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ON THE CONVERGENCE OF $\sum c_n f(nx)$ AND THE LIP 1/2 CLASS

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Abstract. We investigate the almost everywhere convergence of $\sum c_n f(nx)$, where f is a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\,dx = 0.$$

By a known criterion, if f satisfies the above conditions and belongs to the Lip α class for some $\alpha > 1/2$, then $\sum c_n f(nx)$ is a.e. convergent provided $\sum c_n^2 < +\infty$. Using probabilistic methods, we prove that the above result is best possible; in fact there exist Lip 1/2 functions f and almost exponentially growing sequences (n_k) such that $\sum c_k f(n_k x)$ is a.e. divergent for some (c_k) with $\sum c_k^2 < +\infty$. For functions f with Fourier series having a special structure, we also give necessary and sufficient convergence criteria. Finally we prove analogous results for the law of the iterated logarithm.

1. Introduction

Let $f : R \to R$ be a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\,dx = 0, \qquad \int_0^1 f^2(x)\,dx < +\infty. \tag{1}$$

The purpose of the present paper is to study the almost everywhere convergence of the series

$$\sum c_n f(nx) \tag{2}$$

and of the corresponding lacunary series $\sum c_k f(n_k x)$, where (n_k) is a sequence of integers growing rapidly. Specifically, we shall study under what conditions the system $(f(nx))$ has the property that the series (2) is almost everywhere convergent provided

$$\sum c_n^2 < +\infty. \tag{3}$$

Using the standard terminology of the theory of orthogonal series, in this case the system $(f(nx))$ will be called a *convergence system*. Carleson's convergence theorem (see [5]) states that $(f(nx))$ is a convergence system in the case $f(x) = \sin 2\pi x$. Using this result, Gaposhkin [13] proved the following more general criterion:

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Theorem. *Let f satisfy (1) and assume that f belongs to the Lip α class for some $\alpha > 1/2$. Then $(f(nx))$ is a convergence system.*

In the opposite direction, Nikishin [20] showed that there exists a continuous function f satisfying (1) such that $(f(nx))$ is not a convergence system. For earlier, related examples see Erdős [8], Marstrand [19]; in particular we note that Marstrand's results on the Khinchin conjecure imply the existence of a function f taking only two values and satisfying (1) such that $(f(nx))$ is not a convergence system. Nikishin's function does not satisfy any Lipschitz condition; in fact, its modulus of continuity $\omega(f,\delta)$ fails to satisfy even $\omega(f,\delta) \le (\log 1/\delta)^{-\beta}$ for $\beta > 1$. (This follows from comparing the results of [12], [20].) Hence there is a gap between Gaposhkin's positive result and Nikishin's counterexample, and the precise continuity condition required for $(f(nx))$ to be a convergence system remains open. The purpose of the present paper is to close this gap and to prove the following theorem:

Theorem 1. *There exists a function $f \in Lip\, 1/2$ satisfying (1) and such that $(f(nx))$ is not a convergence system.*

Actually, we shall prove a stronger result, namely:

Theorem 2. *There exists a function $f \in Lip\, 1/2$ satisfying (1) and for any $\varepsilon_k \downarrow 0$ there exists a sequence (n_k) of integers satisfying*

$$n_{k+1}/n_k \ge 1 + \varepsilon_k \qquad (k \ge k_0) \tag{4}$$

such that the series $\sum c_k f(n_k x)$ is a.e. divergent for some (c_k) with $\sum c_k^2 < +\infty$.

In other words, there exists a function $f \in \mathrm{Lip}\, 1/2$ such that $(f(nx))$ not only fails to be a convergence system, but it has almost exponentially fast subsequences which are not convergence systems either.

By a classical theorem of Kac [14], if f is a $\mathrm{Lip}\, \alpha$ function ($\alpha > 0$) satisfying (1) and (n_k) is a sequence of integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1, \tag{5}$$

then $(f(n_k x))$ is a convergence system. Actually, here $f \in \mathrm{Lip}\, \alpha$ can be replaced by the much weaker condition

$$\omega_2(f,\delta) \le (\log 1/\delta)^{-\beta}, \qquad \beta > 1,$$

where $\omega_2(f,\delta)$ is the quadratic modulus of continuity of f (see Gaposhkin [12]). Comparing these results with the fact that $\sum c_n f(nx)$ can be a.e. divergent under (3) even if $f \in \mathrm{Lip}\, 1/2$, we see that the convergence properties of lacunary series $\sum c_k f(n_k x)$ are much better than those of general series $\sum c_n f(nx)$. Several papers have investigated the behavior of $\sum c_k f(n_k x)$ under the Hadamard gap condition (5) (for an extensive survey see [10], Chapter 4) and many interesting results on the convergence, partial sum fluctuations and asymptotic distribution of such sums have been proved. On the other hand, very little is known on the behavior of $\sum c_k f(n_k x)$ if (n_k) grows slower than exponentially. For example, it is not known if the result of Kac formulated above remains valid under any gap condition weaker than (5). Theorem 2 shows that the answer is negative: no matter how slowly ε_k tends to 0, the gap condition (4) does not imply the a.e. convergence of $\sum c_k f(n_k x)$ under $\sum c_k^2 < +\infty$ even if $f \in \mathrm{Lip}\, 1/2$. This behavior is in sharp contrast with the behavior of lacunary trigonometric series, whose typical properties under the

Hadamard gap condition (convergence, central limit theorem, law of the iterated logarithm) remain valid under lacunarity conditions much weaker than (5) (see e.g. [7], [9], [23], [24]).

The following complement of Theorem 1 shows that for $f \in \mathrm{Lip}\,(1/2-\varepsilon)$, $\varepsilon > 0$, $f(nx)$ can fail to be a convergence system in a fairly strong form. In fact, we have

Theorem 3. *For any $0 < \alpha < 1/2$ there exist a function $f \in Lip\,\alpha$ satisfying (1) and a real sequence (c_n) satisfying*

$$\sum c_n^2 (\log n)^\gamma < +\infty \qquad \text{for all} \quad 0 < \gamma < 1 - 2\alpha$$

such that $\sum c_n f(nx)$ is a.e. divergent.

