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SPHERICAL CLASSES AND THE ALGEBRAIC TRANSFER

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Abstract. We study a weak form of the classical conjecture which predicts that there are no spherical classes in Q_0S^0 except the elements of Hopf invariant one and those of Kervaire invariant one. The weak conjecture is obtained by restricting the Hurewicz homomorphism to the homotopy classes which are detected by the algebraic transfer.

Let $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ with $|x_i| = 1$. The general linear group $GL_k = GL(k, \mathbb{F}_2)$ and the (mod 2) Steenrod algebra $\mathcal{A}$ act on P_k in the usual manner. We prove that the weak conjecture is equivalent to the following one: The canonical homomorphism $j_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$ induced by the identity map on P_k is zero in positive dimensions for $k > 2$. In other words, every Dickson invariant (i.e. element of $P_k^{GL_k}$) of positive dimension belongs to $\mathcal{A}^+ \cdot P_k$ for $k > 2$, where $\mathcal{A}^+$ denotes the augmentation ideal of $\mathcal{A}$. This conjecture is proved for $k = 3$ in two different ways. One of these two ways is to study the squaring operation Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} P_k^*)$, the range of j_k^*, and to show it commuting through j_k^* with Kameko's Sq^0 on $\mathbb{F}_2 \underset{GL_k}{\otimes} P(P_k^*)$, the domain of j_k^*. We compute explicitly the action of Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} P_k^*)$ for $k \leq 4$.

1. Introduction

The paper deals with the spherical classes in Q_0S^0, i.e. the elements belonging to the image of the Hurewicz homomorphism

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)\,.$$

Here and throughout the paper, the coefficient ring for homology and cohomology is always $\mathbb{F}_2$, the field of 2 elements.

We are interested in the following classical conjecture.

Conjecture 1.1. (conjecture on spherical classes). There are no spherical classes in Q_0S^0, except the elements of Hopf invariant one and those of Kervaire invariant one.

(See Curtis [9] and Wellington [21] for a discussion.)

Let V_k be an elementary abelian 2-group of rank k. It is also viewed as a k-dimensional vector space over $\mathbb{F}_2$. So, the general linear group $GL_k = GL(k, \mathbb{F}_2)$

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acts on V_k and therefore on $H^*(BV_k)$ in the usual way. Let D_k be the Dickson algebra of k variables, i.e. the algebra of invariants

$$D_k := H^*(BV_k)^{GL_k} \cong \mathbb{F}_2[x_1, \dots, x_k]^{GL_k},$$

where $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ is the polynomial algebra on k generators $x_1, \dots, x_k$, each of dimension 1. As the action of the (mod 2) Steenrod algebra $\mathcal{A}$ and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathcal{A}$.

One way to attack Conjecture 1.1 is to study the Lannes–Zarati homomorphism

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^*,$$

which is compatible with the Hurewicz homomorphism (see [12], [13, p.46]). The domain of φ_k is the E_2-term of the Adams spectral sequence converging to $\pi_*^s(S^0) \cong \pi_*(Q_0S^0)$. Furthermore, according to Madsen's theorem [15] which asserts that D_k is dual to the coalgebra of Dyer–Lashof operations of length k, the range of φ_k is a submodule of $H_*(Q_0S^0)$. By compatibility of φ_k and the Hurewicz homomorphism we mean φ_k is a "lifting" of the latter from the "E_∞-level" to the "E_2-level".

Let h_r denote the Adams element in $Ext_{\mathcal{A}}^{1,2^r}(\mathbb{F}_2, \mathbb{F}_2)$. Lannes and Zarati proved in [13] that φ_1 is an isomorphism with $\{\varphi_1(h_r) \mid r \geq 0\}$ forming a basis of the dual of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_1$ and φ_2 is surjective with $\{\varphi_2(h_r^2) | \, r \geq 0\}$ forming a basis of the dual of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_2$. Recall that, from Adams [1], the only elements of Hopf invariant one are represented by h_1, h_2, h_3 of the stems $i = 2^r - 1 = 1, 3, 7$, respectively. Moreover, by Browder [5], the only dimensions where an element of Kervaire invariant one would occur are $2(2^r - 1)$, for $r > 0$, and it really occurs at this dimension if and only if h_r^2 is a permanent cycle in the Adams spectral sequence for the spheres.

Therefore, Conjecture 1.1 is a consequence of the following:

Conjecture 1.2. $\varphi_k = 0$ in any positive stem i for $k > 2$.

It is well known that the Ext group has intensively been studied, but remains very mysterious. In order to avoid the shortage of our knowledge of the Ext group, we want to restrict φ_k to a certain subgroup of Ext which (1) is large enough and worthwhile to pursue and (2) could be handled more easily than the Ext itself. To this end, we combine the above data with Singer's algebraic transfer.

Singer defined in [20] the algebraic transfer

$$Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_i(BV_k) \to Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2),$$

where $PH_*(BV_k)$ denotes the submodule consisting of all $\mathcal{A}$-annihilated elements in $H_*(BV_k)$. It is shown to be an isomorphism for $k \leq 2$ by Singer [20] and for $k = 3$ by Boardman [4]. Singer also proved that it is an isomorphism for $k = 4$ in a range of internal degrees. But he showed it is *not* an isomorphism for $k = 5$. However, he conjectures that Tr_k is a *monomorphism* for any k.

Our main idea is to study the restriction of φ_k to the image of Tr_k.

Conjecture 1.3. (weak conjecture on spherical classes).

$$\varphi_k \cdot Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) := (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$$

is zero in positive dimensions for $k > 2$.

In other words, there are no spherical classes in Q_0S^0, except the elements of Hopf invariant one and those of Kervaire invariant one, which can be detected by the algebraic transfer.

A natural question is: How can one express $\varphi_k \cdot Tr_k$ in the framework of invariant theory alone, and without using the mysterious Ext group?

Let $j_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$ be the natural homomorphism induced by the identity map on P_k. We have

Theorem 2.1. *$\varphi_k \cdot Tr_k$ is dual to j_k, or equivalently,*

$$j_k = Tr_k^* \cdot \varphi_k^* .$$

By this theorem, Conjecture 1.3 is equivalent to

Conjecture 1.4. $j_k = 0$ in positive dimensions for $k > 2$.

This seems to be a surprise, because by an elementary argument involving taking averages, one can see that if $H \subset GL_k$ is a subgroup of odd order then the similar homomorphism

$$j_H : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^H) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^H$$

is an isomorphism. Furthermore, j_1 is iso and j_2 is mono.

Obviously, $j_k = 0$ if and only if the composite

$$D_k = P_k^{GL_k} \xrightarrow{proj} \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \xrightarrow{j_k} (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k} \xrightarrow{\subseteq} \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k$$

is zero. So, Conjecture 1.4 can equivalently be stated in the following form.

Conjecture 1.5. Let D_k^+, $\mathcal{A}^+$ denote the augmentation ideals in D_k and $\mathcal{A}$, respectively. Then $D_k^+ \subset \mathcal{A}^+ \cdot P_k$ for any $k > 2$.

The domain and range of j_k both are still mysterious. Anyhow, they seem easier to handle than the Ext group. They both are well-known for $k = 1, 2$. Furthermore, on the one hand, $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$ is computed for $k = 3$ by Kameko [11], Alghamdi–Crabb–Hubbuck [3] and Boardman [4]. On the other hand, $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k})$ is determined by Hu'ng–Peterson [18] for $k = 3$ and 4.

