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ASYMPTOTIC ANALYSIS FOR LINEAR DIFFERENCE EQUATIONS

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Abstract. We are concerned with asymptotic analysis for linear difference equations in a locally convex space. First we introduce the profile operator, which plays a central role in analyzing the asymptotic behaviors of the solutions. Then factorial asymptotic expansions for the solutions are given quite explicitly. Finally we obtain Gevrey estimates for the solutions. In a forthcoming paper we will develop the theory of cohomology groups for recurrence relations. The main results in this paper lay analytic foundations of such an algebraic theory, while they are of intrinsic interest in the theory of finite differences.

1. Introduction

Let U be a locally convex linear space over $K = \mathbb{C}$ or $\mathbb{R}$, $P = (P_n)_{n\in\mathbb{Z}}$ an infinite sequence of linear operators $P_n : U \to U$. We are concerned with asymptotic analysis for the linear difference equation

$$\begin{cases} \phi_n = 0 & (n \ll 0), \\ \phi_n = P_n\phi_{n-1} + v_n & (n \in \mathbb{Z}), \end{cases} \tag{1.1}$$

where the known data $v = (v_n)_{n\in\mathbb{Z}}$ is an infinite sequence of vectors in U such that $v_n = 0$ for all $n \ll 0$.

Let V_f be the linear space of all infinite sequences $v = (v_n)_{n\in\mathbb{Z}}$ of vectors in U such that $v_n = 0$ for all $n \ll 0$. The space V_f is referred to as the *formal sequence space* for U. Let $T : V_f \to V_f$ be the translation operator defined by

$$(Tv)_n = v_{n-1} \qquad (n \in \mathbb{Z}). \tag{1.2}$$

We consider $P = (P_n)$ as a linear operator $P : V_f \to V_f$ defined by

$$Pv = (P_n v_n) \qquad \text{for} \quad v = (v_n) \in V_f. \tag{1.3}$$

A linear operator of this form is referred to as a *homogeneous* linear operator on V_f. Then the difference equation (1.1) is simply rewritten as

$$(I - PT)\phi = v, \tag{1.4}$$

where I is the identity operator on V_f.

For any $v \in V_f$ it is clear that the difference equation (1.1) or (1.4) admits a unique solution $\phi \in V_f$ depending linearly on $v \in V_f$. This solution is denoted by

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$\phi(v) = (\phi_n(v))$ and called the *formal solution* for $v \in V_f$. Then $\phi : V_f \to V_f$, $v \mapsto \phi(v)$ defines a linear operator, which is called the *formal solution operator*.

From the observation above, there is no difficulty in the difference equation (1.4) as long as it is considered on the formal sequence space V_f; there always exists a unique solution $\phi(v) = (\phi_n(v)) \in V_f$ for any $v \in V_f$, and there is no constraint on the asymptotic behavior of $\phi_n(v)$ as $n \to +\infty$. However, once (1.4) is considered on those linear subspaces V of V_f which are characterized by some global behaviors of $v = (v_n) \in V$ as $n \to +\infty$, the difference equation (1.4) becomes quite difficult; although there exists a unique *formal* solution $\phi(v) \in V_f$ for any $v \in V$, it is not at all clear whether $\phi(v)$ belongs to V or not. Moreover the formal solution $\phi(v)$ for $v \in V$ might be subject to certain asymptotic conditions, but it is generally difficult to find such conditions effectively. This situation becomes clear if we consider, for example, the cases where V is the linear space of all convergent sequences in U, or the space of all sequences $v = (v_n) \in V_f$ such that $\lim_{n\to+\infty} v_n = 0$, etc.

The discussion above leads us to the problem of finding suitable pairs (V, P) for which the difference equation (1.4) on the space V can be analyzed in great detail. We introduce a nice class of such pairs (V, P) in the following definition.

Definition 1.1. Let V be a linear subspace of V_f, $P = (P_n)_{n\in\mathbb{Z}}$ a homogeneous linear operator on V_f, N a nonnegative integer. Then the pair (V, P) is said to be an *asymptotic pair of order* N if the following conditions hold:

(AP-1) P and T map V into itself,

(AP-2) for any $v = (v_n) \in V$, $v_n \to 0$ as $n \to +\infty$,

(AP-3) there exists a quartet $(c, U_1, \Phi, \{\Psi_k\}_{k=0}^N)$ such that

(1) $c \in K(= \mathbb{C} \text{ or } \mathbb{R})$,

(2) U_1 is a closed linear subspace of U,

(3) $\Phi : V \to U_1$ is a *surjective* linear operator,

(4) $\Psi_k : U_1 \to U$ $(k = 0, 1, \ldots, N)$ are linear operators, Ψ_0 is the *inclusion map*, and

(5) $\phi_n(v)$ admits the asymptotic expansion:

$$\phi_n(v) = \left(\sum_{k=0}^{N} \Psi_k [n + c]_k\right) \Phi v + o\left(\frac{1}{n^N}\right) \quad \text{as } n \to +\infty \qquad (\forall v \in V),$$

where $[x]_k$ is the *lower* factorial monomial of degree k (cf. Definition 2.2.1),

(AP-4) for any $v \in V$, $\phi(v) \in V_f$ belongs to V if and only if $\Phi v = 0$.

The operator Φ is called the *profile operator*. The triple $\mathcal{A} = (c, U_1, \{\Psi_k\}_{k=0}^N)$ is referred to as the *asymptotic data* of (V, P). Moreover the pair (V, P) is said to be an *asymptotic pair of order* ∞ if there exists a quartet $(c, U_1, \Phi, \{\Psi_k\}_{k=0}^\infty)$ such that, for any nonnegative integer N, the pair (V, P) is an asynptotic pair of order N with profile operator Φ and asymptotic data $(c, U_1, \{\Psi_k\}_{k=0}^N)$. The triple $\mathcal{A} = (c, U_1, \{\Psi_k\}_{k=0}^\infty)$ is referred to as the *asymptotic data* of (V, P).

We remark that the profile operator Φ is *not* a member of the asymptotic data $\mathcal{A}$. The reason for this will be mentioned just after Problem 1.2 below. In this paper we always assume $c = 0$ for the sake of simplicity. This restriction causes no loss of generality since, if necessary, we may replace n by $n + c$ without trouble. With this understanding the pair $\mathcal{A} = (U_1, \{\Psi_k\}_{k=0}^N)$ or $(U_1, \{\Psi_k\}_{k=0}^\infty)$ is referred to as the asymptotic data of (V, P). If (V, P) is an asymptotic pair, we may say that the conditions (AP), especially (AP-3) and (AP-4), provide good enough information about the difference equation (1.4) on the space V. The purpose of this paper is to settle the following problem.

Problem 1.2. (1) Construct asymptotic pairs (V, P).

(2) Find algorithms for calculating the asymptotic data $\mathcal{A}$ explicitly only in terms of P.

In (2) of Problem 1.2 we do not require any algorithmic formula for the profile operator Φ; expecting such a formula seems too easygoing. In fact, as mentioned previously, the linear subspace V of V_f should be characterized by some *global* behaviors of $v = (v_n) \in V$ as $n \to +\infty$, and therefore the space V must be a highly *transcendental* object. Hence any asymptotic formula for the formal solution $\phi(v)$ for $v \in V$ cannot be purely algorithmic (nor algebraic); rather it must contain some transcendental terms.

So what we should try to do is to separate the asymptotic formula into two parts; one a transcendental part which perfectly reflects the global, and hence transcendental, nature of the space V; the other an algebraic part in which everything can be determined algorithmically in terms of P. Notice that, while V is a transcendental object, P is a *formal*, and hence algebraic, object since $P : V_f \to V_f$ makes sense on the formal space V_f. So the algebraic part should depend only on P. Now our basic idea is to consider that the profile operator Φ and the asymptotic data $\mathcal{A}$ represent the transcendental part and the algebraic part of the asymptotic formula (5) in (AP-3), respectively. This idea motivates (2) of Problem 1.2 as well as the separate treatment of the profile operator Φ from the asymptotic data $\mathcal{A}$. Finally, we remark that what is important about the profile operator Φ is its very *existence*, on the basis of which the entire theory works out.

In this paper we construct two classes of asymptotic pairs — *rapidly decreasing pairs* and *Gevrey pairs*. To do this we study the global behaviors of the solutions of the difference equation (1.4) as $n \to +\infty$ in fairly detail. The main results of this paper are rigorously stated in Section 3. In this introduction we confine ourselves to roughly sketching them by restricting our attention to asymptotic pairs of *order* ∞.

We assume that $P = (P_n)$ admits an *upper* factorial asymptotic expansion of the form

$$P_n \sim \sum_{i=0}^{\infty} P^{(i)} (n)_i \qquad \text{as} \quad n \to +\infty, \tag{1.5}$$

where $P^{(i)} : U \to U$ are linear operators on U and $(n)_i$ is the upper factorial monomial of degree i (cf. Definition 2.2.1). See (3.4) for the rigorous meaning of (1.5). Let ℓ^∞ be the linear space of all infinite sequences $v = (v_n) \in V_f$ such that $v_n \to 0$ *rapidly* as $n \to +\infty$, i.e., the convergence is faster than any inverse polynomial order (cf. Definition 3.2.2). Then the formal solution $\phi(v) = (\phi_n(v))$ for $v \in \ell^\infty$ of the difference equation (1.4) admits a *lower* factorial asymptotic expansion of the form

$$\phi_n(v) \sim \left(\sum_{i=0}^{\infty} \Psi_i [n]_i \right) \Phi v \qquad (\forall v \in \ell^\infty) \tag{1.6}$$

(cf. Notation 3.4.1), where $\Phi : \ell^\infty \to U_1$ is a continuous linear operator of ℓ^∞ *onto* a closed subspace U_1 of U, and the coefficients Ψ_i of the formal factorial series in (1.6) are continuous linear operators of U_1 into U. The subspace U_1 can be determined explicitly. Moreover an algorithm for determing Ψ_i is obtained explicitly in terms of the coefficients $P^{(i)}$ of the asymptotic expansion (1.5) of P (cf. Theorem II(2) and Definition 3.2.6). The asymptotic formula (1.6) implies that (ℓ^∞, P) is an asymptotic pair of order ∞ with asymptotic data $\mathcal{A} = (U_1, \{\Psi_k\}_{k=0}^{\infty})$, the profile operator being Φ (cf. Corollary 3.4.2). The asymptotic pair (ℓ^∞, P) is referred to as a rapidly decreasing pair.

Another and more important class of asymptotic pairs is obtained by the use of *Gevrey spaces* $\mathcal{G}^{t,a+}$ with appropriate Gevrey indices (t, a). Here the Gevrey space $\mathcal{G}^{t,a+}$ is the linear space of all infinite sequences $v = (v_n) \in V_f$ such that, for any $b > a$ and for any continuous semi-norm $|\cdot|$ of U,

$$|v_n| = O\left(\frac{b^n}{(n!)^t}\right) \qquad \text{as} \quad n \to +\infty \tag{1.7}$$

(cf. Definition 3.5.2). We show that if the index (t, a) is chosen suitably then $(\mathcal{G}^{t,a+}, P)$ becomes an asymptotic pair. The admissible indices (t, a) are determined explicitly in terms of the coefficients $P^{(i)}$ of the asymptotic expansion (1.5) of P (cf. Corollary 3.6.2). The asymptotic pair $(\mathcal{G}^{t,a+}, P)$ is referred to as a Gevrey pair. Some examples of rapidly decreasing and Gevrey pairs are given in Section 7.

Since we are interested in the asymptotic behavior of $\phi_n(v)$ as $n \to +\infty$, it causes no loss of generality to restrict our attention to the following *truncated* difference equation:

$$\begin{cases} \phi_0 = v_0, \\ \phi_n = P_n \phi_{n-1} + v_n \qquad (n \in \mathbb{Z}_{\geq 0}), \end{cases} \tag{1.8}$$

where the negative part $n \in \mathbb{Z}_{<0}$ of the difference equation (1.1) is truncated. In what follows we assume that the suffix n ranges over the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers. So $v = (v_n)$ and $\phi = (\phi_n)$ mean $v = (v_n)_{n \in \mathbb{Z}_{\geq 0}}$ and $\phi = (\phi_n)_{n \in \mathbb{Z}_{\geq 0}}$, respectively. Now the formal sequence space V_f is redefined as

$$V_f = \{\, v = (v_n)_{n \in \mathbb{Z}_{\geq 0}} \; ; \; v_n \in U \,\}. \tag{1.9}$$

The organization of this paper is as follows: In Section 2 we present some background materials from functional analysis and factorial series. This section is a preliminary to the later sections. In Section 3 we state the main results of this paper. In Section 4 we establish the existence of the profile operator Φ. In Section 5 we derive a factorial asymptotic expansion for the solutions $\phi(v)$ of the difference equation (1.8). Section 6 is devoted to the Gevrey estimates which enable us to construct the Gevrey pairs. In the final Section 7, we apply our results to some difference equations which arise from systems of confluent hypergeometric differential equations in two variables. This demonstrates the applicability of our results to a rather different and unexpected area of mathematics.

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2. Preliminaries

2.1. Functional analysis. Let U be a *locally convex* linear space, $\mathcal{N}$ any fixed system of semi-norms on U which determines the locally convex topology of U. For our purpose it is desirable that $\mathcal{N}$ is as small as possible as a set.

Definition 2.1.1. A linear transformation A of U is said to be *strongly bounded* if, for any $|\cdot| \in \mathcal{N}$, there exists a constant C depending on $|\cdot|$ such that

$$|Au| \le C|u| \qquad (\forall u \in U).$$

We denote by $|A|$ the infimum of all such constants C. Let $\mathcal{B} = \mathcal{B}(U)$ be the set of all strongly bounded transformations of U. Then $\mathcal{B}$ is a locally convex algebra with system of semi-norms $\mathcal{N}$ such that

$$|AB| \le |A||B| \qquad (\forall A, B \in \mathcal{B},\ \forall\, |\cdot| \in \mathcal{N}).$$

Remark 2.1.2. Let X and Y be supplementary projections in $\mathcal{B}$, i.e., $X \in \mathcal{B}, X^2 = X$ and $X + Y = I$. For each $|\cdot| \in \mathcal{N}$ we can define a semi-norm $\|\cdot\|$ on U by

$$\|u\| = \sqrt{|Xu|^2 + |Yu|^2}.$$

Note that $|\cdot|$ and $\|\cdot\|$ are equivalent semi-norms. Indeed, we have

$$\frac{1}{\sqrt{2}}|u| \le \|u\| \le \sqrt{|X|^2 + |Y|^2}\ |u|.$$

Hence, without loss of generality, we may assume

$$|u|^2 = |Xu|^2 + |Yu|^2 \qquad (\forall\, |\cdot| \in \mathcal{N}). \tag{2.1}$$

For, otherwise, we can replace $|\cdot|$ by $\|\cdot\|$; the latter satisfies (2.1). If X and Y are given then we always assume (2.1). Under this assumption we have

$$|X|,\ \ |Y| = 0 \ \text{ or } \ 1. \tag{2.2}$$

The use of the Landau symbol O on a locally convex space is sometimes confusing. So we should clarify in what sense the symbol O is used in this paper.

