TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 11, November 1997, Pages 4685–4688
S 0002-9947(97)02049-7

CORRECTION AND EXTENSION OF "MEASURABLE QUOTIENTS OF UNIPOTENT TRANSLATIONS ON HOMOGENEOUS SPACES"

DAVE WITTE

ABSTRACT. The statements of Main Theorem 1.1 and Theorem 2.1 of the author's paper [*Trans. Amer. Math. Soc.* **345** (1994), 577–594] should assume that Γ is discrete and G is connected. (Corollaries 1.3, 5.6, and 5.8 are affected similarly.) These restrictions can be removed if the conclusions of the results are weakened to allow for the possible existence of transitive, proper subgroups of G. In this form, the results can be extended to the setting where G is a product of real and p-adic Lie groups.

There are two errors in the proof of Theorem 2.1 of [W]. To eliminate these mistakes,

> *the statements of Theorem 2.1 and Main Theorem 1.1 should assume that* Γ *is discrete and* G *is connected.*

A few other results are affected: In Corollary 1.3, assume Γ is discrete and U is connected. In Corollary 5.6, assume Λ is discrete. In Corollary 5.8, assume Λ is discrete and N is connected.

Although it includes the main cases of interest, the restriction to connected groups and discrete subgroups is not entirely satisfying. For more general Lie groups, however, there are two problems in the proof: (1) it was tacitly assumed that $\mathrm{Aff}(\Lambda\backslash G)$ is second countable, but this may not be the case if G/G° is not finitely generated; and (2) some proper subgroups of G may be transitive on $\Gamma\backslash G$, so X may not project onto all of G in Step 1 on p. 582. The first mistake can be eliminated by using the topology of convergence in measure. The second problem can be eliminated by hypothesis (as above), but it can be better resolved by weakening the conclusion to account for the transitive proper subgroups (see Corollary A.2 below).

The improved argument that eliminates problem (1) applies to any locally compact group, not just Lie groups (see A.1). Because M. Ratner's Classification of Invariant Measures (1.2) is now known to be true not just for Lie groups, but also for direct products of real and p-adic Lie groups [R], this allows us to extend Main Theorem 1.1 to this more general setting (see A.3).

Before stating our results, let us present a simplified definition of central double-coset quotients. The original definition imposed a more complicated restriction on K, because the author did not realize that noncompact groups are unnecessary (when $\Gamma\backslash G$ has finite volume).

Received by the editors July 1, 1996.
1991 *Mathematics Subject Classification.* Primary 22E40, 28C10, 58F11; Secondary 22D40, 22E35, 28D15.

Definition (cf. [W, p. 578])**.** Let U be a subgroup of a locally compact group G, and let $\Gamma\backslash G$ be a finite-volume homogeneous space of G. Suppose

1. Λ is a closed subgroup of G containing Γ; and
2. K is a compact subgroup of $\operatorname{Aff}(\Lambda\backslash G)$ that centralizes U.

Then the natural U-action on $\Lambda\backslash G/K$ is a quotient of the U-action on $\Gamma\backslash G$. It is called a *central double-coset quotient*.

We should point out that the topology on $\operatorname{Aff}(\Lambda\backslash G)$ here is the topology of convergence in measure. The more natural topology of uniform convergence on compact sets may be used instead if $\operatorname{Aff}(\Lambda\backslash G)$ is second countable and locally compact in this topology. Unfortunately, this is not always the case, although it is true if G is a Lie group whose component group G/G° is finitely generated.

We also present a naive analogue, for non-Lie groups, of the notion of Lie subgroup of a Lie group.

Definition. Let us say that a subgroup H of a locally compact group G is *semi-closed* if, for some n, there is a closed subgroup X of $G^n = G \times G \times \cdots \times G$ such that H is the projection of X into the first factor of G^n.