Let $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P^{GL}) := \bigoplus_{k \geq 0} \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k})$ and $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL} := \bigoplus_{k \geq 0} (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$. They are equipped with canonical coalgebra structures. We get

Proposition 3.1. *$j = \bigoplus j_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P^{GL}) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL}$ is a homomorphism of coalgebras.*

Let $Sq^0 : PH_*(BV_k) \to PH_*(BV_k)$ be Kameko's squaring operation that commutes with the Steenrod operation $Sq^0 : Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2) \to Ext_{\mathcal{A}}^{k,2t}(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer Tr_k (see [11], [3], [4], [17]). Note that Sq^0 is completely different from the identity map. We prove

Proposition 4.2. *There exists a homomorphism*

$$Sq^0 : P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) ,$$

which commutes with Kameko's Sq^0 through the homomorphism j_k^*.

These two propositions lead us to two different proofs of the following theorem.

Theorem 3.2. *$j_k^* = 0$ in positive dimensions for $k = 3$. In other words, there is no spherical class in Q_0S^0 which is detected by the triple algebraic transfer.*

We compute explicitly the action of Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))$ for $k = 3$ and 4 in Propositions 5.2 and 5.4.

The paper contains six sections and is organized as follows.

Section 2 is to prove Theorem 2.1. In Section 3, we assemble the j_k for $k \geq 0$ to get a homomorphism of coalgebras $j = \bigoplus j_k$. By means of this property of j we give there a proof of Theorem 3.2. Section 4 deals with the existence of the squaring operation Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))$ that leads us to an alternative proof for Theorem 3.2. This proof helps to explain the problem. In Section 5, we compute explicitly the action of Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))$ for $k \leq 4$. Finally, in Section 6 we state a conjecture on the Dickson algebra that concerns spherical classes.

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2. Expressing $\varphi_k \cdot Tr_k$ in the framework of invariant theory

First, let us recall how to define the homomorphism j_k.

We have the commutative diagram

$$\begin{array}{ccc}
P_k^{GL_k} & \xrightarrow{\subset} & P_k \\
\downarrow & \overset{p}{\searrow} & \downarrow \\
\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) & \xrightarrow{\widetilde{j_k}} & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k \ ,
\end{array}$$

where the vertical arrows are the canonical projections, and $\widetilde{j_k}$ is induced by the inclusion $P_k^{GL_k} \subset P_k$. Obviously, $p(P_k^{GL_k}) \subset (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$. So, $\widetilde{j_k}$ factors through $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$ to give rise to

$$\begin{array}{rccc}
j_k : & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) & \to & (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}, \\
& j_k(1 \underset{\mathcal{A}}{\otimes} Y) & = & 1 \underset{\mathcal{A}}{\otimes} Y,
\end{array}$$

for any polynomial $Y \in D_k = P_k^{GL_k}$.

The goal of this section is to prove the following theorem.

Theorem 2.1. $j_k = Tr_k^* \cdot \varphi_k^*$.

Now we prepare some data in order to prove the theorem at the end of this section.

First we sketch Lannes–Zarati's work [13] on the derived functors of the destabilization. Let $\mathcal{D}$ be the destabilization functor, which sends an $\mathcal{A}$-module M to the unstable $\mathcal{A}$-module $\mathcal{D}(M) = M/B(M)$, where $B(M)$ is the submodule of M generated by all $Sq^i u$ with $u \in M$, $i > |u|$.

$\mathcal{D}$ is a right exact functor. Let $\mathcal{D}_k$ be its k-th derived functor for $k \geq 0$.

Suppose M_1, M_2 are $\mathcal{A}$-modules. Lannes and Zarati defined in [13, §2] a homomorphism

$$\begin{array}{rccc} \cap: & Ext^r_{\mathcal{A}}(M_1, M_2) \otimes \mathcal{D}_s(M_1) & \to & \mathcal{D}_{s-r}(M_2), \\ & (f, z) & \mapsto & f \cap z\,, \end{array}$$

as follows.

Let $F_*(M_i)$ be a free resolution of M_i, $i = 1, 2$. A class $f \in Ext^r_{\mathcal{A}}(M_1, M_2)$ can be represented by a chain map $F : F_*(M_1) \to F_{*-r}(M_2)$ of homological degree $-r$. We write $f = [F]$. If $z = [Z]$ is represented by $Z \in F_*(M_1)$, then by definition $f \cap z = [F(Z)]$.

Let M be an $\mathcal{A}$-module. We set $r = s = k$, $M_1 = \Sigma^{-k}M$, $M_2 = P_k \otimes M$, where as before $P_k = \mathbb{F}_2[x_1, \dots, x_k]$, and get the homomorphism

$$\cap : Ext^k_{\mathcal{A}}(\Sigma^{-k}M, P_k \otimes M) \otimes \mathcal{D}_k(\Sigma^{-k}M) \to P_k \otimes M\,.$$

Now we need to define the Singer element $e_k(M) \in Ext^k_{\mathcal{A}}(\Sigma^{-k}M, P_k \otimes M)$ (see Singer [20, p. 498]). Let $\widehat{P}_1$ be the submodule of $\mathbb{F}_2[x, x^{-1}]$ spanned by all powers x^i with $i \geq -1$, where $|x| = 1$. The $\mathcal{A}$-module structure on $\mathbb{F}_2[x, x^{-1}]$ extends that of $P_1 = \mathbb{F}_2[x]$ (see Adams [2], Wilkerson [22]). The inclusion $P_1 \subset \widehat{P}_1$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \to P_1 \to \widehat{P}_1 \to \Sigma^{-1}\mathbb{F}_2 \to 0\,.$$

Denote by e_1 the corresponding element in $Ext^1_{\mathcal{A}}(\Sigma^{-1}\mathbb{F}_2, P_1)$.

Definition 2.2. (Singer [20]).

(i) $e_k = \underbrace{e_1 \otimes \cdots \otimes e_1}_{k \text{ times}} \in Ext^k_{\mathcal{A}}(\Sigma^{-k}\mathbb{F}_2, P_k)$.

(ii) $e_k(M) = e_k \otimes M \in Ext^k_{\mathcal{A}}(\Sigma^{-k}M, P_k \otimes M)$, for M an $\mathcal{A}$-module. Here we also denote by M the identity map of M.

The cap-product with $e_k(M)$ gives rise to the homomorphism

$$\begin{array}{rccc} e_k(M): & \mathcal{D}_k(\Sigma^{-k}M) & \to & \mathcal{D}_0(P_k \otimes M) \equiv P_k \otimes M, \\ & e_k(M)(z) & = & e_k(M) \cap z\,. \end{array}$$

As $\mathbb{F}_2$ is an unstable $\mathcal{A}$-module, the following theorem is a special case (but would be the most important case) of the main result in [13].

Theorem 2.3 (Lannes–Zarati [13])**.** *Let $D_k \subset P_k$ be the Dickson algebra of k variables. Then $e_k(\Sigma\mathbb{F}_2) : \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \to \Sigma D_k$ is an isomorphism of internal degree 0.*

Next, we explain in detail the definition of the Lannes–Zarati homomorphism

$$\varphi_k : Ext^k_{\mathcal{A}}(\Sigma^{-k}\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*\,,$$

which is compatible with the Hurewicz map (see [12], [13]).