It is natural to ask if $\sum c_n^2 (\log n)^\gamma < +\infty$ for some $\gamma \geq \gamma(\alpha)$ implies the a.e. convergence of $\sum c_n f(nx)$ if $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leq 1/2)$. For $\alpha = 1/2$ the answer is positive (see Gaposhkin [11], Theorem 3) but for $0 < \alpha < 1/2$ the question remains open.

The results formulated above show that $f \in \mathrm{Lip}\,\alpha$, $\alpha > 1/2$, is a best possible continuity condition for $(f(nx))$ to be a convergence system. On the other hand, this condition is not necessary and sufficient: using Theorem 2 of [13] one can easily construct continuous functions f satisfying no Lipschitz condition such that $(f(nx))$ is a convergence system. Under some (restrictive) arithmetic conditions on the Fourier series of f, the following theorem gives a necessary and sufficient condition:

Theorem 4. *Assume that f satisfies (1) and its Fourier series has the form*

$$f \sim \sum_{k \in H} (a_k \cos 2\pi k x + b_k \sin 2\pi k x),$$

where H is a set of coprime integers. Then $(f(nx))$ is a convergence system iff

$$\sum_{k \in H} (|a_k| + |b_k|) < +\infty. \tag{6}$$

Moreover, if (6) fails then for any $\varepsilon_k \downarrow 0$, there exists a sequence (n_k) of integers satisfying (4) such that $(f(n_k x))$ is not a convergence system.

Our previous results show the significance of the Lip 1/2 class for the convergence of $\sum c_k f(n_k x)$. In what follows, we prove that similar results hold for other asymptotic properties of $\sum c_k f(n_k x)$, e.g., for the law of the iterated logarithm. It is a well known fact that for rapidly increasing (n_k) the sequence $(\sin 2\pi n_k x)$ behaves like a sequence of independent random variables. For example, by results of Erdős [9] and Takahashi [23], [24], if (n_k) satisfies the lacunarity condition

$$n_{k+1}/n_k \geq 1 + c/k^\beta, \qquad \beta < 1/2, \tag{7}$$

then $(\sin 2\pi n_k x)$ satisfies the central limit theorem and the law of the iterated logarithm, i.e.,

$$\lim_{N\to\infty} \lambda \left(0 \leq x \leq 1 : \sum_{k \leq N} \sin 2\pi n_k x < t\sqrt{N/2} \right) = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2}\, du \tag{8}$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k \leq N} \sin 2\pi n_k x = 1 \qquad \text{a.e.,} \tag{9}$$

where λ denotes the Lebesgue measure. Moreover, the gap condition (7) is best possible in the sense that if we require only $n_{k+1}/n_k \ge 1 + c/\sqrt{k}$, both (8) and (9) become false. (See Erdős [9], Berkes [2], [3].) As the example in Kac [15], p. 646, shows, (8) and (9) do not extend to sequences of the form $f(n_k x)$ (with f satisfying (1)) even if f is a trigonometric polynomial and (n_k) satisfies the Hadamard gap condition (5). However, Dhompongsa [6] proved the following result (see also Takahashi [25]):

Theorem. *Let $f \in Lip\,\alpha$, $\alpha > 1/2$ satisfy (1) and let (n_k) be a sequence of integers satifying (7). Then for any bounded sequence (c_k) of real numbers we have*

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \left| \sum_{k\le N} c_k f(n_k x) \right| < +\infty \qquad a.e. \tag{10}$$

Actually, in [6] (10) is proved only for $c_k = 1$, but the proof remains valid for arbitrary bounded sequences (c_k). Moreover, (10) holds uniformly in f in the sense that

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sup_{f\in\Lambda_\alpha} \left| \sum_{k\le N} c_k f(n_k x) \right| < +\infty \qquad \text{a.e.}, \tag{11}$$

where Λ_α is the class of functions satisfying (1) and $|f(x) - f(y)| \le |x - y|^\alpha$ for all $0 \le x, y \le 1$. (Under the Hadamard gap condition (5), relation (11) was proved earlier by Kaufman and Philipp [17].) In what follows, we shall prove that, similarly to the convergence of $\sum c_k f(n_k x)$, the assumption $f \in \mathrm{Lip}\,\alpha$, $\alpha > 1/2$, for the LIL (10) is sharp. In fact, we shall prove the following:

Theorem 5. *There exists a function $f \in Lip\,1/2$ satisfying (1) and for each $\varepsilon_k \downarrow 0$ there exists a sequence (n_k) of integers satisfying (4) such that*

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k\le N} c_k f(n_k x) = +\infty \qquad a.e. \tag{12}$$

for some sequence $c_k = \pm 1$.

Theorem 5 shows that there is a basic difference between the LIL behavior of $f(n_k x)$ for Hadamard and subexponential (n_k): while in the case $n_{k+1}/n_k \ge q > 1$ the LIL (10) holds for a very large class of functions $f \in L_2$ containing all Lipschitz functions and functions of bounded variation (see Philipp [21], Takahashi [22]), if we assume only (4) for some $\varepsilon_k \downarrow 0$ then (10) fails even for $f \in \mathrm{Lip}\,1/2$.

The following version of Theorem 5 shows that if we assume only $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1/2$, $f(n_k x)$ can fail the LIL in a much stronger form than in (12). In fact, we have

Theorem 6. *For any $0 < \alpha < 1/2$ there exist a function $f \in Lip\,\alpha$ satisfying (1), a sequence (n_k) of integers satisfying (7) for all $\beta > 0$ and a sequence $c_k = \pm 1$ such that*

$$\limsup_{N\to\infty} N^{-1/2} (\log N)^{-\rho} \sum_{k\le N} c_k f(n_k x) = +\infty \qquad a.e. \tag{13}$$

for any $0 < \rho < 1/2 - \alpha$.

