TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 11, November 1997, Pages 4635–4657
S 0002-9947(97)01777-7

ON THE KOLYVAGIN CUP PRODUCT

AMNON BESSER

Abstract. We define a new cohomological operation, which we call the Kolyvagin cup product, that is a generalization of the derivative operator introduced by Kolyvagin in his work on Euler systems. We show some of the basic properties of this operation. We also define a higher dimensional derivative in certain cases and a dual operation which we call the Kolyvagin cap product and which generalizes a computation of Rubin.

1. Introduction

In his work on Euler systems [Kol90], Kolyvagin is using a basic, derivative-like, operator to construct cohomology classes from his systems. The operation can be described as follows: let $\Delta = \langle\tau\rangle$ be a cyclic group with n elements and let M be a Δ-module. If $m \in M$ is such that the norm of m with respect to Δ is 0, i.e.,

$$\mathrm{N}\, m = \sum_{k=0}^{n-1} \tau^k m = 0,$$

then the element

$$Dm = \sum_{k=0}^{n-1} k\tau^k m \pmod{nM} \in M/nM$$

is Δ-invariant.

A few cohomological generalizations of this construction have been investigated and used since (Faltings, [Nek92], [Rub93]). Rubin's construction, which is the most general that we know of, goes as follows: Let H be a normal subgroup of a group G, and let $\Delta = G/H$. Let M be a G-module. Define the G-module $M' = \mathrm{Hom}_H(\mathbb{Z}[G], M)$ and let M'' be the kernel of the map

$$\eta(f) = \sum_{\sigma\in\Delta} \sigma^{-1}(f(\sigma)) \text{ from } M' \text{ to } M.$$

Now, associated to a homomorphism $\psi : \Delta \to \mathbb{Z}/n$ is a G-module map

$$\delta_\psi : M'' \to M/n \text{ defined by } \delta_\psi(f) = \sum_{\sigma\in\Delta} \psi(\sigma)\sigma^{-1}(f(\sigma)).$$

The induced maps on cohomology, $\delta_\psi^* : H^i(G, M'') \to H^i(G, M/n)$, are the higher dimensional analogues of the Kolyvagin derivatives. The term $H^i(G, M'')$ projects onto the kernel of the corestriction map $\mathrm{cor}_H^G : H^i(H, M) \to H^i(G, M)$ with kernel

Received by the editors May 16, 1995 and, in revised form, April 11, 1996.
1991 *Mathematics Subject Classification.* Primary 11R34; Secondary 18G15.
Partially supported by an NSF grant.

which is contained in $H^{i-1}(G,M)$, and one hopes that this group is small or that it may be annihilated by a small integer in order to make the operation depend only on the kernel of the corestriction map. When $i=0$ this operation precisely corresponds to sending $m \in \operatorname{Ker} N : M^H \to M^G$ to $\sum_{\sigma\in\Delta} \psi(\sigma)\sigma^{-1}m$. Thus, the Kolyvagin derivative is the special case when $i=0$, Δ is a cyclic group of order n and ψ is the isomorphism of Δ onto $\mathbb{Z}/n$ sending the generator τ to 1.

Our purpose in this work is to give a different, even more general, cohomological extension of the Kolyvagin construction, which coincides with the Rubin construction in the above situation, and also to prove for this construction some of the properties of the Kolyvagin derivative that are used in work on Euler systems. The starting point is to notice that, for $i=0$, the operation described by Rubin can be carried out in greater generality when ψ is a 1-cocycle of Δ with values in any Δ-module A. We will call the map

$$(\psi, m) \to \sum_{\sigma\in\Delta} \psi(\sigma)\otimes\sigma(m) \in (A\otimes M)^G \tag{1.1}$$

the "Kolyvagin operation". Here, G acts on A through its quotient Δ and $A\otimes M$ is given the diagonal action of G. The observation that the above element is G-invariant will be referred to as the "Kolyvagin trick".

One is tempted to try and interpret (1.1) as a cup product of ψ and m. A simple computation shows that as a function of ψ (1.1) does not factor through $H^1(\Delta, A)$. However, a cup product interpretation is possible in the following way: Let I be the augmentation ideal in the group ring $\mathbb{Z}[\Delta]$. The 1-cocycle ψ may be interpreted as a G-homomorphism $\psi : I \to A$ (throughout this paper we do not distinguish between the two ways of viewing ψ). The kernel of the norm map on M may be interpreted as $\operatorname{Hom}_G(\check{I}, M)$, where $\check{I}$ is the quotient of $\mathbb{Z}[\Delta]$ by the $\mathbb{Z}$-module generated by the norm element, $\sum_{\sigma\in\Delta}\sigma$, and the Kolyvagin operation above amounts then to pulling back $\psi\otimes m \in \operatorname{Hom}_G(I\otimes\check{I})A\otimes M$ by the map $\gamma : \mathbb{Z}\to I\otimes\check{I}$ defined by $\gamma(1) = \sum_{\sigma\in\Delta}(\sigma-1)\otimes\sigma$. This operation now carries over almost without any change to the case when m is in $\operatorname{Ext}^i_G(\check{I}, M)$ and gives our generalization to the Kolyvagin derivative, which we call the Kolyvagin cup product (see Definition 4.1).

It's not too hard to see that our Kolyvagin cup product coincides with the operation defined by Rubin in the case that $A=\mathbb{Z}/n$. The extra generality gives some interesting applications, which we hope to discuss in a future paper.

We prove a few fundamental results about our cup product. Two of these, Propositions 5.1 and 5.5, are analogues of the well known rules of composition between the corestriction and restriction. Other results, such as 5.4 and 8.1, allow computations with the Kolyvagin cup product in the case that M is described as the quotient of another G-module C. We should mention that 5.1 and 5.4 are generalizations of results proved by Rubin. The method of proof is different and relies as much as possible on general considerations in homological algebra instead of explicit computations with cocycles. Our hope is that this would allow further generalizations in the future (étale cohomology, K-theory). In fact, the starting point for this work was to try to find more homological-algebra-like proofs of Rubin's results.

The main result, Theorem 8.1, is about the localization of the Kolyvagin cup product. In the applications to Euler systems that we have in mind, it shows that one can carry out the crucial step in the Kolyvagin method of computing the localization of the cohomology classes obtained via the derivative operator by using the Euler system.

We also describe two related operations: One is, in some precise sense (see Propositions 7.3 and 7.6), a dual of the Kolyvagin cup product and is also a generalization of an operation described by Rubin in [Rub93]. The other is a higher order derivative which one can define in certain cases. This type of higher derivatives has appeared in dimension 0 in the work of Darmon [Dar92].

There is an important application of these higher derivative to the work of Nekovář on Euler systems on Kuga-Sato varieties [Nek92]. The result described there is not as sharp as one could hope for, i.e., the p-torsion in the Tate-Šafarevič group is only finite and not 0 for almost all p. The reason for that is that the "Kolyvagin test classes" one derives from the Euler system have to be multiplied by some p-power at a number of places. At one point, this is due to the problem of lifting the naive Kolyvagin derivative in a cohomological setting (see [Nek92], page 111 and remark on page 124). Our treatment of higher derivatives, and in particular Theorem 6.7 allows us to overcome this problem and thus improve the bounds given in [Nek92] (we intend to treat the problem of the finiteness of the Tate-Šafarevič group in a future paper).

Some of this work was done while the author was visiting at MSRI. I would like to thank Jon Rogawski for a number of helpful conversations. I would also like to thank the referee for his careful reading of the manuscript and his helpful and enlightening remarks.

2. Some homological algebra

In this and the next section, we collect together a few results in homological algebra, to be used in later sections. We use $X^\bullet, Y^\bullet, \dots$ to denote chain complexes of objects in abelian categories, usually the category of $\mathbb{Z}[G]$-modules. For a map $i^\bullet : X^\bullet \to Y^\bullet$ we let $i^\bullet_*$ denote the induced map on homology groups. For a map $i : X \to Y$ we will let

$$\mathrm{Ext}^\bullet(Y, M) \xrightarrow{i^*} \mathrm{Ext}^\bullet(X, M)$$

be the pullback map on extension groups.

We recall a few well known facts about cup products:

Definition 2.1. Let X, Y, A and M be $\mathbb{Z}[G]$-modules, for some group G. Let $X^\bullet$ and $Y^\bullet$ be $\mathbb{Z}[G]$-free resolutions of X and Y respectively. If we assume that either X or Y is $\mathbb{Z}$-free, then $X^\bullet \otimes Y^\bullet$ is a resolution of $X \otimes Y$. Given $[f] \in \mathrm{Ext}^i_G(X, A)$ and $[g] \in \mathrm{Ext}^j_G(Y, M)$, we may define $f \cup g \in \mathrm{Ext}^{i+j}_G(X \otimes Y, A \otimes M)$ as follows: Take representatives $f \in \mathrm{Hom}_G(X^i, A)$ and $g \in \mathrm{Hom}_G(Y^i, M)$ for $[f]$ and $[g]$ respectively. We may extend f and g to maps from $X^\bullet$ to A and $Y^\bullet$ to M by making them 0 in all the other degrees. Now we may define $[f] \cup [g]$ to be represented by the homomorphism $f \otimes g : X^\bullet \otimes Y^\bullet \to A \otimes M$.

The next lemma is then obvious:

Lemma 2.2. *If $\phi : X \to X'$ and $\psi : Y \to Y'$ are two maps of G-modules, then the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Ext}^i_G(X', A) \times \mathrm{Ext}^j_G(Y', M) & \xrightarrow{\phi^* \times \psi^*} & \mathrm{Ext}^i_G(X, A) \times \mathrm{Ext}^j_G(Y, M) \\ \cup \big\downarrow & & \cup \big\downarrow \\ \mathrm{Ext}^{i+j}_G(X' \otimes Y', A \otimes M) & \xrightarrow{(\phi\otimes\psi)^*} & \mathrm{Ext}^{i+j}_G(X \otimes Y, A \otimes M) \end{array}$$

We now describe a map that appears in the group cohomology of torsion modules. We might as well do things in a general homological algebra setting, so let $\mathcal{A}$ be an abelian category with enough projectives (in practice this will be the category of $\mathbb{Z}[G]$-modules for some group G). An object M of $\mathcal{A}$ is said to be n-torsion, for some integer n, if $n\,\mathrm{id}_M = 0$. Given an object X of $\mathcal{A}$, we will denote by X/n the object $\mathrm{Coker}\, n$, where n means $n\,\mathrm{id}_X$. We denote by $\mathrm{Tor}^i(X, \mathbb{Z}/n)$ the i-th left derived functor of $X \to X/n$. If $\mathrm{Tor}^i(X, \mathbb{Z}/n) = 0$ for all $i > 0$, then any projective resolution $X^\bullet$ of X has the property that $X^\bullet/n$ is a resolution of X/n. This is the case, for instance, for a G-module which is free as a $\mathbb{Z}$-module.