Let N be an $\mathcal{A}$-module. By definition of the functor $\mathcal{D}$, we have a natural homomorphism: $\mathcal{D}(N) \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} N$. Suppose $F_*(N)$ is a free resolution of N. Then the above natural homomorphism induces a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{D}F_k(N) & \longrightarrow & \mathcal{D}F_{k-1}(N) & \longrightarrow & \cdots \\
 & & \downarrow i_k & & \downarrow i_{k-1} & & \\
\cdots & \longrightarrow & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} F_k(N) & \longrightarrow & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} F_{k-1}(N) & \longrightarrow & \cdots\ .
\end{array}$$

Here the horizontal arrows are induced from the differential in $F_*(N)$, and

$$i_k[Z] = [1 \underset{\mathcal{A}}{\otimes} Z]$$

for $Z \in F_k(N)$. Passing to homology, we get a homomorphism

$$\begin{array}{rccc}
i_k: & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(N) & \to & Tor_k^{\mathcal{A}}(\mathbb{F}_2, N), \\
 & 1 \underset{\mathcal{A}}{\otimes} [Z] & \mapsto & [1 \underset{\mathcal{A}}{\otimes} Z]\ .
\end{array}$$

Taking $N = \Sigma^{1-k}\mathbb{F}_2$, we obtain a homomorphism

$$i_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(\Sigma^{1-k}\mathbb{F}_2) \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2)\ .$$

Note that the suspension $\Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \Sigma D_k$ and the desuspension

$$\Sigma^{-1} : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2) \xrightarrow{\cong} Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2)$$

are isomorphisms of internal degree 1 and (-1), respectively. This leads us to

Definition 2.4. (Lannes–Zarati [13]). The homomorphism φ_k of internal degree 0 is the dual of

$$\varphi_k^* = \Sigma^{-1} i_k \left(1 \underset{\mathcal{A}}{\otimes} e_k^{-1}(\Sigma\mathbb{F}_2) \right) \Sigma : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2)\ .$$

Now we recall the definition of the algebraic transfer. Consider the cap-product

$$\begin{array}{ccc}
Ext_{\mathcal{A}}^r(\Sigma^{-k}\mathbb{F}_2, P_k) \otimes Tor_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) & \to & Tor_{s-r}^{\mathcal{A}}(\mathbb{F}_2, P_k), \\
(e, z) & \mapsto & e \cap z\ .
\end{array}$$

Taking $r = s = k$ and $e = e_k$ as in Definition 2.2, we obtain the homomorphism

$$\begin{array}{rccccc}
Tr_k^* : & Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) & \to & Tor_0^{\mathcal{A}}(\mathbb{F}_2, P_k) & \equiv & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k, \\
 & Tr_k^*[1 \underset{\mathcal{A}}{\otimes} Z] & = & e_k \cap [Z] = [1 \underset{\mathcal{A}}{\otimes} E(Z)] & \equiv & 1 \underset{\mathcal{A}}{\otimes} E(Z),
\end{array}$$

for $Z \in F_k(\Sigma^{-k}\mathbb{F}_2)$, where $e_k = [E]$ is represented by a chain map $E : F_*(\Sigma^{-k}\mathbb{F}_2) \to F_{*-k}(P_k)$.

Singer proved in [20] that e_k is GL_k-invariant, hence $Im(Tr_k^*) \subset (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$. This gives rise to a homomorphism, which is also denoted by Tr_k^*,

$$Tr_k^* : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-k}\mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}\ .$$

Definition 2.5. (Singer [20]). The k-th algebraic transfer $Tr_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) \to Ext_{\mathcal{A}}^{k,k+*}(\mathbb{F}_2, \mathbb{F}_2)$ is the homomorphism dual to Tr_k^*.

We have finished the preparation of the needed data.

Proof of Theorem 2.1. Note that the usual isomorphism

$$Ext^k_{\mathcal{A}}(\Sigma^{-k}\mathbb{F}_2, P_k) \xrightarrow{\cong} Ext^k_{\mathcal{A}}(\Sigma^{1-k}\mathbb{F}_2, \Sigma P_k)$$

sends $e_k(\mathbb{F}_2)$ to $e_k(\Sigma\mathbb{F}_2) = e_k(\mathbb{F}_2) \otimes \Sigma\mathbb{F}_2$. Moreover, if $e_k(\mathbb{F}_2) = [E]$ is represented by a chain map $E : F_*(\Sigma^{-k}\mathbb{F}_2) \to F_{*-k}(P_k)$ then $e_k(\Sigma\mathbb{F}_2) = [E_\Sigma]$ is represented by the induced chain map $E_\Sigma : F_*(\Sigma^{1-k}\mathbb{F}_2) \to F_{*-k}(\Sigma P_k)$, which is defined by $E_\Sigma = \Sigma E \Sigma^{-1}$.

By Theorem 2.3, $e_k(\Sigma\mathbb{F}_2)$ is an isomorphism. So, for any $Y \in D_k$, there exists a representative of $e_k^{-1}(\Sigma\mathbb{F}_2)\Sigma Y$, which is denoted by $E_\Sigma^{-1}\Sigma Y \in F_k(\Sigma^{1-k}\mathbb{F}_2)$, such that $E_\Sigma\left(E_\Sigma^{-1}\Sigma Y\right) = \Sigma Y$.

The cap-product with $e_k(\Sigma\mathbb{F}_2) = [E_\Sigma]$ induces the homomorphism

$$\begin{array}{rclcl} \widetilde{Tr}_k^*: \ Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2) & \to & Tor_0^{\mathcal{A}}(\mathbb{F}_2, \Sigma P_k) & \equiv & \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \Sigma P_k, \\ \widetilde{Tr}_k^*[1 \underset{\mathcal{A}}{\otimes} Z] & = & e_k(\Sigma\mathbb{F}_2) \cap [Z] = [1 \underset{\mathcal{A}}{\otimes} E_\Sigma(Z)] & \equiv & 1 \underset{\mathcal{A}}{\otimes} E_\Sigma(Z), \end{array}$$

for $Z \in F_k(\Sigma^{1-k}\mathbb{F}_2)$. It is easy to check that $Tr_k^* = \Sigma^{-1}\widetilde{Tr}_k^*\Sigma$. Moreover, set

$$\widetilde{\varphi}_k^* = \Sigma\varphi_k^*\Sigma^{-1} = i_k\left(1 \underset{\mathcal{A}}{\otimes} e_k^{-1}(\Sigma\mathbb{F}_2)\right) : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} \Sigma D_k \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-k}\mathbb{F}_2)\,.$$

Obviously, $Tr_k^* \cdot \varphi_k^* = \Sigma^{-1}\widetilde{Tr}_k^* \cdot \widetilde{\varphi}_k^*\Sigma$. Now, for any $Y \in D_k$, we have

$$\begin{aligned} Tr_k^* \cdot \varphi_k^*(1 \underset{\mathcal{A}}{\otimes} Y) &= \Sigma^{-1}\widetilde{Tr}_k^* \cdot \widetilde{\varphi}_k^*\Sigma(1 \underset{\mathcal{A}}{\otimes} Y) \\ &= \Sigma^{-1}\widetilde{Tr}_k^* \cdot \widetilde{\varphi}_k^*(1 \underset{\mathcal{A}}{\otimes} \Sigma Y) \\ &= \Sigma^{-1}\widetilde{Tr}_k^*\left[1 \underset{\mathcal{A}}{\otimes} E_\Sigma^{-1}\Sigma Y\right] \\ &= \Sigma^{-1}\left(1 \underset{\mathcal{A}}{\otimes} E_\Sigma(E_\Sigma^{-1}\Sigma Y)\right) \\ &= 1 \underset{\mathcal{A}}{\otimes} \Sigma^{-1}(\Sigma Y) \\ &= 1 \underset{\mathcal{A}}{\otimes} Y\,. \end{aligned}$$

By definition of j_k, we also have $j_k(1 \underset{\mathcal{A}}{\otimes} Y) = 1 \underset{\mathcal{A}}{\otimes} Y$. The theorem is proved.