Theorem A.1. *Let U be a subgroup of a locally compact, second countable group G, and let $\Gamma\backslash G$ be a finite-volume homogeneous space of G, such that every semi-closed subgroup of G that contains U, and is transitive on $\Gamma\backslash G$, is open. Assume that the U-action on $\Gamma\backslash G$ is ergodic, and that every ergodic invariant probability measure for the diagonal U-action on $(\Gamma\backslash G) \times (\Gamma\backslash G)$ is homogeneous for a subgroup of $G \times G$. Then each quotient of the U-action on $\Gamma\backslash G$ is isomorphic to a central double-coset quotient of $(\Gamma \cap H)\backslash H$, for some open subgroup H of G that contains U, and is transitive on $\Gamma\backslash G$.*

If G is a Lie group, then, by replacing it with a Lie subgroup of the smallest dimension among those with the property that they are transitive and contain U, we can arrange for the theorem's hypothesis on semi-closed subgroups to be fulfilled. Then, by combining the theorem with Ratner's Theorem (1.2), we obtain the following version of Main Theorem 1.1 that does not assume Γ is discrete or that G is connected, but has a weaker conclusion.

Corollary A.2. *Let U be a nilpotent, unipotent subgroup of a Lie group G, and let $\Gamma\backslash G$ be a finite-volume homogeneous space of G. If the U-action on $\Gamma\backslash G$ is ergodic, then each quotient of the U-action on $\Gamma\backslash G$ is isomorphic to a central double-coset quotient of $(\Gamma \cap H)\backslash H$, for some Lie subgroup H of G that contains U, and is transitive on $\Gamma\backslash G$.* □

If G is a direct product of real Lie groups and p-adic Lie groups (for different p's), then any closed subgroup of G contains an open, normal subgroup that is a product of closed subgroups of the factors of G [R, Prop. 1.5, p. 289]. Thus, the dimension of any closed subgroup (or of any semi-closed subgroup of any semi-closed subgroup of … of G) can be defined as the sum of the dimensions of its intersections with the direct factors of G. Thus, by replacing G with an appropriate subgroup of minimal dimension, we can again arrange for the theorem's hypothesis on semi-closed subgroups to be fulfilled. So, because Ratner's Theorem has been generalized to many groups of this type [R] (or [MT] in the special case where G is a product of algebraic groups), we obtain the following classification of quotients for homogeneous spaces of these product groups. (See [R, p. 276] for the definition of *regular*; every algebraic group is regular.)

Corollary A.3. *Let S be a finite set of places of $\mathbb{Q}$ and, for each $p \in S$, let G_p be a regular Lie group over $\mathbb{Q}_p$ (where $\mathbb{Q}_p = \mathbb{R}$ if $p = \infty$). Let G be a closed subgroup of the direct product $\bigtimes_{p\in S} G_p$. Let U be a subgroup of G that is generated by one-parameter* Ad*-unipotent subgroups of $\bigtimes_{p\in S} G_p$, and let Γ be a lattice in G. If the U-action on $\Gamma\backslash G$ is ergodic, then each quotient of the U-action on $\Gamma\backslash G$ is isomorphic to a central double-coset quotient of $(\Gamma\cap H)\backslash H$, for some semi-closed subgroup H of G that contains U, and is transitive on $\Gamma\backslash G$.* □

Definition. Let $\Gamma\backslash G$ be a homogeneous space of a locally compact group G, and let $A, B \subset G$. We say that a Borel function $\phi\colon \Gamma\backslash G \to \Gamma\backslash G$ is *affine for A via B* if, for each $a \in A$, there is some $b \in B$ such that $\phi(xa) = \phi(x)b$ for all $x \in \Gamma\backslash G$. Note that if the subgroup Γ contains no nontrivial, normal subgroup of G, then the element b is unique, for a given $a \in A$. Furthermore, if ϕ is affine for G (via G), then ϕ is an affine map.

Proof of Theorem A.1. Follow the proof of Theorem 2.1 (with "locally compact" in the place of "Lie") until the definition of L, which is a displayed equation on p. 582. (There is a typographical error at the start of line 9 on p. 582: (S,ν) should be (T,ν).)

From the proof of Step 1 (on p. 582), and using the hypothesis on semi-closed subgroups of G, we see that, for a.e. $(\Gamma s, \Gamma t) \in M$, there is a closed subgroup X of $G\times G$ such that *(1)* for almost every $x \in X$, we have $(\Gamma s,\Gamma t)\cdot x \in M$, *(2)* the subgroup X projects onto an open subgroup of each of the two factors of $G \times G$, and *(3)* the subgroup X contains U.