Similarly to the convergence of $\sum c_k f(n_k x)$, we note that although $f \in \operatorname{Lip}\alpha$, $\alpha > 1/2$, is an optimal condition for the LIL (10) in terms of the continuity properties of f, it is not a necessary and sufficient condition. In what follows, we give a sufficient condition for the LIL (10) in terms of the Fourier coefficients of f, and then show that under certain additional assumptions it is also necessary:

Theorem 7. *Let f satisfy (1) and assume that its Fourier series*

$$f \sim \sum (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$$

satisfies

$$\sum (|a_k| + |b_k|) < +\infty. \tag{14}$$

Then for any sequence (n_k) of integers satisfying (7) and for any bounded sequence (c_k) of real numbers we have

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \left| \sum_{k\le N} c_k f(n_k x) \right| < +\infty \qquad a.e.$$

For additional information on the lim sup in Theorem 7, see the Remark before the proof of Theorems 4 and 8 in Section 2.

Theorem 8. *Assume that f satisfies (1) and its Fourier series has the form*

$$f \sim \sum_{k\in H} (a_k \cos 2\pi kx + b_k \sin 2\pi kx),$$

where H is a set of coprime integers. Then the LIL (10) holds for all (n_k) satisfying (7) and all bounded sequences (c_k) if and only if

$$\sum_{k\in H} (|a_k| + |b_k|) < +\infty. \tag{15}$$

As a comparison, note that $f \in \operatorname{Lip}\alpha$, $\alpha > 1/2$, implies not only (14) but also

$$\sum (|a_k| + |b_k|) k^\gamma < +\infty \quad \text{for all} \quad 0 < \gamma < \alpha - 1/2.$$

(Use, e.g., formula (3.4) in Zygmund [26], p. 241). Hence $f \in \operatorname{Lip}\alpha$, $\alpha > 1/2$ misses being a necessary and sufficient condition by at least a polynomial factor.

In conclusion we note that the growth properties of $\sum_{k\le N} f(n_k x)$ for certain (discontinouos) functions f were studied in our previous paper [4] in connection with the uniform distibution of $\{n_k x\}$ mod 1. In particular, the proofs of the results there show that the bad LIL behavior described by Theorem 5 above can hold also if f is the normalized indicator function of an interval or $f(x) = x - [x] - 1/2$.

2. Proof of the theorems

Let (ε_n) be independent r.v.'s taking the values ± 1 with probability 1/2 and let

$$f(x) = \sum_p \frac{\varepsilon_p \sin 2\pi px}{p}, \tag{16}$$

where the sum is extended over all primes p. Letting

$$s_j^2 = \sum_{2^j \le p < 2^{j+1}} p^{-2},$$

it follows from the prime number theorem that

$$s_j^2 \le 2^{-2j} \sum_{p<2^{j+1}} 1 \le \text{const} \cdot 2^{-j} j^{-1},$$

and thus, using Kahane [16], Theorem 2 on p. 66 with $\beta = 1/2$, $\gamma = -1/2$ and Theorem 3 on p. 49, we get that with probability one the series (16) converges uniformly in $[0,1]$ and its sum is a Lip 1/2 function. In the sequel, (ε_n) will denote a fixed (nonrandom) ± 1 sequence having this property.

Let (p_n) be the sequence of primes, and arrange the squarefree integers $p_{i_1} \cdots p_{i_r}$ $(i_1 < i_2 < \cdots < i_r,\ r = 0, 1, \dots)$ as a single sequence (m_k) so that for each $k \ge 1$ the set $\{m_1, \dots, m_{2^k}\}$ is identical with the set of all products $p_{i_1} \cdots p_{i_r}$, $1 \le i_1 < \cdots < i_r \le k$, $r \ge 0$. (For example, let $m_1 = 1$, $m_2 = p_1$, $m_3 = p_2$, $m_4 = p_1 p_2$, and for each $k \ge 1$ define the segment $\{m_j,\ 2^k < j \le 2^{k+1}\}$ as the set of the 2^k products $p_{i_1} \cdots p_{i_r} p_{k+1}$, $1 \le i_1 < \ldots < i_r \le k$, $r \ge 0$, arranged in an arbitrary manner.) Note that the sequence (m_k) is not increasing. Define a ± 1 sequence (δ_k) by

$$\delta_k = \varepsilon_{p_{i_1}} \cdots \varepsilon_{p_{i_r}} \qquad \text{if} \qquad m_k = p_{i_1} \cdots p_{i_r}. \tag{17}$$

Lemma 1. *For any $N \ge 1$ we have*

$$\int_0^1 \left(\sum_{k=1}^N \delta_k f(m_k x) \right)^2 dx \ge \text{const} \cdot N (\log_3 N)^2, \tag{18}$$

where $\log_3$ denotes the three times iterated logarithm.

Proof. Given positive integers a, b and primes $p \ne q$, let us say that $a \overset{p,q}{=} b$ if there exists an integer c such that $a = pc$ and $b = qc$, i.e. the prime factorizations of a and b are identical up to p and q. To prove (18) we first observe that for any integers a, b we have

$$\int_0^1 f(ax) f(bx)\, dx = \begin{cases} \sum_p \frac{1}{2p^2} & \text{if } a = b, \\ \frac{\varepsilon_p \varepsilon_q}{2pq} & \text{if } a \overset{p,q}{=} b \text{ with primes } p \ne q, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Indeed, by (16)

$$\int_0^1 f(ax) f(bx)\, dx = \sum_{ap=bq} \frac{\varepsilon_p \varepsilon_q}{2pq},$$

where p, q run through the primes. Clearly, for $a \ne b$ the relation $ap = bq$ is equivalent to $a \overset{q,p}{=} b$, and thus (19) is valid. (The interchange of p and q in $a \overset{p,q}{=} b$ causes no problem since $\varepsilon_p \varepsilon_q / 2pq$ is symmetric in p and q.) From (19) we immediately get

$$\int_0^1 \delta_i \delta_j f(m_i x) f(m_j x)\, dx = \begin{cases} \sum_p \frac{1}{2p^2} & \text{if } m_i = m_j, \\ \frac{1}{2pq} & \text{if } m_i \overset{p,q}{=} m_j \text{ with primes } p \ne q, \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

upon noticing that if $m_i \overset{p,q}{=} m_j$ then $m_i = p_{i_1} \cdots p_{i_r} p$, $m_j = p_{i_1} \cdots p_{i_r} q$ with different primes $p_{i_1}, \dots, p_{i_r}$, p, q, and thus $\delta_i = \varepsilon_{p_{i_1}} \cdots \varepsilon_{p_{i_r}} \varepsilon_p$, $\delta_j = \varepsilon_{p_{i_1}} \cdots \varepsilon_{p_{i_r}} \varepsilon_q$, i.e. $\delta_i \delta_j = \varepsilon_p \varepsilon_q$.