3. The homomorphism of coalgebras $j = \bigoplus j_k$

The canonical isomorphism $V_k \cong V_\ell \times V_m$, for $k = \ell + m$, induces the usual inclusion $GL_k \supset GL_\ell \times GL_m$ and the usual diagonal $\Delta : P_k \to P_\ell \otimes P_m$. Therefore, it induces two homomorphisms

$$\begin{aligned} \overline{\Delta}_D &: \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \to \left(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_\ell^{GL_\ell})\right) \otimes \left(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_m^{GL_m})\right)\,, \\ \overline{\Delta}_P &: (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k} \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_\ell)^{GL_\ell} \otimes (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_m)^{GL_m}\,. \end{aligned}$$

Here and in what follows, $\otimes$ means the tensor product over $\mathbb{F}_2$, except when otherwise specified.

Set

$$\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D = \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P^{GL}) := \bigoplus_{k\geq 0} \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) ,$$

$$(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL} := \bigoplus_{k\geq 0} (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k} .$$

It is easy to see that $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P^{GL})$ and $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL}$ are endowed with the structure of a cocommutative coalgebra by $\overline{\Delta}_D$ and $\overline{\Delta}_P$, respectively. The coalgebra structure of $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL}$ was first given by Singer [20].

Proposition 3.1. *$j = \bigoplus j_k : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P^{GL}) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P)^{GL}$ is a homomorphism of coalgebras.*

Proof. This follows immediately from the commutative diagram

$$\begin{array}{ccc}
\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k & \xrightarrow{\quad j_k \quad} & (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k} \\
\Big\downarrow {\scriptstyle \overline{\Delta}_D} & & \Big\downarrow {\scriptstyle \overline{\Delta}_P} \\
(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_\ell) \otimes (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_m) & \xrightarrow{\quad j_\ell \otimes j_m \quad} & (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_\ell)^{GL_\ell} \otimes (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_m)^{GL_m} .
\end{array}$$

Remark. According to Singer [20], $Tr^* = \bigoplus Tr_k^*$ is a homomorphism of coalgebras. One can see that $\varphi^* = \bigoplus \varphi_k^*$ is also a homomorphism of coalgebras. Then, so is $j = Tr^* \cdot \varphi^*$. This is an alternative proof for Proposition 3.1.

Now let

$$\mathbb{F}_2 \underset{GL}{\otimes} PH_*(BV) := \bigoplus_{k\geq 0} \left(\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) \right) \cong \bigoplus_{k\geq 0} \left((\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k} \right)^* ,$$

$$P(\mathbb{F}_2 \underset{GL}{\otimes} H_*(BV)) := \bigoplus_{k\geq 0} P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) \cong \bigoplus_{k\geq 0} \left(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \right)^* .$$

Passing to the dual, we obtain the homomorphism of algebras

$$j^* : \mathbb{F}_2 \underset{GL}{\otimes} PH_*(BV) \to P(\mathbb{F}_2 \underset{GL}{\otimes} H_*(BV)) .$$

As an application of j^*, we give here a proof for Conjecture 1.4 with $k = 3$.

Theorem 3.2. *$j_3 : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_3^{GL_3}) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_3)^{GL_3}$ is zero in positive dimensions.*

Proof. We equivalently show that

$$j_3^* : \mathbb{F}_2 \underset{GL_3}{\otimes} PH_*(BV_3) \to P(\mathbb{F}_2 \underset{GL_3}{\otimes} H_*(BV_3))$$

is a trivial homomorphism in positive dimensions.

$\mathbb{F}_2 \underset{GL_3}{\otimes} PH_*(BV_3)$ is described by Kameko [11], Alghamdi–Crabb–Hubbuck [3] and Boardman [4] as follows. $\mathbb{F}_2 \underset{GL_1}{\otimes} PH_*(BV_1)$ has a basis consisting of h_r, $r \geq 0$, where h_r is of dimension $2^r - 1$ and is sent by the isomorphism Tr_1 to the Adams element, denoted also by h_r, in $Ext_{\mathcal{A}}^{1,2^r}(\mathbb{F}_2, \mathbb{F}_2)$. According to [11], [3],

[4], $\mathbb{F}_2 \underset{GL_3}{\otimes} PH_*(BV_3)$ has a basis consisting of some products of the form $h_r h_s h_t$, where r, s, t are non-negative integers (but not all such appear), and some elements c_i $(i \geq 0)$ with $\dim(c_i) = 2^{i+3} + 2^{i+1} + 2^i - 3$.

We will show in Lemma 3.3 that any decomposable element in $P(\mathbb{F}_2 \underset{GL_3}{\otimes} H_*(BV_3))$ is zero. Then, since j^* is a homomorphism of algebras, j_3^* sends any element of the form $h_r h_s h_t$ to zero.

On the other hand, by Hu'ng–Peterson [18], $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_3$ is concentrated in the dimensions $2^{s+2} - 4$ $(s \geq 0)$ and $2^{r+2} + 2^{s+1} - 3$ $(r > s > 0)$. Obviously, these dimensions are different from $\dim(c_i)$ for any i. Then j_3^* also sends c_i to zero. ☐

To complete the proof of the theorem, we need to show the following lemma.

Lemma 3.3. *Let $\overline{D}_k = \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k$. Then the diagonal*

$$\overline{\Delta}_D : \overline{D}_3 \to \overline{D}_1 \otimes \overline{D}_2 \oplus \overline{D}_2 \otimes \overline{D}_1$$

is zero in positive dimensions.

Proof. Let us recall some informations on the Dickson algebra D_k. Dickson proved in [10] that $D_k \cong \mathbb{F}_2[Q_{k-1}, Q_{k-2}, \ldots, Q_0]$, a polynomial algebra on k generators, with $|Q_s| = 2^k - 2^s$. Note that Q_s depends on k, and when necessary, will be denoted $Q_{k,s}$. An inductive definition of $Q_{k,s}$ is given by

Definition 3.4.

$$Q_{k,s} = Q_{k-1,s-1}^2 + v_k \cdot Q_{k-1,s}\ ,$$

where, by convention, $Q_{k,k} = 1$, $Q_{k,s} = 0$ for $s < 0$ and

$$v_k = \prod_{\lambda_i \in \mathbb{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{k-1} x_{k-1} + x_k)\,.$$

Dickson showed in [10] that

$$v_k = \sum_{s=0}^{k-1} Q_{k-1,s}\, x_k^{2^s}\ .$$

Now we turn back to the lemma.

Since $\overline{\Delta}_D$ is symmetric, we need only to show that the diagonal

$$\overline{\Delta} : \overline{D}_3 \to \overline{D}_2 \otimes \overline{D}_1$$

is zero in positive dimensions.

