5) and z_0 is a 2π-periodic solution of (7.1). Then z_0 is continuous (and therefore bounded). Hence for each $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that

$$|G_z(z_0(t) + w, t) - G_z(z_0(t), t) - G_{zz}(z_0(t), t)w| \leq \varepsilon|w| + C(\varepsilon)|w|^2$$

for all $t \in \mathbf{R}$ and $w \in \mathbf{R}^{2N}$. So

$$|\langle \nabla\varphi(z_0+v) - \nabla\varphi(z_0), y\rangle - \int_0^{2\pi} G_{zz}(z_0,t)v\cdot y\,dt| \le (\varepsilon\|v\| + C'(\varepsilon)\|v\|^2)\|y\|$$

for all $v, y \in E$. Dividing by $\|v\|$ and letting $v \to 0$, we see that $\nabla\varphi$ is Fréchet differentiable at z_0 and

$$\Phi(z_0+v) = \Phi(z_0) + \frac{1}{2}\langle \Phi''(z_0)v, v\rangle + \psi(z_0+v),$$

where $\nabla\psi(z_0+v) = o(\|v\|)$ as $v \to 0$. Note that we made no assumption that H satisfies (7.3), and therefore Φ may not be of class C^2. We will call the solution z_0 *nondegenerate* if $\Phi''(z_0)$ is invertible.

Remark 7.7. Suppose that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies the hypotheses of Theorem 7.4. If the nontrivial solution z_0 of (7.1) is nondegenerate, then (7.1) has a second nontrivial solution. Indeed, suppose 0 and z_0 are the only solutions. According to Theorem 5.3, their contribution to the Morse polynomial is respectively $t^{j^-(A_0)}$ and t^{q_0}, where $q_0 \in \mathbf{Z}$. So the Morse inequalities give

$$t^{j^-(A_0)} + t^{q_0} = t^{j^-(A)} + (1+t)Q(t),$$

a contradiction upon setting $t = 1$. In the framework of Theorem 7.5 the above conclusion remains valid if $j^0(A_0) = 0$ (note that (7.3) is not needed here).

The nondegeneracy condition for z_0 in Remark 7.7 is in general not easy to verify. However, if the difference between the indices $j^-(A)$ and $j^-(A_0)$ is sufficiently large, this condition can be avoided.

Theorem 7.8. *Suppose that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies (7.3), (7.5), (7.6) and $j^0(A_0) = 0$. Then (7.1) has at least two nontrivial 2π-periodic solutions in each of the following cases:*

(i) $|j^-(A) - j^-(A_0)| \ge 2N$ and $j^0(A) = 0$;

(ii) $|j^-(A) - j^-(A_0)| \ge 2N$, G_z is bounded and $G(z,t) \to -\infty$ uniformly in t as $|z| \to \infty$;

(iii) $|j^+(A) - j^+(A_0)| \ge 2N$, G_z is bounded and $G(z,t) \to \infty$ uniformly in t as $|z| \to \infty$.

Proof. Assume that (i) or (ii) is satisfied. Let z_0 be the nontrivial solution we already know exists, and suppose there are no other ones. According to Theorem 5.4,

$$c^q_{\mathcal{E}'}(\Phi, z_0) = [c^{q-r_0}(\widetilde{\psi}_0, 0)]$$

for some $r_0 \in \mathbf{Z}$ and some functional $\widetilde{\psi}_0$ defined on a space Z of dimension $\le 2N$. So $c^q_{\mathcal{E}'}(\Phi, z_0)$ can be nonzero only for $0 \le q - r_0 \le \dim Z \le 2N$. Moreover, if $\widetilde{\psi}_0$ has a local minimum at 0, then $c^{q-r_0}(\widetilde{\psi}_0, 0) \ne 0$ if and only if $q - r_0 = 0$; if it has a local maximum, then $c^{q-r_0}(\widetilde{\psi}_0, 0) \ne 0$ if and only if $q - r_0 = \dim Z$; and in other cases $c^0(\widetilde{\psi}_0, 0) = c^{\dim Z}(\widetilde{\psi}_0, 0) = 0$ (see [15] or [34, Theorem 8.6 and Corollary 8.4]). Consequently, the Morse inequalities give

$$t^{j^-(A_0)} + \sum_{i=0}^{2N-2} b_i t^{\alpha+i} = t^{j^-(A)} + (1+t)Q(t),$$

where $b_i \in [\mathbf{Z}]$ and some (or all) b_i may be zero and $\alpha \in \mathbf{Z}$. Since there is an exponent $j^-(A)$ on the right-hand side above, $\alpha + i = j^-(A)$ for some i. The left-hand side contains the exponent $j^-(A_0)$. Therefore $Q(t)$ must have a nonzero term with exponent $j^-(A_0)$ or $j^-(A_0)-1$, and it follows that there is a nonzero term with exponent $j^-(A_0)-1$ or $j^-(A_0)+1$ on the left-hand side. Hence there exists j, $0 \le j \le 2N-2$, such that $\alpha + j = j^-(A_0)+1$ or $j^-(A_0)-1$. So $|j^-(A) - j^-(A_0)| = |i - j \pm 1| \le 2N-1$, a contradiction.

Finally, if (iii) is satisfied, the conclusion is obtained by applying the same argument to $-\Phi$. □

Corollary 7.9. *Suppose that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ satisfies (7.3), H_z is bounded and $H_z(z,t) = o(z)$ uniformly in t as $z \to 0$. Then (7.1) has at least two nontrivial 2π-periodic solutions in each of the following two cases:*

(i) $H(z,t) \to -\infty$ uniformly in t as $|z| \to \infty$ and there is a $\delta > 0$ such that $H(z,t) \ge 0$ whenever $|z| \le \delta$;

(ii) $H(z,t) \to \infty$ uniformly in t as $|z| \to \infty$ and there is a $\delta > 0$ such that $H(z,t) \le 0$ whenever $|z| \le \delta$.

Proof. (i) Note that H satisfies (7.5) and (7.6) with $A = A_0 \equiv 0$. Suppose that z_0 is the only nontrivial solution of (7.1). Since $j^-(0) = i^-(0) = -N$, it follows from (7.10) (with $\mathcal{E}' = \mathcal{E}$) that $c^q_{\mathcal{E}}(\Phi, K) = [\mathcal{F}]$ if $q = -N$ and [0] otherwise. Furthermore, $j^0(0) = 2N$, so according to Corollary 5.5, $c^q_{\mathcal{E}}(\Phi, 0) = [\mathcal{F}]$ if $q = N$ and [0] otherwise (observe that φ in Corollary 5.5 corresponds to $-\psi$ here, cf. (7.2)). Hence we obtain from the Morse inequalities that

$$t^N + \sum_{i=0}^{2N-2} b_i t^{\alpha+i} = t^{-N} + (1+t)Q(t),$$

which leads to a contradiction in the same way as in the proof of Theorem 7.8.

(ii) The argument is similar except that now $c^N_{\mathcal{E}}(\Phi, K) = [\mathcal{F}]$ and $c^{-N}_{\mathcal{E}}(\Phi, 0) = [\mathcal{F}]$ according to (ii) of Lemma 7.6 and Corollary 5.5. □

If G_0 has constant sign for z in a neighbourhood of the origin in $\mathbf{R}^{2N}$, a better result than Theorem 7.5 can be obtained.