We can now easily prove (18). Since by (20) the integral in (18) increases with N and for $2^k \le N < 2^{k+1}$ the quantity $N(\log_3 N)^2$ changes at most by a factor ≤ 3 for $k \ge k_0$, it suffices to prove (18) for $N = 2^k$. Fix a pair (p,q) $(p \ne q)$ from the first k primes $p_1, \dots, p_k$. Since $m_i \overset{p,q}{=} m_j$ means that $m_i = p_{i_1} \cdots p_{i_r} p$, $m_j = p_{i_1} \cdots p_{i_r} q$ with different primes $p_{i_1}, \dots, p_{i_r}$, p, q, the number of pairs (m_i, m_j), $1 \le i, j \le k$, such that the condition in the second line of (20) holds is equal to $2^{k-2} = N/4$. Thus

$$\int_0^1 \left(\sum_{i=1}^N \delta_i f(m_i x) \right)^2 dx \ge \frac{N}{4} \sum_{1 \le i < j \le k} \frac{1}{2 p_i p_j}$$
$$= \frac{N}{16} \left[\left(\sum_{i=1}^k \frac{1}{p_i} \right)^2 - \sum_{i=1}^k \frac{1}{p_i^2} \right] \ge \text{const} \cdot N \left[(\log \log k)^2 - 0(1) \right]$$
$$\ge \text{const} \cdot N (\log_3 N)^2,$$

using the fact that $p_k \sim k \log k$. □

Lemma 2. *Let $1 \le p_1 < q_1 < p_2 < q_2 < \dots$ be integers such that $p_{k+1} \ge 4 q_k$; let $I_1, I_2, \dots$ be sets of integers such that $I_k \subset [2^{p_k}, 2^{q_k}]$ and each element of I_k is divisible by 2^{p_k}. Let (ρ_j) be an arbitrary ± 1 sequence and set*

$$X_k = X_k(\omega) = \sum_{j \in I_k} \rho_j f(j\omega) \qquad (k = 1, 2, \dots, \ \ \omega \in (0,1)).$$

Then there exist independent r.v.'s $Y_1, Y_2, \dots$ on the probability space $((0,1), \mathcal{B}, \lambda)$ such that $EY_k = 0$ and

$$|X_k - Y_k| \le 2^{-k} \qquad (k \ge k_0).$$

Proof. Let $\mathcal{F}_k$ denote the σ-field generated by the dyadic intervals

$$U_\nu = \left[\nu 2^{-4q_k}, (\nu+1) 2^{-4q_k} \right], \qquad 0 \le \nu < 2^{4q_k}, \tag{21}$$

and set

$$\xi_j = \xi_j(\cdot) = E(f(j\cdot)|\mathcal{F}_k), \qquad j \in I_k,$$
$$Y_k = Y_k(\omega) = \sum_{j \in I_k} \rho_j \xi_j(\omega).$$

By $|f(x) - f(y)| \le C|x-y|^{1/2}$ we have

$$|\xi_j(\omega) - f(j\omega)| \le C_1 (j 2^{-4q_k})^{1/2}, \qquad j \in I_k,$$

and since I_k has at most 2^{q_k} elements, we get

$$|X_k - Y_k| \le C_2 \cdot 2^{-q_k/2} \le 2^{-k} \qquad \text{for} \qquad k \ge k_0.$$

Since $p_{k+1} \ge 4q_k$ and since each $j \in I_{k+1}$ is a multiple of $2^{p_{k+1}}$, each interval U_ν in (21) is a period interval for all $f(jx)$, $j \in I_{k+1}$, and thus also for ξ_j, $j \in I_{k+1}$. Hence Y_{k+1} is independent of the σ-field $\mathcal{F}_k$, and since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ and Y_k is $\mathcal{F}_k$ measurable, the r.v.'s $Y_1, Y_2, \dots$ are independent. Finally, $E\xi_j = 0$ by the second relation of (1), and thus $EY_k = 0$. □

Proof of Theorem 2. Let $\psi(k) = 2^{2^{2^k}}$ and let (r_k) be a nondecreasing sequence of integers satisfying $r_k \ge \psi(k)^3$. We define sets

$$I_1^{(1)}, I_2^{(1)}, \dots, I_{r_1}^{(1)}, I_1^{(2)}, \dots, I_{r_2}^{(2)}, \dots, I_1^{(k)}, \dots, I_{r_k}^{(k)}, \dots \tag{22}$$

of positive integers by

$$I_j^{(k)} = 2^{c_j^{(k)}} \left\{ m_1^{(k)}, \dots, m_{\psi(k)}^{(k)} \right\}, \qquad 1 \le j \le r_k,\ k \ge 1,$$

where the $c_j^{(k)}$ are suitable positive integers and $m_1^{(k)}, \dots, m_{\psi(k)}^{(k)}$ are the numbers $m_1, \dots, m_{\psi(k)}$ arranged in increasing order. (Here for any set $\{a, b, \dots\} \subset R$ and $\lambda \in R$, $\lambda\{a, b, \dots\}$ denotes the set $\{\lambda a, \lambda b, \dots\}$.) Clearly we can choose the integers $c_j^{(k)}$ inductively so that the smallest element of any set $I_j^{(k)}$ in (22) exceeds 16 times the 4th power of the largest element of the preceding set. As the left hand side of (18) does not change if we replace m_k by am_k for any integer $a \ge 1$, by Lemma 1 there exist numbers $\delta_\nu^* = \pm 1$ ($\nu = 1, 2, \dots$) such that, setting