For abbreviation, we denote x_1, x_2, x_3 by x, y, z, respectively, $Q_i = Q_{3,i}(x, y, z)$ for $i = 0, 1, 2$, $q_i = Q_{2,i}(x, y)$ for $i = 0, 1$. As is well known, $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_1$ has the basis $\{z^{2^s - 1} |\ s \geq 0\}$, and $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_2$ has the basis $\{q_1^{2^s-1} |\ s \geq 0\}$. By Hu'ng–Peterson [18], $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_3$ has the basis

$$\{Q_2^{2^s-1}\ (s \geq 0), Q_2^{2^r - 2^s - 1} Q_1^{2^s - 1} Q_0\ (r > s > 0)\}\,.$$

For $k \leq 3$, every monomial in $Q_0, \ldots, Q_{k-1}$ which does not belong to the given basis is zero in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k$. Note that the analogous statement is not true for $k \geq 4$ (see [18]).

Using the above inductive definitions of $Q_{k,s}$ and v_k, we get

$$Q_0 = q_0^2 z + q_0 q_1 z^2 + q_0 z^4 \,,$$

or

$$\Delta(Q_0) = q_0^2 \otimes z + q_0 q_1 \otimes z^2 + q_0 \otimes z^4 \,.$$

This implies easily that every term in $\Delta(Q_2^{2^r-2^s-1} Q_1^{2^s-1} Q_0)$ is divisible by q_0, so it equals zero in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_2$ as shown above. In other words,

$$\overline{\Delta}(Q_2^{2^r-2^s-1} Q_1^{2^s-1} Q_0) = 0 \,.$$

Similarly,

$$Q_2 = q_1^2 + v_3 = q_1^2 + q_0 z + q_1 z^2 + z^4 \,,$$

or

$$\begin{aligned}\Delta(Q_2) &= q_1^2 \otimes 1 + q_0 \otimes z + q_1 \otimes z^2 + 1 \otimes z^4 \,,\\ \Delta(Q_2^{2^s-1}) &= (q_1^2 \otimes 1 + q_0 \otimes z + q_1 \otimes z^2 + 1 \otimes z^4)^{2^s-1} \,.\end{aligned}$$

By the same argument as above, we need only to consider terms in $\Delta(Q_2^{2^s-1})$ which are not divisible by q_0. Such a term is some product of powers of $q_1^2 \otimes 1$, $q_1 \otimes z^2$, $1 \otimes z^4$. If it contains a positive power of z then this power is even and it equals zero in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_1$. Otherwise, it should be $q_1^{2(2^s-1)} \otimes 1$. Obviously, $q_1^{2(2^s-1)}$ equals zero in $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_2$. So, $\overline{\Delta}(Q_2^{2^s-1}) = 0$ for $s > 0$.

In summary, $\overline{\Delta} = 0$ in positive dimensions. The lemma is proved. Then, so is Theorem 3.2.

As Tr_3 is an isomorphism (see Boardman [4]), we have an immediate consequence.

Corollary 3.5. *$\varphi_3 : Ext_{\mathcal{A}}^{3,3+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_3)_i^*$ is zero in every positive stem i.*

4. The squaring operation: the existence

Liulevicius was perhaps the first person who noted in [14] that there are squaring operations $Sq^i : Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2) \to Ext_{\mathcal{A}}^{k+i,2t}(\mathbb{F}_2, \mathbb{F}_2)$, which share most of the properties with Sq^i on cohomology of spaces. In particular, $Sq^i(\alpha) = 0$ if $i > k$, $Sq^k(\alpha) = \alpha^2$ for $\alpha \in Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2)$, and the Cartan formula holds for the Sq^i's. However, Sq^0 is not the identity. In fact,

$$\begin{aligned} Sq^0 : \quad Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2) &\to Ext_{\mathcal{A}}^{k,2t}(\mathbb{F}_2, \mathbb{F}_2),\\ [b_1|\dots|b_k] &\mapsto [b_1^2|\dots|b_k^2], \end{aligned}$$

in terms of the cobar resolution (see May [16]).

Recall that $H_*(BV_k)$ is a divided power algebra

$$H_*(BV_k) = \Gamma(a_1, \dots, a_k)$$

generated by $a_1, \dots, a_k$, each of degree 1, where a_i is dual to $x_i \in H^1(BV_k)$. Here and in what follows, the duality is taken with respect to the basis of $H^*(BV_k)$ consisting of all monomials in $x_1, \dots, x_k$.

Let γ_t be the t-th divided power in $H_*(BV_k)$ and for any $a \in H_*(BV_k)$ let $a^{(t)} = \gamma_t(a)$. So $a_i^{(t)}$ is the element dual to x_i^t. One has

$$a_i^{(2^r)} a_i^{(2^r)} = 0\,,$$

and

$$a_i^{(t)} = a_i^{(2^{r_1})} \cdots a_i^{(2^{r_m})}$$

if $t = 2^{r_1} + \cdots + 2^{r_m}$, $0 \le r_1 < \cdots < r_m$.

In [11] Kameko defined a GL_k-homomorphism

$$\begin{aligned} Sq^0 : \quad PH_*(BV_k) \quad &\to \quad PH_*(BV_k), \\ a_1^{(i_1)} \cdots a_k^{(i_k)} \quad &\mapsto \quad a_1^{(2i_1+1)} \cdots a_k^{(2i_k+1)}\,, \end{aligned}$$

where $a_1^{(i_1)} \cdots a_k^{(i_k)}$ is dual to $x_1^{i_1} \cdots x_k^{i_k}$. (See also [3].)

Crabb and Hubbuck gave in [8] a definition of Sq^0 that does not depend on the chosen basis of $H_*(BV_k)$ as follows. The element $a(V_k) = a_1 \cdots a_k$ is nothing but the image of the generator of $\Lambda^k(V_k)$ under the (skew) symmetrization map

$$\Lambda^k(V_k) \to H_k(BV_k) = \Gamma_k(V_k) = (\underbrace{V_k \otimes \cdots \otimes V_k}_{k \text{ times}})_{\Sigma_k}\,.$$

Let $F : H^*(BV_k) \to H^*(BV_k)$ be the Frobenius homomorphism defined by $F(x) = x^2$ for any x, and let $c : H_*(BV_k) \to H_*(BV_k)$ be the degree-halving dual homomorphism. It is obviously a surjective ring homomorphism. Then Sq^0 can be defined by

$$Sq^0(c(y)) = a(V_k)y\,.$$

Since $y \in \ker c$ if and only if $a(V_k)y = 0$, Sq^0 is a monomorphism of GL_k-modules. Further, it is easy to see that $cSq_*^{2i+1} = 0$, $cSq_*^{2i} = Sq_*^i c$. So Sq^0 maps $PH_*(BV_k)$ to itself.

Using a result of Carlisle and Wood [6] on the boundedness conjecture, Crabb and Hubbuck also noted in [8] that for any d, there exists t_0 such that

$$Sq^0 : PH_{2^t d + (2^t - 1)k}(BV_k) \to PH_{2^{t+1} d + (2^{t+1} - 1)k}(BV_k)$$

is an isomorphism for every $t \ge t_0$.

Kameko's Sq^0 is shown to commute with Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer Tr_k by Boardman [4] for $k = 3$ and by Minami [17] for general k.