Theorem 7.10. *Suppose that H satisfies (7.5) and (7.6) with A and A_0 independent of t and either $j^0(A) = 0$ or G_z is bounded and $G(z,t) \to -\infty$ uniformly in t as $|z| \to \infty$. Then (7.1) has a nontrivial 2π-periodic solution in each of the following two cases:*

(i) $j^-(A) \ne j^-(A_0) + j^0(A_0)$ and there is a $\delta > 0$ such that $G_0(z,t) \ge 0$ whenever $|z| \le \delta$;

(ii) $j^-(A) \ne j^-(A_0)$ and there is a $\delta > 0$ such that $G_0(z,t) \le 0$ whenever $|z| \le \delta$.

If A, A_0 are t-dependent, the same conclusion remains valid provided $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ and satisfies (7.3).

Proof. (i) Suppose that 0 is the only critical point of Φ and let $E = Y \oplus Z$ be the decomposition corresponding to the nonpositive and the positive part of the spectrum of L_0. It has been shown in [26], cf. also [28, 29], that Φ satisfies the local linking condition at 0 with Y and Z as above. Moreover, since A_0 is independent of t, $L_0(F_k) \subset F_k$ for each k and therefore $E_n = (Y \cap E_n) \oplus (Z \cap E_n)$. Also, since $N(L_0) \subset E_n$, it is easy to see that $\dim(Y \cap E_n) = j^-(A_0) + j^0(A_0) + d_n$ for almost all n. Hence in view of Theorem 5.6, $c^{j^-(A_0)+j^0(A_0)}_{\mathcal{E}}(\Phi, 0) \ne [0]$. On the other hand, $c^q_{\mathcal{E}}(\Phi, K) \ne [0]$ if and only if $q = j^-(A)$ (this follows from the proof of Theorem 7.4

if $j^0(A) = 0$ and from (7.10) otherwise). Since $j^-(A) \neq j^-(A_0) + j^0(A_0)$, Φ must have a nontrivial critical point.

If A, A_0 are t-dependent and H satisfies (7.3), we use the filtration $\mathcal{E}'$ and obtain from Corollary 5.5 that $c_{\mathcal{E}'}^{j^-(A_0)+j^0(A_0)}(\Phi, 0) = [\mathcal{F}]$ (φ in corollary 5.5 corresponds to $-\varphi_0$ here). This contradicts the fact that $c_{\mathcal{E}'}^q(\Phi, 0) = c_{\mathcal{E}'}^q(\Phi, K) \neq [0]$ if and only if $q = j^-(A)$.

(ii) The proof is the same except that now Y corresponds to the negative and Z to the nonnegative part of the spectrum of L_0. So $\dim(Y \cap E_n) = j^-(A_0) + d_n$ for almost all n. □

If $G(z, t) \to \infty$ as $|z| \to \infty$, a similar result can be formulated in terms of $j^+(A)$ and $j^+(A_0)$.

Suppose now that

$$\begin{aligned} H(z,t) &= \frac{1}{2}A(t)p \cdot p + G(z,t), \\ &\text{where } G_z(z,t) = o(|z|) \text{ uniformly in } t \text{ as } |z| \to \infty, \end{aligned} \tag{7.13}$$

$z = (p, q) \in \mathbf{R}^N \times \mathbf{R}^N$, $A(t)$ is a symmetric $N \times N$ matrix with 2π-periodic entries and G is 2π-periodic in q and t. If z is a 2π-periodic solution of (7.1), so are all $\widetilde{z} = (p, \widetilde{q})$ with $\widetilde{q} \equiv q \pmod{2\pi}$. Hence to each solution z there corresponds an orbit $\mathcal{O}(z) := \{z + (0, 2\pi\mathbf{Z}^N)\}$. Two solutions z_1, z_2 are called *geometrically distinct* if $\mathcal{O}(z_1) \cap \mathcal{O}(z_2) = \emptyset$. Let $E = \widetilde{E} \oplus N$, where $N := \{(p, q) : p = 0,\ q \in \mathbf{R}^N\}$ and $\widetilde{E} = N^\perp$. So N is the subspace of constant functions (p, q) such that $p = 0$ and $\widetilde{E}$ is the subspace of functions in E whose q-coordinates have mean value zero. Let $z = x + v \in \widetilde{E} \oplus N$ and define (cf. (7.7))

$$\Phi(x, v) := \frac{1}{2}\int_0^{2\pi} (-J\dot{x} \cdot x - A(t)p \cdot p)\, dt - \int_0^{2\pi} G(z, t)\, dt = \frac{1}{2}\langle Lx, x\rangle - \varphi(x, v).$$

Then $\Phi : \widetilde{E} \times N \to \mathbf{R}$. Since $\Phi(x, v) = \Phi(x, \widetilde{v})$ if $\widetilde{v} \equiv v \pmod{2\pi}$, v may be regarded as an element of the torus $T^N := \mathbf{R}^N/2\pi\mathbf{Z}^N$ and Φ maps $M := E \times T^N$ into $\mathbf{R}$. Moreover, distinct critical points of Φ on M correspond to geometrically distinct 2π-periodic solutions of (7.1).

One sees that $Lx = 0$ if and only if $\dot{p} = 0$ and $\dot{q} = A(t)p$. So $N(L)$ consists of $(p, q) \in \widetilde{E}$ such that $p \in \mathbf{R}^N$, $\dot{q} = A(t)p$ and $A(t)p$ has mean value zero. In what follows we assume for simplicity that L is invertible on $\widetilde{E}$. As in [41], this assumption may be relaxed by requiring that if $N(L)$ is nontrivial, then G_z is bounded and $G(p, q, t) \to \infty$ (or $-\infty$) uniformly in (q, t) as $|p| \to \infty$, $p \in P_1 N(L)$ (P_1 is the projector onto the first component of $x = (p, q)$).

Theorem 7.11. *Suppose that $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic in q, t and satisfies* (7.13). *If L is invertible on $\widetilde{E}$ and all 2π-periodic solutions of (7.1) are nondegenerate, then the number of geometrically distinct ones is at least 2^N.*

Proof. Let $\mathcal{E} := \{M_n, d_n\}$, where $M_n := (\widetilde{E} \cap E_n) \times T^N$ (cf. Remark 2.15). Suppose that Φ has finitely many critical points (otherwise there is nothing to prove). Using Propositions 2.12 and 2.14 in the same way as in the proof of Theorem 7.4, we see that $c_{\mathcal{E}}^*(\Phi, K(\Phi)) = c_{\mathcal{E}}^*(\Phi_1, K(\Phi_1))$, where $\Phi_1(x, v) := \frac{1}{2}\langle Lx, x\rangle$. Moreover, if $(\widetilde{W}, \widetilde{W}^-)$ is a bounded admissible pair for $x \mapsto \frac{1}{2}\langle Lx, x\rangle$ and 0 on $\widetilde{E}$, then $(W, W^-) := (\widetilde{W}, \widetilde{W}^-) \times T^N$ is admissible for Φ_1 and $K(\Phi_1)$. Since L is invertible on

$\widetilde{E}$, there exists a unique q_0 such that for almost all n, $H^{q+d_n}(\widetilde{W}\cap E_n, \widetilde{W}^-\cap E_n) = \mathcal{F}$ if $q = q_0$ and $= 0$ otherwise. So it follows from the Künneth formula [17, Proposition VI.12.16], [39, Theorem 5.6.1] that

$$H^{q+d_n}(W\cap E_n, W^-\cap E_n) = \{H^*(\widetilde{W}\cap E_n, \widetilde{W}^-\cap E_n)\otimes H^*(T^N)\}^{q+d_n} = H^{q-q_0}(T^N).$$

