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MONGE-AMPÈRE EQUATIONS RELATIVE TO A RIEMANNIAN METRIC

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ABSTRACT. We prove that in a bounded strictly convex open set Ω in $\mathbb{R}^n$, the problem

$$\begin{cases} \det \nabla^2 u = f(x), \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

where $f > 0, f \in C^\infty(\overline{\Omega}), \varphi \in C^\infty(\partial\Omega)$, has a unique strictly convex solution $u \in C^\infty(\overline{\Omega})$. This result extends to an arbitrary metric a theorem which has been proved by Caffarelli-Nirenberg-Spruck in the case of the Euclidean metric.

I. INTRODUCTION

This work deals with the Dirichlet problem for Monge-Ampère equations relative to an arbitrary smooth Riemannian metric, and its goal is to provide an exact analogue to a result of Caffarelli-Nirenberg-Spruck [CNS] in the case of the flat metric. More precisely let g be a C^∞ Riemannian metric on a smooth bounded open set Ω in $\mathbb{R}^n$, $n \geq 2$. Let ∇u and $\nabla^2 u$ denote the covariant derivative and the Hessian of u relative to the Levi-Civita connexion of the metric g. We shall call the functions u for which the Hessian $(\nabla^2 u)$ is positive definite *strictly convex*.

We shall assume that Ω is strictly convex in the following sense:

$$\text{there exists } h \in C^\infty(\overline{\Omega}) \text{ strictly convex in } \overline{\Omega} \text{ such that } h = 0 \text{ on } \partial\Omega. \tag{I.1}$$

The main result of this paper is then

Theorem I.1. *Let Ω be a C^∞ strictly convex bounded open set in $\mathbb{R}^n$, $n \geq 2$. Given $f \in C^\infty(\overline{\Omega})$, $f > 0$ on $\overline{\Omega}$, and $\varphi \in C^\infty(\partial\Omega)$, the problem*

$$\begin{cases} \det \nabla^2 u(x) = f(x) \text{ in } \Omega, \\ u_{|\partial\Omega} = \varphi, \end{cases} \tag{I.2}$$

has a unique strictly convex solution $u \in C^\infty(\overline{\Omega})$.

The interest in such equations is related to the fact that a few problems in differential geometry (as Minkowski and Weyl problems; see also [GS]) lead to equations of this type. The case of the flat metric in $\mathbb{R}^n$ has been widely investigated during the past years, and Theorem I.1 in this case has been proved by Caffarelli-Nirenberg-Spruck [CNS].

In the two dimensionnal case, for general metrics, this result has been proved by Corona [C], who assumes moreover the existence of an upper solution, by Hong [H]

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in the case $\varphi = 0$ and by Atallah [A] in the general case. Recently Guan-Spruck [GS] investigated related equations for metrics on the sphere when subsolutions exist. Finally we would like to mention the recent results by Atallah [A] in the case where $f = f(x, u, \nabla u)$, assuming the existence of upper and subsolutions.

The proof, as usual, uses the continuity method, which leads to uniform bounds for the $C^{2+\alpha}(\overline{\Omega})$ norms of smooth solutions of (I.2) (see [CNS] or [GT]). By results of Evans [E] and Krylov [KR] such bounds can be deduced from C^2 uniform bounds. The interior estimates and the estimates of the tangential and mixed second derivatives on the boundary, which rely on the previous work mentioned above, appear already in [A], and we give them here for the sake of completeness. The main part of the paper is then devoted to bounding the pure normal derivative on the boundary, which leads to the construction of a local upper solution for the problem (I.1). This construction, which is achieved under an appropriate set of coordinates, is inspired by [CNS] but requires here a careful and more technical proof.

II. Proof of Theorem I.1

A constant C will be called *under control* if it depends only on n, Ω, g, f, φ. A function u is said to be under control if it is bounded by such a constant. Moreover, we shall denote by C a constant which may vary from line to line but is always under control.

1. C^0 and C^1 bounds

First of all, by (I.1), if λ is a large enough positive constant, the function

$$\psi = \lambda h + \varphi \tag{II.1}$$

is a subsolution for problem (I.2). By the maximum principle we get

$$\psi \leq u \leq \max_{\partial\Omega} \varphi \text{ in } \overline{\Omega}. \tag{II.2}$$

On the other hand it is easy to see that $|\nabla u|^2 = \sum g^{ij} \nabla_i u \nabla_j u$ achieves its maximum on the boundary. Since tangential derivatives of u on $\partial\Omega$ are obviously under control, the C^1 bounds in $\overline{\Omega}$ will follow from the control of the normal derivative $\frac{\partial u}{\partial \nu}$ on the boundary; here ν is the interior normal. By the Hopf maximum principle and (II.2) we get

$$\frac{\partial \psi}{\partial \nu} \leq \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega. \tag{II.3}$$

For the upper bound we follow [C]. A geodesic γ normal to $\partial\Omega$ at p intersects $\partial\Omega$ at one other point q, by the strict convexity of Ω. Let w be the linear function on γ equal to φ at p and q, and $\dot{w}$ its gradient, which is a continuous function. Since u is strictly convex on γ, the maximum principle implies that $u \leq w$ on $\gamma \cap \Omega$ and $\frac{\partial u}{\partial \nu}(p) \leq \dot{w}(p) \leq \inf_{\partial\Omega} \dot{w}$.

2. C^2 estimates on the boundary

We shall prove pointwise bounds for the second derivatives on the boundary. Let p be a point on $\partial\Omega$; it can be taken as the origin of coordinates. Using the function h given by (I.1), the maximum principle and a system of normal coordinates near the origin, one see easily that in a neighborhood V of the origin, the boundary is given by

$$\partial\Omega \cap V = \{(x_1, \dots, x_n) : x_n = \rho(x_1, \dots, x_{n-1})\}, \tag{II.4}$$

where ρ is a smooth function such that

$$\rho(x') = \frac{1}{2}\sum_{i=1}^{n-1} k_i x_i^2 + O(|x'|^3), \quad k_i > 0, \quad x' = (x_1, \dots, x_{n-1}). \tag{II.5}$$

Estimates for the pure tangential derivatives. The vector field

$$X_\alpha = \frac{\partial}{\partial x_\alpha} + \frac{\partial \rho}{\partial x_\alpha}\frac{\partial}{\partial x_n}$$

is tangent to the boundary if $\alpha = 1, 2, \dots, n-1$, and

$$\nabla_{\alpha\beta} u(0) = (X_\alpha X_\beta \varphi)(0) - \frac{\partial^2 \rho}{\partial x_\alpha \partial x_\beta}(0)\frac{\partial u}{\partial x_n}(0).$$

This shows, using the C^1 bounds, that $\nabla_{\alpha\beta} u(0)$ is under control.

Estimates for the mixed second derivatives. Differentiating the identity $\operatorname{Log}\det \nabla_{ij} u = \operatorname{Log} f = \Phi$, one gets

$$u^{ij}\nabla_{ijk} u = \nabla_k \Phi, \quad k = 1, \dots, n, \tag{II.6}$$

where $(u^{ij}) = (\nabla_{ij} u)^{-1}$ and the summation convention has been used. Let us set $L = u^{ij}\nabla_{ij}$ and let $X = \sum_{k=1}^n b_k \nabla_k$ be a smooth vector field. Then

$$L(Xu) = u^{ij}(\nabla_{ij} b_k)\nabla_k u + 2u^{ij}\nabla_i b_k \nabla_{kj} u + u^{ij} b_k \nabla_{ijk} u + u^{ij} b_k(\nabla_{kij} u - \nabla_{ijk} u).$$

By the well known Ricci formulas and (II.6) one gets

$$L(Xu) = u^{ij}\underbrace{(\nabla_{ij} b_k)\nabla_k u}_{(1)} + \underbrace{2u^{ij}\nabla_i b_k \nabla_{kj} u}_{(2)} + \underbrace{b_k \nabla_k \Phi}_{(3)} + \underbrace{u^{ij} b_k R^s_{jki}\nabla_s u}_{(4)}$$

where R^s_{jki} are the components of the Riemann curvature tensor, which depends only on the metric g. By the C^1 estimates we can write

$$(1) + (4) = u^{ij} A_{ij} \tag{II.7}$$

where the A_{ij} are under control.