One denotes also by Sq^0 the operation

$$Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k)$$

induced by Kameko's Sq^0. It preserves the product. Further, for $k = 3$, it satisfies

$$Sq^0(h_r h_s h_t) = h_{r+1} h_{s+1} h_{t+1}\,, \quad Sq^0(c_i) = c_{i+1}$$

(see Boardman [4]).

Lemma 4.1. *$Sq_*^{2r+1} Sq^0 = 0$, $Sq_*^{2r} Sq^0 = Sq^0 Sq_*^r$.*

Proof. We need a formal notation. Namely, for $a \in H_1(BV_k)$, set $(a^{(t)})^{[2]} = a^{(2t)}$. In general, $(a^{(t)})^{[2]} \neq \gamma_2(\gamma_t(a)) = \binom{2t-1}{t} a^{(2t)}$ (see Cartan [7]).

We start with a simple remark.

Let $x \in H^1(BV_k)$; then $Sq^r(x^s) = \binom{s}{r} x^{s+r}$. Let a denote the dual element of x. Then, by dualizing,

$$Sq^r_*(a^{(t)}) = \binom{t-r}{r} a^{(t-r)}.$$

As a consequence, $Sq^{2r+1}_*(a^{(2t+1)}) = 0$ and

$$Sq^{2r}_*(a^{(2t)}) = \binom{2t-2r}{2r} a^{(2t-2r)} = \binom{t-r}{r} \left(a^{(t-r)}\right)^{[2]} = \left(Sq^r_* a^{(t)}\right)^{[2]}.$$

Let $\alpha = a_1^{(i_1)} \cdots a_k^{(i_k)}$. By the Cartan formula, we have

$$\begin{aligned} Sq^r_* Sq^0(\alpha) &= Sq^r_*(a_1^{(2i_1+1)} \cdots a_k^{(2i_k+1)}) \\ &= \sum_{r_1+\cdots+r_k=r} Sq^{r_1}_*(a_1^{(2i_1+1)}) \cdots Sq^{r_k}_*(a_k^{(2i_k+1)}). \end{aligned}$$

The term corresponding to $(r_1, \ldots, r_k)$ equals 0 if at least one of $r_1, \ldots, r_k$ is odd. Hence $Sq^{2r+1}_* Sq^0(\alpha) = 0$. Furthermore,

$$\begin{aligned} Sq^{2r}_* Sq^0(\alpha) &= \sum_{r_1+\cdots+r_k=r} Sq^{2r_1}_*(a_1^{(2i_1+1)}) \cdots Sq^{2r_k}_*(a_k^{(2i_k+1)}) \\ &= \sum_{r_1+\cdots+r_k=r} \left\{ Sq^{2r_1}_*(a_1^{(2i_1)}) \cdots Sq^{2r_k}_*(a_k^{(2i_k)}) \right\} a_1 \cdots a_k \\ &= \sum_{r_1+\cdots+r_k=r} \left\{ Sq^{r_1}_*(a_1^{(i_1)}) \right\}^{[2]} \cdots \left\{ Sq^{r_k}_*(a_k^{(i_k)}) \right\}^{[2]} a_1 \cdots a_k \\ &= Sq^0 Sq^r_*(\alpha). \end{aligned}$$

The lemma is proved.

Proposition 4.2. *For every positive integer k, there exists a homomorphism*

$$Sq^0 : P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))$$

that sends an element of degree n to an element of degree $2n+k$ and makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) & \xrightarrow{j_k^*} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) \\ \downarrow{\scriptstyle Sq^0} & & \downarrow{\scriptstyle Sq^0} \\ \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k) & \xrightarrow{j_k^*} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)). \end{array}$$

Proof. Since $Sq^0 : H_*(BV_k) \to H_*(BV_k)$ is a GL_k-homomorphism, we can define $Sq^0_D = 1 \underset{GL_k}{\otimes} Sq^0$ and get a commutative diagram

$$\begin{array}{ccc} H_*(BV_k) & \longrightarrow & \mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k) \\ \big\downarrow Sq^0 & & \big\downarrow Sq^0_D \\ H_*(BV_k) & \longrightarrow & \mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)\ , \end{array}$$

where the horizontal arrows are the canonical projections.

Next, we show that Sq^0_D sends the primitive part to itself. In other words, suppose $\alpha \in H_*(BV_k)$ satisfies

$$Sq^r_*(1 \underset{GL_k}{\otimes} \alpha) = 1 \underset{GL_k}{\otimes} Sq^r_*\alpha = 0$$

for any $r > 0$; we want to show that

$$Sq^r_*(Sq^0(1 \underset{GL_k}{\otimes} \alpha)) = 0$$

for any $r > 0$. By definition of Sq^0 and Lemma 4.1, we have for every $r > 0$

$$\begin{aligned} Sq^r_*(Sq^0(1 \underset{GL_k}{\otimes} \alpha)) &= 1 \underset{GL_k}{\otimes} Sq^r_* Sq^0(\alpha) \\ &= \begin{cases} 1 \underset{GL_k}{\otimes} Sq^0(Sq^{r/2}_*(\alpha)), & r \text{ even}, \\ 0, & r \text{ odd}, \end{cases} \\ &= \begin{cases} Sq^0 Sq^{r/2}_*(1 \underset{GL_k}{\otimes} \alpha), & r \text{ even}, \\ 0, & r \text{ odd}, \end{cases} \\ &= 0. \end{aligned}$$

Therefore, the above commutative diagram gives rise to a commutative diagram

$$\begin{array}{ccc} PH_*(BV_k) & \xrightarrow{\widetilde{j_k^*}} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k)) \\ \big\downarrow Sq^0 & & \big\downarrow Sq^0_D \\ PH_*(BV_k) & \xrightarrow{\widetilde{j_k^*}} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))\ . \end{array}$$

By definition of j_k, the homomorphism $\widetilde{j_k^*}$ factors through $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k)$ and the previous diagram induces the commutative diagram stated in the proposition, in which Sq^0_D is re-denoted by Sq^0 for short. The proposition is proved.

As an application of Proposition 4.2, we give an alternative proof of Theorem 3.2.

By Kameko [11], Alghamdi et al. [3] and Boardman [4], $\mathbb{F}_2 \underset{GL_3}{\otimes} PH_*(BV_3)$ has a basis consisting of some products of the form $h_r h_s h_t$ and certain elements c_i $(i \geq 0)$ with $Sq^0(c_i) = c_{i+1}$ for any $i \geq 0$.

By Lemma 3.3, j_3^* vanishes on any product $h_r h_s h_t$. Making use of Proposition 4.2, one has

$$j_3^*(c_i) = j_3^*(Sq^0)^i(c_0) = (Sq^0)^i(j_3^*(c_0)).$$

One needs only to show that $j_3^*(c_0) = 0$. Recall that $\dim(c_0) = 8$. The only element of dimension 8 in D_3 is Q_2^2. Obviously, $Q_2^2 = Sq^4 Q_2$. So $P(\mathbb{F}_2 \underset{GL_3}{\otimes} H_*(BV_3))_8 = (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_3)_8^* = 0$. Therefore, $j_3^*(c_0) = 0$. Theorem 3.2 is proved.

5. The squaring operation: an explicit formula for $k \leq 4$

Let $d_{(i_{k-1},\dots,i_0)}$ be the dual element of $Q_{k-1}^{i_{k-1}} \cdots Q_0^{i_0} \in D_k$, where the duality is taken with respect to the basis of D_k consisting of all monomials in the Dickson invariants $Q_{k-1}, \dots, Q_0$.