Definition 2.3. Let X and M be objects of $\mathcal{A}$ and assume M is n-torsion and $\mathrm{Tor}^i(X, \mathbb{Z}/n) = 0$ for all $i > 0$. We then define

$$s_{X,n} : \mathrm{Ext}^\bullet(X, M) \to \mathrm{Ext}^\bullet(X/n, M)$$

as follows: Let $Q^\bullet$ be a projective resolutions of X/n. It follows from the assumptions that there is a map of complexes $h^\bullet : Q^\bullet \to X^\bullet/n$. $s_{X,n}$ is then the map induced on the homology from the map

$$\mathrm{Hom}(X^\bullet, M) \cong \mathrm{Hom}(X^\bullet/n, M) \xrightarrow{h^{\bullet *}} \mathrm{Hom}(Q^\bullet, M).$$

It is immediately verified that $s_{X,n}$ is a section of the canonical map

$$\mathrm{Ext}^\bullet(X/n, M) \xrightarrow{r^*_{X,n}} \mathrm{Ext}^\bullet(X, M)$$

coming from the projection $r_{X,n} : X \to X/n$. As an interesting example, let M be an n-torsion G-module. From the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

one gets the short exact sequences

$$0 \to H^{i-1}(G, M) \to \mathrm{Ext}^i_G(\mathbb{Z}/n, M) \to H^i(G, M) \to 0$$

by noticing that the map $H^i(G, M) \xrightarrow{n} H^i(G, M)$ is 0. The map $s_{\mathbb{Z},n} : H^i(G, M) \to \mathrm{Ext}^i_G(Z/n, M)$ gives a splitting of these sequences.

Lemma 2.4. *Let X and Y be G-modules which are $\mathbb{Z}$-free, and let A and M be G-modules killed by an integer n. Let $s_{Y,n} : \mathrm{Ext}^i_G(Y, M) \to \mathrm{Ext}^i_G(Y/n, M)$ be the canonical map constructed in Definition* 2.3. *Let $r_{?,n} : ? \to ?/n$ be the reduction map for $? = X, Y$. Then, the following diagram commutes*:

$$\begin{array}{ccc} \mathrm{Ext}^j_G(X/n, A) \times \mathrm{Ext}^i_G(Y, M) & \xrightarrow{r^*_{X,n} \times s_{Y,n}} & \mathrm{Ext}^j_G(X, A) \times \mathrm{Ext}^i_G(Y/n, M) \\ \cup \big\downarrow & & \cup \big\downarrow \\ \mathrm{Ext}^{i+j}_G(X/n \otimes Y, A \otimes M) & \xrightarrow{\cong} & \mathrm{Ext}^{i+j}_G(X \otimes Y/n, A \otimes M) \end{array}$$

where the isomorphism in the bottom row is induced by the canonical isomorphism $X \otimes Y/n \cong X/n \otimes Y$.

Proof. This is an immediate consequence of the fact that $s_{Y,n}$ is a section of $r^*_{Y,n}$ and of Lemma 2.2 with the maps ϕ and ψ being the reduction maps $r_{X,n}$ and $r_{Y,n}$. □

We now describe another map, closely linked with $s_{X,n}$, which will come up again in the next section.

Definition 2.5. A *balanced sequence* $\mathcal{E}$ is a short exact sequence of chain complexes,

$$0 \to E^\bullet(\mathcal{E}) \xrightarrow{i} F^\bullet(\mathcal{E}) \xrightarrow{p} G^\bullet(\mathcal{E}) \to 0,$$

together with a map of complexes $\varepsilon : G^\bullet(\mathcal{E}) \to F^\bullet(\mathcal{E})$ with $p\varepsilon = 0$.

Definition 2.6. Given a balanced sequence

$$\mathcal{E} = \left(0 \to E^\bullet \xrightarrow{i} F^\bullet \xrightarrow{p} G^\bullet \to 0,\ \varepsilon\right),$$

we let $u_{\mathcal{E}} : H(G^\bullet) \to H(E^\bullet)$ be the map on homology induced from the unique map of complexes $\bar{u} : G^\bullet \to E^\bullet$ satisfying $i\bar{u} = \varepsilon$.

One clearly has $i_* \circ u_{\mathcal{E}} = \varepsilon_*$.

As mentioned before, there is a connection between the maps s and u defined in 2.3 and 2.6 which we now explain. Let $\mathcal{A}$ be an abelian category with enough projectives, n an integer and $M \in \mathcal{A}$ an n-torsion object. Suppose we have an exact sequence

$$0 \to X \xrightarrow{\tilde{i}} Y \xrightarrow{\tilde{p}} Z \to 0$$

and that we have a map $\tilde{\varepsilon} : Y \to X$ such that $\tilde{\varepsilon}\tilde{i}$ is multiplication by n on X (remark: this is all we'll need for this work, but the same considerations will apply in the more general case where $\tilde{\varepsilon}\tilde{i}$ is divisible by n in $\mathrm{End}(X)$). We then get a map

$$\tilde{u} : Z \to X/n \text{ defined by } \tilde{u}(z) = \tilde{\varepsilon}(y) \pmod{nX},\ \tilde{p}(y) = z. \tag{2.1}$$

Lemma 2.7. *In the situation described above:*

1. *There are projective resolutions $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ of X, Y and Z respectively, and liftings $\tilde{i}^\bullet$, $\tilde{p}^\bullet$ and $\tilde{\varepsilon}^\bullet : Y^\bullet \to X^\bullet$ to the maps $\tilde{i}$, $\tilde{p}$ and $\tilde{\varepsilon}$ respectively, such that:*
   (a) *there is an exact sequence of complexes*
   $$0 \to X^\bullet \xrightarrow{\tilde{i}^\bullet} Y^\bullet \xrightarrow{\tilde{p}^\bullet} Z^\bullet \to 0,$$
   (b) *the map $\tilde{\varepsilon}^\bullet\tilde{i}^\bullet$ is multiplication by n on $X^\bullet$*
2. *If, in addition, $X^\bullet/n$ is a resolution of X/n, then the map $u_{\mathcal{E}}$ associated with the balanced sequence $\mathcal{E}$ composed of*

$$0 \to \mathrm{Hom}(Z^\bullet, M) \xrightarrow{i = \circ\tilde{p}^\bullet} \mathrm{Hom}(Y^\bullet, M) \xrightarrow{p = \circ\tilde{i}^\bullet} \mathrm{Hom}(X^\bullet, M) \to 0 \tag{2.2}$$

together with the map $\varepsilon = \circ\tilde{\varepsilon}^\bullet$, is the composition

$$\mathrm{Ext}^\bullet(X, M) \xrightarrow{s_{X,n}} \mathrm{Ext}^\bullet(X/n, M) \xrightarrow{\tilde{u}^*} \mathrm{Ext}^\bullet(Z, M). \tag{2.3}$$

Proof. Most of part 1, except for the existence of $\tilde{\varepsilon}^\bullet$, is of course standard homological algebra. As usual, it is enough to construct the 0 terms and the maps between them: we choose surjections $X^0 \xrightarrow{e_X} X$ and $Z^0 \xrightarrow{e_Z} Z$ and let $Y^0 = X^0 \oplus Z^0$ with $\tilde{i}^0$ and $\tilde{p}^0$ the obvious maps. We take $e_Y : Y^0 \to Y$ such that the diagrams

$$\begin{array}{ccc} X^0 & & \\ {\scriptstyle e_X}\downarrow & \searrow^{e_Y|_{X^0}} & \\ X & \xrightarrow{\tilde{i}} & Y, \end{array} \qquad\qquad \begin{array}{ccc} & & Z^0 \\ & {}^{e_Y|_{Z^0}}\swarrow & \downarrow{\scriptstyle e_Z} \\ Y & \xrightarrow{\tilde{p}} & Z \end{array}$$

commute (the map $e_Y|_{Z^0}$ exists since Z^0 is projective). Finally, one takes $\tilde{\varepsilon}^0|_{X^0}$ to be multiplication by n and $\tilde{\varepsilon}^0|_{Z^0}$ to make

$$\begin{array}{ccc} X^0 & \xleftarrow{\tilde{\varepsilon}^0|_{Z^0}} & Z^0 \\ {\scriptstyle e_X}\downarrow & & \downarrow{\scriptstyle e_Y|_{Z^0}} \\ X & \xleftarrow{\tilde{\varepsilon}} & Y \end{array}$$

commute. For part 2, one notices that the map in (2.3) corresponds to the pullback under a map of resolutions $Z^\bullet \to X^\bullet/n$ and that there is an obvious choice of such a map (analogous to (2.1)) for which the resulting pullback is exactly $u_{\mathcal{E}}$. □

3. A technical lemma

The purpose of this section is to prove Lemma 3.1. The proof is done using a rather nasty diagram chase. It would be nice to find a simpler and more conceptual proof for this lemma (I am told by the referee that the first part may be proved by an argument using spectral sequences).

To make it easier to follow the computation, we will use a different notational convention then the one used in section 2:

- Chain complexes are denoted by bold capital letters;
- The differential of all complexes is denoted d and is raising degrees. It is assumed to commute with all the other maps whenever that makes sense;
- If $\mathbf{C}$ denotes a complex, $\mathbf{C}[h]$ denotes the same complex with degrees shifted by h;
- Elements in a chain complex are denoted by bold letters matching the complex from which they are taken, e.g., $\mathbf{b}^\mathbf{x}$, $\mathbf{b}_0^\mathbf{x} \in \mathbf{B}^\mathbf{X}$;
- The kernel of d on the complex $\mathbf{C}$ is denoted by $Z(\mathbf{C})$ and the homology of the complex by $H(\mathbf{C})$;
- The homology class of $\mathbf{c} \in \mathbf{C}$ is denoted by $[\mathbf{c}]$;
- Maps between complexes are always denoted by either i for injection or p for projection, with an upper indexing.

Lemma 3.1. *Suppose we are given the following commuting diagram of chain complexes, with exact rows and columns:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{B}^{\mathbf{Z}} & \xrightarrow{i^z} & \mathbf{C}^{\mathbf{Z}} & \xrightarrow{p^z} & \mathbf{M}^{\mathbf{Z}} & \longrightarrow & 0 \\
 & & i^b\downarrow & & i^c\downarrow & & i^m\downarrow & & \\
0 & \longrightarrow & \mathbf{B}^{\mathbf{Y}} & \xrightarrow{i^y} & \mathbf{C}^{\mathbf{Y}} & \xrightarrow{p^y} & \mathbf{M}^{\mathbf{Y}} & \longrightarrow & 0 \\
 & & p^b\downarrow & & p^c\downarrow & & p^m\downarrow & & \\
0 & \longrightarrow & \mathbf{B}^{\mathbf{X}} & \xrightarrow{i^x} & \mathbf{C}^{\mathbf{X}} & \xrightarrow{p^x} & \mathbf{M}^{\mathbf{X}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{3.1}
$$