Hence

$$c^*_{\mathcal{E}}(\Phi, K(\Phi)) = H^*_{\mathcal{E}}(W, W^-) = [H^{*-q_0}(T^N)],$$

and since $H^*(T^N) = H^*(S^1)\otimes\cdots\otimes H^*(S^1)$ (N times),

$$\beta^q_{\mathcal{E}}(W, W^-) \equiv \dim_{\mathcal{E}} H^q_{\mathcal{E}}(W, W^-) = \left[\binom{N}{q-q_0}\right] \quad \text{for } q_0 \le q \le q_0 + N$$

and $\beta^q_{\mathcal{E}}(W, W^-) = [0]$ otherwise. We have assumed that all critical points of Φ are nondegenerate. Locally we may identify Φ on $M = \widetilde{E}\times T^N$ with the same functional on E. It follows therefore as in Remark 7.7 that each critical point (x_i, v_i) of Φ on M contributes with a term t^{q_i} to the Morse polynomial. Since all coefficients a_q in the polynomial Q in the Morse inequalities are nonnegative (in the sense that $a_q \in [\mathbf{Z}_+]$), it follows from Theorem 3.1 that

$$M^q_{\mathcal{E}}(\hat{W}, \hat{W}^-) \ge \beta^q_{\mathcal{E}}(\hat{W}, \hat{W}^-),$$

where the relation $\ge$ has the obvious meaning and $(\hat{W}, \hat{W}^-)$ is a (globally) admissible pair for Φ and $K(\Phi)$. So if m is the number of critical points of Φ, then

$$[m] = \sum_{q\in\mathbf{Z}} M^q_{\mathcal{E}}(\hat{W}, \hat{W}^-) \ge \sum_{q\in\mathbf{Z}} \beta^q_{\mathcal{E}}(\hat{W}, \hat{W}^-) = \sum_{q=q_0}^{q_0+N}\left[\binom{N}{q-q_0}\right] = \left[2^N\right]. \quad \square$$

Remark 7.12. (i) Theorems 7.4, 7.5 and 7.10 extend different results contained in [1, 2, 13, 26, 32, 41]. In [1, 2] it was assumed that H_{zz} is bounded, A, A_0 are time-independent and $j^0(A) = 0$, in [13] H_{zz} is bounded and $j^0(A) = j^0(A_0) = 0$, in [26] A, A_0 are time-independent and $j^0(A) = 0$, in [32] H_{zz} is bounded and $j^0(A) = 0$, and in [41] A, A_0 are time-independent. Theorem 7.4, with a different proof, may also be found in [10, p. 186]. Theorem 7.8 extends a result by Bertotti [7], where it was assumed that H_{zz} is bounded and $j^0(A) = 0$ (see also [32]), and Theorem 7.11 is a generalization of a result in [19] and [41] (in [19] H_{zz} is bounded and A time-independent, and in [41] A is time-independent).

The assumption that H_{zz} is bounded—which has been made in some of the work mentioned above—was needed in order to reduce the problem to a finite dimensional one. After this reduction the usual Morse theory was employed.

(ii) If $A = 0$, G is 2π-periodic in all variables and all solutions of (7.1) are nondegenerate, then the number of geometrically distinct ones is at least 2^{2N}. This follows by an easy modification of the proof of Theorem 7.11. Thus we recover the celebrated result by Conley and Zehnder on Arnold's conjecture [12], see also [10].

(iii) If the nondegeneracy assumption in Theorem 7.11 is removed and if $H \in C^1(\mathbf{R}^{2N}\times\mathbf{R}, \mathbf{R})$, then (7.1) has at least $N+1$ geometrically distinct 2π-periodic solutions [18, 30, 40].

8. Wave equation

In this section we are concerned with the existence of nontrivial solutions of the wave equation (0.3) satisfying the boundary and the periodicity conditions (0.4). More precisely, we consider the problem

$$\begin{cases} \Box u := u_{tt} - u_{xx} = f(x,t,u), & 0 < x < \pi, t \in \mathbf{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbf{R}, \\ u(x,t+2\pi) = u(x,t), & 0 < x < \pi, t \in \mathbf{R}, \end{cases} \tag{8.1}$$

with f satisfying the following hypotheses:

$$f \in C([0,\pi] \times \mathbf{R}^2, \mathbf{R}), \text{ and } f(x,t+2\pi,\xi) = f(x,t,\xi) \text{ for all } x,t,\xi, \tag{8.2}$$

$$\text{There exists an } \varepsilon > 0 \text{ such that } (f(x,t,\xi) - f(x,t,\eta))(\xi - \eta) \geq \varepsilon(\xi-\eta)^2 \text{ for all } x,t,\xi,\eta, \tag{8.3}$$

$$f(x,t,\xi) = b\xi + g(x,t,\xi), \quad \text{where } g(x,t,\xi) = o(|\xi|) \text{ uniformly in } (x,t) \text{ as } |\xi| \to \infty, \tag{8.4}$$

$$f(x,t,\xi) = b_0\xi + g_0(x,t,\xi), \text{ where } g_0(x,t,\xi) = o(|\xi|) \text{ uniformly in } (x,t) \text{ as } \xi \to 0. \tag{8.5}$$

It follows from (8.2)–(8.5) that the constants b_0, b are positive and $u = 0$ is a solution of (8.1) (the trivial solution).

Let $\Omega := (0,\pi) \times (0, 2\pi)$ and let E be the space of functions

$$u(x,t) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} \sin jx \, \mathrm{e}^{ikt}, \qquad c_{j,-k} = \overline{c}_{jk}, \tag{8.6}$$

such that $\sum_{j\neq|k|} |j^2 - k^2|\, |c_{jk}|^2 + \sum_{j=|k|} |c_{jk}|^2 < \infty$. Then E is a Hilbert space with inner product

$$\langle u, u' \rangle := \pi^2 \sum_{j \neq |k|} |j^2 - k^2| c_{jk}\, \overline{c}'_{jk} + \pi^2 \sum_{j=|k|} c_{jk}\, \overline{c}'_{jk}.$$

Observe that the basis $\{\sin jx \, \mathrm{e}^{ikt}\}$ for E consists of eigenfunctions of the wave operator $\Box$. In what follows $\| \ \|$ will denote the norm in E and $\| \ \|_p$ the norm in $L^p(\Omega)$, $1 \leq p \leq \infty$.

Let

$$N := \{u \in E : u(x,t) = \sum_{j=|k|} c_{jk} \sin jx \, \mathrm{e}^{ikt}\}$$

and denote the orthogonal complement of N in E by $N^\perp$. Let $u \in E$. Then $\Box u = 0$ (in the sense of distributions) if and only if $u \in N$. So N is the (generalized) nullspace of the operator $\Box$ subject to our boundary and periodicity conditions. It follows from the Fourier series representation (8.6) that for each $h \in L^2(\Omega)$ such that h is L^2-orthogonal to N there exists a unique $v \in N^\perp$ satisfying $\Box v = h$. Moreover [8, 9],

$$\|v\|_\infty \leq C \|h\|_2, \tag{8.7}$$

where the constant C is independent of h. Since the quotient of the norms of $\sin jx \, \mathrm{e}^{ikt}$ in E and in $L^2(\Omega)$ is $|j^2 - k^2|$ and since $|j^2 - k^2| \to \infty$ as $j^2 + k^2 \to \infty$, $j \neq |k|$, it follows from a standard argument that the embedding $N^\perp \hookrightarrow L^2(\Omega)$ is compact. On the other hand, for each $u \in N$, $\|u\| = \|u\|_2$. So N is not compactly embedded in $L^2(\Omega)$.