It is well-known that

$$P(\mathbb{F}_2 \underset{GL_1}{\otimes} H_*(BV_1)) = \text{ Span}\left\{d_{(2^s-1)} \,|\, s \geq 0\right\},$$
$$P(\mathbb{F}_2 \underset{GL_2}{\otimes} H_*(BV_2)) = \text{ Span}\left\{d_{(2^s-1,0)} \,|\, s \geq 0\right\}.$$

By means of the definition of Sq^0 one can easily show that

$$Sq^0(d_{(2^s-1)}) = d_{(2^{s+1}-1)}\,,$$
$$Sq^0(d_{(2^s-1,0)}) = d_{(2^{s+1}-1,0)}\,.$$

In this section we compute Sq^0 explicitly on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(BV_k))$ for $k = 3$ and 4.

Theorem 5.1 (Hu'ng–Peterson [18]). *$PD_3^* := P(\mathbb{F}_2 \underset{GL_3}{\otimes} H_*(BV_3))$ has a basis consisting of*

$$d_{(2^s-1,0,0)}, \qquad s \geq 0\,,$$
$$d_{(2^r-2^s-1,2^s-1,1)}, \qquad r > s > 0\,.$$

They are of dimensions $2^{s+2} - 4$ and $2^{r+2} + 2^{s+1} - 3$, respectively.

Remark. It is easy to check that PD_3^* has at most one non-zero element of any dimension.

Proposition 5.2. *$Sq^0 : PD_3^* \to PD_3^*$ is given by*

$$Sq^0(d_{(2^s-1,0,0)}) = 0\,,$$
$$Sq^0(d_{(2^r-2^s-1,2^s-1,1)}) = d_{(2^{r+1}-2^{s+1}-1,2^{s+1}-1,1)}\,.$$

Proof. For brevity, we denote x_1, x_2, x_3 by x, y, z and a_1, a_2, a_3 by a, b, c, respectively.

The first part of the proposition is an immediate consequence of dimensional information. To prove the second part we start by recalling that, from Definition 3.4, we have

$$Q_{3,0} = Q_0 = x^4 y^2 z^1 + \text{ (symmetrized)}\,.$$

Suppose m, n are non-negative integers. Let $x^\alpha y^\beta z^\gamma$ be the biggest monomial in $Q_2^m Q_1^n$ with respect to the lexicographic order on (α, β, γ). We claim that $x^{\alpha+4} y^{\beta+2} z^{\gamma+1}$ appears exactly one time in $Q_2^m Q_1^n Q_0$, or equivalently

(a) $$Q_2^m Q_1^n Q_0 = x^{\alpha+4} y^{\beta+2} z^{\gamma+1} + \text{ (other terms).}$$

Indeed, suppose to the contrary that it appears more than once in $Q_2^m Q_1^n Q_0$. That means there exists a monomial $x^{\alpha'} y^{\beta'} z^{\gamma'}$ in $Q_2^m Q_1^n$, which is different from $x^\alpha y^\beta z^\gamma$, and a permutation σ on the set $\{4, 2, 1\}$ such that

$$x^{\alpha+4} y^{\beta+2} z^{\gamma+1} = x^{\alpha'+\sigma(4)} y^{\beta'+\sigma(2)} z^{\gamma'+\sigma(1)}\,.$$

Since $\alpha + 4 = \alpha' + \sigma(4)$ and $4 \geq \sigma(4)$, this implies $\alpha \leq \alpha'$. Combining this with the fact that (α, β, γ) is the biggest monomial in $Q_2^m Q_1^n$ with respect to the lexicographic order on (α, β, γ), one gets $\alpha = \alpha'$ and $\sigma(4) = 4$. Similarly, $\beta = \beta'$, $\gamma = \gamma'$ and σ is the identity permutation. This contradiction proves (a), or equivalently

(b) $$d_{(m,n,1)} = 1 \otimes a^{(\alpha+4)} b^{(\beta+2)} c^{(\gamma+1)} + (\text{other terms}).$$

Here and throughout the proof, $\otimes$ means the tensor product over GL_3.

By definition of the squaring operation

(c) $$Sq^0(1 \otimes a^{(\alpha+4)} b^{(\beta+2)} c^{(\gamma+1)}) = 1 \otimes a^{(2\alpha+9)} b^{(2\beta+5)} c^{(2\gamma+3)}.$$

Now a direct computation using Definition 3.4 shows that

$$\begin{aligned} Q_2 Q_1 Q_0 &= x^{12} y^4 z + x^{10} y^6 z + x^{10} y^5 z^2 + x^{10} y^4 z^3 \\ &\quad + x^9 y^6 z^2 + x^9 y^5 z^3 + x^8 y^6 z^3 + x^8 y^5 z^4 + \text{ (symmetrized)}\,. \end{aligned}$$

Note that $x^9 y^5 z^3$ and its symmetrized terms are the only terms of the form $x^{\text{odd}} y^{\text{odd}} z^{\text{odd}}$ in $Q_2 Q_1 Q_0$. On the other hand,

$$Q_2^{2m} Q_1^{2n} = x^{2\alpha} y^{2\beta} z^{2\gamma} + (\text{other terms})\,,$$

where $x^{2\alpha} y^{2\beta} z^{2\gamma}$ is the biggest monomial in this polynomial with respect to the lexicographic order on $(2\alpha, 2\beta, 2\gamma)$. Focusing on monomials of the form $x^{\text{odd}} y^{\text{odd}} z^{\text{odd}}$ and using the same argument as in the proof of (a), we have

$$Q_2^{2m+1} Q_1^{2n+1} Q_0 = x^{2\alpha+9} y^{2\beta+5} z^{2\gamma+3} + (\text{other terms})\,.$$

This is equivalent to

(d) $$1 \otimes a^{(2\alpha+9)} b^{(2\beta+5)} c^{(2\gamma+3)} = d_{(2m+1,2n+1,1)} + (\text{other terms}).$$

Combining (b), (c) and (d), we get

$$Sq^0(d_{(m,n,1)}) = d_{(2m+1,2n+1,1)} + \text{ (other terms)}\,.$$

Applying this for $(m, n, 1) = (2^r - 2^s - 1, 2^s - 1, 1)$, we obtain

$$Sq^0(d_{(2^r-2^s-1,2^s-1,1)}) = d_{(2^{r+1}-2^{s+1}-1,2^{s+1}-1,1)} + \text{ (other terms)}\,.$$

In addition, Sq^0 maps PD_3^* to itself (by Proposition 4.2) and PD_3^* consists of at most one non-zero element of any dimension. So the proposition is proved.

Theorem 5.3 (Hu'ng–Peterson [18]). *$PD_4^* := P(\mathbb{F}_2 \underset{GL_4}{\otimes} H_*(BV_4))$ has a basis consisting of*

$$\begin{array}{ll} d_{(2^s-1,0,0,0)}, & s \geq 0\,, \\ d_{(2^r-2^s-1,2^s-1,1,0)}, & r > s > 0\,, \\ d_{(2^t-2^r-1,2^r-2^s-1,2^s-1,2)}, & t > r > s > 1\,, \\ d_{(2^r-2^{s+1}-2^s-1,2^s-1,2^s-1,2)}, & r > s+1 > 2\,. \end{array}$$

They are of dimensions $2^{s+3} - 8$, $2^{r+3} + 2^{s+2} - 6$, $2^{t+3} + 2^{r+2} + 2^{s+1} - 4$ and $2^{r+3} + 2^{s+1} - 4$, respectively.