Now, since $u^{ij}\nabla_{kj} u = \delta_{ik}$, we get

$$(2) = 2\nabla_i b_i. \tag{II.8}$$

It follows that $LXu = A_0 + A_{ij} u^{ij}$, where A_0 and A_{ij} are under control; therefore $|L(X(u-\varphi))| \le C_1 + C_2 \sum_{i=1}^n u^{ii}$ since $(u^{ij}) > 0$. Now

$$\sum_{i=1}^n u^{ii} \ge n(\det u^{ij})^{1/n} = n f^{-1/n} \ge C_3 > 0,$$

which implies $|L(X(u-\varphi))| \le C_4 \sum_{i=1}^n u^{ii}$.

Now by the strict convexity of the function h given by (I.1) we get $u^{ij}\nabla_{ij}h \geq C_5 \sum_{i=1}^{n} u^{ii}$. It follows that, if λ is large enough,

$$\lambda Lh - |L(X(u-\varphi))| > 0 \text{ in } \overline{\Omega}.$$

Let us assume now that X is tangent to the boundary; then $X(u-\varphi)|_{\partial\Omega} = h|_{\partial\Omega} = 0$. It follows from the Hopf maximum principle that

$$\Big|\frac{\partial}{\partial\nu}(X(u-\varphi))\Big| \leq -\lambda \frac{\partial h}{\partial \nu} \text{ on } \partial\Omega.$$

Taking $X = \partial_\alpha + \rho_\alpha \partial_n$, we get

$$|\nabla_{n\alpha}u(0)| \leq C, \qquad \alpha = 1, \dots, n-1.$$

Estimates for the pure normal derivative. As before, let p be a point on $\partial\Omega$ that we take as the origin. We shall work in a special set of coordinates in a neighborhood V of the origin. Let $\nu(p)$ be the interior unit normal (with respect to g) to $\partial\Omega$ at $p \in V \cap \partial\Omega$. Let us consider first the case $n \geq 3$. Let $X_1, \dots, X_{n-1}$ be a basis of the tangent space to $\partial\Omega$ at the origin. A point $p \in \partial\Omega \cap V$ will be described by its geodesic coordinates, i.e. $p = \exp_0 \sum_{i=1}^{n-1} x_i X_i$. Now let us consider the map $H : \mathbb{R}^{n-1} \times \mathbb{R}_+ \to V$,

$$H(x_1, \dots, x_{n-1}, x_n) = \exp_p x_n \nu(p), \qquad p = \exp_0 \sum_{i=1}^{n-1} x_i X_i.$$

This map is a diffeomorphism from a neighborhood $W_0 \times [0, \varepsilon]$ of the origin in $\mathbb{R}^{n-1} \times \mathbb{R}_+$ to a neighborhood V of the origin in $\overline{\Omega}$. In this system of coordinates $\partial\Omega \cap V = \{(x_1, \dots, x_n) : x_n = 0\}$. If $n = 2$ let the boundary be given by a curve $\alpha(t)$ parametrized by the arc length; then we consider $H(t, x_2) = \exp_{\alpha(t)} x_2 \nu(t)$. In these coordinates the metric g and the Christoffel symbols satisfy the set of relations

$$\text{(II.9)} \quad \begin{cases} \text{i)} & g = dx_n^2 + \sum_{i,j=1}^{n-1} g_{ij}\, dx_i\, dx_j, \\ \text{ii)} & \Gamma_{ij}^k(0) = 0, \quad 1 \leq i, j, k \leq n-1, \\ \text{iii)} & \Gamma_{ij}^n(0) = k_i \delta_{ij}, \quad 1 \leq i, j \leq n-1, \text{ where } k_i \text{ are the principal} \\ & \text{curvatures of the boundary at the origin; } k_i > 0 \text{ by (I.1)}, \\ \text{iv)} & \Gamma_{in}^k(0) = -k_i \delta_{ik}, \quad 1 \leq i, k \leq n-1, \\ \text{v)} & \Gamma_{in}^n(0) = 0, \quad i = 1, \dots, n-1, \\ \text{vi)} & \Gamma_{nn}^k(x) = 0, \quad 1 \leq k \leq n, \quad x \in V. \end{cases}$$

In these coordinates

$$\frac{\partial^2 u}{\partial \nu^2}(0) = \nabla_{nn}u(0) = \frac{\partial^2 u}{\partial x_n^2}(0) \quad \text{by vi)}.$$

Now our equation (I.2), which is invariant, can be written

$$\nabla_{nn}u(0) \cdot \det\Big((\nabla_{ij}u(0))_{1 \leq i,j \leq n-1}\Big) + G\Big((\nabla_{ij}u(0))_{(i,j)\neq(n,n)}\Big) = f.$$

By the preceding estimates, G is under control. Therefore a uniform upper bound for $\nabla_{nn}u(0)$ will follow from a uniform lower bound for $\det(\nabla_{ij}u(0))_{1\le i,j\le n-1}$. We shall show that

$$\sum_{i,j=1}^{n-1} \nabla_{ij}u(0)\,\xi_i\,\xi_j \ge C_0\,|\xi|^2, \tag{II.10}$$

where $C_0 > 0$ depends only on Ω, n, g, φ, f. This will follow from $\nabla_{ii}u(0) \ge C_1 > 0$, $i = 1, \dots, n-1$. By symetry it is therefore enough to show that $\nabla_{11}u(0)$ is uniformly bounded below. Now by (II.9) iii),

$$\nabla_{11}u(0) = u_{11}(0) - k_1\,u_n(0) = \varphi_{11}(0) - k_1\,u_n(0)\,.$$

Since we do not have any control of the lower bound of $\varphi_{11}(0)$ we are going to replace u by $v = u + w$, where w will be chosen such that $\nabla_{11}v(0) = \nabla_{11}u(0) = -k_1\,v_n(0)$, and we shall work with v.

Lemma II.1. *One can find, in a neighborhood of the origin, a smooth convex function w such that $w(0) = -\varphi(0)$ and*

i) $\nabla_{11}w(0) = 0$,
ii) $w_i(0) = -\varphi_i(0)$, $i = 1, \dots, n-1$,
iii) $w_{11}(0) = -\varphi_{11}(0)$,
iv) $\nabla_{ij}w(0) = \delta_{ij}$, $2 \le i, j \le n$,
v) $\nabla_{1j}w(0) = 0$, $2 \le j \le n$.

Proof. Recall that

$$\nabla_{11} = \frac{\partial^2}{\partial x_1^2} - \sum_{i=1}^{n} \Gamma_{11}^{i}\,\frac{\partial}{\partial x_i}.$$

Let a_i be the quadratic part of the Taylor expansion of Γ_{11}^{i} at the origin, i.e., for small $|x|$,

$$\Gamma_{11}^{i}(x) = a_i(x) + O(|x|^3)\,, \qquad i = 1, \dots, n.$$

Then by (II.9) iii), $a_n(0) = k_1$, $a_k(0) = 0$, $k = 1, \dots, n-1$.