which gives rise to a doubly long exact sequence of homology groups

$$
\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots & H\left(\mathbf{M}^{\mathbf{Y}}[-2]\right) & \xrightarrow{\delta^y} & H\left(\mathbf{B}^{\mathbf{Y}}[-1]\right) & \xrightarrow{i^y_*} & H\left(\mathbf{C}^{\mathbf{Y}}[-1]\right) & \xrightarrow{p^y_*} & H\left(\mathbf{M}^{\mathbf{Y}}[-1]\right) & \xrightarrow{\delta^y} & H\left(\mathbf{B}^{\mathbf{Y}}\right) & \cdots \\
 & p^m_*\downarrow & & p^b_*\downarrow & & p^c_*\downarrow & & p^m_*\downarrow & & p^b_*\downarrow & \\
\cdots & H\left(\mathbf{M}^{\mathbf{X}}[-2]\right) & \xrightarrow{\delta^x} & H\left(\mathbf{B}^{\mathbf{X}}[-1]\right) & \xrightarrow{i^x_*} & H\left(\mathbf{C}^{\mathbf{X}}[-1]\right) & \xrightarrow{p^x_*} & H\left(\mathbf{M}^{\mathbf{X}}[-1]\right) & \xrightarrow{\delta^x} & H\left(\mathbf{B}^{\mathbf{X}}\right) & \cdots \\
 & \delta^m\downarrow & & \delta^b\downarrow & & \delta^c\downarrow & & \delta^m\downarrow & & \delta^b\downarrow & \\
\cdots & H\left(\mathbf{M}^{\mathbf{Z}}[-1]\right) & \xrightarrow{\delta^z} & H\left(\mathbf{B}^{\mathbf{Z}}\right) & \xrightarrow{i^z_*} & H\left(\mathbf{C}^{\mathbf{Z}}\right) & \xrightarrow{p^z_*} & H\left(\mathbf{M}^{\mathbf{Z}}\right) & \xrightarrow{\delta^z} & H\left(\mathbf{B}^{\mathbf{Z}}[+1]\right) & \cdots \\
 & i^m_*\downarrow & & i^b_*\downarrow & & i^c_*\downarrow & & i^m_*\downarrow & & i^b_*\downarrow & \\
\cdots & H\left(\mathbf{M}^{\mathbf{Y}}[-1]\right) & \xrightarrow{\delta^y} & H\left(\mathbf{B}^{\mathbf{Y}}\right) & \xrightarrow{i^y_*} & H\left(\mathbf{C}^{\mathbf{Y}}\right) & \xrightarrow{p^y_*} & H\left(\mathbf{M}^{\mathbf{Y}}\right) & \xrightarrow{\delta^y} & H\left(\mathbf{B}^{\mathbf{Y}}[+1]\right) & \cdots \\
 & p^m_*\downarrow & & p^b_*\downarrow & & p^c_*\downarrow & & p^m_*\downarrow & & p^b_*\downarrow & \\
\cdots & H\left(\mathbf{M}^{\mathbf{X}}[-1]\right) & \xrightarrow{\delta^x} & H\left(\mathbf{B}^{\mathbf{X}}\right) & \xrightarrow{i^x_*} & H\left(\mathbf{C}^{\mathbf{X}}\right) & \xrightarrow{p^x_*} & H\left(\mathbf{M}^{\mathbf{X}}\right) & \xrightarrow{\delta^x} & H\left(\mathbf{B}^{\mathbf{X}}[+1]\right) & \cdots \\
 & \delta^m\downarrow & & \delta^b\downarrow & & \delta^c\downarrow & & \delta^m\downarrow & & \delta^b\downarrow & \\
\cdots & H\left(\mathbf{M}^{\mathbf{Z}}\right) & \xrightarrow{\delta^z} & H\left(\mathbf{B}^{\mathbf{Z}}[+1]\right) & \xrightarrow{i^z_*} & H\left(\mathbf{C}^{\mathbf{Z}}[+1]\right) & \xrightarrow{p^z_*} & H\left(\mathbf{M}^{\mathbf{Z}}[+1]\right) & \xrightarrow{\delta^z} & H\left(\mathbf{B}^{\mathbf{Z}}[+2]\right) & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

1. *Assume that we have* $[\mathbf{m^z}] \in H\left(\mathbf{M^Z}\right)$ *and* $[\mathbf{c^y}] \in H\left(\mathbf{C^Y}\right)$ *such that*

$$i^m_*[\mathbf{m^z}] = p^y_*[\mathbf{c^y}] \in H\left(\mathbf{M^Y}\right).$$

Then there exist $[\mathbf{b^x}] \in H\left(\mathbf{B^X}\right)$ *which satisfies the two equalities:*
(a) $i^x_*[\mathbf{b^x}] = p^c_*[\mathbf{c^y}] \in H\left(\mathbf{C^X}\right)$*; and*
(b) $\delta^b[\mathbf{b^x}] = -\delta^z[\mathbf{m^z}]$.

2. *Assume, in addition, that there are maps of complexes*

$$\varepsilon^b : \mathbf{B^X} \to \mathbf{B^Y},\ \varepsilon^c : \mathbf{C^X} \to \mathbf{C^Y},\ \varepsilon^m : \mathbf{M^X} \to \mathbf{M^Y},$$

commuting with the horizontal maps between the complexes in the obvious way, that there is an endomorphism π *of all the complexes, commuting with all the other maps, and such that*

$$p^b \circ \varepsilon^b = \pi|_{\mathbf{B^X}},\ p^c \circ \varepsilon^c = \pi|_{\mathbf{C^X}},\ p^m \circ \varepsilon^m = \pi|_{\mathbf{M^X}},$$

and

$$\pi|_{\mathbf{M}^{\mathbf{X}}} = 0.$$

Finally, assume that there exist some $[\mathbf{c}_0^{\mathbf{x}}] \in H\left(\mathbf{C}^{\mathbf{X}}\right)$ *with* $[\mathbf{c}^{\mathbf{y}}] = \varepsilon_*^c [\mathbf{c}_0^{\mathbf{x}}]$. *Let* $u_M : H\left(\mathbf{M}^{\mathbf{X}}\right) \to H\left(\mathbf{M}^{\mathbf{Z}}\right)$ *be the map* $u_{\mathcal{E}}$ *corresponding to the balanced sequence* $\mathcal{E}$ *formed of the rightmost column of the diagram* (3.1) *and the map* ε^m. *Then, there exist* $[\mathbf{b}_0^{\mathbf{y}}] \in H\left(\mathbf{B}^{\mathbf{Y}}\right)$ *and* $[\mathbf{m}_0^{\mathbf{x}}] \in H\left(\mathbf{M}^{\mathbf{X}}[-1]\right)$ *such that*

(a) $i^y [\mathbf{b}_0^{\mathbf{y}}] = \pi_* [\mathbf{c}^{\mathbf{y}}]$;
(b) $\delta^y \varepsilon_*^m [\mathbf{m}_0^{\mathbf{x}}] = \varepsilon_*^b [\mathbf{b}_0^{\mathbf{x}}] - [\mathbf{b}_0^{\mathbf{y}}]$; *and*
(c) $\delta^m [\mathbf{m}_0^{\mathbf{x}}] + u_M p_*^x [\mathbf{c}_0^{\mathbf{x}}] = [\mathbf{m}^{\mathbf{z}}]$.

Proof. 1. We begin by choosing representatives $\mathbf{m}^{\mathbf{z}} \in Z\left(\mathbf{M}^{\mathbf{Z}}\right)$ and $\mathbf{c}^{\mathbf{y}} \in Z\left(\mathbf{C}^{\mathbf{Y}}\right)$ for $[\mathbf{m}^{\mathbf{z}}]$ and $[\mathbf{c}^{\mathbf{y}}]$ respectively. The condition $i_*^m [\mathbf{m}^{\mathbf{z}}] = p_*^y [\mathbf{c}^{\mathbf{y}}]$ implies

$$i^m \mathbf{m}^{\mathbf{z}} = p^y \mathbf{c}^{\mathbf{y}} + d\mathbf{m}^{\mathbf{y}} \text{ for some } \mathbf{m}^{\mathbf{y}} \in \mathbf{M}^{\mathbf{Y}}[-1].$$

We can choose $\bar{\mathbf{c}}^{\mathbf{y}} \in \mathbf{C}^{\mathbf{Y}}[-1]$ s.t. $p^y \bar{\mathbf{c}}^{\mathbf{y}} = \mathbf{m}^{\mathbf{y}}$. Replacing $\mathbf{c}^{\mathbf{y}}$ by $\mathbf{c}^{\mathbf{y}} + d\bar{\mathbf{c}}^{\mathbf{y}}$, which still represents $[\mathbf{c}^{\mathbf{y}}]$, we may assume

$$i^m \mathbf{m}^{\mathbf{z}} = p^y \mathbf{c}^{\mathbf{y}}.$$

The class $p_*^c [\mathbf{c}^{\mathbf{y}}]$ is represented by $p^c \mathbf{c}^{\mathbf{y}} \in Z\left(\mathbf{C}^{\mathbf{X}}\right)$, and

$$p^x p^c \mathbf{c}^{\mathbf{y}} = p^m p^y \mathbf{c}^{\mathbf{y}} = p^m i^m \mathbf{m}^{\mathbf{z}} = 0 \Rightarrow p^c \mathbf{c}^{\mathbf{y}} = i^x \mathbf{b}^{\mathbf{x}} \text{ for some } \mathbf{b}^{\mathbf{x}} \in Z\left(\mathbf{B}^{\mathbf{X}}\right).$$

We choose $[\mathbf{b}^{\mathbf{x}}]$ to be the cohomology class represented by $\mathbf{b}^{\mathbf{x}}$. To find $\delta^b [\mathbf{b}^{\mathbf{x}}]$ we choose $\mathbf{b}^{\mathbf{y}} \in \mathbf{B}^{\mathbf{Y}}$ s.t. $p^b \mathbf{b}^{\mathbf{y}} = \mathbf{b}^{\mathbf{x}}$. Then

$$p^b d\mathbf{b}^{\mathbf{y}} = d\mathbf{b}^{\mathbf{x}} = 0 \Rightarrow d\mathbf{b}^{\mathbf{y}} = i^b \mathbf{b}^{\mathbf{z}} \text{ with } \mathbf{b}^{\mathbf{z}} \in Z\left(\mathbf{B}^{\mathbf{Z}}[1]\right),$$

and we have $\delta^b [\mathbf{b}^{\mathbf{x}}] = [\mathbf{b}^{\mathbf{z}}]$. We find that

$$p^c i^y \mathbf{b}^{\mathbf{y}} = i^x p^b \mathbf{b}^{\mathbf{y}} = i^x \mathbf{b}^{\mathbf{x}} = p^c \mathbf{c}^{\mathbf{y}} \Rightarrow p^c(\mathbf{c}^{\mathbf{y}} - i^y \mathbf{b}^{\mathbf{y}}) = 0.$$

Therefore,

$$\mathbf{c}^{\mathbf{y}} - i^y \mathbf{b}^{\mathbf{y}} = i^c \mathbf{c}^{\mathbf{z}} \text{ with } \mathbf{c}^{\mathbf{z}} \in \mathbf{C}^{\mathbf{Z}},$$

and

$$i^m p^z \mathbf{c}^{\mathbf{z}} = p^y i^c \mathbf{c}^{\mathbf{z}} = p^y(\mathbf{c}^{\mathbf{y}} - i^y \mathbf{b}^{\mathbf{y}}) = p^y \mathbf{c}^{\mathbf{y}} = i^m \mathbf{m}^{\mathbf{z}} \Rightarrow p^z \mathbf{c}^{\mathbf{z}} = \mathbf{m}^{\mathbf{z}}.$$

We may therefore use $\mathbf{c}^{\mathbf{z}}$ to compute $\delta^z [\mathbf{m}^{\mathbf{z}}]$: we have

$$d\mathbf{c}^{\mathbf{z}} = i^z \bar{\mathbf{b}}^{\mathbf{z}} \text{ with } \bar{\mathbf{b}}^{\mathbf{z}} \in Z\left(\mathbf{B}^{\mathbf{Z}}[1]\right),$$

and the class of $\bar{\mathbf{b}}^{\mathbf{z}}$ is

$$\left[\bar{\mathbf{b}}^{\mathbf{z}}\right] = \delta^z [\mathbf{m}^{\mathbf{z}}].$$

We are done if we show that $[\mathbf{b}^{\mathbf{z}}] = -\left[\bar{\mathbf{b}}^{\mathbf{z}}\right]$. In fact,

$$i^c i^z \bar{\mathbf{b}}^{\mathbf{z}} = i^c d\mathbf{c}^{\mathbf{z}} = d i^c \mathbf{c}^{\mathbf{z}} = d(\mathbf{c}^{\mathbf{y}} - i^y \mathbf{b}^{\mathbf{y}}) = -d i^y \mathbf{b}^{\mathbf{y}} = -i^y d\mathbf{b}^{\mathbf{y}} = -i^y i^b \mathbf{b}^{\mathbf{z}} = -i^c i^z \mathbf{b}^{\mathbf{z}}$$

and therefore $\mathbf{b}^{\mathbf{z}} = -\bar{\mathbf{b}}^{\mathbf{z}}$.