Notation 2.1.3. (i) On a normed space: Let S be a normed space with the norm $|\cdot|$, e.g., $S = \mathbb{C}$ or $\mathbb{R}$, etc. Let $\{s_i\}_{i\in I}$ be a subset of S, $\{b_i\}_{i\in I}$ a set of nonnegative numbers, both indexed by $i \in I$. Then we write

$$a_i = O(b_i) \qquad (i \in I)$$

if there exists a constant C such that $|a_i| \le Cb_i$ for all $i \in I$. There is no ambiguity in this case, since the norm $|\cdot|$ is unique.

(ii) On a locally convex space: Let S be a locally convex space with system of semi-norms $\mathcal{N}$, e.g., $S = U, \mathcal{B}$ or any other locally convex spaces that appear later. Let $\{s_i\}_{i\in I}$ be a subset of S, $\{b_i\}_{i\in I}$ a set of nonnegative numbers. Then it is possible to provide the following two meanings for the expression

$$s_i = O(b_i) \qquad (i \in I). \tag{2.3}$$

(1) For each semi-norm $|\cdot| \in \mathcal{N}$ there exists a constant C such that $|s_i| \le Cb_i$ for all $i \in I$, where C may depend on $|\cdot|$.
(2) There exists a constant C such that $|s_i| \le Cb_i$ for all $i \in I$ and $|\cdot| \in \mathcal{N}$, where C is independent of $|\cdot|$.

Clearly (2) implies (1), but the converse is not always true if $\mathcal{N}$ is an infinite set. In this paper the expression (2.3) means (1). For (2) we use the following expression:

$$s_i = O_u(b_i) \qquad (i \in I).$$

The subscript u stands for the *uniformity* with respect to the semi-norms.

(iii) Let S be as in (ii), $a_i(|\cdot|)$ and $b_i(|\cdot|)$ nonnegative numbers depending on $i \in I$ and $|\cdot| \in \mathcal{N}$. First we write

$$a_i(|\cdot|) = O(b_i(|\cdot|)) \qquad (i \in I),$$

if, for each $|\cdot| \in \mathcal{N}$, there exists a constant C depending on $|\cdot|$ such that $a_i(|\cdot|) \le C b_i(|\cdot|)$ for all $i \in I$. Secondly we write

$$a_i(|\cdot|) = O_u(b_i(|\cdot|)) \qquad (i \in I),$$

if there exists a constant C independent of $|\cdot|$ such that $a_i(|\cdot|) \le C b_i(|\cdot|))$ for all $i \in I$ and $|\cdot| \in \mathcal{N}$.

The above notation is quite convenient in what follows.

2.2. Factorial series.

Definition 2.2.1. The *upper factorial monomial* of degree i is defined by

$$(n)_i = \frac{(-1)^i (i-1)!}{n(n+1)(n+2)\cdots(n+i-1)} \qquad (i \ge 1).$$

Similarly the *lower factorial monomial* of degree i is defined by

$$[n]_i = \frac{(-1)^i (i-1)!}{n(n-1)(n-2)\cdots(n-i+1)} \qquad (i \ge 1).$$

By convention we set $(n)_0 = [n]_0 = 1$. For a linear space S, an *upper* (resp. a *lower*) *factorial series* over S is an expression of the form

$$\sum_{i=0}^{\infty} s_i (n)_i \qquad \left(\text{resp. } \sum_{i=0}^{\infty} s_i [n]_i\right),$$

where $s_i \in S$. An *upper* (resp. a *lower*) *factorial polynomial* over S is an upper (resp. a lower) factorial series such that $s_i \ne 0$ for at most finitely many i.

In this paper a factorial series is always a *formal* factorial series; the adjective "formal" is omitted. If S is a ring then the set of all upper (or lower) factorial series admits a ring structure. We refer to [3][6][12][13] for more detailed information about factorial series.

Notation 2.2.2.

$$\begin{bmatrix} i \\ j \end{bmatrix} = \frac{i(i+1)(i+2)\cdots(i+j-1)}{j!} \qquad (j \ge 1),$$

$$[i,j] = \frac{(i-1)!(j-1)!}{(i+j-1)!} \qquad (i,j \ge 1).$$

By convention we set

$$\begin{bmatrix} i \\ 0 \end{bmatrix} = [j,0] = [0,k] = 1 \qquad (j,k \ge 0).$$

The following lemmas are used in the later sections.

Lemma 2.2.3. *For any* $i \geq 1$,

$$[n]_i - [n-1]_i = [n]_{i+1}.$$

Lemma 2.2.4. *For any* $N \geq i \geq 1$,

$$[n+m]_i = \sum_{j=i}^{N} \begin{bmatrix} m \\ j-i \end{bmatrix} [n]_j + O\left(\frac{1}{n^{N+1}}\right) \qquad (n \geq N).$$

Lemma 2.2.5. *For any* $N \geq i+j \geq 1$,

$$\begin{aligned}(n)_i[n-1]_j &= [i,j][n+i-1]_{i+j} \\ &= [i,j] \sum_{k=i+j}^{N} \begin{bmatrix} i-1 \\ k-i-j \end{bmatrix} [n]_k + O\left(\frac{1}{n^{N+1}}\right) \qquad (n \geq N).\end{aligned}$$

The proof of these lemmas is omitted, since it is rather standard.

3. Main Results

3.1. Assumptions. In the statement of the main results we make the following assumptions.

Assumption A. U is a *sequentially* complete locally convex space. $P = (P_n)$ is an infinite sequence in $\mathcal{B}$. X and Y are supplementary projections in $\mathcal{B}$.

$$XP_nX = X + O\left(\frac{1}{n^2}\right), \qquad XP_nY = O\left(\frac{1}{n}\right), \tag{3.1}$$
$$YP_nX = O\left(\frac{1}{n}\right) \qquad (n \geq 1),$$

and, for each $|\cdot| \in \mathcal{N}$, there exists a constant ρ $(0 \leq \rho < 1)$ such that

$$|YP_nY| \leq \rho + O\left(\frac{1}{n^2}\right) \qquad (n \geq 1). \tag{3.2}$$

The constant ρ may depend on the semi-norm $|\cdot| \in \mathcal{N}$.

Assumption A$_u$. All assumptions are the same as in Assumption A except for the following alteration: The Landau symbol O in (3.1) and (3.2) is replaced by O_u (cf. Notation 2.1.3), and the constant ρ in (3.2) is *independent* of $|\cdot| \in \mathcal{N}$.

Remark 3.1.1. If U is sequentially complete then $\mathcal{B}$ is also sequentially complete.

Assumption B$_N$. N is a nonnegative integer. U is a *sequentially complete* locally convex space. $P = (P_n)$ is an infinite sequence in $\mathcal{B}$. X and Y are *supplementary projections* in $\mathcal{B}$. There exist strongly bounded operators $P^{(i)} \in \mathcal{B}$ $(i = 0, 1, \ldots, N+1)$ such that

$$P_n = \sum_{i=0}^{N+1} P^{(i)}(n)_i + O\left(\frac{1}{n^{N+2}}\right) \qquad (n \geq N+1) \tag{3.3}$$

in the space $\mathcal{B}$, and

(1) $XP^{(0)} = P^{(0)}X = X$,
(2) for each $|\cdot| \in \mathcal{N}$, $|Z| < 1$, where $Z := YP^{(0)}Y$, and
(3) $XP^{(1)}X = O$.

In case $N = \infty$ the assumption must be sligntly modified.

Assumption B_∞. All assumptions are the same as in Assumption B_N except for (3.3). Condition (3.3) is replaced by the following one: There exist $P^{(i)} \in \mathcal{B}$ $(i \geq 0)$ such that (3.3) holds for all $N \geq 0$. We express this condition as

$$P_n \sim \sum_{i=0}^{\infty} P^{(i)}(n)_i. \tag{3.4}$$

3.2. Notation. In order to state the main results we introduce some notation.

Notation 3.2.1. We set

$$U_\nu = \{\, u \in U \;;\; Xu = \nu u \,\} \qquad (\nu = 0, 1).$$

Then U_ν $(\nu = 0, 1)$ are closed linear subspaces of U such that $U = U_1 \oplus U_0$. We denote by I_ν the identity map on U_ν.

Definition 3.2.2. For each $|\cdot| \in \mathcal{N}$ and $v = (v_n) \in V_f$, we set

$$|v|_0 = \sum_{n=0}^{\infty}(n+1)|Xv_n| + \sup_{n\geq 0}\{(n+1)|Yv_n|\},$$
$$|v|_N = \sup_{n\geq 0}\,(n+1)^{N+2}|Xv_n| + \sup_{n\geq 0}\left\{(n+1)^{N+1}|Yv_n|\right\} \qquad (N \geq 1).$$

Let ℓ^N be the linear space of all $v \in V_f$ such that $|v|_N < \infty$ for all $|\cdot| \in \mathcal{N}$. Note that ℓ^N is a locally convex space having semi-norms $|\cdot|_N$ with $|\cdot| \in \mathcal{N}$, and that there exists a bounded inclusion $\ell^{N+1} \hookrightarrow \ell^N$. Moreover we set

$$\ell^\infty = \bigcap_{N\geq 0} \ell^N.$$

Then ℓ^∞ is a locally convex space having semi-norms $|\cdot|_N$ with $|\cdot| \in \mathcal{N}$ and $N \in \mathbb{Z}_{\geq 0}$.

Definition 3.2.3. For $m \in \mathbb{Z}_{\geq 0}$ let ι_m be the bounded linear map defined by

$$\iota_m : U_1 \to \ell^0, \quad u \mapsto (\delta_{nm}u)_{n\geq 0},$$

where δ_{nm} is the Kronecker symbol.

Notation 3.2.4. Using Notation 2.2.2, we set

$$P_{ij} = \sum_{k=1}^{i-j}[j,k]\begin{bmatrix} k-1 \\ i-j-k \end{bmatrix}P^{(k)} \qquad (0 \leq j < i \leq N+1).$$

For example, for $1 \leq i - j \leq 3$, we have

$$P_{i+1,i} = \frac{P^{(1)}}{i_+}, \qquad P_{i+2,i} = \frac{P^{(2)}}{i_+(i+1)},$$
$$P_{i+3,i} = \frac{P^{(2)}}{i_+(i+1)} + \frac{P^{(3)}}{i_+(i+1)(i+2)} \qquad (i \geq 0),$$

where $i_+ = \max\{i, 1\}$.

Notation 3.2.5. Since $|Z| < 1$ for any $|\cdot| \in \mathcal{N}$ (cf. (2) of Assumption B_N), the sequential completeness of $\mathcal{B}$ (cf. Remark 3.1.1) implies that $I_0 - Z : U_0 \to U_0$ has an inverse in $\mathcal{B}(U_0)$, which is denoted by $(I_0 - Z)^{-1}$. Indeed the inverse map is given by the C. Neumann series $\sum_{n=0}^{\infty} Z_0^n$. We set

$$A_{ij} = XP_{i+1,j} + \left(I + \frac{1}{i}XP^{(1)}\right)(I_0 - Z)^{-1}(YP_{ij} - \delta_{i,j+1}Z) \qquad (0 \leq j < i \leq N),$$

where δ_{ij} is the Kronecker symbol.

Definition 3.2.6. For any $i \in \mathbb{Z}_{\geq 2}$ let $\mathbf{S}_i$ be the set of all subsets of $\{1, 2, \dots, i-1\}$ (including the empty set $\emptyset$). For each $J \in \mathbf{S}_i$ we set

$$A_J = \begin{cases} A_{i0} & (J = \emptyset), \\ A_{ij_k} A_{j_k j_{k-1}} \cdots A_{j_2 j_1} A_{j_1 0} & (J \neq \emptyset), \end{cases}$$

where $J = \{j_1, j_2, \dots, j_k\}$ with $j_1 < j_2 < \cdots < j_k$ in the latter case. We define the operators $\Psi_i \in \mathcal{B}$ $(i = 0, 1, \dots, N)$ by $\Psi_0 = I$, $\Psi_1 = A_{10}$ and

$$\Psi_i = \sum_{J \in \mathbf{S}_i} A_J \qquad (i \geq 2).$$

Lemma 3.2.7. *The operators Ψ_i $(i = 0, 1, \dots, N)$ are determined by $\Psi_0 = I$ and the recurrence relation*

$$\Psi_i = \sum_{j=0}^{i-1} A_{ij} \Psi_j \qquad (1 \leq i \leq N).$$

This lemma is easily established by induction on i.

Remark 3.2.8. For $i = 1, 2, \dots, N$, the operator Ψ_i depends only on the first $(j+2)$ coefficients $P^{(0)}, P^{(1)}, \dots, P^{(i+1)}$ of P (cf. (3.3)).

3.3. Profile operator.

Theorem I. (1) *Under Assumption* A, *there exist a bounded linear operator Φ : $\ell^0 \to U_1$ such that*

$$|\phi_n(v) - \Phi v| = O\left(\frac{|v|_0}{n+1}\right) \qquad (v \in \ell^0,\ n \geq 0). \tag{3.5}$$

(2) *Under Assumption* A_u, *the Landau symbol O in* (3.5) *can be replaced by O_u (cf. Notation 2.1.3). Moreover there exists an integer m_0 such that $\Phi \circ \iota_m : U_1 \to U_1$ is a topological isomorphism for all $m \geq m_0$ (cf. Definition 3.2.3).*

Remark 3.3.1. The operator Φ is called the *profile operator* (cf. Definition 4.8.1). The author does not know whether Assertion (2) of Theorem I holds under Assumption A (cf. Remark 4.8.4). This assertion is closely related to Condition (3) of (AP-3) in Definition 1.1. So we use it essentially to construct asymptotic pairs.

Theorem I is established in 4.8 (cf. Theorem 4.8.3).

3.4. Asymptotic expansions.

Theorem II. *Let Φ be the profile operator in Theorem* I, *Ψ_i the operators defined by Definition 3.2.6.*

(1) *Under Assumption* B_N *we have*

$$\left| \phi_n(v) - \left(\sum_{i=0}^{N} \Psi_i [n]_i \right) \Phi v \right| = O\left(\frac{|v|_N}{(n+1)^{N+1}} \right) \qquad (v \in \ell^N,\ n \geq N).$$

(2) *Under Assumption* B_∞, *for any $N \geq 0$, we have*

$$\left| \phi_n(v) - \left(\sum_{i=0}^{N} \Psi_i [n]_i \right) \Phi v \right| = O\left(\frac{|v|_N}{(n+1)^{N+1}} \right) \qquad (v \in \ell^\infty,\ n \geq N).$$

Notation 3.4.1. Regarding $v \mapsto \phi_n(v)$ as an operator $\phi_n : \ell^\infty \to U$, we express Assertion (2) of Theorem II as

$$\phi_n \sim \left(\sum_{i=0}^{\infty} \Psi_i [n]_i \right) \Phi \qquad \text{in } \ \ell^\infty.$$

Theorem II is established in Section 5 (cf. Theorem 5.1.1). From Theorem I and Theorem II we obtain the following corollary.