Let us solve the analytic Cauchy problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial x_1^2} - \displaystyle\sum_{k=1}^{n} a_k(x)\,\frac{\partial w}{\partial x_k} = \lambda\,|x|^2, \\ w|_{x_1=0} = -\varphi(0) - \displaystyle\sum_{i=2}^{n-1} \varphi_i(0)\,x_i - \frac{1}{k_1}\,\varphi_{11}(0)\,x_n \\ \qquad + \dfrac{1}{2}\displaystyle\sum_{i=2}^{n-1}\Big(1 - \frac{k_i}{k_1}\varphi_{11}(0)\Big)\,x_i^2 \\ \qquad + \displaystyle\sum_{i=2}^{n-1} k_i\,\varphi_i(0)\,x_i\,x_n + \frac{1}{2}\,x_n^2, \\ \dfrac{\partial w}{\partial x_1}\Big|_{x_1=0} = -\varphi_1(0) + k_1\,\varphi_1(0)\,x_n\,. \end{cases} \tag{II.11}$$

Then for any λ the conditions i) to v) are satisfied. We just have to show that w is convex, i.e., $(\nabla_{ij}w) \ge 0$ if λ is large enough. We have $\nabla_{11}w = \lambda|x|^2 + O(|x|^3)\cdot O(|\nabla w|)$. Now for $(i,j) \ne (1,1)$ $\nabla_{ij}w - \delta_{ij}$ vanishes at the origin; its linear part is $\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(\nabla_{ij} w)(0)\, x_k$. Since λ occurs in the equation with a quadratic term $|x|^2$ and the data are independant of λ, this linear part is independant of λ. Therefore

$$|\nabla_{ij} w - \delta_{ij}| \le C_1\, |x| + C_2(\lambda)|x|^2\,, \qquad (i,j) \ne (1,1),$$

where C_1 depends only on the data in (II.11). Taking λ large enough and then $|x|$ small, we get easily

$$\sum_{i,j=1}^{n} \nabla_{ij} w(x)\, \xi_i\, \xi_j \ge \frac{\lambda}{2} |x|^2\, \xi_1^2 + \frac{1}{2} \sum_{i=2}^{n} \xi_i^2 \ge 0\,. \quad \diamondsuit$$

Recall that we had to prove that $\nabla_{11} u(0) \ge C_0 > 0$. Now by Lemma II.1

$$\nabla_{11} u(0) = \nabla_{11} v(0) = \Big(v_{11}(0) - \sum_{k=1}^{n} \Gamma_{11}^{k}(0)\, v_k(0)\Big) = v_{11}(0) - k_1\, v_n(0) = -k_1\, v_n(0)$$

since $v_{11}(0) = \varphi_{11}(0) + w_{11}(0) = 0$ by iii).

Therefore we are lead to prove the uniform bound

$$v_n(0) \le -\varepsilon_0 < 0\,. \tag{II.12}$$

This will be proved by constructing an appropriate local upper solution.

Lemma II.2. *For small $\delta > 0$ one can find $\varepsilon_0 > 0$ and a smooth function ρ in $V = \Big\{(x_1, \dots, x_n) : 0 \le x_n \le \delta, 0 \le x_n + \sum_{i=1}^{n-1} \frac{k_i}{4} x_i^2 \le \delta\Big\}$ such that:*
i) $\rho(0) = 0$, $\frac{\partial \rho}{\partial x_n}(0) = -\varepsilon_0$,
ii) $\begin{cases} \det \nabla_{ij} \rho < \det \nabla_{ij} v \ \text{in } V, \\ v \le \rho \ \text{on } \partial V. \end{cases}$

Let us first show how Lemma II.2 implies (II.12). We use this weak form of the maximum principle.

Lemma II.3. *In a bounded open set V let there be given a smooth convex function v (i.e. $(\nabla_{ij} v) \ge 0$) and a smooth function ρ such that*

$$\begin{cases} \det \nabla_{ij} v(x) > \det \nabla_{ij} \rho(x) \ \text{in } V, \\ v \le \rho \ \text{on } \partial V. \end{cases}$$

Then $v \le \rho$ in V.

Proof. Note that ρ is not assumed to be convex. We prove this lemma by contradiction. Suppose there is $\overline{x}$ in V such that $\rho(\overline{x}) < v(\overline{x})$. The continuous function $\rho - v$, which is nonnegative on $\partial\Omega$ and < 0 at $\overline{x}$, will achieve its absolute minimum at $x_0 \in V$. It follows that $((\nabla_{ij}\rho - \nabla_{ij} v)(x_0)) \ge 0$, so $(\nabla_{ij}\rho)(x_0) \ge (\nabla_{ij} v)(x_0)$. But this implies $\det(\nabla_{ij}\rho(x_0)) \ge \det(\nabla_{ij} v(x_0))$, which is a contradiction. $\diamondsuit$

Assuming Lemma II.2, it follows from Lemma II.3 that $v(0, x_n) \le \rho(0, x_n)$ for $x_n > 0$, and since $v(0,0) = \rho(0,0) = 0$ we deduce that $\frac{\partial v}{\partial x_n}(0) \le \frac{\partial \rho}{\partial x_n}(0) = -\varepsilon_0$, and (II.12) is proved.

Proof of Lemma II.2. We can write, with $x' = (x_2, \dots, x_{n-1})$,

$$\begin{aligned} v(x',0) &= \varphi(x') + w(x',0) \\ &= \varphi(0) + w(0) + \sum_{i=1}^{n-1} (\varphi_i(0) + w_i(0))\, x_i + \frac{1}{2}(\varphi_{11}(0) + w_{11}(0))\, x_1^2 \\ &\quad + \sum_{j=2}^{n-1} a_j\, x_1\, x_j + O\Big(\sum_{j=2}^{n-1} x_j^2\Big) + \frac{bk_1}{3} x_1^3 + O\Big(\sum_{j=1}^{n-1} x_j^4\Big). \end{aligned}$$

It follows from Lemma II.1 that

$$v(x',0) = \sum_{j=2}^{n-1} a_j\, x_1\, x_j + \frac{1}{3} bk_1\, x_1^3 + O\Big(\sum_{j=2}^{n-1} x_j^2\Big) + O\Big(\sum_{j=1}^{n-1} x_j^4\Big), \tag{II.13}$$

where a_j and b depend only on the data and the O are under control. We shall set, with $\varepsilon_0 = \delta^{3/2}$,

$$\begin{aligned} \rho(x', x_n) = -\varepsilon_0\, x_n + \sum_{j=2}^{n-1} a_j x_1\, x_j + B \sum_{j=2}^{n-1} x_j^2 \\ + \frac{\delta}{2K}\Big(bx_1 + \frac{K}{\delta}\Big(x_n + \sum_{i=1}^{n-1} \frac{k_i}{4} x_i^2\Big)\Big)^2 + \frac{bk_1}{12} x_1^3, \end{aligned} \tag{II.14}$$

where the small letters b, a_j are fixed by (II.13) and the big B, K are to be chosen.

First claim. $v \le \rho$ on ∂V.

- If $x_n = 0$ and $\sum_{i=1}^{n-1} \frac{k_i}{4} x_i^2 \le \delta$, using (II.13) and (II.14) we get

$$\rho - v \ge (B - C) \sum_{j=2}^{n-1} x_j^2 + \frac{K}{2\delta}\Big(\sum_{j=1}^{n-1} \frac{k_j}{4} x_j^2\Big)^2 - C \sum_{j=1}^{n-1} x_j^4,$$

since for $j = 2, \dots, n-1$

$$|x_1\, x_j^2| \le \frac{1}{2}\Big(\sum_{j=2}^{n-1} x_j^2 + \sum_{j=1}^{n-1} x_j^4\Big).$$

Taking $B \ge 2C$, $\delta < 1$ and $\frac{k_j^2}{16} K \ge 3C$, we get $\rho - v \ge 0$ on this part of ∂V.