2. We have

$$p^y \pi \mathbf{c}^{\mathbf{y}} = \pi p^y \mathbf{c}^{\mathbf{y}} = 0 \Rightarrow \pi \mathbf{c}^{\mathbf{y}} = i^y \mathbf{b}_0^{\mathbf{y}} \text{ with } \mathbf{b}_0^{\mathbf{y}} \in Z\left(\mathbf{B}^{\mathbf{Y}}\right).$$

We let $[\mathbf{b}_0^{\mathbf{y}}]$ be the homology class of $\mathbf{b}_0^{\mathbf{y}}$. The first property is obviously satisfied. The assumption $[\mathbf{c}^{\mathbf{y}}] = \varepsilon_*^c [\mathbf{c}_0^{\mathbf{x}}]$ implies that

$$\mathbf{c}^{\mathbf{y}} = \varepsilon^c \mathbf{c}_0^{\mathbf{x}} + d\tilde{\mathbf{c}}^y \text{ with } \tilde{\mathbf{c}}^y \in \mathbf{C}^{\mathbf{Y}}[-1].$$

Therefore

$$i^x \mathbf{b}^{\mathbf{x}} = p^c \mathbf{c}^{\mathbf{y}} = p^c(\varepsilon^c \mathbf{c}_0^{\mathbf{x}} + d\tilde{\mathbf{c}}^y) = \pi \mathbf{c}_0^{\mathbf{x}} + dp^c \tilde{\mathbf{c}}^y,$$

and

$$i^y \mathbf{b}_0^{\mathbf{y}} = \pi \mathbf{c}^{\mathbf{y}} = \pi \varepsilon^c \mathbf{c}_0^{\mathbf{x}} + d\pi \tilde{\mathbf{c}}^y.$$

It follows that

$$\begin{aligned} i^y(\varepsilon^b \mathbf{b}^{\mathbf{x}} - \mathbf{b}_0^{\mathbf{y}}) = \varepsilon^c i^x \mathbf{b}^{\mathbf{x}} - i^y \mathbf{b}_0^{\mathbf{y}} &= (\varepsilon^c \pi \mathbf{c}_0^{\mathbf{x}} + d\varepsilon^c p^c \tilde{\mathbf{c}}^y) - (\pi \varepsilon^c \mathbf{c}_0^{\mathbf{x}} + d\pi \tilde{\mathbf{c}}^y) \\ &= d(\varepsilon^c p^c - \pi)\tilde{\mathbf{c}}^y \end{aligned}$$

and that therefore a preimage of $\varepsilon_*^b [\mathbf{b}^{\mathbf{x}}] - [\mathbf{b}_0^{\mathbf{y}}]$ is given by the class of

$$p^y(\varepsilon^c p^c - \pi)\tilde{\mathbf{c}}^y = p^y \varepsilon^c p^c \tilde{\mathbf{c}}^y = \varepsilon^m p^m p^y \tilde{\mathbf{c}}^y.$$

It is easy to check that $dp^m p^y \tilde{\mathbf{c}}^y = 0$. Therefore, we may take $\mathbf{m}_0^{\mathbf{x}} = p^m p^y \tilde{\mathbf{c}}^y$ to represent the cohomology class $[\mathbf{m}_0^{\mathbf{x}}]$, and the second property holds. Finally, we compute $\delta^m [\mathbf{m}_0^{\mathbf{x}}]$. It is represented by $\mathbf{m}_0^{\mathbf{z}} \in Z\left(\mathbf{M}^{\mathbf{Z}}\right)$, which satisfies

$$i^m \mathbf{m}_0^{\mathbf{z}} = dp^y \tilde{\mathbf{c}}^y = p^y d\tilde{\mathbf{c}}^y = p^y(\mathbf{c}^{\mathbf{y}} - \varepsilon^c \mathbf{c}_0^{\mathbf{x}}) = i^m \mathbf{m}^{\mathbf{z}} - \varepsilon^m p^x \mathbf{c}_0^{\mathbf{x}}.$$

Thus $i^m(\mathbf{m}^{\mathbf{z}} - \mathbf{m}_0^{\mathbf{z}}) = \varepsilon^m p^x \mathbf{c}_0^{\mathbf{x}}$, and so by the definition of u_M we have

$$[\mathbf{m}^{\mathbf{z}}] - [\mathbf{m}_0^{\mathbf{z}}] = u_M p_*^x [\mathbf{c}_0^{\mathbf{x}}].$$

□

4. Definition of the cup product

Let G be a group and H a normal subgroup with a finite index, so that we have the short exact sequence

$$1 \to H \to G \to \Delta \to 1, \tag{4.1}$$

with Δ a finite group. Let $\mathbb{Z}[\Delta]$ be the group ring of Δ, with the obvious (left) G action. The standard augmentation map, $\varepsilon : \mathbb{Z}[\Delta] \to \mathbb{Z}$, sits inside a short exact sequence,

$$0 \to I \xrightarrow{i} \mathbb{Z}[\Delta] \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \tag{4.2}$$

in which I is the augmentation ideal. The dual exact sequence is

$$0 \to \mathbb{Z} \xrightarrow{\mathrm{N}} \mathbb{Z}[\Delta] \xrightarrow{p} \check{I} \to 0. \tag{4.3}$$

Here, N is the norm (or trace) map that sends $1 \in \mathbb{Z}$ to $\sum_{g\in\Delta} g$, and $\check{I} = \mathbb{Z}[\Delta]/\mathbb{Z} \sum_{g\in\Delta} g$. The duality is given by the $\mathbb{Z}[\Delta]$ map $\check{\gamma} : \mathbb{Z}[\Delta] \otimes \mathbb{Z}[\Delta] \to \mathbb{Z}$ defined by the pairing

$$\langle \sum_{\sigma\in\Delta} n_\sigma \sigma, \sum_{\sigma\in\Delta} m_\sigma \sigma \rangle = \sum_{\sigma\in\Delta} m_\sigma n_\sigma, \tag{4.4}$$

under which $\mathbb{Z}\sum_{g\in\Delta} g$ and I are exact annihilators. We define $\gamma : \mathbb{Z} \to I \otimes \check{I}$ to be the dual map to the map induced from $\check{\gamma}$ on $\check{I} \otimes I$. A simple computation shows that γ is the unique map satisfying

$$\gamma(1) = \sum_{\sigma\in\Delta} (\sigma - 1) \otimes p(\sigma). \tag{4.5}$$

For future use we record the following two commutative diagrams with exact rows. The first is

(4.6)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & I\otimes \check{I} & \xrightarrow{i\otimes \mathrm{id}} & \mathbb{Z}[\Delta]\otimes \check{I} & \xrightarrow{\varepsilon\otimes \mathrm{id}} & \check{I} & \longrightarrow & 0\\
 & & \gamma\big\uparrow & & \beta\big\uparrow & & \big\| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{N} & \mathbb{Z}[\Delta] & \xrightarrow{p} & \check{I} & \longrightarrow & 0.
\end{array}$$

The first row is the sequence (4.2) tensored with $\check{I}$, the second is just (4.3), and the map β is defined by

$$\beta(\sigma)=\sigma\otimes p(\sigma).$$

In the second diagram the first row is (4.3) tensored with I while the second row is (4.2):

(4.7)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \xrightarrow{\mathrm{id}\otimes \mathrm{N}} & I\otimes \mathbb{Z}[\Delta] & \xrightarrow{\mathrm{id}\otimes p} & I\otimes \check{I} & \longrightarrow & 0\\
 & & \big\| & & \beta'\big\uparrow & & \gamma\big\uparrow & & \\
0 & \longrightarrow & I & \xrightarrow{i} & \mathbb{Z}[\Delta] & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Here

$$\beta'(\sigma)=\sigma\left(\sum_{\tau\in\Delta}(\tau-1)\otimes\tau\right).$$

Let M and A be G-modules.

Definition 4.1. The Kolyvagin cup product is the map

$$\cup_k:\mathrm{Ext}^j_G(I,A)\times\mathrm{Ext}^i_G(\check{I},M)\to H^{i+j}(G,M\otimes A)$$

which is the composition

$$\begin{aligned}\mathrm{Ext}^j_G(I,A)\times\mathrm{Ext}^i_G(\check{I},M)&\xrightarrow{\cup}\mathrm{Ext}^{i+j}_G(I\otimes\check{I},M\otimes A)\\ &\xrightarrow{\gamma^*}\mathrm{Ext}^{i+j}_G(\mathbb{Z},M\otimes A)\cong H^{i+j}(G,M\otimes A).\end{aligned}$$

We now give a more concrete interpretation for the domain of the Kolyvagin cup product.

Lemma 4.2. *In the situation described above we have the following long exact sequences*:

(4.8)
$$\cdots\to H^{i-1}(G,M)\xrightarrow{\delta_M^{(4.3)}}\mathrm{Ext}^i_G(\check{I},M)\xrightarrow{p^*}H^i(H,M)\xrightarrow{\mathrm{cor}_H^G}H^i(G,M)\to\cdots$$

Here we use the notation $\delta_M^{(4.3)}$ to denote the boundary map on the long exact sequence of $\mathrm{Ext}(\bullet,M)$ associated to the short exact sequence (4.3), and we use similar notation for other long exact sequences.

Proof. Since $\mathrm{Ext}^i_G(\mathbb{Z}[\Delta],M)\cong H^i(H,M)$, (4.8) is just the long exact sequence associated with the short exact sequence (4.3). □

Remark 4.3. If $H^{i-1}(G,M)=0$ (in particular if $i=0$), the sequence (4.8) implies

$$\operatorname{Ext}^i_G(\check{I},M)\cong \operatorname{Ker} H^i(H,M)\xrightarrow{\operatorname{cor}^G_H} H^i(G,M).$$

We will often assume that this is the situation. In case it is not, we may define a "Kolyvagin cup product" for

$$\psi\times m\in \operatorname{Ext}^j_G(I,A)\times\left(\operatorname{Ker} H^i(H,M)\xrightarrow{\operatorname{cor}^G_H} H^i(G,M)\right)$$

by using the following procedure: We choose a surjection $pr: M\twoheadrightarrow M'$ in such a way that pr_* kills $H^{i-1}(G,M)$ (for instance we can choose $M'=M/M^G$ if $i=1$). It is readily seen that for any $z\in\operatorname{Ext}^i_G(\check{I},M)$ mapping to m,

$$(pr\otimes\operatorname{id})_*(\psi \mathbin{\textcircled{\scriptsize k}} z)\in H^{i+j}(G,A\otimes M')$$

will only depend on ψ and m. All of the results which are stated here under the assumption $H^{i-1}(G,M)=0$ can also be applied in this more general case after the above modification.

Remark 4.4. With the exception of 7.3, we will always assume that H acts trivially on A. This allows us to identify $\operatorname{Hom}_G(I,A)$ with the group of cocycles of Δ with values in A.