$$X_j^{(k)} = X_j^{(k)}(\omega) = \sum_{\nu \in I_j^{(k)}} \delta_\nu^* f(\nu\omega),$$

we have

$$E\left(X_j^{(k)}\right)^2 \ge \text{const} \cdot \psi(k)(\log_3 \psi(k))^2.$$

By Lemma 2 there exist independent r.v.'s $Y_j^{(k)}$ ($1 \le j \le r_k,\ k = 1, 2, \dots$) such that $EY_j^{(k)} = 0$ and

$$\sum_{k,j} |X_j^{(k)} - Y_j^{(k)}| \le K \tag{23}$$

for some constant $K > 0$. Hence by the Minkowski inequality

$$E(Y_j^{(k)})^2 \ge \text{const} \cdot \psi(k)(\log_3 \psi(k))^2. \tag{24}$$

Also $|Y_j^{(k)}| \le |X_j^{(k)}| + K \le \text{const} \cdot \psi(k)$, and thus, setting

$$Z_k = \frac{1}{\sqrt{r_k \psi(k)} \log_3 \psi(k)} \sum_{j=1}^{r_k} Y_j^{(k)},$$

$$\sigma_k^2 = E\left(\sum_{j=1}^{r_k} Y_j^{(k)}\right)^2,$$

we get from the central limit theorem with remainder (see e.g. [18], p. 288), (24) and $r_k \ge \psi(k)^3$,

$$\begin{aligned} P(Z_k \ge 1) &\ge P\left(\sum_{j=1}^{r_k} Y_j^{(k)} \ge \text{const} \cdot \sigma_k\right) \\ &\ge (1 - \Phi(c)) - \text{const} \cdot \frac{r_k \psi(k)^3}{(r_k \psi(k)(\log_3 \psi(k))^2)^{3/2}} \\ &\ge 1 - \Phi(c) - o(1) \ge c' > 0 \qquad (k \ge k_0), \end{aligned}$$

where Φ denotes the Gaussian distribution function and c and c' are positive absolute constants. Since the Z_k are independent, the Borel–Cantelli lemma implies

$P(Z_k \geq 1 \text{ i.o.}) = 1$, i.e. $\sum_{k\geq 1} Z_k$ is a.e. divergent, which, in view of (23), yields that

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{r_k \psi(k)} \log_3 \psi(k)} \sum_{j=1}^{r_k} X_j^{(k)} \qquad \text{is a.e. divergent.} \tag{25}$$

Now let

$$(n_i) = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{r_k} I_j^{(k)}. \tag{26}$$

Then the sum in (25) is of the form $\sum_{i=1}^{\infty} c_i f(n_i x)$, where $|c_i|$ is decreasing and

$$\sum_{i=1}^{\infty} c_i^2 = \sum_{k=1}^{\infty} \frac{1}{r_k \psi(k)(\log_3 \psi(k))^2} r_k \psi(k) = \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty.$$

Finally, denote by $1+\rho_k$ the smallest of the ratios $m_{j+1}^{(k)}/m_j^{(k)}$, $1 \leq j \leq \psi(k)-1$; clearly $\rho_k > 0$. Given $\varepsilon_k \downarrow 0$, one can choose r_k growing so rapidly that

$$\rho_k \geq \varepsilon_{r_{k-1}} \qquad k = 1, 2, \ldots. \tag{27}$$

Now if n_s and n_{s+1} belong to the same set $I_j^{(k)}$, then clearly $s \geq r_{k-1}$, and thus by (27) we get $n_{s+1}/n_s \geq 1 + \rho_k \geq 1 + \varepsilon_{r_{k-1}} \geq 1 + \varepsilon_s$. Since $n_{s+1}/n_s \geq 2$ if n_s and n_{s+1} belong to different $I_j^{(k)}$'s, we proved that (n_k) satisfies

$$n_{k+1}/n_k \geq 1 + \varepsilon_k \qquad (k \geq k_0). \tag{28}$$

This completes the proof of Theorem 2. □

Proof of Theorem 5. Let $f, (m_k), (\delta_k)$ be the same as in the proof of Theorem 2 and let $p(k)$ be a sequence of positive integers with $p(k) \leq k^{1/4}$. Define sets $I_1, I_2, \ldots$ of integers by

$$I_k = 2^{c_k}\left\{m_1^{(k)}, \ldots, m_{p(k)}^{(k)}\right\} \qquad (k = 1, 2, \ldots),$$

where the c_k are suitable positive integers and $m_1^{(k)}, \ldots, m_{p(k)}^{(k)}$ are the numbers $m_1, \ldots, m_{p(k)}$, arranged in increasing order. Clearly we can choose the c_k inductively so that the smallest integer of I_k is greater than 16 times the 4th power of the largest integer of I_{k-1}. By Lemma 1, there exist numbers $\delta_\nu^* = \pm 1 \quad (\nu = 1, 2, \ldots)$ such that for

$$X_k = X_k(\omega) = \sum_{\nu \in I_k} \delta_\nu^* f(\nu\omega)$$

we have

$$E(X_k^2) \geq \text{const} \cdot p(k)(\log_3 p(k))^2.$$

By Lemma 2 one can find independent r.v.'s Y_k with $EY_k = 0$ such that

$$|X_k - Y_k| \leq 2^{-k} \qquad (k \geq k_0). \tag{29}$$

Thus, setting $P(k) = \sum_{j\leq k} p(j)$ and using the Minkowski inequality, we get

$$s_k^2 := \sum_{j\leq k} E(Y_j^2) \geq \text{const} \cdot \sum_{j\leq k} E(X_j^2) \geq P(k)\omega_k,$$

where $\omega_k \to +\infty$. Since $|Y_k| \le |X_k| + 1 \le \text{const} \cdot p(k) = O(k^{1/4}) = O(s_k^{1/2})$, we get by Kolmogorov's LIL (see e.g. Loève [18], p. 260)

$$\varlimsup_{k\to\infty} \frac{\sum_{j\le k} Y_j}{(P(k)\log\log P(k))^{1/2}} = +\infty \qquad \text{a.e.} \tag{30}$$