- On $x_n + \sum_{i=1}^{n-1} \frac{k_i}{4} x_i^2 = \delta$, since $v = u + w$ and we have uniform bounds for the first derivatives of u on $\overline{\Omega}$, we can write

$$v(x', x_n) = v(x', 0) + x_n\, \tilde{v}, \qquad 0 \le x_n \le \delta,$$

where $\tilde{v}$ is under control. Therefore

$$\begin{aligned} \rho - v \ge -\varepsilon_0\, x_n + B \sum_{j=2}^{n-1} x_j^2 + \frac{\delta}{2K}(bx_1 + K)^2 \\ + \frac{bk_1}{12} x_1^3 - \frac{b\,k_1}{3} x_1^3 - C \sum_{j=2}^{n-1} x_j^2 - C \sum_{j=1}^{n-1} x_j^4 - C\delta. \end{aligned}$$

Now since $|x_1| = O(\delta^{1/2})$ we can take δ so small that $|bx_1| \le 1$. Taking $K \ge 2$ we get $(K+bx_1)^2 \frac{\delta}{2K} \ge \frac{K}{8}\delta$. Now we take $B \ge 2C$, $\varepsilon_0 < 1$. Since in V $|x_1|^3 + \sum_{j=1}^{n-1} x_j^4 \le C\delta$, one can find a constant under control such that $\rho - v \ge \frac{K}{8}\delta - C\delta$. This is positive if $K \ge 8C$.

Second claim. We have, for δ small enough and B large,

$$\det \nabla_{ij}\rho \le C\,\delta^{1/2} \text{ in } V. \tag{II.15}$$

Let us set $\lambda = \frac{\delta}{K}$ (which is small) and $\theta = \lambda\rho$. Then

$$\begin{aligned}\theta = -\varepsilon_0\lambda x_n + \lambda \sum_{j=2}^{n-1} a_j\, x_1\, x_j + \lambda B \sum_{j=2}^{n-1} x_j^2 \\ + \frac{1}{2}\Big(\lambda b x_1 + x_n + \sum_{j=1}^{n-1} \frac{k_j}{4} x_j^2\Big)^2 + \frac{1}{12}\lambda b k_1 x_1^3.\end{aligned} \tag{II.16}$$

Then

$$\text{(II.15)} \Longleftrightarrow \det \nabla_{ij}\theta \le C\lambda^n\,\delta^{1/2}.$$

Let us introduce some notation. We shall denote by a dot the derivative with respect to λ. If $I = (i_1 j_1, i_2 j_2, \dots, i_k j_k)$ is a multi-index of length $|I| = k$, we set

$$\nabla_I\theta = \nabla_{i_1 j_1}\theta \dots \nabla_{i_k j_k}\theta;$$

therefore

$$\nabla_I\dot\theta = \nabla_{i_1 j_1}\dot\theta \dots \nabla_{i_k j_k}\dot\theta.$$

We denote by F the determinant function

$$F(u_{ij}) = \det u_{ij}$$

and set as usual

$$\frac{\partial^{|I|}F}{\partial u_I} = \frac{\partial}{\partial u_{i_1 j_1}} \dots \frac{\partial F}{\partial u_{i_k j_k}}.$$

Let us mention an important thing. Since F is the determinant function, $\frac{\partial F}{\partial u_{ij}}$ is independent of $u_{i\ell}$ and $u_{\ell j}$ for $\ell = 1, \dots, n$. Therefore $\frac{\partial^{|I|}F}{\partial u_I}$ vanishes unless $i_1 \ne i_2 \ne \dots \ne i_k$, $j_1 \ne j_2 \ne \dots \ne j_k$. Now by the Taylor formula we have

$$\det \nabla_{ij}\theta = \sum_{k=0}^{n} \frac{1}{k!}\Big(\frac{d}{d\lambda}\Big)^k \Big[F(\nabla_{ij}\theta(\lambda))\Big]_{\lambda=0}\lambda^k + \lambda^{n+1}G(x,\lambda); \tag{II.17}$$

after that δ, B will be chosen such that the first term of the right hand side of (II.17) will be $O(\lambda^n\,\delta^{1/2})$, and we shall have $G(x,\lambda) = O(1)$. Therefore

$$G(x,\lambda)\,\lambda^{n+1} = \frac{\delta}{K} G(x,\lambda)\,\lambda^n = O(\lambda^n\,\delta^{1/2}).$$

Lemma II.4.

$$\left(\frac{d}{d\lambda}\right)^k \big[F(\nabla_{ij}\theta(\lambda))\big]_{\lambda=0} = \sum_{|I|=k} \frac{\partial^k F}{\partial u_I}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0)$$
$$+\frac{k(k-1)}{2}\nabla_{11}\ddot\theta(0)\sum_{|I|=k-2}\frac{\partial^{k-1}F}{\partial u_I\,\partial u_{11}}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0)$$
$$+\sum_{\substack{q=q_1+q_2\\ q_1+2q_2=k\\ q_2\geq 1}}\frac{k!}{q_1!\,q_2!}\sum_{\substack{|I_j|=q_j\\ I_2\neq(11)}}\frac{\partial^q F}{\partial u_{I_1}\partial u_{I_2}}(\nabla_{ij}\theta(0))\,\nabla_{I_1}\dot\theta(0)\,\frac{\nabla_{I_2}\ddot\theta(0)}{2^{q_2}}\,.$$

Proof. Let us recall the Faa di Bruno formula. If we set $(1) = \left(\frac{d}{d\lambda}\right)^k\big[F(\nabla_{ij}\theta(\lambda))\big]$, then

$$(1) = \sum \frac{k!}{q_1!\dots q_k!}\sum_{|I_j|=q_j}\frac{\partial^q F}{\partial u_{I_1}\dots\partial u_{I_k}}(\nabla_{ij}\theta)\,\frac{\nabla_{I_1}\dot\theta}{1!^{q_1}}\,\frac{\nabla_{I_2}\ddot\theta}{2!^{q_2}}\dots\frac{\nabla_{I_k}\ddot\theta^{(k)}}{k!^{q_k}}$$

where $q = q_1+\dots+q_k$ and the first sum is taken for $q_1+2q_2+\dots+kq_k = k$. But $\left(\frac{d}{d\lambda}\right)^\ell\theta \equiv 0$ for $\ell\geq 3$, and therefore

$$(1) = \sum_{\substack{q=q_1+q_2\\ q_1+2q_2=k}}\frac{k!}{q_1!\,q_2!}\sum_{|I_j|=q_j}\frac{\partial^q F}{\partial u_{I_1}\partial u_{I_2}}(\nabla_{ij}\theta)\,\nabla_{I_1}\dot\theta\cdot\frac{\nabla_{I_2}\ddot\theta}{2^{q_2}}\,.$$

If $|I_2| = q_2 = 0$, the corresponding term in (1) is $\sum\limits_{|I|=k}\frac{\partial^k F}{\partial u_I}(\nabla_{ij}\theta)\nabla_I\dot\theta$. If $|I_2| = q_2 = 1$ and $I_2 = (11)$, then $q_1 = k-2$ and the corresponding term in (1) is

$$\sum_{|I|=k-2}\frac{k!}{(k-2)!}\,\frac{\partial^{k-1}F}{\partial u_I\,\partial u_{11}}(\nabla_{ij}\theta)\,\nabla_I\dot\theta\,\frac{\nabla_{11}\ddot\theta}{2}.$$

The proof is complete. ♢

It follows from (II.16) that

(II.18)
$$\begin{cases}\theta(0) = \dfrac12\Big(x_n + \displaystyle\sum_{j=1}^{n-1}\frac{k_j}{4}x_j^2\Big)^2,\\ \dot\theta(0) = -\varepsilon_0\,x_n + \displaystyle\sum_{j=2}^{n-1}a_j\,x_1\,x_j + B\sum_{j=2}^{n-1}x_j^2 + bx_1\Big(x_n + \sum_{j=1}^{n-1}\frac{k_j}{4}x_j^2\Big) + \frac{1}{12}bk_1\,x_1^3,\\ \ddot\theta(0) = b^2\,x_1^2\,.\end{cases}$$

Moreover using (II.9) we get in V

$$\text{(II.19)}\qquad\begin{cases}\Gamma_{ij}^k = O(\delta^{1/2}), \quad 1\leq i,j,k\leq n-1,\\ \Gamma_{ij}^n = k_i\,\delta_{ij} + O(\delta^{1/2}), \quad 1\leq i,j\leq n-1,\\ \Gamma_{in}^k = O(1), \quad 1\leq i,k\leq n-1,\\ \Gamma_{in}^n = O(\delta^{1/2}), \quad 1\leq i\leq n-1,\\ \Gamma_{nn}^k \equiv 0, \quad 1\leq k\leq n\,.\end{cases}$$

Finally we recall that $\nabla_{ij} = \frac{\partial^2}{\partial x_i\,\partial x_j} - \sum\limits_{k=1}^{n}\Gamma_{ij}^k\,\frac{\partial}{\partial x_k}$.