Corollary 3.4.2. *Under Assumptions* A_u *and* B_∞, (ℓ^∞, P) *is an asymptotic pair of order* ∞. *The asymptotic data of* (ℓ^∞, P) *is given by* $(U_1, \{\Psi_i\}_{i=0}^\infty)$, *where* U_1 *and* Ψ_i *are defined in Notation 3.2.1 and Definition 3.2.6, respectively.*

Corollary 3.4.2 is established in 5.5.

3.5. Gevrey spaces. We introduce Gevrey spaces and establish some notation.

Definition 3.5.1. Let $t \geq 0$ and $a > 0$. For $v = (v_n) \in V_f$ and $|\cdot| \in \mathcal{N}$ we set

$$|v|_{t,a} = \sup_{n \geq 0} \left\{ \frac{(n!)^t}{a^n} |v_n| \right\}.$$

Let $\mathcal{G}^{t,a}$ be the set of all $v \in V_f$ such that $|v|_{t,a} < \infty$ for all $|\cdot| \in \mathcal{N}$. Then $\mathcal{G}^{t,a}$ is a locally convex space having semi-norms $|\cdot|_{t,a}$ with $|\cdot| \in \mathcal{N}$. $\mathcal{G}^{t,a}$ is called the *Gevrey space* with index (t, a). An index (t, a) is said to be *of convergent type* if either $t = 0, 0 < a < 1$ or $t > 0, a > 0$ holds.

Definition 3.5.2. Notice that if $a < b$ then $\mathcal{G}^{t,a} \subset \mathcal{G}^{t,b}$. For $t, a \geq 0$ we set

$$\mathcal{G}^{t,a+} = \bigcap_{b > a} \mathcal{G}^{t,b}.$$

Then $\mathcal{G}^{t,a+}$ is a locally convex space having semi-norms $|\cdot|_{t,b}$ with $|\cdot| \in \mathcal{N}$ and $b > a$. $\mathcal{G}^{t,a+}$ is called the *Gevrey space* with index $(t, a+)$. An index $(t, a+)$ is said to be of *convergent type* if either $t = 0, 0 \leq a < 1$ or $t > 0, a \geq 0$ holds.

Any Gevrey space $\mathcal{G}^{t,a}$ (resp. $\mathcal{G}^{t,a+}$) of convergent type is contained in ℓ^∞, and hence the profile operator Φ acts on $\mathcal{G}^{t,a}$ (resp. $\mathcal{G}^{t,a+}$). Thus we can introduce the following spaces.

Definition 3.5.3. For any index (t, a) or $(t, a+)$ of convergent type, we set

$$\mathcal{G}_0^{t,a} = \{ \, v \in \mathcal{G}^{t,a} \, ; \, \Phi v = 0 \, \}, \qquad \mathcal{G}_0^{t,a+} = \{ \, v \in \mathcal{G}^{t,a+} \, ; \, \Phi v = 0 \, \}.$$

In addition to Assumption B_N we make the following assumption.

Assumption C. Let p, q and r be integers such that $1 \leq p, q \leq N + 2$ and $0 \leq r \leq N + 2$. We assume

$$\begin{aligned} &XP^{(i)}Y = O \qquad (0 \leq i < p), \qquad YP^{(i)}X = O \qquad (0 \leq i < q), \\ &YP^{(i)}Y = O \qquad (0 \leq i < r). \end{aligned}$$

If $r = 0$ then the third condition should be ignored.

Notation 3.5.4. Let p, q, r be as in Assumption C. The following integer s plays an important role in what follows:

$$s = \min \{p + q - 1, r\}.$$

Definition 3.5.5. Under Assumptions B_N and C, for each $|\cdot| \in \mathcal{N}$, we introduce the following constants:

$$c_1(|\cdot|) = \overline{\lim}_{n\to\infty}\{\, n(n+1)|XP_nX - X|\,\}, \qquad c_2(|\cdot|) = \overline{\lim}_{n\to\infty}\{\, n^p|XP_nY|\,\},$$
$$c_3(|\cdot|) = \overline{\lim}_{n\to\infty}\{\, n^q|YP_nX|\,\}, \qquad c_4(|\cdot|) = \overline{\lim}_{n\to\infty}\{\, n^r|YP_nY|\,\}.$$

By Assumptions B_N and C, they are well-defined finite numbers. Using these constants, we define the constant $c_0(|\cdot|)$ by

$$c_0(|\cdot|) = \begin{cases} c_2(|\cdot|)c_3(|\cdot|) & (p+q-1<r), \\ c_2(|\cdot|)c_3(|\cdot|) + c_4(|\cdot|) & (p+q-1=r), \\ c_4(|\cdot|) & (p+q-1>r). \end{cases}$$

Finally we introduce the following (*possibly infinite*) constant:

$$a_0 = \sup\{\, c_0(|\cdot|)\ ;\ |\cdot| \in \mathcal{N}\,\}.$$

Remark 3.5.6. In the following cases, (3.3) in Assumption B_N implies

$$\begin{aligned} c_1(|\cdot|) &= |XP^{(2)}X| && (N \ge 1), \\ c_2(|\cdot|) &= (p-1)!|XP^{(p)}Y| && (1 \le p \le N+1), \\ c_3(|\cdot|) &= (q-1)!|YP^{(q)}X| && (1 \le q \le N+1), \\ c_4(|\cdot|) &= \begin{cases} |Z| & (r=0), \\ (r-1)!|YP^{(r)}Y| & (1 \le r \le N+1). \end{cases} \end{aligned}$$

If $r=0$ then $p+q-1 \ge 1 > 0 = r$, and hence $c_0(|\cdot|) = c_4(|\cdot|) = |Z|$. In this case, (2) of Assumption B_N implies $c_0(|\cdot|) < 1$.

3.6. Gevrey estimates.

Theorem III. *Under Assumptions* B_N *and* C*, if the integer* s *(cf. Notation 3.5.4) and the index* (t,a) *satisfy one of the following conditions:*

(1) $t = s = 0,\ c_0(|\cdot|) < a < 1$ *(cf. Remark 3.5.6),*
(2) $t = 0 < s,\ 0 < a < 1,$
(3) $0 < t < s,\ a > 0,$
(4) $t = s > 0,\ a > c_0(|\cdot|),$

then there exists a nonnegative constant $M(t,a,|\cdot|)$*, depending only on* (t,a) *and* $|\cdot| \in \mathcal{N}$*, such that*

$$|\phi_n(v)| \le M(t,a,|\cdot|)\frac{(n+1)^{p+1}a^n}{(n!)^t}|v|_{t,a} \qquad (v \in \mathcal{G}_0^{t,a},\ n \ge 0),$$

where $\mathcal{G}_0^{t,a}$ *is defined in Definition 3.5.3.*

Theorem III is established in 6.6 (cf. Theorem 6.6.2).

Corollary 3.6.1. *Under Assumptions* B_N *and* C*, if* (s, a_0) *(cf. Notation 3.5.4 and Definition 3.5.5) and* (t,a) *satisfy one of the following conditions:*

(1) $t = s = 0,\ a_0 \le a < 1,$
(2) $t = 0 < s,\ 0 \le a < 1,$
(3) $0 < t < s,\ a \ge 0,$
(4) $t = s > 0,\ a_0 < \infty$ *and* $a \ge a_0,$

then the linear map

$$\phi : \mathcal{G}_0^{t,a+} \to \mathcal{G}^{t,a+}, \quad v \mapsto \phi(v)$$

is well-defined and bounded, where $\mathcal{G}_0^{t,a+}$ is defined in Definition 3.5.3.

Corollary 3.6.1 immediately follows from Theorem III. Moreover we have the following corollary.

Corollary 3.6.2. *Under Assumptions* A_u, B_N *and* C, *if one of the conditions* (1)–(4) *in Corollary 3.6.1 holds, then the pair $(\mathcal{G}^{t,a+}, P)$ is an asymptotic pair of order N. The asymptotic data of $(\mathcal{G}^{t,a+}, P)$ is given by $(U_1, \{\Psi_i\}_{i=0}^N)$, where U_1 and Ψ_i are defined in Notation 3.2.1 and Definition 3.2.6, respectively.*

Corollary 3.6.2 is established in 6.7. We refer to Remark 3.3.1 for the reason why Assumption A_u is added to Assumptions B_N and C in Corollary 3.6.2.

4. Profile Operator

4.1. Propagators.

Definition 4.1.1. We set $P_{mm} = I$ $(m \geq 0)$ and

$$P_{nm} = P_n P_{n-1} \cdots P_{m+2} P_{m+1} \qquad (n > m \geq 0).$$

P_{nm} is called the *propagator* (from time m to time n).

Lemma 4.1.2. *For any $v = (v_n) \in V_f$ the solution $\phi(v)$ of the difference equation (1.8) is given by*

$$\phi_n(v) = \sum_{m=0}^{n} P_{nm} v_m.$$

Theorem 4.1.3. *Under Assumption* A *we have*

$$X(P_{km} - P_{nm})X = O\left(\frac{1}{n+1}\right) \qquad (k \geq n \geq m \geq 0),$$

$$X(P_{km} - P_{nm})Y = O\left(\frac{1}{n+1}(\rho^{n-m} + \frac{1}{m+1})\right) \qquad (k \geq n \geq m \geq 0),$$

$$Y P_{nm} X = O\left(\frac{1}{n+1}\right) \qquad (n \geq m \geq 0),$$

$$Y P_{nm} Y = O\left(\rho^{n-m} + \frac{1}{(n+1)(m+1)}\right) \qquad (n \geq m \geq 0).$$

Under Assumption A_u *the Landau symbol O can be replaced by O_u.*

Remark 4.1.4. For a sufficiently large *fixed* m, many authors have studied asymptotic representations of P_{nm} with respect to n (cf. [1][2][4][5][11]). In their studies U is assumed to be a finite-dimensional complex or real vector space. In this paper, however, we need an asymptotic representation with respect to both n and m. Theorem 4.1.3 is by no means the best possible result; it is just simple and sufficient for our purpose. After some preliminaries, Theorem 4.1.3 will be established in 4.6.

4.2. Two-dimensional problem. Theorem 4.1.3 will be established by the method of majorants. In order to construct the majorants, we consider the following two-dimensional problem: Let

$$M_n = \begin{pmatrix} 1 & \frac{a}{n+1} \\ \frac{b}{n+1} & \rho \end{pmatrix} + O\left(\frac{1}{(n+1)^2}\right) \qquad (n \geq 0), \tag{4.1}$$

where $a, b > 0$ and $0 \leq \rho < 1$. Put $M_{mm} = I$ $(m \geq 0)$ and

$$M_{nm} = M_n M_{n-1} \cdots M_{m+1} \qquad (n > m \geq 0).$$

The problem is to obtain an asymptotic representation of M_{nm} with respect to n and m.

We follow the method of Z. Benzaid and D.A. Lutz [1] with some additional elaboration. We set

$$S_n = \begin{pmatrix} 1 & -\frac{\alpha}{n+1} \\ \frac{\beta}{n+1} & 1 \end{pmatrix} \qquad (n \geq 0),$$

where $\alpha = \frac{a}{1-\rho}$ and $\beta = \frac{b}{1-\rho}$. Moreover we set $N_n = (S_n)^{-1} M_n S_{n-1}$. Then we have $N_n = D + E_n$, where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \qquad E_n = O\left(\frac{1}{(n+1)^2}\right) \quad (n \geq 1). \tag{4.2}$$

We set $N_{mm} = I$ $(m \geq 0)$ and $N_{nm} = N_n N_{n-1} \cdots N_{m+1}$ $(n > m \geq 0)$. Then $M_{nm} = S_n N_{nm} (S_m)^{-1}$. So the problem is reduced to the same problem for N_{nm}.

4.3. Contraction mapping. For each fixed $m \geq 0$, let $\mathcal{L}$ be the linear space of all sequences $L = (L_n)_{n \geq m}$ in $M_2(\mathbb{R})$. We define a linear transformation $K : \mathcal{L} \to \mathcal{L}$ by $(KL)_m = O$ and

$$(KL)_n = \sum_{i=m+1}^{n} D^{n-i} E_i L_{i-1} \qquad (n > m), \tag{4.3}$$

where D and E_i are given by (4.2). Moreover we define $N^{(m)}, D^{(m)} \in \mathcal{L}$ by

$$N^{(m)} = (N_{nm})_{n \geq m}, \qquad D^{(m)} = (D^{n-m})_{n \geq m}. \tag{4.4}$$

Lemma 4.3.1. (1) *For any $\Lambda \in \mathcal{L}$ the equation* (4.5) *below has a unique solution $L(\Lambda) \in \mathcal{L}$:*

$$L = \Lambda + KL \tag{4.5}$$

(2) *$N^{(m)} = L(D^{(m)})$, where $N^{(m)}$ and $D^{(m)}$ are given by (4.4).*

Proof. Assertion (1) follows from the fact that K is an operator of Volterra type. Assertion (2) is a consequence of the very definition of K (not immediate).

Let $\mathcal{L}^\infty$ be the linear subspace of $\mathcal{L}$ consisting of all $L = (L_n) \in \mathcal{L}$ such that $\|L\| := \sup_{n \geq m} |L_n| < \infty$, where $|\cdot|$ is the operator norm in $M_2(\mathbb{R})$. We set

$$c_m = \sum_{i=m+1}^{\infty} |E_i|. \tag{4.6}$$

Since $E_i = O(\frac{1}{i^2})$, we have $c_m = O(\frac{1}{m+1})$. Using $|D| = 1$, we can easily show the following lemma.

Lemma 4.3.2. *Let c_m be defined by (4.6). The linear transformation $K : \mathcal{L} \to \mathcal{L}$ induces a bounded linear transformation $K : \mathcal{L}^\infty \to \mathcal{L}^\infty$, whose operator norm $\|K\|$ is estimated as*

$$\|K\| \leq c_m.$$

Lemma 4.3.3. *Let m_0 be the smallest number such that $c_{m_0} < 1$ (cf. (4.6)). Then,*

$$N_{nm} - D^{n-m} = O\left(\frac{1}{m+1}\right) \qquad (n \geq m \geq m_0).$$

Proof. Lemma 4.3.2 implies that, for all $m \geq m_0$, $K : \mathcal{L}^\infty \to \mathcal{L}^\infty$ is a contraction mapping. By the contraction mapping principle, for any $\Lambda \in \mathcal{L}^\infty$, the equation (4.5) has a unique solution $L^\infty(\Lambda) \in \mathcal{L}^\infty$. Lemma 4.3.1.(1) implies that if $\Lambda \in \mathcal{L}^\infty$ then $L^\infty(\Lambda) = L(\Lambda)$. Applying this fact to $D^{(m)} \in \mathcal{L}^\infty$ and using Lemma 4.3.1(2), we obtain $N^{(m)} = L^\infty(D^{(m)}) \in \mathcal{L}^\infty$. Explicitly, $N^{(m)}$ is given by the C. Neumann series:

$$N^{(m)} = D^{(m)} + \sum_{i=1}^{\infty} K^i D^{(m)}.$$

By Lemma 4.3.2 and $\|D^{(m)}\| = 1$, we have

$$\begin{aligned} \|N^{(m)} - D^{(m)}\| &\leq \sum_{i=1}^{\infty} \|K\|^i \|D^{(m)}\| \leq \sum_{i=1}^{\infty} (c_m)^i \\ &\leq \frac{c_m}{1 - c_m} = O\left(\frac{1}{m+1}\right). \end{aligned}$$

This means $|N_{nm} - D^{n-m}| = O(\frac{1}{m+1})$ $(n \geq m \geq m_0)$.