In view of (29), relation (30) remains valid if we replace Y_j by X_j, and thus, defining a sequence (n_k) by

$$(n_k) = \bigcup_{j=1}^{\infty} I_j, \tag{31}$$

we get

$$\varlimsup_{k\to\infty} \frac{\sum_{j\le P(k)} \delta_j^{**} f(n_j\omega)}{\sqrt{P(k)\log\log P(k)}} = +\infty \qquad \text{a.e.} \tag{32}$$

where δ_j^{**} $(j = 1, 2, \dots)$ denotes the ± 1 sequence defined by $\delta_j^{**} = \delta_{n_j}^{*}$. Finally, set

$$1 + \rho_k := \min\{m_i/m_j : 1 \le i, j \le k,\ m_i > m_j\}. \tag{33}$$

Clearly $\rho_k > 0$ and ρ_k is decreasing. Hence, given $\varepsilon_k \downarrow 0$, we can choose $p(k)$ growing so slowly that

$$\rho_{p(k)} \ge \varepsilon_k \qquad (k \ge k_0). \tag{34}$$

Now if n_s and n_{s+1} belong to I_k, then clearly $s \ge k$ and thus $n_{s+1}/n_s \ge 1 + \rho_{p(k)} \ge 1 + \varepsilon_k \ge 1 + \varepsilon_s$. Since $n_{s+1}/n_s \ge 2$ if n_s and n_{s+1} belong to different I_j's, we have proved that (n_k) satisfies (28). □

Proof of Theorem 7. We will need the following fact.

Lemma 3. *Let (n_k) be a sequence of integers satisfying (7). Then for any $N \ge N_0$ (where N_0 is an absolute constant), all numbers $|c_j| \le 1$, and any even p satisfying $\log\log N \le p \le 4\log\log N$ we have*

$$\int_0^1 \left(\sum_{\sqrt{N}\le j\le N} c_j \cos 2\pi n_j x \right)^p dx \le \frac{p!}{(p/2)!} N^{p/2}. \tag{35}$$

This result follows from Lemma 8 of [3] with $\delta = 1$, $c = (\sqrt{N})^{1/2-\beta}$, $\varepsilon = 1/2$, $a = \sqrt{N}$. (See the last statement of the lemma involving a.) Note that in [3] we have $c_j = 1$ $(j = 1, 2, \dots)$, but the proof in the general case requires no change.

Lemma 4. *Let (n_k) be a sequence of positive integers satisfying (7) and let $S_N = \sum_{k\le N} c_k \cos 2\pi n_k x$, where $|c_k| \le 1$ $(k = 1, \dots, N)$. Then we have*

$$P\left(\sup_{N\ge N_1} \frac{|S_N|}{\sqrt{N\log\log N}} > t \right) \le \frac{1}{t^2} \qquad (t \ge t_1), \tag{36}$$

where P denotes the Lebesgue measure in $(0, 1)$ and N_1, t_1 are positive absolute constants.

Proof. We first show that for $t \geq 4$, $N \geq N_0$ (where N_0 is the constant in Lemma 3) group we have

$$P\left(|S_N| \geq 2t\sqrt{N \log\log N}\right) \leq 2\exp\left(-\frac{1}{2}\log t \cdot \log\log N\right). \tag{37}$$

To see this let $p = 2[\log\log N]$; then by Lemma 3, $p!/(p/2)! \leq 2(2p)^{p/2}$ and the Markov inequality, we get

$$\begin{aligned}
&P\left(|S_N| \geq 2t\sqrt{N \log\log N}\right)\\
&\leq P\left(\left|\sum_{\sqrt{N}\leq i\leq N} c_i \cos 2\pi n_i x\right| \geq t\sqrt{N\log\log N}\right)\\
&\leq \left(t\sqrt{N\log\log N}\right)^{-p}\frac{p!}{(p/2)!}N^{p/2} \leq t^{-p}(p/2)^{-p/2}2(2p)^{p/2}\\
&\leq 2t^{-p/2} \leq 2\exp\left(-\frac{1}{2}\log t \cdot \log\log N\right),
\end{aligned}$$

proving (37). Let $d \geq 2$ be a positive integer; then using (37) and a maximal inequality for trigonometric sums (see Berkes [1], Lemma 6 with $Q = 0$; the extension for sums with coefficients c_j is again automatic) we get

$$\begin{aligned}
&P\left(\sup_{N\geq 2^d}|S_N|/\sqrt{N\log\log N} > t\right)\\
&\leq \sum_{k=d}^{\infty} P\left(\max_{2^k\leq N<2^{k+1}}|S_N| > t\sqrt{2^k\log\log 2^k}\right)\\
&\leq \sum_{k=d}^{\infty} 2P\left(|S_{2^{k+1}}| \geq \frac{t}{2}\sqrt{2^k\log\log 2^k}\right)\\
&\leq \sum_{k=d}^{\infty} 4\exp\left(-\frac{1}{4}\log\frac{t}{8}\cdot\log k\right) = 4\sum_{k=d}^{\infty} k^{-\frac{1}{4}\log\frac{t}{8}}\\
&\leq 4\int_{d-1}^{\infty} x^{-\frac{1}{4}\log\frac{t}{8}}\,dx \leq 4(d-1)^{-\frac{1}{8}\log\frac{t}{8}} \leq t^{-2}
\end{aligned}$$

provided that d and t are sufficiently large. Thus (36) is proved. □

We can now easily prove Theorem 7. Let $f = \sum(a_k \cos 2\pi kx + b_k \sin 2\pi kx)$, where $\sum(|a_k| + |b_k|) < +\infty$. Let

$$S_N(x) = \sum_{k\leq N} c_k f(n_k x),$$

$$S_N^{(1)}(x) = \sum_{k\leq N} c_k \cos 2\pi n_k x, \qquad S_N^{(2)}(x) = \sum_{k\leq N} c_k \sin 2\pi n_k x.$$