Now it is easy to see that

$$\text{(II.20)}\qquad \begin{cases} \nabla_{ij}\theta(0) = \dfrac{k_i k_j}{4} x_i x_j - \dfrac{k_i}{2}\delta_{ij}\Gamma + \sigma_{ij}\,, \quad 1 \le i, j \le n-1, \\ \nabla_{in}\theta(0) = \dfrac{k_i}{2} x_i + \sigma_{in}\,, \quad 1 \le i \le n-1, \\ \nabla_{nn}\theta(0) = 1, \end{cases}$$

where

$$\text{(II.21)}\qquad \begin{cases} \Gamma = x_n + \displaystyle\sum_{j=1}^{n-1} \frac{k_j}{4} x_j^2, \\ \sigma_{ij} = O(\Gamma^{3/2}). \end{cases}$$

It is easy to simplify $\det \nabla_{ij}\theta(0)$. Let us set

$$\text{(II.22)}\qquad \begin{cases} c_{ij} = \nabla_{ij}\theta(0) - \Big(\dfrac{k_i}{2} x_i + \sigma_{in}\Big)\nabla_{nj}\theta(0)\,, \quad 1 \le i \le n-1\,, \quad 1 \le j \le n, \\ c_{nj} = \nabla_{nj}\theta(0)\,, \quad 1 \le j \le n; \end{cases}$$

then

$$\text{(II.23)}\qquad \det \nabla_{ij}\theta(0) = \det c_{ij}\,.$$

Now we set

$$\text{(II.24)}\qquad \begin{cases} d_{ij} = c_{ij} - \Big(\dfrac{k_j}{2} x_j + \sigma_{nj}\Big) c_{in}\,, \quad 1 \le i \le n\,, \quad 1 \le j \le n-1, \\ d_{in} = c_{in}\,, \quad 1 \le i \le n; \end{cases}$$

then

$$\text{(II.25)}\qquad \det d_{ij} = \det c_{ij} = \det \nabla_{ij}\theta(0)\,.$$

Using the chain rule and an induction over $|I|$, we easily prove

Lemma II.5.

$$\frac{\partial^{|I|}F}{\partial u_I}(\nabla_{ij}\theta(0)) = \frac{\partial^{|I|}F}{\partial u_I}(d_{ij}) + \sum_{|J|=|I|} c_{IJ}(x)\,\frac{\partial^{|J|}F}{\partial u_J}(d_{ij}),$$

where c_{IJ} are smooth functions satisfying $c_{IJ} = O(\Gamma^{1/2})$.

Using (II.20), (II.21), (II.22) and (II.24) we see easily that

$$\text{(II.26)}\qquad \begin{cases} d_{ij} = -\dfrac{k_i}{2}\delta_{ij}\Gamma + \sigma_{ij}\,, \quad 1 \le i, j \le n-1\,, \\ d_{in} = d_{nj} = 0\,, \quad 1 \le i, j \le n-1\,, \\ d_{nn} = 1\,. \end{cases}$$

Now we have

Lemma II.6. *For $|I| \le n-2$ one has*

$$\frac{\partial^{|I|}F}{\partial u_I}(\nabla_{ij}\theta(0)) = c_I\,\Gamma^{n-|I|-1} + O\big(\Gamma^{n-|I|-1+1/2}\big),$$

where c_I is a real constant.

Moreover, if $|I| = n-2$ we have $c_I < 0$.

Proof. By Lemma II.5 the above formula will be implied by the same formula for $\frac{\partial^{|I|}F}{\partial u_I}(d_{ij})$. Let us multiply the last line of the matrix (d_{ij}) by Γ. We get a matrix $(\tilde{d}_{ij})$, where $\tilde{d}_{ij} = d_{ij}$ if $(i,j) \neq (n,n)$ and $\tilde{d}_{nn} = \Gamma$. Now $F(\tilde{d}_{ij})$ is homogeneous of order n in the $\tilde{d}_{ij}$'s; therefore $\frac{\partial^{|I|}F}{\partial u_I}(\tilde{d}_{ij}) = (1)$ is homogeneous of order $n-|I|$. A term of (1) is either made of diagonal term or it has at least one nondiagonal term. In the first case it is equal to $\mathrm{cte}\,\Gamma^{n-|I|} + O(\Gamma^{n-|I|+1/2})$; in the second case it is bounded by $\mathrm{cte}\,\Gamma^{n-|I|-1}\Gamma^{3/2} = O(\Gamma^{n-|I|+1/2})$. Dividing by Γ, we get the result. If $|I| = n-2$, $\frac{\partial^{|I|}F}{\partial u_I}(d_{ij})$ is homogeneous of order two; its principal term is of the form $d_{ii}\,d_{jj}$; if $1 \le i,j \le n-1$ this term is $O(\Gamma^2)$, so the main term is $d_{ii}\,d_{nn} = -\frac{k_i}{2}\Gamma$. ♢

Next, using (II.18), we get

$$\text{(II.27)} \qquad \begin{cases} \nabla_{11}\dot{\theta}(0) = bk_1\,x_1 + \gamma_{11}, \\ \nabla_{1j}\dot{\theta}(0) = a_j + \dfrac{1}{2}bk_j\,x_j + \gamma_{1j}\,, \quad 1 \le j \le n-1, \\ \nabla_{1n}\dot{\theta}(0) = b + BO(\Gamma^{1/2}), \\ \nabla_{in}\dot{\theta}(0) = BO(\Gamma^{1/2}), \quad 2 \le i \le n-1, \\ \nabla_{ij}\dot{\theta}(0) = \left[2B - \dfrac{1}{2}bk_j\,x_1\right]\delta_{ij} + \gamma_{ij}\,, \quad 2 \le i,j \le n-1, \\ \nabla_{nn}\dot{\theta}(0) = 0, \end{cases}$$

where

$$\text{(II.28)} \qquad \gamma_{ij} = BO(\Gamma) + O(\delta^{3/2})\,.$$

Now we use (II.17), Lemmas II.4 and II.6 and (II.27) to compute $\det \nabla_{ij}\theta(\lambda)$. The integer k will refer to (II.17). We set

$$A_k = \lambda^k \Bigg[\sum_{|I|=k} \frac{\partial^k F}{\partial u_I}(\nabla_{ij}\theta(0))\,\nabla_I\dot{\theta}(0) + k(k-1)[b^2 + O(\Gamma^{1/2})] \sum_{|I|=k-2} \frac{\partial^{k-1}F}{\partial u_I\,\partial u_{11}}(\nabla_{ij}\theta(0))\,\nabla_I\dot{\theta}(0)\Bigg]$$

since by (II.18) $\nabla_{11}\ddot{\theta}(0) = 2b^2 + O(\Gamma^{1/2})$.

Case 1. $k \le n-3$.

Taking $\Gamma \le 1$ and $B \ge 1$, we get from (II.27) $|\nabla_{ij}\dot{\theta}(0)| \le c_0 B$. Then Lemma II.6 implies

$$\text{(II.29)} \qquad |A_k| \le C_k\,\lambda^k\,B^k\,\Gamma^{n-k-1}\,, \quad 0 \le k \le n-3\,.$$

Case 2. $k = n-2$.