5. Basic properties of the Kolyvagin cup product

In the situation (4.1), the well known action of Δ on $H^i(H,M)$ may be described as follows: If $\sigma\in\Delta$ and $m\in H^i(H,M)\cong\operatorname{Ext}^i_G(\mathbb{Z}[\Delta],M)$, then $\sigma(m)$ is the pullback of m under the automorphism of $\mathbb{Z}[\Delta]$ given by $\tau\to\tau\sigma$. With the assumption that $A=H^0(H,A)$, we also have a cup product $A\times H^i(H,M)\to H^i(H,A\otimes M)$ which may be described by sending $a\times m\in\operatorname{Ext}^0_G(\mathbb{Z}[\Delta],A)\times\operatorname{Ext}^i_G(\mathbb{Z}[\Delta],M)$ to the pullback of $a\cup m\in\operatorname{Ext}^i_G(\mathbb{Z}[\Delta]\otimes\mathbb{Z}[\Delta],A\otimes M)$ by the map $\mathbb{Z}[\Delta]\to\mathbb{Z}[\Delta]\otimes\mathbb{Z}[\Delta]$ which sends τ to $\tau\otimes\tau$.

Proposition 5.1. *Let $\psi:\Delta\to A$ be a one-cocycle, viewed as an element of $\operatorname{Ext}^0_G(I,A)$. Then the following diagram commutes:*

$$\begin{array}{ccc}
\operatorname{Ext}^i_G(\check{I},M) & \xrightarrow{\psi \mathbin{\textcircled{\scriptsize k}} \bullet} & H^i(G,A\otimes M)\\
{\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle \operatorname{res}^G_H}\\
H^i(H,M) & \xrightarrow{\textcircled{1}} & H^i(H,A\otimes M),
\end{array}$$

where the map $\textcircled{1}$ *is*

$$x\in H^i(H,M)\xrightarrow{\textcircled{1}}\sum_{\sigma\in\Delta}\psi(\sigma)\cup\sigma(x).$$

Proof. We start with $\psi\in\operatorname{Hom}_G(I,A)$ and $m\in\operatorname{Ext}^i_G(\check{I},M)$, and form

$$\psi\cup m\in\operatorname{Ext}^i_G(I\otimes\check{I},A\otimes M).$$

Going along the upper and right sides of the diagram,

$$\operatorname{res}^G_H(\psi \mathbin{\textcircled{\scriptsize k}} m)\in H^i(H,A\otimes M)\cong\operatorname{Ext}^i_G(\mathbb{Z}[\Delta],A\otimes M)$$

is the pullback of $\psi\cup m$ by the map

$$\mathbb{Z}[\Delta]\xrightarrow{\varepsilon}\mathbb{Z}\xrightarrow{\gamma} I\otimes\check{I}. \tag{5.1}$$

Going the other way, for each $\sigma \in \Delta$, $\psi(\sigma) \in H^0(H, A)$ is the pullback of ψ by $f_\sigma : \mathbb{Z}[\Delta] \to I$ defined by $f_\sigma(\tau) = \tau(\sigma - 1)$. Therefore, $\textcircled{1}(p^*(m))$ is the pullback of $\psi \cup m$ along the map $\mathbb{Z}[\Delta] \to \mathbb{Z}[\Delta] \otimes \mathbb{Z}[\Delta] \to I \otimes \mathbb{Z}[\Delta] \to I \otimes \check{I}$ given by

$$\tau \to \tau \otimes \tau \to \sum_{\sigma\in\Delta} \tau(\sigma - 1) \otimes \tau\sigma \to \sum_{\sigma\in\Delta} \tau(\sigma - 1) \otimes p(\tau\sigma). \tag{5.2}$$

To see that (5.1) and (5.2) are the same it is enough to check for $\tau = 1$. But this equality is clear. ☐

Remark 5.2. If $i = 0$, the restriction map above is just the injection $(A \otimes M)^G \hookrightarrow (A \otimes M)^H$. It follows that in this case the Kolyvagin cup product is the same as the operation (1.1).

Proposition 5.3. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Ext}^j_G(I, A) \times H^{i-1}(G, M) & \xrightarrow{\mathrm{id}\times\delta_M^{(4.3)}} & \mathrm{Ext}^j_G(I, A) \times \mathrm{Ext}^i_G(\check{I}, M) \\
\delta_A^{(4.2)}\times\mathrm{id}\downarrow & & \cup_K\downarrow \\
H^{j+1}(G, A) \times H^{i-1}(G, M) & \xrightarrow{\cup} & H^{i+j}(G, A \otimes M)
\end{array}$$

Proof. The commutativity will follow if we show the commutativity of the following two diagrams:

$$\begin{array}{ccc}
\mathrm{Ext}^j_G(I, A) \times H^{i-1}(G, M) & \xrightarrow{\cup} & \mathrm{Ext}^{i+j-1}(I, A \otimes M) \\
\delta_A^{(4.2)}\times\mathrm{id}\downarrow & & \delta_{A\otimes M}^{(4.2)}\downarrow \\
H^{j+1}(G, A) \times H^{i-1}(G, M) & \xrightarrow{\cup} & H^{i+j}(G, A \otimes M)
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Ext}^j_G(I, A) \times H^{i-1}(G, M) & \xrightarrow{\mathrm{id}\times\delta_M^{(4.3)}} & \mathrm{Ext}^j_G(I, A) \times \mathrm{Ext}^i_G(\check{I}, M) \\
\cup\downarrow & & \cup\downarrow \\
\mathrm{Ext}^{i+j-1}_G(I, A \otimes M) & \xrightarrow{\delta^{(4.7)}} & \mathrm{Ext}^{i+j}_G(\check{I} \otimes I, A \otimes M) \\
\Vert & & \gamma^*\downarrow \\
\mathrm{Ext}^{i+j-1}_G(I, A \otimes M) & \xrightarrow{\delta_{A\otimes M}^{(4.2)}} & H^{i+j}(G, A \otimes M)
\end{array}$$

since the composition of the two maps on the right hand side is $\cup_K$. The map $\delta^{(4.7)}$ is the boundary map derived from the first row of (4.7). The commutativity of the first diagram and the top square of the second diagram follows from a standard argument in homological algebra. The commutativity of the bottom square follows from the commutativity of (4.7). ☐

Let

$$0 \to B \xrightarrow{i'} C \xrightarrow{p'} M \to 0 \tag{5.3}$$

be a short exact sequence of G-modules. Let B' be the kernel of $A \otimes C \to A \otimes M$, so that there are a surjection $\mathrm{pr} : A \otimes B \twoheadrightarrow B'$ and a short exact sequence

$$0 \to B' \to A \otimes C \to A \otimes M \to 0, \tag{5.4}$$

and therefore a long exact sequence with a boundary map

$$H^{i+j}(G, A \otimes M) \xrightarrow{\delta^{(5.4)}} H^{i+j+1}(G, B'). \tag{5.5}$$

Let $m \in \operatorname{Ker} H^i(H, M) \xrightarrow{\operatorname{cor}_H^G} H^i(G, M)$ and let $z \in \operatorname{Ext}_G^i(\check{I}, M)$ satisfy $p^*(z) = m$. Assume there is $c \in H^i(H, C)$ s.t. $p'_*(c) = m$. It follows that there is $b \in H^i(G, B)$ s.t. $i'_*(b) = \operatorname{cor}_H^G(c)$.

Theorem 5.4. *In the situation described above, we may choose the class $b \in H^i(G, B)$ such that the following holds: For any $\psi \in \operatorname{Ext}_G^j(I, A)$,*

$$\delta^{(5.4)}(\psi \mathbin{\cup\!\!\!|}\, z) = -\operatorname{pr}_*(\delta_A^{(4.2)}(\psi) \cup b).$$

Proof. Choose projective resolutions $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ of $\mathbb{Z}$, $\mathbb{Z}[\Delta]$ and $\check{I}$ respectively in such a way that we may lift the short exact sequence (4.3) to a short exact sequence of complexes

$$0 \to X^\bullet \xrightarrow{i} Y^\bullet \xrightarrow{p} Z^\bullet \to 0.$$

If we apply part 1 of Lemma 3.1 to the doubly short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_G(Z^\bullet, B) & \xrightarrow{i'\circ} & \operatorname{Hom}_G(Z^\bullet, C) & \xrightarrow{p'\circ} & \operatorname{Hom}_G(Z^\bullet, M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \circ p} & & \downarrow{\scriptstyle \circ p} & & \downarrow{\scriptstyle \circ p} & & \\
0 & \longrightarrow & \operatorname{Hom}_G(Y^\bullet, B) & \xrightarrow{i'\circ} & \operatorname{Hom}_G(Y^\bullet, C) & \xrightarrow{p'\circ} & \operatorname{Hom}_G(Y^\bullet, M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \circ i} & & \downarrow{\scriptstyle \circ i} & & \downarrow{\scriptstyle \circ i} & & \\
0 & \longrightarrow & \operatorname{Hom}_G(X^\bullet, B) & \xrightarrow{i'\circ} & \operatorname{Hom}_G(X^\bullet, C) & \xrightarrow{p'\circ} & \operatorname{Hom}_G(X^\bullet, M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we find that we may choose $b \in H^i(G, B) = H(\operatorname{Hom}_G(X^\bullet, B))$ in such a way that

$$\delta_B^{(4.3)} b = -\delta^{(5.3)} z \text{ in } \operatorname{Ext}_G^{i+1}(\check{I}, B).$$

Therefore,

$$-\psi \mathbin{\cup\!\!\!|}\, \delta^{(5.3)} z = \psi \mathbin{\cup\!\!\!|}\, \delta_B^{(4.3)} b = \delta_A^{(4.2)}(\psi) \cup b,$$

where the second equality follows from proposition 5.3. A general argument about cup products gives

$$\delta^{(5.4)}(\psi \mathbin{\cup\!\!\!|}\, z) = \operatorname{pr}_*(\psi \mathbin{\cup\!\!\!|}\, \delta^{(5.3)} z),$$

and the result follows. □

The next result is an analogue of the known formula for the composition of corestriction and restriction: $\operatorname{cor}_H^G \circ \operatorname{res}_H^G(x) = |G/H| x$. Let $n = |\Delta|$ and assume that n annihilates both M and A. Let $\psi : \Delta \to A$ be a 1-cocycle. Since $nA = 0$, the Kolyvagin trick shows that the element

$$a(\psi) := \sum_{\sigma \in \Delta} \psi(\sigma)$$

is G-invariant. Recall that we have a map $p^* : \mathrm{Ext}^i_G(\check{I}, M) \to H^i(H, M)$ induced from the map $p : \mathbb{Z}[\Delta] \to \check{I}$ of (4.3).