A function u is said to be a *weak solution* of (8.1) if $u \in E$ and

$$\int_\Omega u\Box\varphi\,dxdt = \int_\Omega f(x,t,u)\varphi\,dxdt$$

for all smooth $\varphi \in E$. If f is sufficiently smooth and satisfies (8.3), then weak solutions of (8.1) are known to be classical ones [9, 37].

Let

$$F(x,t,\xi) := \int_0^\xi f(x,t,s)\,ds,$$

where f satisfies (8.2)–(8.5), and consider the functional

$$\Phi(u) := \frac{1}{2}\int_\Omega (u_t^2 - u_x^2)\,dxdt + \int_\Omega F(x,t,u)\,dxdt. \tag{8.8}$$

It is easy to see [38, Appendix B] that $\Phi \in C^1(E, \mathbf{R})$ and critical points of Φ are weak solutions of (8.1). Since the E- and the L^2-norm coincide on N, we cannot expect Φ to be of class C^2 even if f is smooth (the second term of the right-hand side of (8.8) is in fact in $C^2(L^2(\Omega), \mathbf{R})$ if and only if F is quadratic with respect to ξ, cf. [3, Example 1.4.6]). Therefore Theorem 5.4 and Corollary 5.5 cannot be applied to the study of (8.1).

Let $0 < \lambda_1 < \lambda_2 \le ...$ be the positive eigenvalues of $\Box$ and let $e_1, e_2, ...$ be the corresponding eigenfunctions chosen in such a way that $e_n = \sin jx \cos kt$ or $e_n = \sin jx \sin kt$ for some j, k with $j^2 - k^2 = \lambda_n$. Then $\langle e_n, e_m\rangle = 0$ if $m \ne n$. Define

$$E_0 := \{u \in E : u(x,t) = \sum_{k^2-j^2 \ge 0} c_{jk} \sin jx\, \mathrm{e}^{ikt}\} \tag{8.9}$$

and

$$E_n := E_0 \oplus \operatorname{span}\{e_1, ..., e_n\}.$$

Note that the first term on the right-hand side of (8.8) is positive semidefinite on E_0 and negative definite on $\operatorname{span}\{e_1, ..., e_n\}$. Clearly, $(E_n)_{n=1}^\infty$ is a filtration of E. Let $\mathcal{E} := \{E_n, n\}_{n=1}^\infty$. We will show that Φ satisfies (PS)* under suitable assumptions on f. Since $E = N \oplus N^\perp$, each $u \in E$ has the representation $u = z + v$, where $z \in N$ and $v \in N^\perp$.

Lemma 8.1. *(i) For each fixed $v \in N^\perp$,*

$$\inf_{z\in N} \int_\Omega F(x,t,z+v)\,dxdt$$

is attained at a unique $z := z(v)$.

(ii) $z(v_j) \to z(\overline{v})$ (in E) whenever $v_j \to \overline{v}$ in $L^2(\Omega)$.

Proof. (i) Since $F(x,t,\xi) \ge \varepsilon\xi^2/2$ according to (8.3),

$$\int_\Omega F(x,t,z+v)\,dxdt \ge \frac{\varepsilon}{2}\|z+v\|_2^2. \tag{8.10}$$

Hence the functional $z \mapsto \int_\Omega F(x,t,z+v)\,dxdt$ is coercive (recall that $\|z\| = \|z\|_2$). Since it is also strictly convex, the conclusion follows.

(ii) Let $v_j \to \overline{v}$ in $L^2(\Omega)$ and let (v_m) be a subsequence of (v_j). Since

$$\int_\Omega F(x,t,z(v_m)+v_m)\,dxdt \le \int_\Omega F(x,t,v_m)\,dxdt,$$

it follows from (8.10) that $z(v_m)$ is bounded. We may assume (taking a subsequence if necessary) that $z(v_m) \to \overline{z}$ weakly. Since $z(v)$ is a critical point of the functional $z \mapsto \int_\Omega (F(x,t,z+v)\,dxdt$,

$$\int_\Omega f(x,t,z(v)+v)\varphi\,dxdt = 0 \qquad \text{for all } \varphi \in N. \tag{8.11}$$

This and (8.3) imply

$$\begin{aligned}\varepsilon\|\overline{z}-z(v_m)\|_2^2 &\le \int_\Omega (f(x,t,\overline{z}+v_m)-f(x,t,z(v_m)+v_m))(\overline{z}-z(v_m))\,dxdt \\ &= \int_\Omega f(x,t,\overline{z}+v_m)(\overline{z}-z(v_m))\,dxdt.\end{aligned}$$

Since $f(x,t,\overline{z}+v_m) \to f(x,t,\overline{z}+\overline{v})$ in $L^2(\Omega)$ and $z(v_m) \to \overline{z}$ weakly in $L^2(\Omega)$, the second integral above tends to zero. So $z(v_m) \to \overline{z}$ strongly. Moreover,

$$\int_\Omega F(x,t,z(\overline{v})+v_m)\,dxdt \ge \int_\Omega F(x,t,z(v_m)+v_m)\,dxdt$$

according to the minimizing property of $z(v)$. Passing to the limit we obtain

$$\int_\Omega F(x,t,z(\overline{v})+\overline{v})\,dxdt \ge \int_\Omega F(x,t,\overline{z}+\overline{v})\,dxdt.$$

Hence $\overline{z} = z(\overline{v})$.

We have shown that each subsequence of $(z(v_j))$ contains a subsequence converging to $z(\overline{v})$. It follows that $z(v_j) \to z(\overline{v})$. □

A similar result, for superlinear f, has been obtained by Tanaka [43, Lemma 1.1].

Denote the spectrum of the operator $\Box$ subject to the boundary and the periodicity conditions in (8.1) by $\sigma(\Box)$. Let

$$G(x,t,\xi) := \int_0^\xi g(x,t,s)\,ds.$$

Proposition 8.2. *The functional Φ (given by (8.8)) satisfies (PS)* if either $b \notin \sigma(\Box)$ or g is bounded and $G(x,t,\xi) \to \infty$ (or $G(x,t,\xi) \to -\infty$) uniformly in (x,t) as $|\xi| \to \infty$. Moreover, under these conditions $\Phi|_{E_n}$ satisfies (PS) for each n.*

Proof. Let (u_j) be a (PS)*-sequence. Since $F(x,t,\xi) = \frac{1}{2}b\xi^2 + G(x,t,\xi)$,

(8.12)
$$\Phi(u) = \frac{1}{2}\int_\Omega (u_t^2 - u_x^2 + bu^2)\,dxdt + \int_\Omega G(x,t,u)\,dxdt =: \frac{1}{2}\langle Lu,u\rangle + \psi(u).$$

Suppose $b \in \sigma(\Box)$. Let

$$\begin{aligned} E^+ &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b>0} c_{jk} \sin jx\, \mathrm{e}^{ikt}\}, \\ E^- &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b<0} c_{jk} \sin jx\, \mathrm{e}^{ikt}\}, \\ E^0 &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b=0} c_{jk} \sin jx\, \mathrm{e}^{ikt}\}, \end{aligned}$$

and write $u = u^+ + u^0 + u^- \in E^+ \oplus E^0 \oplus E^-$. Note that $L(E^\pm) \subset E^\pm$ and $L(E_n) \subset E_n$. Since the quadratic form $u \mapsto \langle Lu, u\rangle$ is positive definite on E^+, negative definite on E^- and since

$$P_{n_j}\nabla\Phi(u_j) = Lu_j^+ + Lu_j^- + P_{n_j}\nabla\psi(u_j) \to 0 \tag{8.13}$$

and $\nabla\psi(E)$ is bounded (because g is), it follows that the sequence $(u_j^+ + u_j^-)$ is bounded, and so is $\psi(u_j)$. By the mean value theorem,

$$|\psi(u_j^0) - \psi(u_j)| \le \sup_{u\in E} \|\nabla\psi(u)\|\|u_j^+ + u_j^-\|.$$

Hence

$$\psi(u_j^0) = \int_\Omega G(x,t,u_j^0)\, dxdt$$

is bounded. Since E^0 is finite dimensional and $G(x,t,\xi) \to \infty$ (or $-\infty$) as $|\xi| \to \infty$, the sequence (u_j^0), and therefore also (u_j), is bounded.