Let us consider the first term in A_k. It contains $I = (22, 33, \dots, n-1\,n-1)$, for which

$$\nabla_I\dot{\theta}(0) = \prod_{j=2}^{n-1}(2B + O(\Gamma^{1/2}) + \gamma_{jj}) = (2B)^{n-2} + C(B)\,O(\Gamma^{1/2}) + C(B)\,\delta^{3/2}$$

and

$$\frac{\partial^{n-2}F}{\partial u_I}(\nabla_{ij}\theta(0)) = -\frac{k_1}{2}\Gamma + O(\Gamma^{3/2}).$$

For different I we have $|\nabla_I\dot{\theta}(0)| \le C\,B^{n-3}$ and $\frac{\partial^{n-2}F}{\partial u_I}(\nabla_{ij}\theta(0)) = O(\Gamma)$.

Therefore the first term in A_{n-2} is equal to

$$\lambda^{n-2}\Big[(n-2)!\Big(-\frac{k_1}{2}\Gamma+O(\Gamma^{3/2})\Big)\Big((2B)^{n-2}+C(B)\,O(\Gamma^{1/2})+C(B)\,\delta^{3/2}\Big) \\ +O(B^{n-3})\,O(\Gamma)\Big]$$
$$=-\frac{k_1}{2}\,(n-2)!\,\lambda^{n-2}\,\Gamma\,(2B)^{n-2}\Big(1+O\Big(\frac{1}{B}\Big)+C(B)\,O(\Gamma^{1/2})+C(B)\,\delta^{3/2}\Big).$$

The second term in A_{n-2} is bounded by $C\,B^{n-4}\,\Gamma^2\,\lambda^{n-2}$. It follows that

(II.30) $$A_{n-2}=-c_n\,\lambda^{n-2}\,B^{n-2}\,\Gamma\Big(1+O\Big(\frac{1}{B}\Big)+C(B)\,O(\Gamma^{1/2})+C(B)\,\delta^{3/2}\Big),$$

where c_n is a strictly positive constant.

To deal with the cases where $k=n-1$ and n we use the following facts.

Lemma II.7.

i) $$\sum_{|I|=n-1}\frac{\partial^{n-1}F}{\partial u_I}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0)=(n-1)!\sum_{i,j=1}^{n}\frac{\partial F}{\partial u_{ij}}(\nabla_{ij}\dot\theta(0))\,\nabla_{ij}\theta(0).$$

ii) $$\sum_{|I|=n}\frac{\partial^{n}F}{\partial u_I}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0)=n!\det(\nabla_{ij}\dot\theta(0)).$$

Proof. Identify the coefficients of λ^{n-1} and λ^n in the equality

$$F\Big(\nabla_{ij}\theta(0)+\lambda\nabla_{ij}\dot\theta(0)\Big)=\lambda^n F\Big(\nabla_{ij}\dot\theta(0)+\frac{1}{\lambda}\nabla_{ij}\theta(0)\Big).$$

♢

Case $k=n-1$**.** Here we may neglect the terms which are $O(\lambda^{n-1}\,\delta^{3/2})$. We consider the first term in A_{n-1} and we use i) of Lemma II.7. By (II.20) we have $\nabla_{ij}\theta(0)=O(\Gamma)$ if $1\le i,j\le n-1$ and $\big|\frac{\partial F}{\partial u_{ij}}(\nabla_{ij}\dot\theta(0))\big|\le C(B)$; the corresponding term is then bounded by $\lambda^{n-1}C(B)\Gamma$. We are left with

$$(1)=2\sum_{i=1}^{n-1}\frac{\partial F}{\partial u_{in}}(\nabla_{ij}\dot\theta(0))\,\nabla_{in}\theta(0)+\frac{\partial F}{\partial u_{nn}}(\nabla_{ij}\dot\theta(0))\,\nabla_{nn}\theta(0).$$

We have $\frac{\partial F}{\partial u_{1n}}(\nabla_{ij}\dot\theta(0))=\operatorname{cof}(\nabla_{1n}\dot\theta(0))$. If we develop $\operatorname{cof}(\nabla_{1n}\dot\theta(0))$ with respect to the first column, all the determinants will be $O(\Gamma^{1/2})$ because the last line is $O(\Gamma^{1/2})$, except for the last one, which is equal to

$$(-1)^n\,(b+O(\Gamma^{1/2}))((2B)^{n-2}+C(B)\,O(\Gamma^{1/2})).$$

Therefore

$$\frac{\partial F}{\partial u_{1n}}=-b\,(2B)^{n-2}+C(B)\,O(\Gamma^{1/2})+C(B)\,\delta^{3/2}$$

and

(II.31) $$2\,\frac{\partial F}{\partial u_{1n}}\,(\nabla_{ij}\dot\theta(0))\,\nabla_{1n}\theta(0)=-bk_1\,x_1(2B)^{n-2}+C(B)\,O(\Gamma).$$

For $j = 2, \dots, n-1$ we have $\frac{\partial F}{\partial u_{jn}}(\nabla_{ij}\dot\theta(0)) = b\,a_j(2B)^{n-3} + C(B)\,O(\Gamma^{1/2})$, so

$$2\,\frac{\partial F}{\partial u_{in}}\,(\nabla_{ij}\dot\theta(0))\,\nabla_{in}\theta(0)(2B)^{n-3}\,bk_i\,a_i\,x_i + C(B)\,O(\Gamma). \tag{II.32}$$

Now $\nabla_{nn}\theta(0) = 1$ and $\frac{\partial F}{\partial u_{nn}}(\nabla_{ij}\dot\theta(0)) = \mathrm{cof}(\nabla_{nn}\dot\theta(0))$. We get

$$\frac{\partial F}{\partial u_{nn}} = b\,k_1 x_1 (2B)^{n-2} - \sum_{j=2}^{n-1}\Big(a_j + \frac{1}{2}bk_j\,x_j\Big)^2 (2B)^{n-3} + C(B)\,O(\Gamma).$$

It follows from (II.31) and (II.32) that

$$\begin{aligned}(1) = -bk_1\,x_1(2B)^{n-2} + (2B)^{n-3}\sum_{j=2}^{n-1} bk_j\,x_j\,a_j + bk_1\,x_1(2B)^{n-2}\\ - (2B)^{n-3}\sum_{j=2}^{n-1}\Big(a_j + \frac{1}{2}bk_j\,x_j\Big)^2 + C(B)\,O(\Gamma),\end{aligned}$$

$$(1) = -(2B)^{n-3}\sum_{j=2}^{n-1}\Big(a_j^2 + \frac{1}{4}b^2\,k_j^2\,x_j^2\Big) + C(B)\,O(\Gamma).$$

Let us consider the second term in A_{n-1}, which is

$$\sum_{|I|=n-3}\frac{\partial^{n-2}F}{\partial u_I\,\partial u_{11}}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0);$$

it is bounded by $C\,B^{n-3}\,\Gamma^{n-(n-2)-1} = C\,B^{n-3}\,\Gamma$, by Lemma II.6. Therefore we get

$$A_{n-1} = -(n-1)!\,\lambda^{n-1}(2B)^{n-3}\sum_{j=2}^{n-1}\Big(a_j^2 + \frac{1}{4}b^2\,k_j^2\,x_j^2\Big) + \lambda^{n-1}\,C(B)\,O(\Gamma). \tag{II.33}$$

Case $k = n$. By Lemma II.7

$$\begin{aligned}A_n = \lambda^n\Big[&\underbrace{n!\,\det(\nabla_{ij}\dot\theta(0))}_{(1)}\\ &\underbrace{+\,n(n-1)(b^2 + O(\Gamma^{1/2}))\sum_{|I|=n-2}\frac{\partial^{n-1}F}{\partial u_I\,\partial u_{11}}(\nabla_{ij}\theta(0))\,\nabla_I\dot\theta(0)}_{(2)}\Big].\end{aligned}$$