Proposition 5.5. *In the above situation, there is a map*

$$u_{M,G} : H^i(G, M) \to \mathrm{Ext}^i_G(\check{I}, M)$$

satisfying

1. $p^* \circ u_{M,G} = \mathrm{res}^G_H$ *as maps from* $H^i(G, M)$ *to* $H^i(H, M)$;
2. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Hom}_G(I, A) \times H^i(G, M) & \xrightarrow{\mathrm{id} \times u_{M,G}} & \mathrm{Hom}_G(I, A) \times \mathrm{Ext}^i_G(\check{I}, M) \\
{\scriptstyle a \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \cup} \\
H^0(G, A) \times H^i(G, M) & \xrightarrow{\cup} & H^i(G, A \otimes M).
\end{array}$$

Proof. Let $u_{M,G}$ be the map $u_{\mathcal{E}}$ with the balanced sequence $\mathcal{E}$ associated to the short exact sequence (4.3) and the usual augmentation map as described in the second part of Lemma 2.7. For this short exact sequence we denote the map $\tilde{u}$ of (2.1) by $\bar{\varepsilon}$. The first assertion is then clear from the remark at the end of Definition 2.6. Since A is killed by n, the reduction map $r_{I,n} : I \to I/n$ induces an isomorphism $r^*_{I,n} : \mathrm{Hom}_G(I/n, A) \to \mathrm{Hom}_G(I, A)$. If ϕ is in $\mathrm{Hom}_G(I/n, A)$, it is easy to see that $a(r_{I,n}(\phi)) \in H^0(G, A)$ is the pullback of ϕ under the map

$$w : \mathbb{Z} \to I/n \text{ defined by } w(1) = \sum_{\sigma \in \Delta} (\sigma - 1)$$

(notice that this element is invariant in I/n). According to Lemma 2.7, $u_{M,G} = \bar{\varepsilon}^* \circ s_{\mathbb{Z},n}$. We may therefore reformulate the proposition as saying that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_G(I/n, A) \times \mathrm{Ext}^i_G(\mathbb{Z}, M) & \xrightarrow{r^*_{I,n} \times s_{\mathbb{Z},n}} & \mathrm{Hom}_G(I, A) \times \mathrm{Ext}^i_G(\mathbb{Z}/n, M) \\
{\scriptstyle w^* \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \bar{\varepsilon}^*} \\
\mathrm{Hom}_G(\mathbb{Z}, A) \times \mathrm{Ext}^i_G(\mathbb{Z}, M) & & \mathrm{Hom}_G(I, A) \times \mathrm{Ext}^i_G(\check{I}, M) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathrm{Ext}^i_G(\mathbb{Z} \otimes \mathbb{Z}, A \otimes M) & & \mathrm{Ext}^i_G(I \otimes \check{I}, A \otimes M) \\
{\scriptstyle \cong} \downarrow & \swarrow {\scriptstyle \gamma^*} & \\
\mathrm{Ext}^i_G(\mathbb{Z}, A \otimes M) & &
\end{array}$$

commutes. In view of Lemma 2.4, both ways of getting to $\mathrm{Ext}^i_G(\mathbb{Z}, A \otimes M)$ involve pulling back the same element via the two maps

$$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{w \otimes \mathrm{id}} I/n \otimes \mathbb{Z} \text{ and } \mathbb{Z} \xrightarrow{\gamma} I \otimes \check{I} \xrightarrow{\mathrm{id} \otimes \bar{\varepsilon}} I \otimes \mathbb{Z}/n.$$

One checks directly that these maps are identical; hence the result. $\square$

6. Higher derivatives

In this section we define, under certain restrictions, a notion of a derivative of order k,

$$\frac{\partial^k}{\partial f} m \in H^i(G,M), \quad m \in \operatorname{Ker} \operatorname{cor}_H^G : H^i(H,M) \to H^i(G,M),$$

for the cohomology class m with respect to "a function f of order k on Δ" in such a way that the Kolyvagin cup product defined in section 4 may be identified with the derivative $\frac{\partial^1}{\partial \psi} m$ w.r.t. the cocycle ψ. Things become more complicated as more and more choices are involved. It is for this reason that we restrict to the following situation:

- G and H are the same as in section 4;
- R is a ring with a trivial G action;
- M is an $R[G]$-module;
- We fix $i \geq 0$ and assume that for any finite R-module L with a trivial G action, $\operatorname{Ext}^{i-1}_{R[G]}(L,M) = 0$ (in the most interesting case, in which $i = 1$, this condition amounts to $M^G = 0$).

We may replace $\mathbb{Z}$ by R in the setup of section 4. In particular, we redefine I (resp. $\check{I}$) to be the kernel (resp. the cokernel) of the obvious map $\varepsilon : R[\Delta] \to R$ (resp. $\mathrm{N} : R \to R[\Delta]$). Since

$$H^\bullet(G,M) = \operatorname{Ext}^\bullet_{R[G]}(R,M)$$

and

$$H^\bullet(H,M) = \operatorname{Ext}^\bullet_{R[H]}(R,M) \cong \operatorname{Ext}^\bullet_{R[G]}(R[\Delta],M),$$

the long exact sequence (4.8) is isomorphic to the sequence

$$\text{(6.1)} \quad \ldots \to H^{i-1}(G,M) \to \operatorname{Ext}^i_{R[G]}(\check{I},M) \to H^i(H,M) \xrightarrow{\operatorname{cor}_H^G} H^i(G,M) \to \ldots$$

We will need two different multiplications of $R[\Delta]$. If we think of $R[\Delta]$ as the set of functions $f : \Delta \to R$, the first is the convolution product, which is the one used to define the group ring structure on $R[\Delta]$, and will be denoted by $*$. The second is pointwise multiplication of functions. As before, we have a pairing

$$R[\Delta] \times R[\Delta] \to R, \ \langle f, g \rangle = \sum_{\sigma \in \Delta} f(\sigma) g(\sigma).$$

Definition 6.1. We define an increasing filtration

$$0 = F^{-\infty} = \cdots = F^{-1} \subset F^0 \subset \cdots \subset F^\infty := \bigcup_{r \in \mathbb{N}} F^r$$

on $R[\Delta]$ by setting, for $r \geq 1$, $F^{r-1}R[\Delta]$ to be the annihilator with respect to the above pairing of the ideal I^{*r} (I^{*r} is the r-th power, with respect to the usual product in the group ring, of the augmentation ideal).

Lemma 6.2. *With respect to the filtration above:*

1. *The function $f \in R[\Delta]$ is in $F^{r+1}R[\Delta]$ if and only if its image in $R[\Delta]/F^r$ is Δ-invariant.*
2. *With respect to the pointwise product, $F^r F^s \subset F^{r+s}$.*

Proof. (1) Suppose $f \in F^{r+1}$ and let $g \in I^{*r}$. Then

$$\langle \sigma(f) - f, g\rangle = \langle \sigma(f), g\rangle - \langle f, g\rangle = \langle f, \sigma^{-1}(g)\rangle - \langle f, g\rangle = \langle f, (\sigma^{-1} - 1) * g\rangle = 0,$$

the last equality because $(\sigma^{-1} - 1) * g \in I^{*r+1}$. Hence $\sigma(f) - f \in F^r$. Conversely, if $\sigma(f) - f \in F^r$, then the same equation shows that f annihilates every element of the form $(\sigma^{-1} - 1) * g$ with $g \in I^{*r}$. Since I^{*r+1} is generated by these elements, the result follows.

(2) The case $r = s = 0$ is easy, as F^0 is just the set of constant functions. Given $f \in F^r$ and $g \in F^s$, we find that

$$\sigma(fg) - fg = \sigma(f)\sigma(g) - fg = (\sigma(f) - f)\sigma(g) + (\sigma(g) - g)f.$$

The result therefore follows by induction on $r + s$ and part 1. □

To $f \in F^k$ we associate the left ideal $R[\Delta] * f$ filtered by

$$F^r(R[\Delta] * f) = F^r \cap R[\Delta] * f$$

with $r \le k$. For such an r we denote the inclusion $F^r(R[\Delta] * f) \subset R[\Delta]$ by $\mathrm{N}_{f,r}$. We then have a pullback map

$$H^i(H, M) \cong \mathrm{Ext}^i_{R[G]}(R[\Delta], M) \xrightarrow{\mathrm{N}^*_{f,r}} \mathrm{Ext}^i_{R[G]}(F^r(R[\Delta] * f), M).$$

We now define the higher derivatives. We first remark that by our assumptions it follows from part 1 of Lemma 6.2 that $\mathrm{Ext}^{i-1}_{R[G]}\left(F^k(R[\Delta] * f), M\right) = 0$ for all k.

Definition 6.3. Let $m \in H^i(H, M)$ and let $f \in F^k$. Assume that $\mathrm{N}^*_{f,k-1}(m) = 0$. By the last remark, $\mathrm{N}^*_{f,k}(m)$ pulls back uniquely to an element

$$z \in \mathrm{Ext}^{i-1}_{R[G]}\left(F^k(R[\Delta] * f)/F^{k-1}(R[\Delta] * f), M\right).$$

We define

$$\frac{\partial^k}{\partial f} m \in \mathrm{Ext}^i_{R[G]}(R, M) \cong H^i(G, M)$$

to be the pullback of z by the canonical G-map $R \to F^k(R[\Delta] * f)/F^{k-1}(R[\Delta] * f)$ sending 1 to $f \pmod{F^{k-1}(R[\Delta] * f)}$

We leave it to the reader to verify the following two propositions:

Proposition 6.4. *There is an inclusion $H^1(\Delta, R) \cong \mathrm{Hom}(\Delta, R) \hookrightarrow F^1$ which induces an isomorphism $F^1/F^0 \cong H^1(\Delta, R)$. If $z \in \mathrm{Ext}^i_G(\check{I}, M)$ is the unique pullback of $m \in H^i(G, M)$ and $\psi \in \mathrm{Hom}(\Delta, R)$, then $\frac{\partial^1}{\partial \psi} m$ is the push forward of $\psi \cup z \in H^i(G, R \otimes M)$ by the canonical map $R \otimes M \to M$.*

Proposition 6.5. *We have*

$$\mathrm{res}^G_H \frac{\partial^k}{\partial f} m = \sum_{\sigma \in \Delta} f(\sigma)\sigma(m),$$

where Δ acts on $H^i(H, M)$ in the usual way.

It is possible to check the conditions for defining the k-th derivative by verifying the vanishing of all lower order derivatives. To be precise, we have

Proposition 6.6. *If* $\mathrm{N}^*_{f,k-1}(m) = 0$*, then for any* $r < k$ *and any* $g \in F^r(R[\Delta] * f)$ *we have*

$$\frac{\partial^r}{\partial g} m = 0.$$

Conversely, let, for $r = 0, \ldots, k-1$*,*

$$f_{r,s} \in F^r(R[\Delta] * f), \quad s = 1, \ldots, n(r),$$

be functions whose images generate $F^r(R[\Delta] * f)/F^{k-1}(R[\Delta] * f)$*. If* $\frac{\partial^r}{\partial f_{r,s}} m = 0$ *for all* r *and* s*, then* $\mathrm{N}^*_{f,k-1}(m) = 0$*.*

We now discuss a few special cases:

1. Let $f_1, \ldots, f_k \in \mathrm{Hom}(\Delta, R)$. The R-module generated by all the monomials in $f_1, \ldots, f_k$ of degree $\leq k$ contains $R[\Delta] * f_1 \cdot f_2 \cdots f_k$, and the r-th step in its filtration is generated by all the monomials of degree $\leq r$. Thus, we may define inductively the derivative $\frac{\partial^k}{\partial f_1 \cdot f_2 \cdots f_k} m$ if all the lower order derivatives $\frac{\partial^r}{\partial f_{i_1} \cdots f_{i_r}} m$ vanish for $r < k$.