4.4 Technical lemma. We list the technical lemmas that are used later.

Lemma 4.4.1. *Let r and N be constants such that $0 \leq \rho < 1$ and $N > 0$.*

$$\sum_{i=1}^{n} \frac{\rho^{n-i}}{i^N} = O\left(\frac{1}{n^N}\right) \qquad (n \geq 1).$$

Proof.

$$\begin{aligned} \sum_{i=1}^{n} \frac{\rho^{n-i}}{i^N} &\leq \sum_{i=1}^{[n/2]} \frac{\rho^{n-i}}{i^N} + \sum_{i=[n/2]+1}^{n} \frac{\rho^{n-i}}{i^N} \\ &\leq \sum_{i=1}^{[n/2]} \rho^{n-i} + \frac{1}{([n/2]+1)^N} \sum_{i=[n/2]+1}^{n} \rho^{n-i} \\ &= \rho^{n-[n/2]} \frac{1 - \rho^{[n/2]}}{1 - \rho} + \frac{1}{([n/2]+1)^N} \frac{1 - \rho^{n-[n/2]}}{1 - \rho} = O\left(\frac{1}{n^N}\right). \end{aligned}$$

In order to obtain a precise estimate for N_{nm}, we use the following lemma.

Lemma 4.4.2. *Let ρ be a constant such that $0 \le \rho < 1$.*

$$(1)\quad \sum_{i=m+1}^{n} \frac{1}{i^2} = O\left(\frac{1}{m+1}\right), \qquad (2)\quad \sum_{i=m+1}^{n} \frac{\rho^{i-m-1}}{i^2} = O\left(\frac{1}{(m+1)^2}\right),$$

$$(3)\quad \sum_{i=m+1}^{n} \frac{\rho^{n-i}}{i} = O\left(\frac{1}{n+1}\right), \qquad (4)\quad \sum_{i=m+1}^{n} \frac{\rho^{n-i}}{i^2} = O\left(\frac{1}{(n+1)(m+1)}\right),$$

$$(5)\quad \frac{\rho^{n-m}}{m+1} = O\left(\frac{1}{n+1}\right) \qquad (n \ge m \ge 0).$$

Proof. Assertions (1) and (2) are easy; (3) is an immediate consequence of Lemma 4.4.1; (4) easily follows from (3). Finally (5) is shown from the fact that $(n+1)\rho^n$ is nonincreasing for n sufficiently large.

4.5. Estimates. Let us obtain an estimate for N_{nm} and then for M_{nm}.

Lemma 4.5.1. *Let m_0 be the integer defined in Lemma 4.3.3. Then,*

$$N_{nm} = \begin{pmatrix} 1 + O(\frac{1}{m+1}) & O(\frac{1}{(m+1)^2}) \\ O(\frac{1}{(n+1)(m+1)}) & \rho^{n-m} + O\left(\frac{1}{m+1}(\rho^{n-m+1} + \frac{1}{(m+1)(n+1)})\right) \end{pmatrix}$$
$$(n \ge m \ge m_0).$$

Proof. Since $N_{mm} = I$ we may assume $n > m \ge m_0$. By Lemma 4.3.1(2) we have

$$N_{nm} = D^{n-m} + \sum_{i=m+1}^{n} D^{n-i} E_i N_{i-1,m}. \tag{4.7}$$

Using Lemma 4.3.3 we can easily show that

(4.8)

$$D^{n-i} E_i N_{i-1,m} = \begin{pmatrix} O(\frac{1}{i^2}) & O\left(\frac{1}{i^2}(\rho^{i-m-1} + \frac{1}{m+1})\right) \\ O(\frac{\rho^{n-i}}{i^2}) & O\left(\frac{1}{i^2}(\rho^{i-m-1} + \frac{\rho^{n-i}}{m+1})\right) \end{pmatrix} \quad (n \ge i > m \ge m_0).$$

Applying (1),(2),(4) and (5) in Lemma 4.4.2 to (4.7)-(4.8), we establish the lemma.

Proposition 4.5.2.

$$M_{nm} = \begin{pmatrix} O(1) & O(\frac{1}{m+1}) \\ O(\frac{1}{m+1}) & O(\rho^{n-m} + \frac{1}{(n+1)(m+1)}) \end{pmatrix} \qquad (n \ge m \ge 0).$$

Proof. Applying Lemma 4.5.1 to $M_{nm} = S_n N_{nm} (S_m)^{-1}$, we obtain

$$M_{nm} = \varepsilon_m \begin{pmatrix} 1 + O(\frac{1}{m+1}) & \alpha_{nm} + O(\frac{1}{(m+1)^2}) \\ \beta_{nm} + O(\frac{1}{m+1}(\frac{1}{n+1} + \frac{\rho^{n-m}}{m+1})) & \gamma_{nm}(1 + O(\frac{1}{m+1})) \end{pmatrix} \quad (n \ge m \ge m_0),$$

where m_0 is defined in Lemma 4.3.3 and

$$\alpha_{nm} = \alpha\left(\frac{1}{m+1} - \frac{\rho^{n-m}}{n+1}\right), \qquad \beta_{nm} = \beta\left(\frac{1}{n+1} - \frac{\rho^{n-m}}{m+1}\right),$$

$$\gamma_{nm} = \rho^{n-m} + \frac{\alpha\beta}{(n+1)(m+1)}, \qquad \varepsilon_m = \frac{1}{1 + \frac{\alpha\beta}{(m+1)^2}}.$$

Using Lemma 4.4.2.(5) we obtain the proposition for $n \ge m \ge m_0$. Next we consider the case $0 \le m < m_0$. The subcase $0 \le m \le n < m_0$ is trivial. In the other subcase $n \geq m_0 > m \geq 0$, we divide M_{nm} as $M_{nm} = M_{nm_0} M_{m_0 m}$. Applying the proposition (for $n \geq m = m_0$) to M_{nm_0}, we establish the proposition in this subcase. In conclusion, the proposition holds for all $n \geq m \geq 0$.

4.6. Majorants. Now we are in a position to establish Theorem 4.1.3. We use the method of majorants. For $G = (g_{ij}), H = (h_{ij}) \in M_2(\mathbb{R})$, we write $G \leq H$ if $g_{ij} \leq h_{ij}$ for all $i, j = 1, 2$. We fix any $|\cdot| \in \mathcal{N}$. To each $A \in \mathcal{B}$ we associate the following 2×2-matrix:

$$\|A\| = \begin{pmatrix} |XAX| & |XAY| \\ |YAX| & |YAY|. \end{pmatrix}$$

The following lemma is easy to see.

Lemma 4.6.1.

$$\|AB\| \leq \|A\| \|B\| \qquad (\forall A, B \in \mathcal{B}).$$

Now we complete the proof of Theorem 4.1.3. By Assumption A we can choose the constants $a, b > 0$ in (4.1) so that

$$\|P_n\| \leq M_n \qquad (n \geq 1),$$

where the matrix M_n is given by (4.1). Under Assumption A_u the constants a, b and the minor term $O((n+1)^{-2})$ in (4.1) are uniform with respect to $|\cdot| \in \mathcal{N}$. Hence, by Lemma 4.6.1, we have for $n > m \geq 0$,

$$\|P_{nm}\| \leq \|P_n\| \|P_{n-1}\| \cdots \|P_{m+1}\| \leq M_n M_{n-1} \cdots M_{m+1} = M_{nm}.$$

This is also valid for $n = m$. Proposition 4.5.2 implies Theorem 4.1.3.

4.7. Limit propagators. As a collorary to Theorem 4.1.3 we obtain the following theorem.

Theorem 4.7.1. *Under Assumption* A *there exists an infinite sequence* $\{R_m\}$ *of operators in* $\mathcal{B}$ *such that*

(1) $\quad XR_m = R_m,$

(2) $\quad R_m X - XP_{nm}X = O\left(\dfrac{1}{n+1}\right) \qquad (n \geq m \geq 0),$

(3) $\quad R_m Y - XP_{nm}Y = O\left(\dfrac{1}{n+1}(\rho^{n-m} + \dfrac{1}{m+1})\right) \qquad (n \geq m \geq 0).$

Under Assumption A_u *the Landau symbol* O *can be replaced by* O_u.

Proof. Theorem 4.1.3 implies that, for any fixed m, $\{XP_{nm}\}_{n \geq m}$ is a Cauchy sequence in $\mathcal{B}$. Since $\mathcal{B}$ is sequentially complete (cf. Remark 3.1.1), there exists an $R_m \in \mathcal{B}$ such that $XP_{nm} \to R_m$ in $\mathcal{B}$ as $n \to \infty$. Then Assertion (1) is clear. Letting $k \to \infty$ in Theorem 4.1.3, we also obtain (2) and (3).

Putting $n = m$ in (2)(3) of Theorem 4.7.1, we obtain the following corollary.

Corollary 4.7.2. *Under Assumption* A*,*

$$R_m X = X + O\left(\frac{1}{m+1}\right), \qquad R_m Y = O\left(\frac{1}{m+1}\right) \qquad (m \geq 0).$$

Under Assumption A_u *the Landau symbol* O *can be replaced by* O_u.

Definition 4.7.3. The operator R_m in Theorem 4.7.1 is called the *limit propagator* (from time m).

4.8. Profile operator. Now we are in a position to define the profile operator Φ.

Definition 4.8.1. For $n \geq 0$ we define a bounded linear operator $\Phi_n : \ell^0 \to U_1$ by

$$\Phi_n v = \sum_{m=n}^{\infty} R_m v_m \qquad (v = (v_n) \in \ell^0). \tag{4.9}$$

Lemma 4.8.2 below shows that Φ_n is well-defined. We set $\Phi = \Phi_0$ and call it the *profile operator*.

Lemma 4.8.2. *Under Assumption* A, Φ_n *is well-defined, i.e., the infinite series in (4.9) is (absolutely) convergent and* $\Phi_n v$ *admits the following estimate:*

$$|\Phi_n v| = O\left(\frac{|v|_0}{n+1}\right) \qquad (v \in \ell^0,\ n \geq 0).$$

Under Assumption A_u *the Landau symbol* O *can be replaced by* O_u*.*

Proof. We only consider the case of Assumption A. In the case of Assumption A_u we have only to replace O by O_u in the proof. Corollary 4.7.2 implies

$$\begin{aligned} |R_m v_m| &\leq |R_m X||X v_m| + |R_m Y||Y v_m| \\ &= O\left(\frac{1}{m+1}(m+1)|X_m| + \frac{1}{(m+1)^2}\sup_{k\geq 0}\,(k+1)|v_k|\right) \qquad (m \geq 0), \end{aligned}$$

and hence

$$\begin{aligned} \sum_{m=n}^{k} |R_m v_m| &= O\left(\sum_{m=n}^{k}\frac{1}{m+1}(m+1)|X v_m| + \sup_{k\geq 0}\,(k+1)|v_k|\sum_{m=n}^{k}\frac{1}{(m+1)^2}\right) \\ &= O\left(\frac{1}{n+1}\left\{\sum_{m=n}^{\infty}(m+1)|X_m| + \sup_{k\geq 0}\,(k+1)|v_k|\right\}\right) \\ &= O\left(\frac{|v|_0}{n+1}\right) \qquad (k \geq n \geq 0). \end{aligned}$$

Therefore the sequential completeness of U implies that (4.9) is converegent.

Now we can state the main theorem in this section.

Theorem 4.8.3 (Theorem I). (1) *Under Assumption* A *there exists a bounded linear operator* $\Phi : \ell^0 \to U_1$ *(the profile operator) such that*

$$|\phi_n(v) - \Phi v| = O\left(\frac{|v|_0}{n+1}\right) \qquad (v \in \ell^0,\ n \geq 0).$$

(2) *Under Assumption* A_u *the Landau symbol* O *can be replaced by* O_u*. Moreover there exists an integer* m_0 *such that* $\Phi \circ \iota_m : U_1 \to U_1$ *is a topological isomorphism for all* $m \geq m_0$*, where* ι_m *is defined in Definition 3.2.3.*

Proof. By Lemma 4.1.2 and Definition 4.8.1, we have $\Phi v - \phi_n = J + \Phi_{n+1} v$, where

$$J = \sum_{m=0}^{n} (R_m - P_{nm}) v_m.$$

By using Theorem 4.1.3 and Theorem 4.7.1, J is estimated as follows:

$$\begin{aligned}
|J| &\le \sum_{m=0}^{n} (|R_m X - X P_{nm} X| + |Y P_{nm} X|)|X v_m| \\
&\quad + \sum_{m=0}^{n} |R_m Y - X P_{nm} Y||Y v_m| + \sum_{m=0}^{n} |Y P_{nm} Y||Y v_m| \\
&\le O\left(\frac{1}{n+1} \sum_{m=0}^{n} |X v_m| + \frac{1}{n+1} \sum_{m=0}^{n} \{ \rho^{n-m} + \frac{1}{m+1} \}|Y v_m| \right) \\
&\quad + O\left(\sum_{m=0}^{n} \{ \rho^{n-m} + \frac{1}{(n+1)(m+1)} \}|Y v_m| \right) \\
&= O\left(\frac{|v|_0}{n+1} + \sum_{m=0}^{n} \{ \rho^{n-m} + \frac{1}{(n+1)(m+1)} \}|Y v_m| \right) \\
&= O\left(\frac{|v|_0}{n+1} + \{ \sum_{m=0}^{n} \frac{\rho^{n-m}}{m+1} + \frac{1}{n+1} \sum_{m=0}^{n} \frac{1}{(m+1)^2} \}|v|_0 \right) \\
&= O\left(\frac{|v|_0}{n+1} \right).
\end{aligned}$$

Here Lemma 4.4.1 ($N = 1$) is used in the last equality. This estimate and Lemma 4.8.2 establish (1) of the theorem. The first assertion of (2) is established in a similar manner.

It remains to establish the second assertion of (2). We set $\Phi \circ \iota_m = I_1 - F_m$. Definition 4.8.1 implies $\Phi \circ \iota_m = R_m|_{U_1}$. By Corollary 4.7.2 we have $R_m|_{U_1} = I_1 + O_u(\frac{1}{m+1})$. So there exist $m_0 \ge 0$ and ρ_1 ($0 \le \rho_1 < 1$) such that $|F_m| \le \rho_1$ for any $m \ge m_0$ and $|\cdot| \in \mathcal{N}$. By the sequential completeness of $\mathcal{B}$, the C. Neumann series $\sum_{i=0}^{\infty}(F_m)^i$ converges in $\mathcal{B}$ and gives the inverse of $\Phi \circ \iota_m$. Hence, for any $m \ge m_0$, $\Phi \circ \iota_m$ is a linear isomorphism.