Remark. PD_4^*, as well as PD_3^*, has at most one non-zero element of any dimension.

Proposition 5.4. *$Sq^0 : PD_4^* \to PD_4^*$ is given by*

$$Sq^0(d_{(2^s-1,0,0,0)}) = Sq^0(d_{(2^r-2^s-1,2^s-1,1,0)}) = 0\ ,$$
$$Sq^0(d_{(2^t-2^r-1,2^r-2^s-1,2^s-1,2)}) = d_{(2^{t+1}-2^{r+1}-1,2^{r+1}-2^{s+1}-1,2^{s+1}-1,2)}\ ,$$
$$Sq^0(d_{(2^r-2^{s+1}-2^s-1,2^s-1,2^s-1,2)}) = d_{(2^{r+1}-2^{s+2}-2^{s+1}-1,2^{s+1}-1,2^{s+1}-1,2)}\ .$$

Proof. We denote x_1, x_2, x_3, x_4 by x, y, z, t and a_1, a_2, a_3, a_4 by a, b, c, d, respectively, for brevity.

The first part of the proposition is an immediate consequence of dimensional information.

We claim that $Q_0 = x^8y^4z^2t +$ (symmetrized). It can be checked by a routine computation using Definition 3.4. Here we give an alternative argument. Indeed, the Dickson algebra $D_4 \cong \mathbb{F}_2[Q_3, Q_2, Q_1, Q_0]$ has exactly one non-zero element of dimension 15. To check the equality we need only to show that the right hand side is GL_4-invariant. Recall that GL_4 is generated by the symmetric group Σ_4 and the transformation $x \mapsto x+y$, $y \mapsto y$, $z \mapsto z$, $t \mapsto t$. So, it suffices to check that the right hand side is invariant under this transformation. We leave it to the reader.

Suppose m, n, p, q are non-negative integers with $q > 0$. Let $x^\alpha y^\beta z^\gamma t^\delta$ be the biggest monomial in $Q_3^m Q_2^n Q_1^p Q_0^{q-1}$ with respect to the lexicographic order on $(\alpha, \beta, \gamma, \delta)$. By the same argument as in the proof of Proposition 5.2 we have

$$Q_3^m Q_2^n Q_1^p Q_0^q = x^{\alpha+8}y^{\beta+4}z^{\gamma+2}t^{\delta+1} + \text{ (other terms)}\,.$$

In other words,

(a) $$d_{(m,n,p,q)} = 1 \otimes a^{(\alpha+8)}b^{(\beta+4)}c^{(\gamma+2)}d^{(\delta+1)} + \text{ (other terms)}.$$

Here and throughout this proof, $\otimes$ denotes the tensor product over GL_4.

By definition of the squaring operation

(b) $$Sq^0(1 \otimes a^{(\alpha+8)}b^{(\beta+4)}c^{(\gamma+2)}t^{(\delta+1)}) = 1 \otimes a^{(2\alpha+17)}b^{(2\beta+9)}c^{(2\gamma+5)}d^{(2\delta+3)}.$$

Using the same method that we used to compute Q_0 above, we can show that

$$Q_3Q_2 = \sum_{\substack{s_1+s_2+s_3+s_4=20\\ s_i=0 \text{ or a power of } 2}} x^{s_1}y^{s_2}z^{s_3}t^{s_4}\ ,$$

$$Q_1 = \sum_{\substack{s_1+s_2+s_3+s_4=14\\ s_i=0 \text{ or a power of } 2}} x^{s_1}y^{s_2}z^{s_3}t^{s_4}\ .$$

In particular, we have

$$Q_3Q_2 = (x^{16}y^2zt + \text{ symmetrized}) + \text{ (other terms)}\,,$$
$$Q_1 = (x^8y^4zt + \text{ symmetrized}) + \text{ (other terms)}\,.$$

Here, in both cases, any other term is of the form $x^{\text{even}}y^{\text{even}}z^{\text{even}}t^{\text{even}}$. So

$$Q_3Q_2Q_1 = (x^{17}y^9z^5t^3 + \text{symmetrized}) + \text{ (other terms)}\,,$$

where $x^{17}y^9z^5t^3$ and its symmetrized terms are the only terms of the form $x^{\text{odd}}y^{\text{odd}}\ z^{\text{odd}}\ t^{\text{odd}}$ in $Q_3Q_2Q_1$. On the other hand,

$$Q_3^{2m}Q_2^{2n}Q_1^{2p}Q_0^{2q-2} = x^{2\alpha}y^{2\beta}z^{2\gamma}t^{2\delta} + \text{ (other terms)}\,,$$

where $x^{2\alpha}y^{2\beta}z^{2\gamma}t^{2\delta}$ is the biggest monomial in the polynomial with respect to the lexicographic order on $(2\alpha, 2\beta, 2\gamma, 2\delta)$. Again, we focus on monomials of the form

$x^{\text{odd}}y^{\text{odd}}z^{\text{odd}}t^{\text{odd}}$ and use the same argument as in the proof of Proposition 5.2 to get

$$Q_3^{2m+1}Q_2^{2n+1}Q_1^{2p+1}Q_0^{2q-2} = x^{2\alpha+17}y^{2\beta+9}z^{2\gamma+5}t^{2\delta+3} + \text{(other terms)},$$

or equivalently

(c)

$$1 \otimes a^{(2\alpha+17)}b^{(2\beta+9)}c^{(2\gamma+5)}d^{(2\delta+3)} = d_{(2m+1,2n+1,2p+1,2q-2)} + \text{(other terms)}.$$

Combining (a), (b) and (c), we get

$$Sq^0(d_{(m,n,p,q)}) = d_{(2m+1,2n+1,2p+1,2q-2)} + \text{(other terms)}.$$

Apply this for $(m,n,p,q) = (2^t-2^r-1, 2^r-2^s-1, 2^s-1, 2)$ and $(2^r-2^{s+1}-2^s-1, 2^s-1, 2^s-1, 2)$. Combining the resulting formulas and the facts that Sq^0 maps PD_4^* to itself (by Proposition 4.2) and that PD_4^* has at most one non-zero element of any dimension, we obtain the last two formulas of the proposition.

6. Final remark

Recall that $\overline{D}_k := \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k$. Let

$$\overline{\Delta}_D : \overline{D}_k \to \bigoplus_{\substack{\ell+m=k \\ \ell,m>0}} \overline{D}_\ell \otimes \overline{D}_m$$

be the diagonal defined at the beginning of Section 3.

Conjecture 6.1. (Hu'ng–Peterson [19]). The diagonal $\overline{\Delta}_D$ is zero in positive dimensions for any $k > 2$.

This conjecture is proved in Lemma 3.3 for $k = 3$ and has been proved for $2 < k < 10$ in [19]. It implies that j_k^* (respectively, φ_k) vanishes on the decomposable elements in $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(BV_k)$ with respect to the product given by Singer [20] and discussed in Section 3 (respectively, in $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ with respect to the cup product) for $2 < k < 10$.

Note added in proof. Conjecture 6.1 has been established by F. Peterson and the author in the final version of [19].

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