2. We have the following theorem:

Theorem 6.7. *Assume as usual that* $\Delta = G/H$*. Assume in addition that*

- $\Delta = \Delta_1 \times \cdots \Delta_k$ *with projections* $\pi_r : \Delta \to \Delta_r$ *and* $\Pi_r : \Delta \to \tilde{\Delta}_r := \Delta_1 \times \cdots \times \hat{\Delta}_r \times \cdots \times \Delta_k$*. We let* G_r *be the preimage of* Δ_r *under the projection* $G \to \Delta$*.*
- $m \in H^i(H, M)$ *is killed by all the partial corestriction, i.e.,* $\mathrm{cor}^H_{G_r} m = 0$*.*
- *The* $\Delta_r = \langle \sigma_r \rangle$*, for* $r = 1, \ldots, k$*, are all cyclic with order divisible by some power* p^n *and with a preferred generator* σ_r*.*
- *The module* M *is killed by* p^n*.*

In this situation, let $D = D_1 \cdots D_k \in \mathbb{Z}[\Delta]$*, where* $D_r = \sum l\sigma_r^l$ *is the Kolyvagin operator that was described in the introduction. Then,* $Dm \in H^i(H, M)^{\Delta}$ *is in the image of the restriction map* $\mathrm{res}^G_H : H^i(G, M) \to H^i(H, M)^{\Delta}$*.*

Proof. More generally, let M be any R module, and let $g_r \in \mathrm{Hom}(\Delta_r, R)$ and $f_r = g_r \circ \pi_r$ for $r = 1, \ldots, k$. The conditions of the theorem assure us that all the derivatives with respect to a proper subset of $\{f_1, \ldots, f_k\}$ vanish. Indeed, the partial corestrictions $\mathrm{cor}^H_{G_r} m$ may be described as pulling $m \in \mathrm{Ext}^i_{R[G]}(R[\Delta], M)$ by the map $\Pi_r^* : R[\tilde{\Delta}_r] \to R[\Delta]$, and the inclusion of the R-module generated by $f_1, \ldots, \hat{f}_s, \ldots, f_k$ in $R[\Delta]$ may be factored through Π_s. Thus, the derivative $\frac{\partial^k}{\partial f_1 \cdots f_k} m$ exists by Proposition 6.6. We now restrict to the case $R = \mathbb{Z}/p^n$, define g_r by $g_r(\sigma_r) = 1$, and conclude easily by Proposition 6.5 □

7. The Kolyvagin cap product

In this section we construct a new operation which is dual to the Kolyvagin cup product we have constructed in section 4. The setup is the same as that of section 4.

Definition 7.1. The *Kolyvagin cap product* is the cup product

$$\mathrm{Ext}^j_G(I, A) \times H^i(G, M) \to \mathrm{Ext}^{i+j}_G(I, A \otimes M).$$

We denote this product by $\psi \times x \to \psi \Cap x$.

As usual, we'll be mostly interested in the case $j = 0$, where ψ is just a one-cocycle with values in A.

The short exact sequence (4.2) gives rise to

$$
\begin{aligned}
(7.1)\quad 0 \to \operatorname{Coker}\left[H^i(G, A\otimes M) \xrightarrow{\operatorname{res}_H^G} H^i(H, A\otimes M)\right] &\xrightarrow{i^*} \operatorname{Ext}^i_G(I, A\otimes M)\\
\xrightarrow{\delta^{(4.2)}_{A\otimes M}} \operatorname{Ker}\left[H^{i+1}(G, A\otimes M) \xrightarrow{\operatorname{res}_H^G} H^{i+1}(H, A\otimes M)\right] &\to 0.
\end{aligned}
$$

We now show that the operation just defined coincides with the one defined by Rubin in [Rub93, §3] when $i = 0$ as follows: Let $x \in M^G = H^0(G, M)$ and let $\psi \in \operatorname{Ext}^0_G(I, A)$ be a one-cocycle $\psi : \Delta \to A$. Assume there is $y \in M^H$ satisfying $\mathrm{N}_{G/H}\, y = x$. One can now form

$$\psi \Cap x = \sum_{\sigma\in\Delta} \psi(\sigma)\otimes\sigma(y) \pmod{(A\otimes M)^G} \in (A\otimes M)^H/(A\otimes M)^G.$$

If $\mathrm{N}_{G/H}\, y = 0$, then the basic Kolyvagin trick shows that $\sum_{\sigma\in\Delta}\psi(\sigma)\otimes\sigma(y)$ is G-invariant; hence the operation is well defined.

The next proposition shows that the above operation, when defined, is given by the Kolyvagin cap product.

Proposition 7.2. *The following diagram commutes:*

$$
\begin{array}{ccc}
H^i(H, M) & \xrightarrow{\operatorname{cor}_H^G} & H^i(G, M)\\
\text{②}\downarrow & & \psi\Cap\bullet\downarrow\\
H^i(H, A\otimes M) & \xrightarrow{i^*} & \operatorname{Ext}^i_G(I, A\otimes M)
\end{array}
$$

where ② *is the map*

$$y \xrightarrow{\text{②}} -\sum_{\sigma\in\Delta}\psi(\sigma)\cup\sigma(y).$$

Proof. As in the proof of Proposition 5.1, we start with $m \in \operatorname{Ext}^i_G(\mathbb{Z}[\Delta], M)$ and form $\psi\otimes m \in \operatorname{Ext}^i_G(I\otimes\mathbb{Z}[\Delta], A\otimes M)$. Going along the upper and right sides of the diagram (resp. the left and lower sides) corresponds to pulling back $\psi\cup m$ by

$$
\begin{array}{ccccc}
I & \xrightarrow{\cong} & I\otimes\mathbb{Z} & \xrightarrow{\mathrm{id}\otimes\mathrm{N}} & I\otimes\mathbb{Z}[\Delta]\\
\tau-1 & \longrightarrow & (\tau-1)\otimes 1 & \longrightarrow & (\tau-1)\otimes(\sum_{\sigma\in\Delta}\sigma)
\end{array}
$$

(resp.

$$
\begin{array}{ccccc}
I & \xrightarrow{i} & \mathbb{Z}[\Delta] & \longrightarrow & I\otimes\mathbb{Z}[\Delta]\\
 & & \tau & \longrightarrow & -\sum_{\sigma\in\Delta}\tau(\sigma-1)\otimes\tau\sigma).
\end{array}
$$

It thus remains to observe that

$$(\tau-1)\otimes(\sum_{\sigma\in\Delta}\sigma) = -\sum_{\sigma\in\Delta}\tau(\sigma-1)\otimes\tau\sigma.$$

□

We now show that the Kolyvagin cap product is dual, in a sense we shall define soon, to the Kolyvagin cup product defined in section 4. To motivate what follows, recall from [Mil86, ch. 1] that Tate duality is defined by the pairing

$$\langle\ ,\ \rangle : H^k(G,M)\times H^i(G,\widehat{M}(1)) \xrightarrow{\cup} H^2(G,\mu_{p^\infty}) \xrightarrow{\mathrm{inv}_G} \mathbb{Q}_p/\mathbb{Z}_p. \tag{7.2}$$

Here, G is the Galois group of $\bar{K}/K$, where K is a nonarchimedean local field (we'll refer to this type of groups as local Galois groups), k and i are nonnegative integers summing to 2, μ_{p^∞} is the G-module of p-power roots of unity in K, $\widehat{M}(1) := \mathrm{Hom}(M,\mu_{p^\infty})$ is the Kummer dual of M and inv_G is the invariant map of local class field theory. When H is a normal group of finite index in G, it is also a local Galois group, so the same duality holds for H. It is known that the invariant commutes with corestriction, i.e., if $x\in H^2(H,\mu_{p^\infty})$, then

$$\mathrm{inv}_G(\mathrm{cor}_H^G x) = \mathrm{inv}_H(x).$$

Our next result shows that the Kolyvagin cup product may be considered as giving a new pairing which is closely connected to the pairing in the Tate duality.

Proposition 7.3. *Let A and M be G-modules (we are not assuming now that A has a trivial H action!). Let $p^* : \mathrm{Ext}^i_G(\check{I},M)\to H^i(H,M)$ and $i^* : H^k(H,A)\to \mathrm{Ext}^k_G(I,A)$ be the maps induced on cohomology from the natural maps p and i in (4.2) and (4.3). Then, p^* and i^* are "adjoint" with respect to the usual cup product and the Kolyvagin cup product, i.e., for $x\in \mathrm{Ext}^i_G(\check{I},M)$ and $y\in H^k(H,A)$ one has*

$$i^*(y)\cup_{\mathrm{K}} x = \mathrm{cor}_H^G(y\cup p^*(x)).$$

Proof. The result follows from the commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\mathrm{N}} & \mathbb{Z}[\Delta] & \xrightarrow{\textcircled{3}} & \mathbb{Z}[\Delta]\otimes\mathbb{Z}[\Delta] \\ {\scriptstyle\gamma}\downarrow & & & & \downarrow{\scriptstyle\mathrm{id}\otimes p} \\ I\otimes\check{I} & & \xrightarrow{i\otimes\mathrm{id}} & & \mathbb{Z}[\Delta]\otimes\check{I} \end{array}$$

where the map $\textcircled{3}$ sends $\sigma\in\mathbb{Z}[\Delta]$ to $\sigma\otimes\sigma$. □

Definition 7.4. Suppose that G is a local Galois group and that M is a G-module. Define the *Kolyvagin pairing*,

$$\langle\ ,\ \rangle_{\mathrm{K}} : \mathrm{Ext}^i_G(I,\widehat{M}(1))\times \mathrm{Ext}^k_G(\check{I},M)\to\mathbb{Q}_p/\mathbb{Z}_p,$$

with $i+k=2$, to be the composition of the Kolyvagin cup product with the invariant map.

Corollary 7.5. *If, in the situation of Proposition 7.3, G is a local Galois group, $i+k=2$ and $A=\widehat{M}(1)$, then one has*

$$\langle i^*(y),x\rangle_{\mathrm{K}} = \langle y,p^*(x)\rangle.$$

We can now state the adjointness of the Kolyvagin cup and cap products:

Proposition 7.6. *Let M, N and A be G-modules and let $\psi \in \mathrm{Ext}_G^j(I, A)$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
H^k(G,N) \times \mathrm{Ext}_G^i(\check{I}, M) & \xrightarrow{\mathrm{id} \times (\psi \cup_{\mathrm{K}} \bullet)} & H^k(G,N) \times H^{i+j}(G, A \otimes M) \\
\Big\downarrow{\scriptstyle (\psi \cap_{\mathrm{K}} \bullet) \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \cup} \\
\mathrm{Ext}_G^{k+j}(I, A \otimes N) \times \mathrm{Ext}_G^i(\check{I}, M) & \xrightarrow{\cup_{\mathrm{K}}} & H^{i+j+k}(G, A \otimes N \otimes M).
\end{array}
$$

Proof. Follows from the commutativity of

$$
\begin{array}{ccc}
\mathbb{Z} \otimes \check{I} \otimes I & \xleftarrow{\mathrm{id} \otimes \gamma} & \mathbb{Z} \otimes \mathbb{Z} \\
\Big\uparrow & & \Big\| \\
\check{I} \otimes I & \xleftarrow{\gamma} & \mathbb{Z}
\end{array}
$$

□

Theorem 7.7. *Let G be a local Galois group and let A and M be G-modules with a trivial action of H on A. Let i and k be two nonnegative integers with $i + k = 2$. Let ψ be a one-cocycle of Δ with values in A. Let $x \in H^k(G, \widehat{A \otimes M}(1))$ and $y \in \mathrm{Ext}_G^i(\check{I}, M)$. Then,*

1. *We have*
$$\langle x, \psi \cup_{\mathrm{K}} y \rangle = \langle \psi \cap_{\mathrm{K}} x, y \rangle_{\mathrm{K}},$$
where we used the identification $A \otimes (\widehat{A \otimes M})(1) \cong \widehat{M}(1)$ to view $\psi \cap_{\mathrm{K}} x$ as an element of $\mathrm{Ext}_G^i(I, \widehat{M}(1))$.
2. *If $\psi \cap_{\mathrm{K}} x = i^*(z)$ with $z \in H^i(H, \widehat{M}(1))$, then*
$$\langle x, \psi \cup_{\mathrm{K}} y \rangle = \langle z, p^*(y) \rangle.$$
3. *In particular, if $x = \mathrm{cor}_H^G(x')$, with $x' \in H^k(H, \widehat{A \otimes M}(1))$, then*
$$\langle x, \psi \cup_{\mathrm{K}} y \rangle = -\langle \sum_{\sigma \in \Delta} \psi(\sigma) \otimes \sigma(x'), p^*(y) \rangle.$$

Proof. All the results are immediate consequences of Propositions 7.6, 7.3 and 7.2. □

8. Localization of the Kolyvagin cup product

In this section, we describe a method which, under certain conditions, allows one to compute the localization of the Kolyvagin cup product of given cohomology classes. The result is inspired by the computations of Nekovář in [Nek92].