Let $u_j = z_j + v_j$, where $z_j \in N$ and $v_j \in N^\perp$. We may assume after passing to a subsequence that $v_j \to \overline{v}$ weakly in $N^\perp$ and strongly in $L^2(\Omega)$. Since $P_{n_j}\nabla\Phi(u_j) \to 0$ and $N \subset E_{n_j}$ (cf. (8.9)),

$$\int_\Omega f(x,t,z_j+v_j)(z_j - z(v_j))\, dxdt = \langle \nabla\Phi(u_j), z_j - z(v_j)\rangle \to 0.$$

By (8.3) and (8.11),

$$\begin{aligned} \varepsilon\|z_j - z(v_j)\|_2^2 &\le \int_\Omega (f(x,t,z_j+v_j) - f(x,t,z(v_j)+v_j))(z_j - z(v_j))\, dxdt \\ &= \int_\Omega f(x,t,z_j+v_j)(z_j - z(v_j))\, dxdt \to 0. \end{aligned}$$

So $z_j - z(v_j) \to 0$ in E. Since $z(v_j) \to z(\overline{v})$ according to Lemma 8.1, $z_j \to z(\overline{v})$ in E. Consequently, $u_j = z_j + v_j \to z(\overline{v}) + \overline{v}$ in $L^2(\Omega)$ and therefore $\nabla\psi(u_j) \to \nabla\psi(z(\overline{v}) + \overline{v})$. Since L is invertible on $E^+ \oplus E^-$, (8.13) shows that $u_j \to z(\overline{v}) + \overline{v}$ in E.

Suppose now that $b \notin \sigma(\Box)$. Then $E^0 = \{0\}$. By (8.4), for each $\varepsilon' > 0$ there is a constant $C = C(\varepsilon')$ such that $|g(x,t,\xi)| \le \varepsilon'|\xi| + C(\varepsilon')$. Hence $\nabla\psi(u) = o(\|u\|)$ as $\|u\| \to \infty$ (cf. (7.8) and the following lines). Since L is now invertible, it follows from (8.13) that (u_j) is a bounded sequence. The remaining part of the proof is the same as above.

Finally, since $N \subset E_n$, an obvious modification of the above argument shows that $\Phi|_{E_n}$ satisfies (PS). □

Let

$$G_0(x,t,\xi) := \int_0^\xi g_0(x,t,s)\, ds$$

and

(8.14)

$$\Phi(u) = \frac{1}{2}\int_\Omega (u_t^2 - u_x^2 + b_0u^2)\,dxdt + \int_\Omega G_0(x,t,u)\,dxdt =: \frac{1}{2}\langle L_0u,u\rangle + \psi_0(u).$$

Then

$$\begin{aligned} F^+ &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b_0>0} c_{jk}\sin jx\,\mathrm{e}^{ikt}\},\\ F^- &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b_0<0} c_{jk}\sin jx\,\mathrm{e}^{ikt}\},\\ F^0 &:= \{u \in E : u(x,t) = \sum_{k^2-j^2+b_0=0} c_{jk}\sin jx\,\mathrm{e}^{ikt}\} \end{aligned}$$

are the subspaces on which $\langle L_0u,u\rangle$ is positive definite, negative definite and zero.

Proposition 8.3. *Φ satisfies the local linking condition at 0 in each of the following cases:*

(i) $b_0 \notin \sigma(\Box)$;

(ii) $b_0 \in \sigma(\Box)$ and there is a $\delta > 0$ such that $G_0(x,t,\xi) \geq 0$ whenever $|\xi| \leq \delta$;

(iii) $b_0 \in \sigma(\Box)$ and there is a $\delta > 0$ such that $G_0(x,t,\xi) \leq 0$ whenever $|\xi| \leq \delta$.

Moreover, $c_{\mathcal{E}}^{-q_0}(\Phi,0) \neq [0]$, where q_0 is the number of eigenvalues of $\Box$ in the interval $(0,b_0]$ (counted with their multiplicity) if (i) or (ii) is satisfied, and in the interval $(0,b_0)$ if (iii) holds.

Proof. Let $E = Y \oplus Z$. If (i) is satisfied, we take $Y = F^-$ and $Z = F^+$, if (ii) holds, $Y = F^-$ and $Z = F^+ \oplus F^0$, and in the remaining case $Y = F^0 \oplus F^-$ and $Z = F^+$. We will show that $\Phi \leq 0$ on $Y \cap B(0,\rho)$ and $\Phi \geq 0$ on $Z \cap B(0,\rho)$ if ρ is small enough. Assuming this, it is easy to obtain the second conclusion. Indeed, suppose that (i) or (ii) is satisfied. Then $Y = F^-$. Since E_n is the subspace of E obtained by taking the sums in (8.6) over all j,k with $j^2 - k^2 \leq \lambda_n$ and Y is obtained by summing over j,k with $j^2 - k^2 > b_0$, $Y \cap E_n$ is spanned by the eigenfunctions e_m such that $b_0 < \lambda_m \leq \lambda_n$. Therefore $\dim(Y \cap E_n) = n - q_0$ for all $n > q_0$. If (iii) is satisfied, then $Y = F^0 \oplus F^-$ and $Y \cap E_n$ is spanned by all e_m such that $b_0 \leq \lambda_m \leq \lambda_n$. So again $\dim(Y \cap E_n) = n - q_0$ for $n > q_0$. Since obviously $E_n = (Y \cap E_n) \oplus (Z \cap E_n)$, $c_{\mathcal{E}}^{-q_0}(\Phi,0) \neq [0]$ according to Theorem 5.6.