Remark 4.8.4. The second assertion of (2) in Theorem 4.8.3 was established by using the estimate

$$R_m|_{U_1} = I_1 + O_u\left(\frac{1}{m+1}\right). \tag{4.10}$$

If O_u is replaced by O in (4.10), then it is not always possible to take m_0 and ρ_1 uniformly with respect to $|\cdot| \in \mathcal{N}$. Hence the C. Neumann series $\sum_{i=0}^{\infty}(F_m)^i$ is not necessarily convergent in $\mathcal{B}$. Our argument fails at this point. Assumption A_u leads to the estimate (4.10), while Assumption A leads only to the estimate (4.10) with O_u replaced by O. For this reason we need Assumption A_u in place of Assumption A.

5. Asymptotic Expansions

5.1. Induction. We establish the following theorem by induction.

Theorem 5.1.1 (Assertion (1) of Theorem II)**.** *Under Assumption* B_N*,*

$$\left| \phi_n(v) - \left(\sum_{i=0}^{N} \Psi_i[n]_i \right) \Phi v \right| = O\left(\frac{|v|_N}{(n+1)^{N+1}} \right) \qquad (v \in \ell^N,\ n \ge N).$$

Assertion (2) of Theorem II is an immediate consequence of Theorem 5.1.1, because Assumption B_∞ implies Assumption B_N for any $N \in \mathbb{Z}_{\geq 0}$. In Assumption B_N, if $N = 0$ then Theorem 5.1.1 immediately follows from Theorem I (under Assumption A). In this section we assume $N \geq 1$. We use the following notation.

Notation 5.1.2. For $m = 0, 1, \dots, N$ and $1 \leq j + k \leq m + 1$, we set

$$F_m(n)v = \phi_n(v) - \left(\sum_{i=0}^{m} \Psi_i [n]_i\right) \Phi v,$$

$$E_m(n) = P_n - \sum_{i=0}^{m+1} P^{(i)}(n)_i,$$

$$\varepsilon_{jkm}(n) = (n)_k [n-1]_j - \sum_{i=j+k}^{m+1} [j,k] \begin{bmatrix} k-1 \\ i-j-k \end{bmatrix} [n]_i,$$

(cf. Definitions 2.2.1, 3.2.6 and Notation 2.2.2).

In order to set up an induction argument, we consider the following claims.

Claims 5.1.3. For any fixed $|\cdot| \in \mathcal{N}$,

$$(5.1.m) \qquad |Y F_m(n) v| = O\left(\frac{|v|_m}{(n+1)^{m+1}}\right) \qquad (v \in \ell^m,\ n \geq m),$$

$$(5.2.m) \qquad |X F_m(n) v| = O\left(\frac{|v|_m}{(n+1)^{m+1}}\right) \qquad (v \in \ell^m,\ n \geq m),$$

$$(5.3.m) \qquad |F_m(n) v| = O\left(\frac{|v|_m}{(n+1)^{m+1}}\right) \qquad (v \in \ell^m,\ n \geq m).$$

First we note that Theorem I (under Assumption A) implies (5.3.0), and hence (5.1.0) and (5.2.0). Under Assumption B_N these claims make sense for $m = 1, \dots, N$ (of course, it is *a priori* not clear whether they are true or not). The induction argument proceeds through the following three steps.

Step 1. (5.3.m) *implies* (5.1.$m+1$) *for* $m = 0, 1, \dots, N-1$.

Step 2. (5.3.m) *and* (5.1.$m+1$) *imply* (5.2.$m+1$) *for* $m = 0, 1, \dots, N-1$.

Step 3. (5.1.m) *and* (5.2.m) *imply* (5.3.m) *for* $m = 0, 1, \dots, N$.

We establish Step 1 and Step 2; Step 3 is trivial. Then Theorem 5.1.1 follows from (5.3.N).

5.2. Technical lemmas. In what follows we use the following lemmas.

Lemma 5.2.1. *Let* $0 \leq \rho < 1$ *and* $N \in \mathbb{Z}_{\geq 0}$. *If* $\{a_n\}_{n \geq 0}$ *is an infinite sequence of nonnegative numbers such that*

$$a_n - \rho a_{n-1} = O\left(\frac{1}{n^N}\right) \qquad (n \geq 1),$$

then

$$a_n = O\left(\frac{1}{(n+1)^N}\right) \qquad (n \geq 0).$$

Proof. We set $b_n = \rho^{-n} a_n$. Then we have $b_i - b_{i-1} = O(\rho^{-i} i^{-N})$. Summing it over $i = 1, 2, \dots, n$, we have $b_n - b_0 = O(\sum_{i=1}^{n} \rho^{-i} i^{-N})$. Hence, using Lemma 4.4.1, we obtain

$$a_n = a_0 \rho^n + O\left(\sum_{k=1}^{n} \frac{\rho^{n-k}}{k^N} \right) = O\left(\frac{1}{(n+1)^N} \right).$$

Lemma 5.2.2. *For $m = 0, 1, \dots, N$ and $1 \le j + k \le m + 1$,*

$$E_m(n) = O\left(\frac{1}{n^{m+2}} \right), \qquad \varepsilon_{jkm}(n) = O\left(\frac{1}{n^{m+2}} \right) \qquad (n \ge m+1).$$

This lemma is an immediate consequence of Lemma 2.2.5 and Assumption B_N.

5.3. Step 1.

Lemma 5.3.1. *If (5.3.m) holds, then we have*

$$(YP_n - Z)\phi_{n-1} = \sum_{i=1}^{m+1} \sum_{j=0}^{i-1} YP_{ij} \Psi_j [n]_i \Phi v + \mathit{\Delta}_m(n) v,$$

with

$$|\mathit{\Delta}_m(n) v| = O\left(\frac{|v|_m}{n^{m+2}} \right) \qquad (v \in \ell^m,\ n \ge m+1).$$

Proof. By (1) of Assumption B_N we have

$$YP_n - Z = \sum_{i=1}^{m+1} YP^{(i)}(n)_i + YE_m(n).$$

Here the point is that the suffix i starts from $i = 1$. By Notation 5.1.2 we have

$$(YP_n - Z)\phi_{n-1} = \sum_{\substack{0 \le j \le m \\ 1 \le k \le m+1 \\ j+k \le m+1}} YP^{(k)} \Psi_j(n)_k [n-1]_j \Phi v + \Theta_m(n) v,$$

where

$$\begin{aligned} \Theta_m(n) = {} & \sum_{i=1}^{m+1} YP^{(i)}(n)_i F_m(n-1) + YE_m(n) \sum_{j=0}^{m} \Psi_j [n-1]_j \Phi \\ & + YE_m(n) \cdot F_m(n-1) + \sum_{\substack{1 \le i \le m+1 \\ 0 \le j \le m \\ i+j \ge m+2}} YP^{(i)} \Psi_j(n)_i [n-1]_j \Phi. \end{aligned}$$

Moreover, by Notation 5.1.2, we have

$$\begin{aligned}
&(YP_n - Z)\phi_{n-1} \\
&\quad = \sum_{\substack{0\le j\le m \\ 1\le k\le m+1 \\ j+k\le m+1}} YP^{(k)}\Psi_j \left(\sum_{i=j+k}^{m+1} [j,k] \begin{bmatrix} k-1 \\ i-j-k \end{bmatrix} [n]_i + \varepsilon_{jkm}(n) \right) \Phi v + \Theta_m(n) v \\
&\quad = \sum_{i=1}^{m+1} \sum_{j=0}^{i-1} Y \sum_{k=1}^{i-j} [j,k] \begin{bmatrix} k-1 \\ i-j-k \end{bmatrix} P^{(k)} \Psi_j [n]_i \Phi v + \mathit{\Delta}_m(n) v \\
&\quad = \sum_{i=1}^{m+1} \sum_{j=0}^{i-1} Y P_{ij} \Psi_j [n]_i \Phi v + \mathit{\Delta}_m(n) v,
\end{aligned}$$

where

$$\mathit{\Delta}_m(n) = \sum_{\substack{0\le j\le m \\ 1\le k\le m+1 \\ j+k\le m+1}} YP^{(k)}\Psi_j \Phi \varepsilon_{jkm}(n) + \Theta_m(n).$$

Applying (5.3.m) and Lemma 5.2.2 to $\mathit{\Delta}_m(n)v$, we obtain the desired estimate.

Proof of Step 1. By using the difference equation (1.8) and Lemma 2.2.3, we have

$$\begin{aligned}
&YF_{m+1}(n)v - ZYF_{m+1}(n-1)v \\
&\qquad = Y\phi_n - ZY\phi_{n-1} - \sum_{i=1}^{m+1} (Y\Psi_i [n]_i - ZY\Psi_i [n-1]_i)\Phi v \\
&\qquad = (YP_n - Z)\phi_{n-1} + Yv_n - \sum_{i=1}^{m+1} \{\, (I_0 - Z)Y\Psi_i [n]_i + Z\Psi_i [n]_{i+1} \,\}\Phi v \\
&\qquad = (YP_n - Z)\phi_{n-1} - \sum_{i=1}^{m+1} \{\, (I_0 - Z)Y\Psi_i + Z\Psi_{i-1} \,\}[n]_i \Phi v \\
&\qquad + Yv_n - Z\Psi_{m+1}[n]_{m+2}\Phi v.
\end{aligned}$$

By Notation 3.2.5 and Definition 3.2.6, we have

$$\begin{aligned}
&YF_{m+1}(n)v - ZYF_{m+1}(n-1)v \\
&\qquad = \sum_{i=1}^{m+1} \left\{ -(I_0 - Z)Y\Psi_i + \sum_{j=0}^{i-1} (YP_{ij} - \delta_{i,j+1} Z)\Psi_j \right\} [n]_i \Phi v + \Xi_m(n) v, \\
&\qquad = \sum_{i=1}^{m+1} (I_0 - Z) Y \left(-\Psi_i + \sum_{j=0}^{i-1} A_{ij} \Psi_j \right) [n]_i \Phi v + \Xi_m(n) v = \Xi_m(n) v,
\end{aligned}$$

where

$$\Xi_m(n) v = Yv_n - Z\Psi_{m+1}[n]_{m+2}\Phi v + \mathit{\Delta}_m(n) v.$$

Lemma 5.3.1 implies

$$|\Xi_m(n) v| = O\left(\frac{|v|_{m+1}}{(n+1)^{m+2}} \right) \qquad (v \in \ell^{m+1},\ n \ge m+1),$$

and hence

$$\begin{aligned}|YF_{m+1}(n)v| &\le |Z||YF_{m+1}(n-1)v| + |\Xi_m(n)v| \\ &\le r|YF_{m+1}(n-1)v| + O\left(\frac{|v|_{m+1}}{(n+1)^{m+2}}\right).\end{aligned}$$

Finally, applying Lemma 5.2.1 to this inequality, we establish Step 1.

5.4. Step 2.

Lemma 5.4.1. *If* (5.3.m) *and* (5.1.$m+1$) *hold, then we have*

$$X\phi_n - X\phi_{n-1} = \sum_{i=1}^{m+1} X\Psi_i[n]_{i+1}\Phi v + \varDelta_m(n)v,$$

with

$$|\varDelta_m(n)v| = O\left(\frac{|v|_{m+1}}{(n+1)^{m+3}}\right) \qquad (v \in \ell^{m+1},\ n \ge m+2).$$

Proof. By Assumption B_N and Notation 5.1.2, we have

$$XP_nX - X = \sum_{i=2}^{m+2} XP^{(i)}X(n)_i + XE_{m+1}(n)X, \tag{5.4}$$

$$XP_nY = \sum_{i=1}^{m+1} XP^{(i)}Y(n)_i + XE_m(n)Y. \tag{5.5}$$

Here the point is that the suffix i in the summation in (5.4) (resp. (5.5)) starts from $i=2$ (resp. $i=1$). By Notation 5.1.2 and $Y\Psi_0 = YX = O$, we have

$$\phi_{n-1} = \sum_{j=0}^{m} \Psi_j[n-1]_j\Phi v + F_m(n-1)v, \tag{5.6}$$

$$Y\phi_{n-1} = \sum_{j=1}^{m+1} Y\Psi_j[n-1]_j\Phi v + YF_{m+1}(n-1)v. \tag{5.7}$$

The difference equation (1.8) yields

$$X\phi_n - X\phi_{n-1} = (XP_nX - X)\phi_{n-1} + XP_nY\cdot Y\phi_{n-1} + Xv_n. \tag{5.8}$$

Substituting the four equalities (5.4)–(5.7) into (5.8), we obtain

$$\begin{aligned}X\phi_n - X\phi_{n-1} = &\sum_{\substack{0\le j\le m\\ 2\le k\le m+2\\ j+k\le m+2}} XP^{(k)}X\Psi_j(n)_k[n-1]_j\Phi v \\ &+ \sum_{\substack{1\le j\le m+1\\ 1\le k\le m+1\\ j+k\le m+2}} XP^{(k)}Y\Psi_j(n)_k[n-1]_j\Phi v + \Theta_m(n)v,\end{aligned}$$

where

$$\begin{aligned}\Theta_m(n) = &\sum_{\substack{2\le i\le m+2\\ 0\le j\le m\\ i+j\ge m+3}} XP^{(i)}X\Psi_j(n)_i[n-1]_j\Phi + XE_{m+1}(n)X\sum_{j=0}^{m}\Psi_j[n-1]_j\Phi\\ &+\sum_{i=2}^{m+2} XP^{(i)}X(n)_i\cdot F_m(n-1) + XE_{m+1}(n)X\cdot F_m(n-1)\\ &+\sum_{\substack{1\le i\le m+1\\ 1\le j\le m+1\\ i+j\ge m+3}} XP^{(i)}Y\Psi_j(n)_i[n-1]_j\Phi + XE_m(n)Y\sum_{j=1}^{m+1}\Psi_j[n-1]_j\Phi\\ &+\sum_{i=1}^{m+1} XP^{(i)}(n)_i\cdot YF_{m+1}(n-1) + XE_m(n)\cdot YF_{m+1}(n-1)\\ &+Xv_n.\end{aligned}$$