Then

$$\begin{aligned} S_N(x) &= \sum_{k\le N}\sum_{\nu=1}^{\infty} c_k a_\nu \cos 2\pi\nu n_k x + \sum_{k\le N}\sum_{\nu=1}^{\infty} c_k b_\nu \sin 2\pi\nu n_k x \\ &= \sum_{\nu=1}^{\infty} a_\nu S_N^{(1)}(\nu x) + \sum_{\nu=1}^{\infty} b_\nu S_N^{(2)}(\nu x), \end{aligned} \tag{38}$$

using the absolute convergence of the double series. From Lemma 4 and its analogue for sine series it follows that the functions

$$\psi_i(x) = \sup_{N\ge N_1} |S_N^{(i)}(x)|/\sqrt{N\log\log N} \qquad (i=1,2)$$

are nonnegative, almost everywhere finite functions on $(0,\infty)$, periodic with period 1, and satisfying $\int_0^1 \psi_i(x)\,dx < +\infty$, $i=1,2$. By the periodicity of ψ_i we have

$$\int_0^1 \psi_i(\nu x)\,dx \le K \qquad (\nu = 1,2,\dots,\ i=1,2)$$

for some constant K, and thus by (14) and the Beppo Levi theorem the function $g(x)\ge 0$ defined by

$$g(x) = \sum_{\nu=1}^{\infty} (|a_\nu|\psi_1(\nu x) + |b_\nu|\psi_2(\nu x))$$

is also a.e. finite. Now for each $\nu\ge 1$, $N\ge N_1$ we have

$$|S_N^{(i)}(\nu x)| \le \psi_i(\nu x)\sqrt{N\log\log N}, \qquad i=1,2,$$

and thus by (38) we get

$$|S_N(x)| \le g(x)\sqrt{N\log\log N}, \qquad N\ge N_1,\ \ x\in(0,1),$$

proving Theorem 7. □

Remark. Note that Lemma 4 remains valid if $1/t^2$ in (36) is replaced by $1/t^\gamma$ for any constant $\gamma>0$ (with N_1 and t_1 depending on γ), and thus the functions ψ_i $(i=1,2)$ defined above satisfy $\int_0^1 \psi_i^p(x)\,dx < +\infty$ for any $p\ge 1$. Hence g and thus the lim sup function in Theorem 7 belong to $L_p(0,1)$ for any $p\ge 1$. Under the stronger assumption $f\in \mathrm{Lip}\,\alpha$, $\alpha>1/2$, the lim sup is actually uniformly bounded (see Takahashi [25]); whether this is true under (14) remains open.

Proof of Theorems 4 and 8. The sufficiency of (6) in Theorem 4 is contained in Theorem 2 of Gaposhkin [13]; note that for this part of the theorem we do not need any arithmetic assumptions on H. Similarly, in the case of Theorem 8, the sufficiency of (15) is contained in Theorem 7, again without any assumptions on H. To prove the converse statements, assume first that H is the set of primes, i.e.

$$f(x) = \sum (a_p\cos 2\pi px + b_p \sin 2\pi px), \qquad \sum (a_p^2+b_p^2) < +\infty.$$

If some of the coefficients a_p, b_p are equal to zero, replace f by $g = f+f^*$, where

$$f^*(x) = \sum_{\{p:\,a_p=0\}} 2^{-p}\cos 2\pi px + \sum_{\{p:\,b_p=0\}} 2^{-p}\sin 2\pi px.$$

Clearly the Fourier series of f^* is absolutely convergent, and thus, by Theorem 2 of [13], $(f^*(n_kx))$ is a convergence system for any increasing sequence (n_k) of integers.

Hence Theorems 4 and 8 are valid for f if and only if they are valid for g, i.e. we can assume without loss of generality that a_p and b_p are all different from 0.

Assume now $\sum(|a_p|+|b_p|)=+\infty$; then e.g. $\sum |a_p|=+\infty$. Let (m_k) be the sequence defined before the formulation of Lemma 1, let $\varepsilon_p=\operatorname{sgn} a_p$, and define the sequence (δ_k) again by (17). Then the following version of Lemma 1 holds:

Lemma 5. *For any $N\ge 1$ we have*

$$\int_0^1 \left(\sum_{i=1}^N \delta_i f(m_i x)\right)^2 dx \ge \frac{N}{32}\left(\left(\sum_{i\le \log N} |a_{p_i}|\right)^2 - \sum_{i\le \log N} a_{p_i}^2\right), \tag{39}$$

where the logarithm is of base 2.

Proof. Let $f_1=\sum a_p\cos 2\pi px$, $f_2=\sum b_p \sin 2\pi px$. Following the proof of Lemma 1, we get

$$\int_0^1 \delta_i\delta_j f(m_i x) f(m_j x)\,dx = \begin{cases} \frac{1}{2}\sum_p |a_p|^2 & \text{if } i=j,\\ \frac{1}{2}|a_p||a_q| & \text{if } m_i \overset{p,q}{=} m_j \text{ with primes } p\ne q,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, following the argument in the proof of Lemma 1 further, for $2^k\le N<2^{k+1}$ we get

$$\int_0^1 \left(\sum_{i=1}^N \delta_i f_1(m_i x)\right)^2 dx \ge \frac{N}{32}\left[\left(\sum_{i=1}^k |a_{p_i}|\right)^2 - \sum_{i=1}^k a_{p_i}^2\right]. \tag{40}$$

But the functions $\sum_{i\le N}\delta_i f_1(m_i x)$ and $\sum_{i\le N} f_2(m_i x)$ are orthogonal, and thus

$$\int_0^1 \left(\sum_{i=1}^N \delta_i f(m_i x)\right)^2 dx = \int_0^1 \left(\sum_{i=1}^N \delta_i f_1(m_i x)\right)^2 dx + \int_0^1 \left(\sum_{i=1}^N \delta_i f_2(m_i x)\right)^2 dx.$$

Hence (40) implies (39).

Since $\sum |a_p|=+\infty$ and $\sum a_p^2<+\infty$, (39) shows that

$$\int_0^1 \left(\sum_{i=1}^N \delta_i f(m_i x)\right)^2 dx \ge N\omega_N, \qquad \text{where} \quad \omega_N\to+\infty.$$

Using this inequality in place of (18), the proofs of Theorems 2 and 5 remain valid and yield Theorems 4 and 8.