We verify the local linking condition only in case (iii) (and make comments on other cases when suitable). Let $u \in Z \equiv F^+$ and write $u = v + z$, $v \in N^\perp \cap F^+$, $z \in N$. First we show that there is an $r > 0$ such that if $D_r := \{u \in Z : \|u\|_2 \leq r\}$, then $\inf_{D_r}\Phi \geq 0$. Let $(u_j) \subset D_r$ be a sequence such that $\Phi(u_j)$ tends to the infimum. Since the quadratic form in (8.14) is positive definite on F^+ and ψ_0 is bounded on D_r, $\Phi(u_j) \to \infty$ if $\|u_j\| \to \infty$ in E. Hence (u_j) is bounded in E and we may assume taking a subsequence that $u_j \to \widetilde{u}$ weakly in E (and $v_j \to \widetilde{v}$ strongly in $L^2(\Omega)$). Furthermore, the function $\xi \mapsto F(x,t,\xi)$ is convex and the quadratic form in (8.8) is positive semidefinite except on the finite dimensional subspace of F^+ on which $-b_0 < k^2 - j^2 < 0$. Therefore Φ is weakly lower semicontinuous on F^+ and $\Phi(\widetilde{u}) = \inf_{D_r}\Phi$ (in particular, the infimum is a finite number). It follows that

$$\langle \nabla\Phi(\widetilde{u}),\varphi\rangle = \lambda\int_\Omega \widetilde{u}\varphi\,dxdt \tag{8.15}$$

for some $\lambda \le 0$ and all $\varphi \in F^+$ (λ is a Lagrange multiplier which takes into account the fact that $\widetilde{u}$ may be on the boundary of D_r). Choosing $\varphi = \widetilde{u}$, we obtain

$$\lambda\|\widetilde{u}\|_2^2 = \int_\Omega (\widetilde{u}_t^2 - \widetilde{u}_x^2)\,dxdt + \int_\Omega f(x,t,\widetilde{u})\widetilde{u}\,dxdt. \tag{8.16}$$

Below $c_1, c_2, \ldots$ will denote different positive constants. Recall that the quadratic form above is negative definite on the finite dimensional subspace on which $-b_0 < k^2 - j^2 < 0$ and positive semidefinite otherwise. Thus $\int_\Omega (\widetilde{u}_t^2 - \widetilde{u}_x^2)\,dxdt \ge -c_1\|\widetilde{u}\|_2^2$. Since $f(x,t,\xi)\xi \ge 0$ for all x,t,ξ, we have

$$\lambda\|\widetilde{u}\|_2^2 \ge -c_1\|\widetilde{u}\|_2^2$$

and $|\lambda| \le c_1$ (if $\widetilde{u} = 0$, then $\lambda = 0$ because $\widetilde{u} \notin \partial D_r$). It follows from (8.15) that

$$\Box\widetilde{v} = Pf(x,t,\widetilde{u}) - \lambda\widetilde{u} \equiv P(f(x,t,\widetilde{u}) - \lambda\widetilde{u})$$

in the sense of distributions, where

$$P\Big(\sum_{j,k} d_{jk}\sin jx\,\mathrm{e}^{ikt}\Big) := \sum_{k^2-j^2+b_0>0} d_{jk}\sin jx\,\mathrm{e}^{ikt}$$

is a bounded projector in $L^2(\Omega)$. Employing (8.7) and (8.4), (8.5) we obtain

$$\|\widetilde{v}\|_\infty \le c_2\|P(f(x,t,\widetilde{u}) - \lambda\widetilde{u})\|_2 \le c_3\|\widetilde{u}\|_2.$$

It has been shown in [27, Lemma 4.3], cf. also [37, Lemma 3.7], that if a function h satisfies (8.3) and

$$\Box v = Ph(x,t,u),$$

where $u = v + z$, then $\|z\|_\infty \le c_4\|v\|_\infty$ (in [27] the setup is slightly different but our conclusion here remains true with the same proof). Since $\lambda \le 0$, $f(x,t,\xi) - \lambda\xi$ satisfies (8.3). Consequently, $\|\widetilde{u}\|_\infty \le c_5\|\widetilde{u}\|_2$. It follows therefore from (8.5) that

$$\Phi(\widetilde{u}) = \frac{1}{2}\langle L_0\widetilde{u}, \widetilde{u}\rangle + \int_\Omega G_0(x,t,\widetilde{u})\,dxdt \ge 0 \tag{8.17}$$

whenever r is sufficiently small. Note that in case (ii) $Z = F^+ \oplus F^0$, the quadratic form above is positive semidefinite on Z, and $G_0(x,t,\widetilde{u}) \ge 0$ if $\|\widetilde{u}\|_\infty \le c_5 r \le \delta$. Hence (8.17) still holds.

We have shown that $\inf_{D_r}\Phi \ge 0$. Since the embedding $E \hookrightarrow L^2(\Omega)$ is continuous, $B(0,\rho) \subset D_r$ for some ρ and $\Phi \ge 0$ on $Z \cap B(0,\rho)$.

Now let $u = u^0 + u^- \in F^0 \oplus F^- \equiv Y$. It suffices to show that there is an $r > 0$ such that $\sup_{C_r}\Phi \le 0$, where $C_r := \{u \in Y : \|u\|_2 \le r\}$ (note that cases (i) and (ii) are simpler because $F^0 = \{0\}$). Let $(u_j) \subset C_r$ be a maximizing sequence for Φ. Since

$$\Phi(u) = \frac{1}{2}\langle L_0u^-, u^-\rangle + \int_\Omega G_0(x,t,u)\,dxdt, \tag{8.18}$$

the first term on the right-hand side above is negative definite on F^- and the second one is bounded on C_r, it follows that $\Phi(u_j) \to -\infty$ as $\|u_j^-\| \to \infty$ in E. So we may assume that $u_j \to \widetilde{u}$ weakly in E and strongly in $L^2(\Omega)$ (we have used that $Y \subset N^\perp$ and the embedding $N^\perp \hookrightarrow L^2(\Omega)$ is compact). Since negative definite quadratic forms are weakly upper semicontinuous, $\lim_{j\to\infty}\Phi(u_j) \le \Phi(\widetilde{u})$. So $\Phi(\widetilde{u}) = \sup_{C_r}\Phi$ and

$$\Box\widetilde{u} = Qf(x,t,\widetilde{u}) - \lambda\widetilde{u} \equiv Q(f(x,t,\widetilde{u}) - \lambda\widetilde{u}) \tag{8.19}$$

in the sense of distributions, where $\lambda \geq 0$ and Q is the projector onto the subspace of $L^2(\Omega)$ on which $k^2 - j^2 + b_0 \leq 0$. The quadratic form on the right-hand side of (8.16) is now negative definite, so

$$0 \leq \lambda \|\widetilde{u}\|_2^2 \leq \int_\Omega f(x,t,\widetilde{u})\widetilde{u}\,dxdt \leq c_6 \|\widetilde{u}\|_2^2$$

and $0 \leq \lambda \leq c_6$. By (8.7) and (8.19), $\|\widetilde{u}\|_\infty \leq c_7 \|\widetilde{u}\|_2$. If r is small enough, then $\|\widetilde{u}\|_\infty \leq \delta$, and it follows from (8.18) that $\Phi(\widetilde{u}) \leq 0$. □

Theorem 8.4. *Suppose that f satisfies* (8.2)–(8.5) *and either $b \notin \sigma(\Box)$ or g is bounded and $G(x,t,\xi) \to \infty$ uniformly in (x,t) as $|\xi| \to \infty$. Then the wave equation (8.1) has a nontrivial weak solution in each of the following cases:*

(i) $b_0 \notin \sigma(\Box)$ and $(0,b_0] \cap \sigma(\Box) \neq (0,b] \cap \sigma(\Box)$;

(ii) $b_0 \in \sigma(\Box)$, $(0,b_0] \cap \sigma(\Box) \neq (0,b] \cap \sigma(\Box)$ and there is a $\delta > 0$ such that $G_0(x,t,\xi) \geq 0$ whenever $|\xi| \leq \delta$;

(iii) $b_0 \in \sigma(\Box)$, $(0,b_0) \cap \sigma(\Box) \neq (0,b] \cap \sigma(\Box)$ and there is a $\delta > 0$ such that $G_0(x,t,\xi) \leq 0$ whenever $|\xi| \leq \delta$.