Moreover, using Notation 5.1.2, we have

$$\begin{aligned}&X\phi_n - X\phi_{n-1}\\ &\quad=\sum_{\substack{0\le j\le m\\ 2\le k\le m+2\\ j+k\le m+2}} XP^{(k)}X\Psi_j\left(\sum_{i=j+k}^{m+2}[j,k]\begin{bmatrix}k-1\\ i-j-k\end{bmatrix}[n]_i+\varepsilon_{jk,m+1}(n)\right)\Phi v\\ &\quad+\sum_{\substack{1\le j\le m+1\\ 1\le k\le m+1\\ j+k\le m+2}} XP^{(k)}Y\Psi_j\left(\sum_{i=j+k}^{m+2}[j,k]\begin{bmatrix}k-1\\ i-j-k\end{bmatrix}[n]_i+\varepsilon_{jk,m+1}(n)\right)\Phi v\\ &\quad+\Theta_m(n)v\\ &\quad=\sum_{i=2}^{m+2}\sum_{j=0}^{i-2}X\left(\sum_{k=2}^{i-j}[j,k]\begin{bmatrix}k-1\\ i-j-k\end{bmatrix}P^{(k)}\right)X\Psi_j[n]_i\Phi v\\ &\quad+\sum_{i=2}^{m+2}\sum_{j=1}^{i-1}X\left(\sum_{k=1}^{i-j}[j,k]\begin{bmatrix}k-1\\ i-j-k\end{bmatrix}P^{(k)}\right)Y\Psi_j[n]_i\Phi v+\mathit{\Delta}_m(n)v,\end{aligned}$$

where

$$\begin{aligned}\mathit{\Delta}_m(n) = \Theta_m(n) &+ \sum_{\substack{0\le j\le m\\ 2\le k\le m+2\\ j+k\le m+2}} XP^{(k)}X\Psi_j\Phi\varepsilon_{jk,m+1}(n)\\ &+\sum_{\substack{1\le j\le m+1\\ 1\le k\le m+1\\ j+k\le m+2}} XP^{(k)}Y\Psi_j\Phi\varepsilon_{jk,m+1}(n).\end{aligned}$$

Using Notation 3.2.4, we have

$$\begin{aligned} X\phi_n - X\phi_{n-1} &= \sum_{i=2}^{m+2}\sum_{j=0}^{i-2} XP_{ij}\Psi_j[n]_i\Phi v \\ &+ \sum_{i=2}^{m+2}[i-1,1]XP^{(1)}Y\Psi_{i-1}[n]_i\Phi v + \mathit{\Delta}_m(n)v \\ &= \sum_{i=1}^{m+1}\left(\sum_{j=0}^{i-1} XP_{i+1,j}\Psi_j + \frac{1}{i}XP^{(1)}Y\Psi_i\right)[n]_{i+1}\Phi v + \mathit{\Delta}_m(n)v. \end{aligned}$$

By Notation 3.2.5 and Definition 3.2.6, we have

$$\begin{aligned} &\sum_{j=0}^{i-1} XP_{i+1,j}\Psi_j + \frac{1}{i}XP^{(1)}Y\Psi_i \\ &\qquad = \sum_{j=0}^{i-1}(XP_{i+1,j} + \frac{1}{i}XP^{(1)}YA_{ij})\Psi_j \\ &\qquad = \sum_{j=0}^{i-1}\left(XP_{i+1,j} + \frac{1}{i}XP^{(1)}(I_0 - Z)^{-1}(YP_{ij} - \delta_{i,j+1}Z)\right)\Psi_j \\ &\qquad = \sum_{j=0}^{i-1} XA_{ij}\Psi_j = X\Psi_i. \end{aligned}$$

Hence we obtain

$$X\phi_n - X\phi_{n-1} = \sum_{i=1}^{m+1} X\Psi_i[n]_{i+1}\Phi v + \mathit{\Delta}_m(n)v.$$

The definition of $\mathit{\Delta}_m(n)$, Lemma 5.2.2, (5.3.m) and (5.1.$m+1$) yield

$$|\mathit{\Delta}_m(n)v| = O\left(\frac{|v|_{m+1}}{(n+1)^{m+3}}\right) \qquad (v \in \ell^{m+1},\ n \geq m+2).$$

Hence the lemma is established.

Proof of Step 2. By Lemma 2.2.3 and Lemma 5.4.1, we have

$$\begin{aligned} XF_{m+1}(n)v - XF_{m+1}(n-1)v &= X\phi_n - X\phi_{n-1} - \sum_{i=1}^{m+1} X\Psi_i([n]_i - [n-1]_i)\Phi v \\ &= \sum_{i=1}^{m+1} X\Psi_i[n]_{i+1}\Phi v + \mathit{\Delta}_m(n)v - \sum_{i=1}^{m+1} X\Psi_i[n]_{i+1}\Phi v \\ &= \mathit{\Delta}_m(n)v. \end{aligned}$$

By Lemma 5.4.1 we have

$$\begin{aligned}|XF_{m+1}(M)v - XF_{m+1}(n)v| &\le \sum_{k=n+1}^{M} |XF_{m+1}(k)v - XF_{m+1}(k-1)v| \\ &= \sum_{k=n+1}^{M} |\Delta_m(k)v| = O\left(\sum_{k=n+1}^{M} \frac{|v|_{m+1}}{(k+1)^{m+3}}\right) \\ &= O\left(\frac{|v|_{m+1}}{(n+1)^{m+2}}\right) \qquad (v \in \ell^{m+1},\ M > n \ge m+1).\end{aligned}$$

Theorem I implies $F_{m+1}(M)v \to 0$ as $M \to \infty$. Letting $M \to \infty$ in the above formula, we obtain

$$|XF_{m+1}(n)v| = O\left(\frac{|v|_{m+1}}{(n+1)^{m+2}}\right) \qquad (v \in \ell^{m+1},\ n \ge m+1).$$

This establishes Step 2.

5.5. Proof of Corollary 3.4.2. To establish Corollary 3.4.2 we show that the pair (ℓ^∞, P) satisfies Conditions (AP-1)-(AP-4) in Definition 1.1. Assumption B_∞ and Definition 3.2.2 imply that P and T map ℓ^∞ into itself, and hence (AP-1). Condition (AP-2) is trivial from Definition 3.2.2. Assertion (2) of Theorem I implies (3) of (AP-3). The other conditions in (AP-3) and (AP-4) readily follow from (2) of Theorem II. Hence (ℓ^∞, P) is an asymptotic pair of order infinity.

6. Gevrey Estimates

6.1. Preliminaries. Let p, q, r and s be as in Assumption C and Notation 3.5.4.

Notation 6.1.1. Let $|\cdot|$ be any semi-norm in $\mathcal{N}$, which is *fixed throughout this section*. Let $C = (C_1, C_2, C_3, C_4)$ be any constants such that $C_i > c_i(|\cdot|)$ $(i = 1, 2, 3, 4)$, where the $c_i(|\cdot|)$ are defined in Definition 3.5.5.

Then Definition 3.5.5 immediately implies the following lemma.

Lemma 6.1.2. *For any constants C as in Notation 6.1.1, there exists a nonegative integer $n_0(C)$ such that*

$$|XP_nX - X| \le \frac{C_1}{n(n+1)}, \qquad |XP_nY| \le \frac{C_2}{n^p},$$
$$|YP_nX| \le \frac{C_3}{n^q}, \qquad |YP_nY| \le \frac{C_4}{n^r} \qquad (n \ge n_0(C)).$$

Definition 6.1.3. For any $v \in \ell^1$ we set

$$f_n(v) = \sum_{k=n}^{\infty} \frac{|X\phi_k(v)|}{(k+1)(k+2)}, \qquad g_n(v) = \sum_{k=n}^{\infty} \frac{|Y\phi_k(v)|}{(k+1)^p},$$

$$\alpha_n(v) = \sum_{k=n}^{\infty} |Xv_k|, \qquad \beta_n(v) = \sum_{k=n}^{\infty} |Yv_k|, \qquad \gamma_n(v) = \sum_{k=n}^{\infty} |v_k|.$$

These are often abbreviated as $f_n, g_n, \alpha_n, \beta_n, \gamma_n$ with v being understood. By Definition 3.2.2 and Theorem II, these infinite series converge and define finite numbers.

Notation 6.1.4. $\ell^1_0 = \{\ v \in \ell^1\ ;\ \Phi v = 0\ \}$.

6.2. Estimate of f_n. The following lemma is easily established.

Lemma 6.2.1. *For any $\varepsilon > 0$ we set $n_1(C, \varepsilon) = \max\{[C_1 + 1/\varepsilon], n_0(C)\}$, where $[\cdot]$ is the Gauss symbol. Then we have*

$$0 < \left(1 - \frac{C_1}{n+1}\right)^{-1} \leq 1 + \varepsilon C_1 \qquad (n \geq n_1(C, \varepsilon)).$$

Lemma 6.2.2. *Let ℓ_0^∞ be as in Notation 6.1.4. For any $\varepsilon > 0$,*

$$f_n(v) \leq \frac{1 + \varepsilon C_1}{n+1} \{C_2 g_n(v) + \alpha_{n+1}(v)\} \qquad (n \geq n_1(C, \varepsilon), v \in \ell_0^1).$$

Proof. The difference equation (1.8) yields, for $k \geq 0$,

$$|X\phi_{k+1}(v) - X\phi_k(v)| \leq |XP_{k+1}X - X||X\phi_k(v)| + |XP_{k+1}Y||Y\phi_k(v)| + |Xv_{k+1}|.$$

Applying Lemma 6.1.2, we have, for $k \geq n_0(C)$,

$$|X\phi_{k+1}(v) - X\phi_k(v)| \leq C_1 \frac{|X\phi_k(v)|}{(k+1)(k+2)} + C_2 \frac{|Y\phi_k(v)|}{(k+1)^p} + |Xv_{k+1}|.$$

Summing over $k = n, n+1, n+2, \dots$ and taking $\lim_{k\to\infty} \phi_k(v) = 0$ into account, we obtain

$$|X\phi_n(v)| \leq C_1 f_n + C_2 g_n + \alpha_{n+1} \qquad (n \geq n_0(C)).$$

Since f_k, g_k and α_k are nonincreasing, we have

$$\frac{|X\phi_k(v)|}{(k+1)(k+2)} \leq \frac{C_1 f_k + C_2 g_k + \alpha_{k+1}}{(k+1)(k+2)} \leq \frac{C_1 f_n + C_2 g_n + \alpha_{n+1}}{(k+1)(k+2)} \quad (k \geq n \geq n_0(C)).$$

Summing over $k = n, n+1, n+2, \dots$, we have

$$f_n \leq \frac{C_1 f_n + C_2 g_n + \alpha_{n+1}}{n+1} \qquad (n \geq n_0(C)).$$

Using Lemma 6.2.1, we obtain

$$f_n \leq \frac{1 + \varepsilon C_1}{n+1} (C_2 g_n + \alpha_{n+1}) \qquad (n \geq n_1(C, \varepsilon)).$$

6.3 Estimate of g_n.

Definition 6.3.1. We set

$$C_0 = \begin{cases} C_2 C_3 & (p+q-1 < r), \\ C_2 C_3 + C_4 & (p+q-1 = r), \\ C_4 & (p+q-1 > r), \end{cases} \qquad B_0 = \sqrt{2} \max\{C_3, 1\}.$$

Lemma 6.3.2. *For any $A > C_0$ and $B > B_0$, there exists a positive constant $\varepsilon = \varepsilon(A, B, C)$ depending only on (A, B, C) such that*

$$\begin{aligned} &\frac{(1 + \varepsilon C_1) C_2 C_3}{(n+1)^{p+q-1}} + \frac{C_4}{(n+1)^r} \leq \frac{A}{(n+1)^s}, \\ &\sqrt{2} \max\{(1 + \varepsilon C_1) C_3, 1\} \leq B \qquad (n \geq m(A, B, C)), \end{aligned}$$

where $m(A, B, C) = n_1(C, \varepsilon(A, B, C))$ and $n_1(C, \varepsilon)$ is defined in Lemma 6.2.1.

Proof. The lemma easily follows from Definition 6.3.1 and the fact that $n_1(C, \varepsilon) \to \infty$ as $\varepsilon \to 0$ (cf. Lemma 6.2.1).

Lemma 6.3.3. *For any $A > C_0$ and $B > B_0$,*

$$g_{n+1}(v) \le \frac{A g_n(v)}{(n+1)^s} + B\gamma_{n+1}(v) \qquad (n \ge m(A,B,C), v \in \ell_0^1),$$

Proof. The difference equation (1.8) yields

$$|Y\phi_{k+1}(v)| \le |YP_{k+1}X||X\phi_k(v)| + |YP_{k+1}Y||Y\phi_k(v)| + |Yv_{k+1}| \quad (k \ge 0).$$

Applying Lemma 6.1.2, we have

$$|Y\phi_{k+1}(v)| \le \frac{C_3}{(k+1)^q}|X\phi_k(v)| + \frac{C_4}{(k+1)^r}|Y\phi_k(v)| + |Yv_{k+1}| \qquad (k \ge n_0(C)),$$

and hence

$$\begin{aligned}\frac{|Y\phi_{k+1}(v)|}{(k+2)^p} &\le \frac{C_3}{(k+1)^{p+q-2}}\frac{|X\phi_k(v)|}{(k+1)(k+2)} + \frac{C_4}{(k+1)^r}\frac{|Y\phi_k(v)|}{(k+1)^p} + \frac{|Yv_{k+1}|}{(k+2)^p}\\ &\le \frac{C_3}{(n+1)^{p+q-2}}\frac{|X\phi_k(v)|}{(k+1)(k+2)} + \frac{C_4}{(n+1)^r}\frac{|Y\phi_k(v)|}{(k+1)^p} + |Yv_{k+1}|\\ &\qquad (k \ge n \ge n_0(C)).\end{aligned}$$

Summing over $k = n, n+1, n+2, \ldots$, we have

$$g_{n+1} \le \frac{C_3 f_n}{(n+1)^{p+q-2}} + \frac{C_4 g_n}{(n+1)^r} + \beta_{n+1} \qquad (n \ge n_0(C)).$$

Applying Lemma 6.2.2, we obtain

$$\begin{aligned}g_{n+1} &\le \frac{(1+\varepsilon C_1)C_3(C_2 g_n + \alpha_{n+1})}{(n+1)^{p+q-1}} + \frac{C_4 g_n}{(n+1)^r} + \beta_{n+1}\\ &= \left\{\frac{(1+\varepsilon C_1)C_2C_3}{(n+1)^{p+q-1}} + \frac{C_4}{(n+1)^r}\right\} g_n + \frac{(1+\varepsilon C_1)C_3\alpha_{n+1}}{(n+1)^{p+q-1}} + \beta_{n+1}\\ &= \left\{\frac{(1+\varepsilon C_1)C_2C_3}{(n+1)^{p+q-1}} + \frac{C_4}{(n+1)^r}\right\} g_n + \max\{(1+\varepsilon C_1)C_3, 1\}(\alpha_{n+1} + \beta_{n+1}),\\ &\qquad (n \ge n_1(C,\varepsilon)).\end{aligned}$$

Finally, using Lemma 6.3.2, we obtain

$$g_{n+1} \le \frac{A g_n}{(n+1)^s} + B\gamma_{n+1} \qquad (n \ge m(A,B,C)).$$

Corollary 6.3.4. *Let A, B and $m = m(A,B,C)$ be as in Lemma 6.3.3. Then,*

$$g_n(v) \le \frac{A^{n-m} g_m(v)}{\{(m+1)(m+2)\cdots(n-1)n\}^s} + \sum_{k=m+1}^{n} \frac{BA^{n-k}\gamma_k(v)}{\{(k+1)(k+2)\cdots(n-1)n\}^s}$$
$$(n \ge m, v \in \ell_0^1).$$

Proof. Induction on $n \ge m$ by using Lemma 6.3.3 establishes this corollary.