Let us consider (1). We use (II.27). We develop the determinant with respect to the first column. Then all the determinants are bounded by $C(B)\,\Gamma^{1/2}$, because the last line left is $O(\Gamma^{1/2})$, except for the last one. For the last determinant we develop with respect to the last column. Then all the determinants will be bounded by $C(B)\,\Gamma^{1/2}$, except for the first one. Then

$$\begin{aligned}(1) &= n!(-1)^{n+1}b\cdot(-1)^n b(2B)^{n-2} + C(B)\,O(\Gamma^{1/2})\\ &= -n!\,b^2(2B)^{n-2} + C(B)\,O(\Gamma^{1/2}).\end{aligned} \tag{II.34}$$

Let us consider (2). In $\frac{\partial F}{\partial u_{11}}(\nabla_{ij}\theta(0))$ the terms u_{1j} and u_{i1} have been cancelled. Let I be a multi-index with $|I| = n-2$. If I contains (i,j) which is different from $(2,2), \dots, (n-1,n-1)$, then $\nabla_I\dot\theta(0)$ is $O(\Gamma^{1/2})$ by (II.27), and the corresponding term in (2) is bounded by $C(B)\Gamma^{1/2}$. When $I = (22, 33, \dots, n-1\,n-1)$ the corresponding term is

$$\frac{\partial^{n-1}F}{\partial u_{22}\dots\partial u_{n-1\,n-1}\partial u_{11}}\nabla_{22}\dot\theta(0)\dots\nabla_{n-1\,n-1}\dot\theta(0) = (2B)^{n-2} + O(\Gamma^{1/2})$$

and there are $(n-2)!$ terms of this form. Therefore

$$(2) = (b^2 + O(\Gamma^{1/2}))\,n(n-1)(n-2)!\big[(2B)^{n-2} + C(B)\,O(\Gamma^{1/2})\big]. \tag{II.35}$$

It follows from (II.34) and (II.35) that

$$A_n = C(B)\,O(\Gamma^{1/2})\,\lambda^n. \tag{II.36}$$

Finally, to compute $\det(\nabla_{ij}\theta)$, me must consider the last term in Lemma 4, which is

$$(1) = \sum_{\substack{q=q_1+q_2\\ q_1+2q_2=k\\ q_2\geq 1}} \frac{k!}{q_1!q_2!} \sum_{\substack{|I_j|=q_j\\ I_2\neq(11)}} \frac{\partial^q F}{\partial u_{I_1}\partial u_{I_2}}(\nabla_{ij}\theta(0))\,\nabla_{I_1}\dot\theta(0)\,\frac{\nabla_{I_2}\ddot\theta(0)}{2^{q_2}}.$$

Since $q_2 \geq 1$, $I_2 \neq (11)$, we have by (II.18) and (II.19) $\nabla_{I_2}\ddot\theta(0) = O(\Gamma^{1/2})$. Then $q = q_1 + q_2 \leq k-1$, and Lemma II.6 implies that $\frac{\partial^q F}{\partial u_{I_1}\partial u_{I_2}}(\nabla_{ij}\theta(0)) = O(\Gamma^{n-q-1}) = O(\Gamma^{n-k})$. We also get, by (II.27), $|\nabla_{I_1}\dot\theta(0)| \leq CB$, so

$$(1) = C(B)\,O\big(\Gamma^{n-k+\frac{1}{2}}\big). \tag{II.37}$$

Summing up, it follows from (II.29), (II.30), (II.33), (II.36) and (II.37) that

$$\begin{aligned}\det(\nabla_{ij}\theta(\lambda)) \leq{}& \underbrace{-c_n\,\lambda^{n-2}\,B^{n-2}\,\Gamma\Big(1 + O\Big(\frac{1}{B}\Big) + C(B)\,O(\Gamma^{1/2}) + C(B)\,\delta^{3/2}\Big)}_{(1)}\\ &+\underbrace{\sum_{k=0}^{n-3}\lambda^k\,C_k\,B^k\,\Gamma^{n-k-1}}_{(2)}\underbrace{-\lambda^{n-1}\,(2B)^{n-3}\sum_{j=2}^{n-1}\Big(a_j^2 + \frac{1}{4}b^2\,k_j^2\,x_j^2\Big)}_{(3)}\\ &+\underbrace{C(B)\,\lambda^{n-1}\,\Gamma}_{(4)}+\underbrace{C(B)\,\lambda^n\,\Gamma^{1/2}}_{(5)}+\underbrace{C(B)\,\lambda^{n+1}}_{(6)}.\end{aligned}$$

Since $\Gamma \leq \delta$ and $\lambda = \frac{\delta}{K}$, the two last terms (5) and (6) are bounded by $C(B)\,\lambda^n\,\delta^{1/2}$. The term (3) is negative. We first choose B so large that $1 + O\big(\frac{1}{B}\big) \geq \frac{1}{2}$. Then we write

$$(4) = \lambda^{n-2}\,B^{n-2}\,\Gamma\cdot\frac{C(B)}{B^{n-2}}\,\lambda.$$

Therefore

$$\det(\nabla_{ij}\theta(\lambda)) \le -c_n\,\lambda^{n-2}\,B^{n-2}\,\Gamma\Big(\frac{1}{2} + C_1(B)\,O(\Gamma^{1/2}) + C_2(B)\,\lambda + C_3(B)\,\delta^{3/2}\Big) + \sum_{k=0}^{n-3} C_k\,\lambda^k\,\Gamma^{n-k-1}\cdot B^{n-3} + C(B)\,\lambda^n\,\delta^{1/2}$$

$$\le -c_n\,\lambda^{n-2}\,B^{n-2}\,\Gamma\Big(\frac{1}{2} + C_1(B)\,O(\Gamma^{1/2}) + C_2(B)\,\lambda + \sum_{k=0}^{n-3} C_k\,\Gamma^{n-k-2}\,\lambda^{k-n+2}\,\frac{1}{B}\Big) + C(B)\,\lambda^n\,\delta^{1/2}.$$

Recall that $\Gamma \le \delta$, $\lambda = \frac{\delta}{K}$, $K \ge 1$, and so

$$C_1(B)\,O(\Gamma^{1/2}) + C_2(B)\,\lambda + \frac{1}{B}\sum_{k=0}^{n-3} C_k\,\Gamma^{n-k-2}\,\lambda^{k-n+2} \le C(B)\,\delta^{1/2} + C_2(B)\,\delta + \frac{1}{B}\sum_{k=0}^{n-3} C_k\,K^{n-k-2}.$$

We take B so large that $\frac{1}{B}\sum_{k=0}^{n-3} C_k\,K^{n-2-k} \le \frac{1}{10}$; then, B being fixed, we take δ so small that $C(B)\,\delta^{1/2} + C_2(B)\,\delta \le \frac{1}{10}$.

It follows that

$$\det(\nabla_{ij}\theta(\lambda)) \le -\frac{3}{10}\,c_n\,\lambda^{n-2}\,B^{n-2}\,\Gamma + C(B)\,\lambda^n\,\delta^{1/2} \le C(B)\,\lambda^n\,\delta^{1/2},$$

which by (II.15) proves our claim. Now, $\det\nabla_{ij}v(x) = \det(\nabla_{ij}u + \nabla_{ij}w) \ge \det\nabla_{ij}u$, since $(\nabla_{ij}w) \ge 0$; so $\det\nabla_{ij}v(x) \ge f(x) \ge c_0 > C\,\delta^{1/2} \ge \det\nabla_{ij}\rho(x)$ in V, which completes the proof of Lemma II.2.

3. Interior estimates of the second derivatives

Let us denote by Δ the Laplace-Beltrami operator, $\Delta = g^{k\ell}\,\nabla_{k\ell}$. We consider

$$w = \operatorname{Log}\Delta u + \lambda h,$$

where λ is a positive constant to be chosen and h the function given by (I.1).