Proof. Suppose that 0 is the only critical point of Φ. If $b \notin \sigma(\Box)$, let $\Phi_\lambda(u) := \frac{1}{2}\langle Lu,u\rangle + (1-\lambda)\psi(u)$ (cf. (8.12)). Since $\nabla\psi(u) = o(\|u\|)$ as $\|u\| \to \infty$, it follows from Propositions 2.12 and 2.14 that $c_{\mathcal{E}}^*(\Phi,0) = c_{\mathcal{E}}^*(\Phi,K(\Phi)) = c_{\mathcal{E}}^*(\Phi_1,0)$ (cf. the proof of Theorem 7.4). Since $E^- \cap E_n$ is spanned by the eigenfunctions e_m such that $b < \lambda_m \leq \lambda_n$, we see that $\dim(E^- \cap E_n) = n - q_\infty$ if $n > q_\infty$, where q_∞ is the number of eigenvalues of $\Box$ in the interval $(0,b]$ (counted with their multiplicity). Hence $M_{\mathcal{E}}^-(L) = -q_\infty$ and

$$c_{\mathcal{E}}^q(\Phi,0) = c_{\mathcal{E}}^q(\Phi,K(\Phi)) = \begin{cases} [\mathcal{F}] & \text{if } q = -q_\infty, \\ [0] & \text{otherwise.} \end{cases} \tag{8.20}$$

Now let $b \in \sigma(\Box)$. Since $\Phi|_{E_n}$ satisfies (PS), we may proceed as in the proof of Lemma 7.6(i) and we obtain (8.20) again (note that here $N(L) \subset E_n$ for almost all n, so $P_nN(L) = N(L)$ and $E_n' = E_n$). According to Proposition 8.3, $c_{\mathcal{E}}^{-q_0}(\Phi,0) \neq [0]$. So $q_0 = q_\infty$. On the other hand, it follows from the definitions of q_0 and q_∞ and from our hypotheses on the intersection with $\sigma(\Box)$ that $q_0 \neq q_\infty$. □

Theorem 8.5. *Suppose that f satisfies* (8.2)–(8.5) *and either $b \notin \sigma(\Box)$ or g is bounded and $G(x,t,\xi) \to -\infty$ uniformly in (x,t) as $|\xi| \to \infty$. Then the wave equation (8.1) has a nontrivial weak solution if the interval $(0,b]$ in the assumptions (i)–(iii) of Theorem 8.4 is replaced by $(0,b)$.*

Proof. The only difference compared to the proof of the preceding theorem is that if $b \in \sigma(\Box)$, we now use the argument of Lemma 7.6(ii) and obtain $c_{\mathcal{E}}^q(\Phi,0) = [\mathcal{F}]$ for $q = M_{\mathcal{E}}^-(L) + M^0(L)$ and [0] otherwise. So (8.20) holds with $q_\infty = -M_{\mathcal{E}}^-(L) - M^0(L)$, and it follows that in the present case q_∞ is the number of eigenvalues of $\Box$ in the interval $(0,b)$. □

Remark 8.6. If $b \notin \sigma(\Box)$, then (8.4) may be replaced by the slightly weaker hypothesis that $|g(x,t,\xi)| \leq \alpha|\xi| + \beta$, where α is less than the distance from b to $\sigma(\Box)$ [27, Corollary 5.2].

Remark 8.7. Theorems 8.4 and 8.5 extend some results of [1, 27]. In [1] it was assumed that $f \in C^1$, the derivative f_ξ is bounded, bounded away from zero and $b \notin \sigma(\Box)$ (on the other hand, if $b_0 \in \sigma(\Box)$, our hypothesis at $\xi = 0$ is different—and rather more restrictive—than the corresponding one in [1]). Theorem 8.4 is a slight generalization of the main result of [27] (in [27] the sign conditions (ii) and (iii) are for $g_0(x,t,\xi)\xi$; here they are for $G_0(x,t,\xi)$). Theorem 8.5 is new.

9. Elliptic system

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary, let $F \in C^1(\overline{\Omega}\times\mathbf{R}^2,\mathbf{R})$ and consider the Dirichlet problem

$$\begin{cases} -\Delta u = F_v(x,u,v) & \text{in } \Omega, \\ -\Delta v = F_u(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.1}$$

Problems of this type have been studied recently in [14] for subquadratic F, in [16, 23] for superquadratic F, and in [42] a bifurcation problem for (9.1) has been considered. Here we assume that F is asymptotically quadratic, or more precisely, that

(9.2)
$$F(x,u,v) = \frac{1}{2}au^2 + buv + \frac{1}{2}cv^2 + G(x,u,v), \quad \text{where } |G_u(x,u,v)| + |G_v(x,u,v)| \\ = o(|u|+|v|) \;\; \text{uniformly} \;\; \text{in } x \;\; \text{as } |u|+|v| \to \infty$$

and

(9.3)
$$F(x,u,v) = \frac{1}{2}a_0u^2 + b_0uv + \frac{1}{2}c_0v^2 + G_0(x,u,v), \\ \text{where } |(G_0)_u(x,u,v)| + |(G_0)_v(x,u,v)| \\ = o(|u|+|v|) \text{ uniformly} \;\; \text{in } x \;\; \text{as } |u|+|v| \to 0.$$

For simplicity a, b, c and a_0, b_0, c_0 are assumed to be constant, though x-dependence could be admitted.

Let $H_0^1(\Omega)$ be the usual Sobolev space (of real-valued functions) and set $E := H_0^1(\Omega) \times H_0^1(\Omega)$. Then E is a Hilbert space with inner product given by

$$\langle (u,v),(u',v')\rangle := \int_\Omega (\nabla u\cdot\nabla u' + \nabla v \cdot \nabla v')\,dx.$$

It is easily seen from [38, Appendix B] and (9.2) that the functional $\Phi : E \to \mathbf{R}$ defined by

$$\Phi(u,v) := \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega F(x,u,v)\,dx$$

is of class C^1 and critical points of Φ correspond to weak solutions of (9.1). Moreover, weak solutions are classical ones if either $N = 1$ or $N \geq 2$ and F_u, F_v are locally Hölder continuous.

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and let $(e_n)_{n=1}^\infty$ be the corresponding orthonormal basis of eigenfunctions. Define

$$F_n := \text{span}\{(e_n,0),(0,e_n)\}, \quad E_n := \text{span}\{(e_i,0),(0,e_j) : 1 \leq i,j \leq n\}$$

and $\mathcal{E} := \{E_n, n\}_{n=1}^{\infty}$. Furthermore, set

$$\begin{aligned}\Phi(u,v) &= \int_\Omega (\nabla u \cdot \nabla v - \frac{1}{2}au^2 - buv - \frac{1}{2}cv^2)\,dx - \int_\Omega G(x,u,v)\,dx \\ &=: \frac{1}{2}\langle L(u,v),(u,v)\rangle - \varphi(u,v)\end{aligned}$$

and

$$\begin{aligned}\Phi(u,v) &= \int_\Omega (\nabla u \cdot \nabla v - \frac{1}{2}a_0u^2 - b_0uv - \frac{1}{2}c_0v^2)\,dx - \int_\Omega G_0(x,u,v)\,dx \\ &=: \frac{1}{2}\langle L_0(u,v),(u,v)\rangle - \varphi_0(u,v).\end{aligned}$$