6.4. Convexity. Given any $v \in \ell_0^1$, consider a sequence $d_n > 0$ $(n \ge 0)$ satisfying the following condition.

Condition 6.4.1. Let s be as in Notation 3.5.4.

(1) $$\frac{d_{n+1}d_{n-1}}{(d_n)^2} \ge \left(\frac{n}{n+1}\right)^s \qquad (n \ge 1),$$

(2) $$\gamma_n(v) \le d_n \qquad (n \ge 0).$$

Lemma 6.4.2. *Let A, B and $m = m(A,B,C)$ be as in Corollary 6.3.4. If $\{d_n\}$ satisfies Condition 6.4.1, then*

$$g_n(v) \le \frac{A^{n-m} g_m(v)}{\{(m+1)(m+2)\cdots(n-1)n\}^s} + B(n-m)\max\left\{\frac{A^{n-m-1}d_{m+1}}{\{(m+2)(m+3)\cdots(n-1)n\}^s}, d_n\right\} \quad (n \ge m+1).$$

Proof. For $n \ge m+1$ we consider the sequence

$$x_k = \frac{A^{n-k}d_k}{\{(k+1)(k+2)\cdots(n-1)n\}^s} \qquad (m+1 \le k \le n).$$

Condition 6.4.1.(1) implies that if $n \ge m+3$, then

$$\frac{x_{k+1}x_{k-1}}{(x_k)^2} = \left(\frac{k+1}{k}\right)^s \frac{d_{k+1}d_{k-1}}{(d_k)^2} \ge 1 \qquad (m+2 \le k \le n-1).$$

This means that the sequence $\{x_k; m+1 \le k \le n\}$ is multiplicatively *convex* (including the cases $n = m+1, m+2$). Hence $\{x_k\}$ attains the minimum at $k = m+1$ or n. By Corollary 6.3.4 and Condition 6.4.1(2), we have

$$g_n \le \frac{A^{n-m} g_m}{\{(m+1)(m+2)\cdots(n-1)n\}^s} + \sum_{k=m+1}^{n} x_k.$$

In the sum $\sum x_k$, we replace each x_k by $\max\{x_k\} = \max\{x_{m+1}, x_n\}$. Then we establish the lemma.

6.5. Preparatory lemmas.

Assumption 6.5.1. Assume that (t,a) satisfies one of the following conditions:

(1) $t = s = 0$, $c_0(|\cdot|) < a < 1$,
(2) $t = 0 < s$, $0 < a < 1$,
(3) $0 < t < s$, $a > 0$,
(4) $t = s > 0$, $a > c_0(|\cdot|)$,

where s and $c_i(|\cdot|)$ are defined in Notation 3.5.4 and Definition 3.5.5, respectively.

Definition 6.5.2. We set

$$L_1(t,a) = \frac{1}{1-a} \qquad (t = 0),$$

$$L_1(t,a) = \max\left\{2, \left(\max_{0\le n\le[(2a)^{1/t}]} \frac{(n!)^t}{a^n}\right)\sum_{k=0}^{\infty}\frac{a^k}{(k!)^t}\right\} \qquad (t > 0).$$

Lemma 6.5.3. *Under Assumption 6.5.1 we have*

$$\gamma_n(v) \le L_1(t,a)|v|_{t,a}\frac{a^n}{(n!)^t} \qquad (n \ge 0, v \in \mathcal{G}^{t,a}).$$

Proof. In case $t = 0$ we have

$$\gamma_n = \sum_{k=n}^{\infty}|v_n| \le |v|_{0,a}\sum_{k=n}^{\infty}a^k = \frac{|v|_{0,a}a^n}{1-a} = L_1(t,a)|v|_{0,a}a^n,$$

which implies the lemma. In case $t > 0$ we easily see that

$$\frac{1}{2} \le 1 - \frac{a}{(k+1)^t} \qquad (k \ge [(2a)^{1/t}]).$$

Hence we have

$$\begin{aligned}|v_n| \le \frac{2|v|_{t,a}a^k}{(k!)^t}\frac{1}{2} &\le \frac{2|v|_{t,a}a^k}{(k!)^t}\left\{1-\frac{a}{(k+1)^t}\right\}\\ &= \frac{2|v|_{t,a}a^k}{(k!)^t} - \frac{2|v|_{t,a}a^{k+1}}{\{(k+1)!\}^t} \qquad (k \ge [(2a)^{1/t}]).\end{aligned}$$

Summing over $k = n, n+1, n+2, \ldots$, we obtain

$$\gamma_n = \sum_{k=n}^{\infty}|v| \le \frac{2|v|_{t,a}a^n}{(n!)^t} \le \frac{|v|_{t,a}L_1(t,a)a^n}{(n!)^t} \qquad (n \ge [(2a)^{1/t}]).$$

As for $0 \le n \le [(2a)^{1/t}]$, we have also

$$\gamma_n \le \sum_{k=0}^{\infty}\frac{|v|_{t,a}a^k}{(k!)^t} = \frac{(n!)^t}{a^n}\sum_{k=0}^{\infty}\frac{a^k}{(k!)^t}\cdot\frac{|v|_{t,a}a^n}{(n!)^t} \le \frac{|v|_{t,a}L_1(t,a)a^n}{(n!)^t}.$$

This establishes the lemma.

Lemma 6.5.4. *Let $\mathcal{G}_0^{t,a+}$ be defined by Definition 3.5.3. There exist constants $L_2(t,a)$ and $L_3(t,a)$, depending only on (t,a), such that*

$$(1) \qquad |\phi_n(v)| \le \frac{L_2(t,a)}{n+1}|v|_{t,a},$$

$$(2) \qquad g_n(v) \le L_3(t,a)|v|_{t,a} \qquad (n \ge 0, v \in \mathcal{G}_0^{t,a}).$$

Proof. By Assumption 6.5.1, the sum

$$L_2'(t,a) = \sum_{n=0}^{\infty}\frac{(n+1)a^n}{(n!)^t} + \sup_{n\ge 0}\frac{(n+1)a^n}{(n!)^t}$$

converges and defines a finite number. By Definition 3.2.2 and Definition 3.5.1, we have $|v|_0 \le L_2'(t,a)|v|_{t,a}$. By Theorem II (or Theorem I) and $\Phi v = 0$, there exists a constant C_5 such that

$$|\phi_n(v)| \le \frac{C_5|v|_0}{n+1} \qquad (n \ge 0, v \in \ell_0^1).$$

Hence, setting $L_2(t,a) = C_5L_2'(t,a)$, we obtain Assertion (1). Next, noting that $p \ge 1$, we set

$$L_3(t,a) = \sum_{n=0}^{\infty}\frac{L_2(t,a)}{(n+1)^{p+1}} < \infty.$$

Then we easily obtain Assertion (2). This establishes the lemma.

Remark 6.5.5. The constant C_5 in the proof depends (only) on $|\cdot| \in \mathcal{N}$. So it is better to say that the constants $L_2(t,a)$ and $L_3(t,a)$ depend only on $(t,a,|\cdot|)$. We did not refer to this dependence explicitly, since $|\cdot|$ is understood to be fixed (cf. Notation 6.1.1).

Lemma 6.5.6. *Let $A > C_0, B > B_0$ and $m = m(A, B, C)$ be as in Lemma 6.4.2. Then*

$$g_n(v) \le L_3(t,a)|v|_{t,a} \frac{A^{n-m}}{\{(m+1)(m+2)\cdots(n-1)n\}^s} + nBL_1(t,a)|v|_{t,a} \max\left\{ \frac{A^{n-m-1}a^{m+1}}{\{(m+1)!\}^t\{(m+2)(m+3)\cdots(n-1)n\}^s}, \frac{a^n}{(n!)^t} \right\}$$
$$(n \ge m+1, v \in \mathcal{G}_0^{t,a}).$$

Proof. For any $v \in \mathcal{G}_0^{t,a}$ we set

$$d_n = L_1(t,a)|v|_{t,a} \frac{a^n}{(n!)^t} \qquad (n \ge 0).$$

Then Condition 6.4.1(1) is easily checked. Moreover, Lemma 6.5.3 implies Condition 6.4.1(2). Hence Lemma 6.4.2 is valid for the sequence $\{d_n\}$. The lemma easily follows from Lemma 6.4.2 and Lemma 6.5.4(2).

6.6. Gevrey estimates. We are now in a position to establish Gevrey estimates.

Lemma 6.6.1. *Under Assumption 6.5.1 there exist a nonnegative integer $m(a)$ depending only on a and nonnegative constants $L_4(t,a)$, $L_5(t,a)$ depending only on (t,a) such that*

$$(1) \qquad g_n(v) \le L_4(t,a)|v|_{t,a} \frac{(n+1)a^n}{(n!)^t} \qquad (n \ge m(t,a)+1, v \in \mathcal{G}_0^{t,a}),$$

$$(2) \qquad f_n(v) \le L_5(t,a)|v|_{t,a} \frac{a^n}{(n!)^t} \qquad (n \ge m(t,a)+1, v \in \mathcal{G}_0^{t,a}).$$

Proof. First we consider the cases (1) and (4) in Assumption 6.5.1. In these cases we have $t = s$ and $c_0(|\cdot|) < a$. By Definitions 3.5.5 and 6.3.1, we can choose the constants $C = (C_1, C_2, C_3, C_4)$ so that $C_i > c_i(|\cdot|)$ $(i = 1,2,3,4)$ and $C_0 < a$. This choice depends only on a. We set $A = a$ $(> C_0)$ and fix $B > B_0$ arbitrary. Then (A, B, C) depends only on a. Lemma 6.5.6 implies

$$g_n(v) \le L_3(t,a)|v|_{t,a} \frac{a^{n-m}}{\{(m+1)(m+2)\cdots(n-1)n\}^t} + nBL_1(t,a)|v|_{t,a} \frac{a^n}{(n!)^t} \qquad (n \ge m+1),$$

where $m = m(A,B,C)$. If we put $m(a) = m(A,B,C)$ and $L_4(t,a) = BL_1(t,a) + (m!)^t a^{-m} L_3(t,a)$, then Assertion (1) holds.

Next we consider the cases (2) and (3) in Assumption 6.5.1. In these cases we have $s < t$. We take any constants $C = (C_1, C_2, C_3, C_4)$ such that $C_i > c_i(|\cdot|)$ $(i = 1,2,3,4)$. Moreover we fix $A > C_0$ and $B > B_0$ arbitrarily. Lemma 6.5.6 implies

$$g_n(v) \le L_3(t,a)|v|_{t,a} \frac{(m!)^s}{A^m} \frac{(A/a)^n}{(n!)^{s-t}} \cdot \frac{a^n}{(n!)^t} + nBL_1(t,a)|v|_{t,a} \max\left\{ \{(m+1)!\}^{s-t} \frac{(A/a)^{n-m-1}}{(n!)^{s-t}},\ 1 \right\} \frac{a^n}{(n!)^t}$$
$$(n \ge m+1),$$

where $m = m(A, B, C)$. So if we set $m(a) = m(A, B, C)$ and define $L_4(t, a)$ as below, then Assertion (1) holds, with

$$L_4(t,a) = \sup_{n \geq m+1} \left[L_3(t,a) \frac{(m!)^s}{A^m} \frac{(A/a)^n}{(n!)^{s-t}} + BL_1(t,a) \left\{ \{(m+1)!\}^{s-t} \frac{(A/a)^{n-m-1}}{(n!)^{s-t}} + 1 \right\} \right].$$

Since $s > t$, $L_4(t, a)$ is well-defined, i.e. $L_4(t, a) < \infty$.

Next, Lemmas 6.2.2, 6.5.3 and Assertion (1) imply

$$\begin{aligned} f_n(v) &\leq \frac{1 + \varepsilon(A, B, C)C_1}{n+1} \{C_2 g_n(v) + \gamma_n(v)\} \\ &\leq \frac{1 + \varepsilon(A, B, C)C_1}{n+1} |v|_{t,a} \{C_2 L_4(t,a)(n+1) + L_1(t,a)\} \frac{a^n}{(n!)^t}. \end{aligned}$$

Assertion (2) easily follows from this inequality.

Strictly speaking, we should say that $m(a)$, $L_4(t, a)$ and $L_5(t, a)$ depend also on $|\cdot|$ (cf. Remark 6.5.5).

Theorem 6.6.2 (Theorem III). *Under Assumptions* B_N *and* C, *if* (t, a) *satisfies Assumption 6.5.1 then there exists a constant* $M(t, a, |\cdot|)$, *depending only on* $(t, a, |\cdot|)$, *such that*

$$|\phi_n(v)| \leq M(t,a,|\cdot|)|v|_{t,a} \frac{(n+1)^{p+1} a^n}{(n!)^t} \qquad (n \geq 0, v \in \mathcal{G}_0^{t,a}, |\cdot| \in \mathcal{N}).$$

Proof. By Definition 6.1.3 and Lemma 6.6.1, we have

$$\begin{aligned} |\phi_n(v)| &\leq |Y\phi_n(v)| + |X\phi_n(v)| \\ &\leq (n+1)^p g_n(v) + (n+1)(n+2) f_n(v) \\ &\leq \{L_4(t,a)(n+1)^{p+1} + L_5(t,a)(n+1)(n+2)\} |v|_{t,a} \frac{a^n}{(n!)^t} \\ &\leq \{L_4(t,a) + 2L_5(t,a)\} |v|_{t,a} \frac{(n+1)^{p+1} a^n}{(n!)^t} \\ &\qquad (n \geq m(a)+1, v \in \mathcal{G}_0^{t,a}). \end{aligned}$$

We set

$$M(t,a,|\cdot|) = \max \left\{ L_4(t,a) + 2L_5(t,a),\ L_2(t,a) \max_{0 \leq n \leq m(a)} \frac{(n!)^t}{(n+1)^{p+2} a^n} \right\}.$$

Then the above inequality and Lemma 6.5.4(1) establish the theorem.

6.7. Proof of Corollary 3.6.2. To establish Corollary 3.6.2 we show that the pair $(\mathcal{G}^{t,a+}, P)$ satisfies Conditions (AP-1)-(AP-4) in Definition 1.1. Assumption B_∞ and Definition 3.5.2 imply that P and T map $\mathcal{G}^{t,a+}$ into itself, and hence (AP-1). Condition (AP-2) is trivial from Definition 3.5.2. Assertion (2) of Theorem I implies (3) of (AP-3). The other conditions in (AP-3) and (AP-4) readily follow from (1) of Theorem II and Corollary 3.6.1. Hence $(\mathcal{G}^{t,a+}, P)$ is an asymptotic pair of order N.