Let p be a point at which the continuous function w is maximun on $\overline{\Omega}$. If $p \in \partial\Omega$ then, by the preceeding estimates of the second derivatives on the boundary, we deduce that all the second derivatives are bounded on $\overline{\Omega}$. Therefore we may assume that $p \in \Omega$. At p (which can be taken as the origin) we consider a set of normal coordinates. Recall that then

$$g_{ij}(p) = \delta_{ij}, \quad \frac{\partial g_{ij}}{\partial x_k}(p) = 0, \quad \Gamma_{ij}^k(p) = 0, \quad 1 \le i, j, k \le n. \tag{II.38}$$

Therefore at p the covariant derivatives agree with the usual one. Without loss of generality we also may assume that at p the matrix $(\nabla_{ij}u)$ is diagonal, i.e.

$$\nabla_{ij}u(p) = 0 \text{ if } i \ne j. \tag{II.39}$$

Then at p the matrix $(u^{ij}) = (\nabla_{ij}u)^{-1}$ is also diagonal.

In the sequel we shall use the Einstein sommation convention. Let us set $L = u^{ij}\nabla_{ij}$. It is then easy to see that at p

$$\Delta u\, Lw = \underbrace{L\,\Delta u}_{(1)} - \underbrace{\frac{u^{ii}(\Delta u)_i^2}{\Delta u}}_{(2)} + \underbrace{\lambda\,\Delta u\, Lh}_{(3)}. \tag{II.40}$$

First of all, by condition (I.1) we have

$$(3) \geq C_0\lambda\,\Delta u \sum_{i=1}^{n} u^{ii}. \tag{II.41}$$

Next we compute the term (1). If we differentiate the equation $\operatorname{Log}\det\nabla_{ij}u = \operatorname{Log} f(x) = \Phi(x)$ twice, we get

$$u^{ij}\,\nabla_{ijk\ell}u + \nabla_\ell(u^{ij})\,\nabla_{ijk}u = \nabla_{k\ell}\Phi;$$

therefore

$$u^{ij}\,\nabla_{k\ell ij}u = \nabla_{k\ell}\Phi - \nabla_\ell(u^{ij})\,\nabla_{ijk}u + u^{ij}(\nabla_{k\ell ij}u - \nabla_{ijk\ell}u).$$

Multiplying by $g^{k\ell}$ and summing, we get at p

$$L\,\Delta u = \underbrace{\Delta\Phi}_{(a)} - \underbrace{g^{kk}\,\nabla_k(u^{ij})\,\nabla_{ijk}u}_{(b)} + \underbrace{u^{ii}\,g^{kk}(\nabla_{kkii}u - \nabla_{iikk}u)}_{(c)}. \tag{II.42}$$

By the well known Ricci formulas, $\nabla_{kkii}u - \nabla_{iikk}u$ is a linear combination of $\nabla_{pq}u$ and $\nabla_\ell u$ with coefficients depending only on the metric g (via the components of the Riemann curvature tensor). Then

$$|(a)+(c)| \leq C\Big(1 + \Delta u \sum_{i=1}^{n} u^{ii} + \sum_{i=1}^{n} u^{ii}\Big), \tag{II.43}$$

where C is under control.

Let us compute the term (b). We have at p

$$\nabla_k(u^{ij}) = -u^{ir}\,u^{sj}\,\nabla_{rsk}u = -u^{ii}\,u^{jj}\,\nabla_{ijk}u;$$

therefore

$$(b) = g^{kk}\,u^{ii}\,u^{jj}(\nabla_{ijk}u)^2. \tag{II.44}$$

Now we use the obvious inequality (for fixed k)

$$u^{ii}\,u^{jj}\Big(\nabla_{ijk}u - \frac{1}{\Delta u}\,u_{ik}(\Delta u)_j\Big)^2 \geq 0.$$

We get using (II.39)

$$u^{ii}\,u^{jj}(\nabla_{ijk}u)^2 \geq 2\,u^{kk}\,u^{jj}\,u_{kk}\,\nabla_{kjk}u\,\frac{(\Delta u)_j}{\Delta u} - \frac{1}{(\Delta u)^2}\,u^{kk}\,u^{jj}\,u_{kk}^2(\Delta u)_j^2.$$

Since $u^{kk}\,u_{kk} = 1$ at p, we get

$$u^{ii}\,u^{jj}(\nabla_{ijk}u)^2 \geq 2\,u^{jj}\,\nabla_{kjk}u\,\frac{(\Delta u)_j}{\Delta u} - u^{jj}\,u_{kk}\,\frac{(\Delta u)_j^2}{(\Delta u)^2}.$$

Multiplying by g^{kk} and taking the sum over k, we get

$$g^{kk}\,u^{ii}\,u^{jj}(\nabla_{ijk}u)^2 \geq 2\,g^{kk}\,u^{jj}\,\nabla_{kjk}u\,\frac{(\Delta u)_j}{\Delta u} - u^{jj}\,\frac{(\Delta u)_j^2}{\Delta u}.$$

By the Ricci formulas

$$g^{kk}\nabla_{kjk}u = g^{kk}\nabla_{kkj}u + R = (\Delta u)_j + R,$$

where R is a linear combination of u_ℓ. Therefore R is under control. It follows that

$$g^{kk}u^{ii}u^{jj}(\nabla_{ijk}u)^2 \geq 2u^{jj}\frac{(\Delta u)_j^2}{\Delta u} + 2Ru^{jj}\frac{(\Delta u)_j}{\Delta u} - u^{jj}\frac{(\Delta u)_j^2}{\Delta u},$$

and so

$$(b) \geq u^{jj}\frac{(\Delta u)_j^2}{\Delta u} + 2Ru^{jj}\frac{(\Delta u)_j}{\Delta u}. \tag{II.45}$$

By (II.42) to (II.45) we can write

$$L\Delta u \geq u^{jj}\frac{(\Delta u)_j^2}{\Delta u} + 2Ru^{jj}\frac{(\Delta u)_j}{\Delta u} - C_0 - C_1\Delta u\sum_{i=1}^n u^{ii} - C_2\sum_{i=1}^n u^{ii}.$$

Now since w is maximum at p we have $\nabla_j w(p) = 0$. Therefore at p

$$\frac{(\Delta u)_j}{\Delta u} + \lambda h_j = 0.$$

It follows that, with a constant C under control, we have

$$L\Delta u \geq u^{jj}\frac{(\Delta u)_j^2}{\Delta u} - C\lambda\sum_{j=1}^n u^{jj} - C\Delta u\sum_{j=1}^n u^{jj} - C\sum_{j=1}^n u^{jj} - C.$$

Coming back to (II.40) and using (II.41), we get, taking $\lambda \geq 1$,

$$\Delta u \cdot Lw \geq -C - C\lambda\sum_{j=1}^n u^{jj} - C\Delta u\sum_{j=1}^n u^{jj} + C_0\lambda\Delta u\sum_{j=1}^n u^{jj}.$$

Now we use the well known inequality

$$\sum_{j=1}^n u^{jj} \geq n(\det u^{ij})^{1/n} = n f^{-1/n} \geq \alpha_0 \text{ in } \overline{\Omega}.$$

It follows that

$$\Delta u \cdot Lw \geq -\frac{C}{\alpha_0}\sum_{j=1}^n u^{jj} - C\lambda\sum_{j=1}^n u^{jj} + (C_0\lambda - C)\Delta u\sum_{j=1}^n u^{jj}.$$

We take λ so large that $C_0\lambda \geq 2C$, and we use the fact that, since the matrix (u^{ij}) is positive, we have $Lw(p) \leq 0$ at p. Therefore

$$\frac{1}{2}C_0\lambda\Delta u(p) \leq C\lambda + \frac{C}{\alpha_0},$$

which shows that $\Delta u(p)$ is under control.

So far we have proved uniform upper bounds for the C^2 norm on $\overline{\Omega}$ of the solutions. It then follows that the eigenvalues are uniformly bounded below, which implies that the linearized operator of equation (I.2) is actually uniformly elliptic. We may apply the results of Evans [E] (for the interior estimates) and of Krylov [KR] (see also [KA]) (for boundary estimates) to get uniform upper bound for the $C^{2+\alpha}(\overline{\Omega})$ norm. This completes the proof of Theorem I.1.

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