Let

$$A := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be the matrix representing the quadratic form in (9.2). Since $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of eigenfunctions, $\int_\Omega e_m e_n\,dx = 0$ if $m \neq n$ and $1 = \int_\Omega |\nabla e_n|^2\,dx = \lambda_n \int_\Omega e_n^2\,dx$. Using this it is easy to see that $L(F_n) \subset F_n$, $L(E_n) \subset E_n$ and the same is true for L_0. Moreover, if $(u,v) = (\alpha e_n, \beta e_n) \in F_n$, then

$$\frac{1}{2}\langle L(u,v),(u,v)\rangle = \alpha\beta - \frac{1}{\lambda_n}\Big(\frac{1}{2}a\alpha^2 + b\alpha\beta + \frac{1}{2}c\beta^2\Big).$$

Hence the linear mapping $L|_{F_n} : F_n \to F_n$ is represented by the matrix

$$T_n(A) := \begin{pmatrix} -\frac{a}{\lambda_n} & 1 - \frac{b}{\lambda_n} \\ 1 - \frac{b}{\lambda_n} & -\frac{c}{\lambda_n} \end{pmatrix}.$$

Let

$$\begin{aligned} i^-(A) &:= \sum_{n=1}^{\infty}(M^-(T_n(A)) - 1), \\ i^+(A) &:= \sum_{n=1}^{\infty}(M^+(T_n(A)) - 1) \end{aligned}$$

and

$$i^0(A) := \sum_{n=1}^{\infty} M^0(T_n(A)).$$

Since $\lambda_n \to \infty$ as $n \to \infty$, $i^{\pm}(A)$ and $i^0(A)$ are well-defined and finite. Moreover, $N(L) \subset E_n$ for some n, $\dim N(L) = i^0(A)$, $Q_nL = L$ (because $L(E_n) \subset E_n$) and $i^-(A) = M^-_{\mathcal{E}}(L)$. It is also easy to see that $i^-(A) + i^+(A) + i^0(A) = 0$.

Lemma 9.1. *Suppose that F satisfies (9.2). Then Φ satisfies (PS)* in each of the following two cases:*

(i) $i^0(A) = 0$;

(ii) G_u, G_v are bounded and $G(x,u,v) \to \infty$ (or $G(x,u,v) \to -\infty$) uniformly in x as $|u| + |v| \to \infty$.

Moreover, under the above assumptions $\Phi|_{E_n}$ satisfies (PS) for each n.

The proof uses the same argument as that of Lemma 7.3 and is therefore omitted (note that $\nabla\varphi$ is compact according to [38, Appendix B]).

Let $F \in C^2(\overline{\Omega} \times \mathbf{R}^2, \mathbf{R})$ and denote the Hessian of F with respect to u, v by D^2. Suppose that there is a constant C such that

$$\|D^2 F(x,u,v)\| \le C(1 + |u| + |v|)^{p-1}, \tag{9.4}$$

where $1 \le p < (N+2)/(N-2)$ if $N > 2$ and $1 \le p < \infty$ if $N = 2$ (if $N = 1$, no growth restriction (9.4) is necessary). Then $\Phi \in C^2(E, \mathbf{R})$ [38, Appendix B].

Denote by A_0 the matrix which represents the quadratic form in (9.3). Below we formulate two sufficient conditions for the existence of a nontrivial solution to (9.1). The proofs are omitted because they are obtained by an easy modification of the arguments in Theorems 7.4 and 7.5 (here in fact the situation is somewhat simpler: since $P_n N(L) \subset N(L)$ for some n, it is not necessary to introduce a modified filtration $\mathcal{E}'$ as in Theorem 7.5).

Theorem 9.2. *Suppose that F satisfies (9.2) and (9.3). If $i^0(A) = i^0(A_0) = 0$ and $i^-(A) \ne i^-(A_0)$, then (9.1) has a nontrivial weak solution.*

Theorem 9.3. *Suppose that $F \in C^2(\overline{\Omega} \times \mathbf{R}^2, \mathbf{R})$ satisfies (9.2), (9.3) and (9.4) ((9.4) may be omitted if $N = 1$). If G_u, G_v are bounded, then (9.1) has a nontrivial weak solution in each of the following two cases:*

(i) $G(x,u,v) \to -\infty$ uniformly in x as $|u| + |v| \to \infty$ and

$$i^-(A) \notin [i^-(A_0), i^-(A_0) + i^0(A_0)];$$

(ii) $G(x,u,v) \to \infty$ uniformly in x as $|u| + |v| \to \infty$ and

$$i^+(A) \notin [i^+(A_0), i^+(A_0) + i^0(A_0)].$$

If $i^0(A_0) = 0$, it is not necessary to assume that $F \in C^2$, and if $i^0(A) = 0$, G_u, G_v need not be bounded and G need not tend to infinity.

Also Theorem 7.10 has a counterpart here. We leave the formulation to the reader and observe only that the local linking condition may be verified e.g. by adapting the argument of [29, Theorem 4].

Suppose that $N = 1$, $\Omega = (a,b)$ and $F \in C^2([a,b] \times \mathbf{R}^2, \mathbf{R})$. If (u_0, v_0) is a solution of (9.1), then $(u,v) \in E$ is in the nullspace of $\Phi''(u_0, v_0)$ if and only if

$$\begin{cases} -u'' = F_{uv}(x, u_0(x), v_0(x))u + F_{vv}(x, u_0(x), v_0(x))v, \\ -v'' = F_{uu}(x, u_0(x), v_0(x))u + F_{uv}(x, u_0(x), v_0(x))v, \\ u(a) = v(a) = u(b) = v(b) = 0. \end{cases}$$

It is well-known that such systems can have at most two linearly independent solutions, so $\dim N(\Phi''(u_0, v_0)) \le 2$ (this is no longer true if $N \ge 2$). Using the argument of Theorem 7.8, we therefore obtain

Theorem 9.4. *Suppose that $N = 1$, $F \in C^2([a,b] \times \mathbf{R}^2, \mathbf{R})$ satisfies (9.2), (9.3) and $i^0(A_0) = 0$. Then (9.1) has at least two nontrivial solutions in each of the following cases:*

(i) $|i^-(A) - i^-(A_0)| \ge 2$ and $i^0(A) = 0$;

(ii) $|i^-(A) - i^-(A_0)| \ge 2$, G_u, G_v are bounded and $G(x,u,v) \to -\infty$ uniformly in x as $|u| + |v| \to \infty$;

(iii) $|i^+(A) - i^+(A_0)| \ge 2$, G_u, G_v are bounded and $G(x,u,v) \to \infty$ uniformly in x as $|u| + |v| \to \infty$.

Remark 9.5. Similar results remain valid for the Neumann problem

$$\begin{cases} -\Delta u = F_v(x,u,v) & \text{in } \Omega, \\ -\Delta v = F_u(x,u,v) & \text{in } \Omega, \\ \partial u/\partial n = \partial v/\partial n = 0 & \text{on } \partial\Omega. \end{cases}$$

The appropriate choice of the space is then $E = H^1(\Omega) \times H^1(\Omega)$ (the inner product and the indices $i^{\pm}(A), i^0(A)$ need to be modified in a rather obvious way). Since the null space of the quadratic form $(u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v \, dx$ in $H^1(\Omega) \times H^1(\Omega)$ consists of constant functions, it is easy to see that also results analogous to Corollary 7.9 (for $N = 1$) and Theorem 7.10 (for $N \geq 1$) are true here (in the latter case we assume F is periodic in one or both variables u, v).

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