7. Examples

We present two simple, but nontrivial, examples to which our theory applies. These examples arise from a different area of mathematics, i.e., algebraic analysis of linear partial differential equations or the D-module theory. In this connection we refer to [9][10]. Let D be a domain in $\mathbb{C}$, $\mathcal{O}(D)$ the linear space of all holomorphic functions in D, $\mathcal{K}$ the set of all compact subsets of D. For each $K \in \mathcal{K}$ we set

$$|f|_K = \sup_{x \in K} |f(x)| \qquad (f \in \mathcal{O}(D)). \tag{7.1}$$

Then $\mathcal{O}(D)$ is a complete locally convex linear space having semi-norms $|\cdot|_K$ with $K \in \mathcal{K}$.

7.1. Example 1. Let D be any *bounded* domain in $\mathbb{C}$. We set $U = \mathcal{O}(D)^2$, where each element in U is regarded as a column vector. Let β and γ be complex constants such that $\gamma \notin \mathbb{Z}$. We consider the difference equation (1.1), where

$$P_n = \begin{pmatrix} 1 & \dfrac{-x}{n-\gamma} \\ \dfrac{-\beta}{n-\gamma} & \dfrac{-x}{n-\gamma} \end{pmatrix}. \tag{7.2}$$

This example arises from the confluent hypergeometric system of two variables known as the Humbert system Φ_2:

$$\begin{cases} [x\partial_x^2 + y\partial_x\partial_y + (\gamma - x)\partial_x - \beta]u = 0, \\ [y\partial_y^2 + x\partial_x\partial_y + (\gamma - y)\partial_y - \beta']u = 0, \end{cases} \tag{Φ_2}$$

where $\beta, \beta', \gamma \in \mathbb{C}$. We refer to [9] for the derivation of (7.2) from (Φ_2). From (7.2) we have $P_n = P^{(0)} + P^{(1)}(n-\gamma)_1$, where

$$P^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^{(1)} = \begin{pmatrix} 0 & x \\ \beta & x \end{pmatrix}, \quad P^{(i)} = O \quad (i \geq 2). \tag{7.3}$$

As the supplementary projections X and Y, we take

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.4}$$

Then we have (cf. Notation 3.2.1)

$$U_\nu = \mathcal{O}(D)\, e_\nu \qquad (\nu = 0, 1), \tag{7.5}$$

where $e_\nu \in U$ are given by

$$e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Any $f \in U$ is expressed as $f = f_0 e_0 + f_1 e_1$ for some $f_0, f_1 \in \mathcal{O}(D)$. For each $K \in \mathcal{K}$ let $|f|_K = \sqrt{|f_0|_K^2 + |f_1|_K^2}$ (cf. (7.1)). We take $\mathcal{N} = \{|\cdot|_K \,;\, K \in \mathcal{K}\}$ as the system of semi-norms on U. Then the condition (2.1) holds. By using the assumption that D is bounded, it is easily checked that Assumption A_u is satisfied for any ρ such that $0 < \rho < 1$. Moreover, Assumption B_∞ is also satisfied.

Notation 3.2.4, (7.3) and (7.4) yield

$$P_{ij} = \begin{cases} \dfrac{P^{(1)}}{j_+} & (i = j+1), \\ O & (i > j+1), \end{cases} \tag{7.6}$$

where $j_+ = \max\{j, 1\}$. Notation 3.2.5, (7.3), (7.4) and (7.6) yield

$$A_{ij} = \begin{cases} \dfrac{1}{j_+}\begin{pmatrix} \dfrac{\beta x}{i} & \dfrac{x^2}{i}, \\ \beta & x \end{pmatrix} & (i = j+1), \\ O & (i > j+1). \end{cases} \tag{7.7}$$

Since $A_{ij} = O$ for $i > j+1$, we have $A_J = O$ for any $J \in \mathbf{S}_i$ other than $J = \{1, 2, \ldots, i-1\}$ (cf. Definition 3.2.6). Hence $\Psi_i = A_{i,i-1}A_{i-1,i-2}\cdots A_{21}A_{10}$ for $i \geq 1$. Using (7.7) we can show that the operators Ψ_i in Definition 3.2.6 are expressed as

$$\Psi_i = \frac{(\beta+1)(\beta+2)\cdots(\beta+i-1)}{[(i-1)!]^2}\begin{pmatrix} \dfrac{\beta x^i}{i} & \dfrac{x^{i+1}}{i} \\ \beta x^{i-1} & x^i \end{pmatrix} \qquad (i \geq 1). \tag{7.8}$$

We set $p = q = r = 1$. Then Assumption C is satisfied and we have $s = 1$ in Notation 3.5.4. Using Remark 3.5.6 we have (cf. Definition 3.5.5)

$$c_2(|\cdot|_K) = c_4(|\cdot|_K) = r_K, \quad c_3(|\cdot|_K) = |\beta| \qquad (K \in \mathcal{K}),$$

where $r_K = \sup_{x\in K}|x|$. Hence $c_0(|\cdot|_K) = r_K(|\beta|+1)$. Therefore the constant a_0 in Definition 3.5.5 is given by

$$a_0 = r_D(|\beta|+1), \qquad \text{where} \quad r_D = \sup_{x\in D}|x|. \tag{7.9}$$

From Corollaries 3.4.2 and 3.6.2 we obtain the following proposition.

Proposition 7.1. *Let $P = (P_n)$ be as in (7.2) and $U = \mathcal{O}(D)^2$, where D is a bounded domain in $\mathbb{C}$. Assume that $V = \ell^\infty$ or $\mathcal{G}^{t,a+}$, where (t, a) satisfies one of the following conditions:*

(1) $t = 0$, $0 \leq a < 1$,
(2) $0 < t < 1$, $a \geq 0$,
(3) $t = 1$, $a \geq a_0$, *where a_0 is given by (7.9).*

Then (V, P) is an asymptotic pair of order infinity with asymptotic data $\mathcal{A} = (-\gamma, U_1, \{\Psi_i\}_{i=0}^\infty)$, where U_1 and Ψ_i are given by (7.5) and (7.8), respectively.

7.2. Example 2. Let $\mathbb{D} = \{x \in \mathbb{C}\,;\, |x| < 1\}$ be the open unit disk in $\mathbb{C}$, D any *relatively compact* domain in $\mathbb{D}$. We set $U = \mathcal{O}(D)^2$, where each element in U is regarded as a column vector. Let α, β and γ be complex constants such that $\gamma \notin \mathbb{Z}$. We consider the difference equation (1.1), where

$$P_n = \begin{pmatrix} 1 + \dfrac{\beta x}{n-\gamma} & \dfrac{x(x-1)}{n-\gamma} \\ \dfrac{\beta(n-\alpha)}{n-\gamma} & \dfrac{x(n-\alpha)}{n-\gamma} \end{pmatrix}. \tag{7.10}$$

This example arises from the confluent hypergeometric system of two variables known as the Humbert system Φ_1:

(Φ_1)

$$\begin{cases} [x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + \{\gamma - (\alpha+\beta+1)x\}\partial_x - \beta y\partial_y - \alpha\beta]u = 0, \\ [y\partial_y^2 + x\partial_x\partial_y + (\gamma - y)\partial_y - x\partial_x - \alpha]u = 0. \end{cases}$$

We refer to [9] for the derivation of (7.10) from (Φ_1). From (7.10) we have $P_n = P^{(0)} + P^{(1)}(n-\gamma)_1$, where

$$P^{(0)} = \begin{pmatrix} 1 & 0 \\ \beta & x \end{pmatrix}, \quad P^{(1)} = \begin{pmatrix} -\beta x & x(1-x) \\ \beta(\alpha-\gamma) & (\alpha-\gamma)x \end{pmatrix}, \quad P^{(i)} = O \quad (i \geq 2). \tag{7.11}$$

As the supplementary projections X and Y, we take

$$X = \begin{pmatrix} 1 & 0 \\ \dfrac{\beta}{1-x} & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ \dfrac{-\beta}{1-x} & 1 \end{pmatrix}. \tag{7.12}$$

Then we have (cf. Notation 3.2.1)

$$U_\nu = \mathcal{O}(D)\, e_\nu \qquad (\nu = 0, 1), \tag{7.13}$$

where $e_\nu \in U$ are given by

$$e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ \dfrac{\beta}{1-x} \end{pmatrix}.$$

Any $f \in U$ is expressed as $f = f_0 e_0 + f_1 e_1$ for some $f_0, f_1 \in \mathcal{O}(D)$. For each $K \in \mathcal{K}$ let $|f|_K = \sqrt{|f_0|_K^2 + |f_1|_K^2}$ (cf. (7.1)). We take $\mathcal{N} = \{|\cdot|_K\,;\, K \in \mathcal{K}\}$ as the system of semi-norms on U. Then the condition (2.1) holds. We set

$$\rho = r_D := \sup_{x \in D} |x|. \tag{7.14}$$

Since D is relatively compact in $\mathbb{D}$, we have $0 \leq \rho < 1$. Assumption A_u is satisfied for ρ defined by (7.14). Moreover Assumption B_∞ is also satisfied. We only check (2) of Assumption B_∞. The operator Z is expressed as

$$Z := Y P^{(0)} Y = \begin{pmatrix} 0 & 0 \\ \dfrac{-\beta x}{1-x} & x \end{pmatrix}. \tag{7.15}$$

This implies $Z e_0 = x e_0$ and $Z e_1 = 0$, i.e., Z is the multiplication by x on U_0 and zero on U_1. In particular we have

$$|Z|_K = r_K := \sup_{x \in K} |x| < 1 \qquad (\forall\, K \in \mathcal{K}). \tag{7.16}$$

This implies (2) of Assumption B_∞.

Notation 3.2.4, (7.11) and (7.12) yield

$$P_{ij} = \begin{cases} \dfrac{P^{(1)}}{j_+} & (i = j+1), \\ O & (i > j+1). \end{cases} \tag{7.17}$$

Notation 3.2.5, (7.11), (7.12), (7.15) and (7.17) yield

$$A_{ij} = \begin{cases} \dfrac{1}{j_+}\left(\dfrac{x}{1-x}\right) Q W Q^{-1} & (i = j+1), \\ O & (i > j+1), \end{cases} \tag{7.18}$$

where the matrices Q and W are given by

$$Q = \begin{pmatrix} 1 & 0 \\ \dfrac{\beta}{1-x} & 1 \end{pmatrix}, \quad W = \begin{pmatrix} \dfrac{\beta(\alpha-\gamma)}{i} & \dfrac{(\alpha-\beta-\gamma-j_+)x(1-x)}{i} \\ \dfrac{\beta(\alpha-\gamma)}{x(1-x)} & \alpha-\beta-\gamma-j_+ \end{pmatrix}.$$

Since $A_{ij} = O$ for $i > j+1$, we have $A_J = O$ for any $J \in \mathbf{S}_i$ other than $J = \{1, 2, \dots, i-1\}$ (cf. Definition 3.2.6). Hence

$$\Psi_i = A_{i,i-1}A_{i-1,i-2}\cdots A_{21}A_{10}$$

for $i \geq 1$. Using (7.18) we can show that the operators Ψ_i in Definition 3.2.6 are expressed as

(7.19)

$$\Psi_i = \left(\frac{x}{1-x}\right)^i \frac{(\beta+1)(\beta+2)\cdots(\beta+i-1)\cdot(\alpha-\gamma-1)(\alpha-\gamma-2)\cdots(\alpha-\gamma-i+1)}{[(i-1)!]^2}$$
$$\times \begin{pmatrix} \dfrac{\beta\{\alpha-\gamma-(\alpha-\beta-\gamma-1)x\}}{i} & \dfrac{(\alpha-\beta-\gamma-1)x(1-x)}{i} \\ \dfrac{\beta\{\alpha-\gamma-(\alpha-\beta-\gamma-1)x\}}{x(1-x)}\left(\dfrac{\beta x}{i}+1\right) & (\alpha-\beta-\gamma-1)\left(\dfrac{\beta x}{i}+1\right) \end{pmatrix}.$$

We set $p = q = 1$ and $r = 0$. Then Assumption C is satisfied and we have $s = 0$ in Notation 3.5.4. By Remark 3.5.6 and (7.16), we have $c_0(|\cdot|_K) = |Z|_K = r_K$ for any $K \in \mathcal{K}$. Hence (7.14) implies that the constant a_0 in Definition 3.5.5 is given by

$$a_0 = r_D < 1. \tag{7.20}$$

From Corollaries 3.4.2, 3.6.2 and (7.20) we obtain the following proposition.

Proposition 7.2. *Let $P = (P_n)$ be as in (7.10) and $U = \mathcal{O}(D)^2$, where D is a relatively compact domain in the open unit disk $\mathbb{D}$. Let a be any constant such that $r_D \leq a < 1$, where r_D is defined by (7.14). Assume that $V = \ell^\infty$ or $\mathcal{G}^{0,a+}$. Then (V, P) is an asymptotic pair of order infinity with asymptotic data $\mathcal{A} = (-\gamma, U_1, \{\Psi_i\}_{i=0}^\infty)$, where U_1 and Ψ_i are given by (7.13) and (7.19), respectively.*

References

1. Benzaid, Z. and Lutz, D.A., *Asymptotic representation of solutions of perturbed systems of linear difference equations*, Studies in Appl. Math. **77** (1987), 195–221. MR **90f:**39003
2. Birkhoff, G.D., *General theory of linear difference equations*, Trans. Amer. Math. Soc. **12** (1911), 243–284.
3. Charrière, H. and Gérard, R., *The rings of formal and convergent inverse factorial series*, Kumamoto J. Math. **5** (1992), 1–20. MR **94b:**39019
4. Coffman, C.V., *Asymptotic behavior of solutions of ordinary difference equations*, Trans. Amer. Math. Soc. **110** (1964), 22-51. MR **27:**6054
5. Evgrafov, M.A., *On the asymptotic behavior of solutions of difference equations* (in Russian), Dokl. Akad. Nauk SSSR **121** (1958), 26–219. MR **21:**5102
6. Gérard, R. and Lutz, D.A., *Convergent factorial series solutions of singular operator equations*, Analysis **10** (1990), 99–145. MR **92f:**39005
7. Hirai, I. and Yanagihara, N., *Difference equations in Banach spaces*, Comm. Math. Univ. St. Paul **28** (1980), 51–61. MR **81k:**39002
8. Ishizuka, S., *On solution complexes of D-modules associated with confluent hypergeometric differential equations*, (in Japanese), Master thesis, Ochanomizu University, Tokyo, 1994.
9. Ishizuka, S., Iwasaki, K. and Majima, H., *Gevrey cohomology groups for the Humbert systems*, in preparation.
10. Iwasaki, K., *Cohomology groups for recurrence relations*, Preprint (1996).
11. Li, Z.-H., *The asymptotic estimates of solutions of difference equations*, J. Math. Anal. Appl. **94** (1983), 181–192. MR **94g:**39001

12. Milne-Thomson, L.M., *The calculus of finite differences*, Chelsea Publ., New York, 1981. MR **13:**245c (2nd ed.)
13. Nielsen, N., *Recherches sur les séries de factorielles*, Ann. Ecole Norm. Sup. **19** (1902), 409–453.

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