If H is an arbitrary sequence of coprime integers, the above argument still applies provided p_n denotes the n-th element of H and p and q in the proof of Lemmas 1 and 5 denote elements of H instead of being primes. Thus Theorems 4 and 8 are also valid in the general case.

Proof of Theorems 3, 6. We follow the proof of Theorems 2, 5 with the definition of f in (16) changed to

$$f(x)=\sum_p \frac{\varepsilon_p \sin 2\pi px}{p^{1/2+\alpha}}, \tag{41}$$

where $0<\alpha<1/2$. Using Theorem 2 in Kahane [16], p. 66, we get, similarly as in the case of (16), that $f \in \operatorname{Lip}\alpha$ almost surely; on the other hand, the proof of Lemma 1 yields in the present case

$$\int_0^1 \left(\sum_{k\le N} \delta_k f(m_k x)\right)^2 dx \ge \text{const}\cdot N(\log N)^{1-2\alpha}(\log\log N)^{-1-2\alpha} \tag{42}$$

instead of (18). Also, Lemma 2 remains valid for this f with $p_{k+1}\ge 4q_k$ replaced by $p_{k+1}\ge Aq_k$, provided A is large enough. Define the intervals $I_j^{(k)}$, $1\le j\le r_k$, $k\ge 1$, as before, with $\psi(k)$ and r_k chosen in the present case as $\psi(k)=2^{2^k}$, $r_k=\psi(k)^3$. Let

$$M_k=\sum_{j\le k} r_j\psi(j)=\sum_{j\le k}\psi(j)^4.$$

Instead of (25) we get in the present case that

$$\sum_{k=1}^\infty \frac{1}{\sqrt{r_k\psi(k)}(\log\psi(k))^{1/2-\alpha}(\log\log\psi(k))^{-1/2-\alpha}}\sum_{j=1}^{r_k} X_j^{(k)} \tag{43}$$

is a.e. divergent, which means that $\sum c_i f(n_i x)$ is a.e. divergent, where (n_i) is defined by (26) and

$$c_i^2=\frac{(\log\log\psi(k))^{1+2\alpha}}{r_k\psi(k)(\log\psi(k))^{1-2\alpha}} \qquad \text{for} \quad M_{k-1}<i\le M_k.$$

Now for $M_{k-1}<i\le M_k$ we have

$$i\le 2\psi(k)^4, \quad \log\log i\ge \log\log\psi(k-1)\sim\log\log\psi(k)$$

by the choice of $\psi(k)$, and thus for $M_{k-1}<i\le M_k$ we get

$$c_i^2\frac{(\log i)^{1-2\alpha}}{(\log\log i)^\tau}\le \text{const}\cdot\frac{1}{r_k\psi(k)}(\log\log\psi(k))^{1+2\alpha-\tau}$$

for any τ. Hence

$$\sum_{i=1}^\infty c_i^2\frac{(\log i)^{1-2\alpha}}{(\log\log i)^{2+2\alpha+\varepsilon}}\le \text{const}\cdot\sum_{k=1}^\infty\frac{1}{(\log\log\psi(k))^{1+\varepsilon}}\le\text{const}\cdot\sum_{k=1}^\infty\frac{1}{k^{1+\varepsilon}}<+\infty$$

for any $\varepsilon>0$, completing the proof of Theorem 3.

To prove Theorem 6 we repeat the proof of Theorem 5 with f in (41) and $p(k)$ chosen as

$$p(k)=[\exp((\log k)^{1-\varepsilon})] \tag{44}$$

for some $0<\varepsilon<1$. Noting that m_k is a product of different primes $p_{i_1},\dots,p_{i_\nu}$ with $i_1,\dots,i_\nu\le 2\log k$, we get by a simple calculation, using $p_n\sim n\log n$,

$$m_k\le\exp((\log k)^{1+\varepsilon/2}), \qquad k\ge k_0,$$

and thus for any $i\ge1$, $j\ge1$, $m_i>m_j$ we have

$$m_i/m_j\ge 1+1/m_j\ge 1+\text{const}\cdot\exp(-(\log j)^{1+\varepsilon/2}).$$

Hence the number ρ_k defined by (33) satisfies

$$\rho_k\ge\text{const}\cdot\exp(-(\log k)^{1+\varepsilon/2}),$$

and therefore, given any $\beta > 0$, (34) will hold with $\varepsilon_k = k^{-\beta}$. Thus the proof of Theorem 5 shows that the sequence (n_k) in (31) satisfies (7) for all $\beta > 0$. On the other hand, (42) and (44) give

$$\begin{aligned} EX_k^2 &\geq \text{const} \cdot p(k)(\log p(k))^{1-2\alpha}(\log\log p(k))^{-1-2\alpha} \\ &\geq \text{const} \cdot p(k)(\log k)^{(1-2\alpha)(1-2\varepsilon)}, \end{aligned} \tag{45}$$

and since $p(k)$ is slowly varying and consequently $P(k) = \sum_{j \leq k} p(j) \sim kp(k)$ and $k \leq P(k) \leq k^2$ for $k \geq k_0$, we get from (45)

$$\begin{aligned} \sum_{j \leq k} EX_j^2 &\geq \text{const} \cdot kp(k)(\log k)^{(1-2\alpha)(1-2\varepsilon)} \\ &\geq \text{const} \cdot P(k)(\log P(k))^{(1-2\alpha)(1-2\varepsilon)}. \end{aligned}$$

Hence, following the proof of Theorem 5 further, we get

$$\limsup_{k\to\infty} \frac{\sum_{j\leq k} Y_j}{P(k)^{1/2}(\log P(k))^{(1/2-\alpha)(1-2\varepsilon)}(\log\log P(k))^{1/2}} > 0 \qquad \text{a.e.} \tag{46}$$

instead of (30). In view of (29), (46) remains valid if we replace Y_j by X_j, and thus (13) follows from (46).

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