The setup is as follows:

- We have the short exact sequence of groups (4.1): $1 \to H \to G \to \Delta \to 1$ with Δ of order n;
- G_l (l for local) is a subgroup of G. We let $H_l = H \cap G_l$ and demand that $G_l/H_l \cong \Delta$;
- A and M are G-modules killed by n, and A has a trivial H action;
- M sits in a short exact sequence of G-modules $0 \to B \xrightarrow{i'} C \xrightarrow{p'} M \to 0$;

- We are given $i > 0$, and cohomology classes $m \in H^i(H, M)$, $c \in H^i(H, C)$ and $c_l \in H^i(G_l, C)$ s.t.
$$m = p'_* c \text{ and } \mathrm{res}^H_{H_l} c = \mathrm{res}^{G_l}_{H_l} c_l;$$
- We have a 1-cocycle $\psi : \Delta \to A$.

As in the setup of Theorem 5.4, we can choose $z \in \mathrm{Ext}^i_G(\check{I}, M)$ s.t. $p^*(z) = m$, and we can form the Kolyvagin cup product $\psi \Cup z \in H^i(G, A \otimes M)$. Our aim in this section is to compute $\mathrm{res}^G_{G_l}(\psi \Cup z)$

As in Theorem 5.4, we find a class $b \in H^i(G, B)$ with $i'_*(b) = \mathrm{cor}^G_H(c)$. In addition, since $p'_*(nc) = nm = 0$, we can also find $b' \in H^i(H, B)$ such that $i'_*(b') = nc$. It follows from the assumption $G/H \cong G_l/G_l$ that $\mathrm{res}^G_{G_l} \circ \mathrm{cor}^G_H = \mathrm{cor}^{G_l}_{H_l} \circ \mathrm{res}^H_{H_l}$, and therefore
$$\begin{aligned}\mathrm{res}^G_{H_l}\mathrm{cor}^G_H c &= \mathrm{res}^{G_l}_{H_l}\mathrm{res}^G_{G_l}\mathrm{cor}^G_H c = \mathrm{res}^{G_l}_{H_l}\mathrm{cor}^{G_l}_{H_l}\mathrm{res}^H_{H_l} c \\ &= \mathrm{res}^{G_l}_{H_l}\mathrm{cor}^{G_l}_{H_l}\mathrm{res}^{G_l}_{H_l} c_l = \mathrm{res}^{G_l}_{H_l} n c_l = n\mathrm{res}^H_{H_l} c.\end{aligned}$$
Let $b_1 = \mathrm{res}^G_H b - b' \in H^i(H, B)$. The above computation shows that
$$i'_*(\mathrm{res}^H_{H_l} b_1) = \mathrm{res}^H_{H_l}(\mathrm{res}^G_H \mathrm{cor}^G_H c - nc) = 0,$$
and so
$$\mathrm{res}^H_{H_l} b_1 = \delta^{(5.3)} m_1 \text{ for some } m_1 \in H^{i-1}(H_l, M).$$

Theorem 8.1. *There exists a choice of b and b' above such that the following holds: There exists $m_2 \in H^{i-1}(G_l, M)$ with*
$$m_1 = \mathrm{res}^{G_l}_{H_l} m_2 \text{ and } \mathrm{res}^G_{G_l}(\psi \Cup z) = a(\psi) \cup p'_*(c_l) + m_2 \cup [\psi],$$
where $a(\psi) = \sum_{\sigma \in \Delta} \psi(\sigma)$ and $[\psi]$ denotes as usual the cohomology class of ψ.

Proof. The condition $G/H \cong G_l/G_l$ shows that when viewing (4.3) as a sequence of G_l-modules it will give the long exact sequence (4.8) with G and H replaced by G_l and H_l. We take the projective resolutions $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ that were used in the proof of Theorem 5.4, and form the same doubly exact sequence of complexes as in that proof but replacing Hom_G by Hom_{G_l}. According to part 1 of Lemma 2.7, we can lift the augmentation map $\varepsilon : \mathbb{Z}[\Delta] \to \mathbb{Z}$ to a map of complexes
$$\varepsilon : Y^\bullet \to X^\bullet \text{ such that } \varepsilon \circ i = \text{ multiplication by } n.$$
It is now easy to check that we may apply part 2 of Lemma 3.1 with the map π being multiplication by n. The lemma gives the existence of m_2 such that
$$m_1 = \mathrm{res}^{G_l}_{H_l} m_2 \text{ and } z = \delta^{(4.3)}_M m_2 + u_{M,G_l} p'_* c_l,$$
with u_{M,G_l} being the map defined in Proposition 5.5. Propositions 5.3 and 5.5 now give the required result. □

To make a more explicit computation we now add further assumptions:

1. $a(\psi) = 0$ (this is for instance true if Δ or A has no 2-torsion and A has a trivial G action);
2. $\mathrm{res}^{G_l}_{H_l}$ is an isomorphism on $H^i(G_l, C)$ and $H^i(G_l, B)$;
3. There is an operator T, acting in a compatible way on B, C and M, $TM = 0$, and we have $x \in H^i(G, C)$ with $\mathrm{cor}^G_H c = Tx$;
4. There is an operator $\mathrm{Fr}(l)$, acting compatibly on the H_l cohomology with coefficients in C, B and M, and $\mathrm{res}^H_{H_l} c = \mathrm{Fr}(l)\mathrm{res}^G_{H_l} x$;

5. The cohomology groups $H^{i-1}(G,M)$ and $H^{i-1}(G_l,C)$ vanish.

Remark 8.2. The reader should have in mind the following situation: K/F is some finite extension of number fields $G=\mathrm{Gal}(\bar{\mathbb{Q}}/F)$, $H=\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. l is some prime of F which is totally ramified in K and G_l is the decomposition group of l (notice that $G/H\cong G_l/H_l$). C is a $\mathbb{Z}_p$-free G-module which has pure weight different from 0 and with good reduction at l. p is assumed not to lie below l. $B=p^kC$. $i=1$. Condition (2) is then satisfied because the weight assumptions imply, according to [Nek92, Lemma 4.1], that $H^1(G_l,C)\cong H^1(G_l/I_l,C)$, with I_l the inertia at l (one should probably be a little more careful and replace C with its quotient by a sufficiently large power of p), and the same is true with G_l replaced by H_l and C replaced by B, and because K_l/F_l is totally ramified. The conditions (3) and (4) are typical conditions for Euler systems of elliptic type (see for instance [Nek92, Proposition 6.1]).

Because of condition (2) we can define $\mathrm{Fr}(l)$ on $H^i(G_l,C)$ and $H^i(G_l,B)$. Denote by $\frac{T}{i'}:C\to B$ the composition $(i')^{-1}\circ T$, which is well defined since by assumptions T has values in the image of i'. Define $\frac{n}{i'}$ similarly. Since $H^{i-1}(G_l,C)=0$, the boundary map $\delta^{(5.3)}:H^{i-1}(G_l,M)\to H^i(G_l,B)$ is injective and we denote by δ^{-1} its inverse when defined. Define χ_ψ from a subspace of $H^i(G_l,C)$ to $H^i(G_l,A\otimes M)$ by

$$\chi_\psi(x_l)=\psi\cup\delta^{-1}\left(\left(\frac{T}{i'}\right)_*-\left(\frac{n}{i'}\right)_*\mathrm{Fr}(l)\right)x_l \text{ whenever it is defined,}$$

i.e., whenever $\left(\frac{T}{i'}\right)_* x_l-\left(\frac{n}{i'}\right)_*\mathrm{Fr}(l)x_l$ is in the image of $\delta^{(5.3)}$. As usual, we can choose $z\in\mathrm{Ext}^i_G(\check{I},M)$ such that $p^*(z)=m$. Since $H^{i-1}(G,M)=0$, z is in fact unique.

Corollary 8.3. *In the situation described above, χ_ψ is defined on $\mathrm{res}^G_{G_l}x$ and we have*

$$\mathrm{res}^G_{G_l}(\psi \Cup z)=\chi_\psi(\mathrm{res}^G_{G_l}x).$$

Proof. The assumption $H^{i-1}(G,M)=0$ implies that the choice of b and b' in Theorem 8.1 is unique. Therefore,

$$b=\left(\frac{T}{i'}\right)_* x \text{ and } b'=\left(\frac{n}{i'}\right)_* c.$$

From (4) we deduce that

$$\mathrm{res}^H_{H_l}b'=\mathrm{res}^H_{H_l}\left(\frac{n}{i'}\right)_* c=\mathrm{Fr}(l)\left(\frac{n}{i'}\right)_*\mathrm{res}^G_{H_l}x,$$

and therefore

$$\delta^{(5.3)}\mathrm{res}^{G_l}_{H_l}m_2=\mathrm{res}^G_{H_l}b-\mathrm{res}^H_{H_l}b'=\left(\frac{T}{i'}\right)_*\mathrm{res}^G_{H_l}x-\mathrm{Fr}(l)\left(\frac{n}{i'}\right)_*\mathrm{res}^G_{H_l}x.$$

Since $\mathrm{res}^{G_l}_{H_l}$ is an isomorphism, we have

$$\delta^{(5.3)}m_2=\left(\left(\frac{T}{i'}\right)_*-\left(\frac{n}{i'}\right)_*\mathrm{Fr}(l)\right)\mathrm{res}^G_{G_l}x,$$

and the result is clear. □

References

[Dar92] Henri Darmon, *A refined conjecture of Mazur-Tate type for Heegner points*, Invent. Math. **110** (1992), no. 1, 123–146. MR **93i:**11066

[Kol90] Victor A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift vol. II, Prog. Math., vol. 87, Birkhäuser, Boston, Basel, Berlin, 1990, pp. 435–483. MR **92g:**11109

[Mil86] James S. Milne, *Arithmetic duality theorems*, Perspectives in Mathematics, Academic Press, 1986. MR **88e:**14028

[Nek92] Jan Nekovář, *Kolyvagin's method for Chow groups of Kuga-Sato varieties*, Invent. Math. **107** (1992), no. 1, 99–125. MR **93b:**11076

[Rub93] Karl Rubin, *Abelian varieties, p-adic heights and derivatives*, Algebra and Number Theory (Essen, 1992; G. Frey and J. Ritter, editors), de Gruyter, Berlin, 1994, pp. 247–266. MR **95i:**11066

Department of Mathematics, University of California at Los Angeles, Box 951555, Los Angeles, California 90095-1555

Current address: Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham, DH1 3LE, United Kingdom