Let $C(\mathcal{X})$ be the semi-group of measure-preserving Borel maps from $\Lambda\backslash G$ to $\Lambda\backslash G$ that commute a.e. with the action of each $u \in U$, where two such maps are identified if they agree a.e. [JR, p. 532]. From the proofs of Steps 2 and 3, we see that a.e. ergodic component of the projection of ρ to $\Lambda\backslash G\times\Lambda\backslash G$ is supported on the graph of some $\phi \in C(\mathcal{X})$. (Furthermore, ϕ is affine for some open subgroup of G, via G.) Then a theorem of Veech [JR, Thm. 1.8.2] states that there is a compact subgroup K of $C(\mathcal{X})$ such that the original quotient (T,ν) is isomorphic to $\Lambda\backslash G/K$ (and, furthermore, we know that a.e. $\phi\in K$ is affine for some open subgroup of G, via G). Thus, all that remains is to show that there is some open subgroup H of G such that K is affine for H via H, for this implies that K is a group of affine maps on $(\Gamma\cap H)\backslash H$.

First, note that it suffices to show that K is affine for some open subgroup of G. For, if this is the case, we may let

$$H = \{a \in G \mid K \text{ is affine for } a\}.$$

For any $\phi\in K$ and $a \in H$, we know that ϕ is affine for a via some $b \in G$. For any $\psi \in K$, because the composition $\psi\phi$ is affine for a, we see that ψ is affine for b. Since ψ is arbitrary, this implies $b \in H$. So ϕ is affine for a via H, as desired.

Second, note that if two maps are each affine for (perhaps different) open subgroups of G (via G), then their composition is also affine for an open subgroup of G (via G). Thus, because K has no proper conull subgroups, we conclude that each $\phi \in K$ is affine for some open subgroup of G. The problem is to find a single open subgroup of G that works uniformly for all $\phi\in K$.

Let $A_1 \supset A_2 \supset \cdots$ be a chain of compact neighborhoods of e in G whose intersection is $\{e\}$, and let $B_1 \subset B_2 \subset \cdots$ be a chain of compact sets whose interiors exhaust G. For each $N \in \mathbb{Z}^+$, define

$$K_N = \{\phi \in K \mid \phi \text{ and } \phi^{-1} \text{ are affine for } A_N \text{ via } B_N\}.$$

We claim that K_N is closed. Suppose $\phi_n \to \phi \in K$. Given $a \in A_N$, each ϕ_n is affine for a via some $b_n \in B_N$. Because B_N is compact, we may assume $\{b_n\}$ converges, to some $b \in B_N$. Then

$$\phi(sa) \leftarrow \phi_n(sa) = \phi_n(s)b_n \to \phi(s)b \qquad \text{(convergence in measure).}$$

So ϕ is affine for a via b. The same argument applies to ϕ^{-1}, so $\phi \in K_N$, as desired.

Because every open subset of G contains some A_n, we know that each $\phi \in K$ is affine for some A_n (via G). Then the compactness of A_n implies that ϕ is affine for A_n via some B_m. Therefore, $\phi \in K_{\max(m,n)}$. Hence, the Baire Category Theorem implies there is some N_1 such that K_{N_1} has nonempty interior. Let $C = K_{N_1}$; the compactness of K (and the fact that $C^{-1} = C$) implies there is some $r \in \mathbb{Z}^+$ such that C^r is an open subgroup of K. This implies that K/C^r is finite. Thus, by enlarging N_1, we may assume that K_{N_1} contains coset representatives of K/C^r. Hence, $C^r = K$.

By modding out the largest normal subgroup of G that is contained in Λ, let us assume, for simplicity, that Λ contains no nontrivial, normal subgroup of G. Then, for any $\phi \in C$ and $a \in A_{N_1}$, there is a unique $b(\phi, a) \in B_{N_1}$ such that ϕ is affine for a via b. We claim that the map $b\colon C \times A_{N_1} \to B_{N_1}$ is continuous. Suppose $\phi_n \to \phi$ and $a_n \to a$, and let $b = b(\phi, a)$. Assume $b_n = b(\phi_n, a_n)$ converges, say to $b' \in B_{N_1}$. We have

$$\phi(s)b = \phi(sa) \leftarrow \phi_n(sa_n) = \phi_n(s)b_n \to \phi(s)b' \qquad \text{(convergence in measure).}$$

So $b' = b$, as desired.

From the conclusion of the preceding paragraph, and the compactness of C, we see that there is some N_2 such that C is affine for A_{N_2} via A_{N_1}. Continuing, we recursively construct a sequence $\{N_i\}$ such that C is affine for A_{N_i} via $A_{N_{i-1}}$. By composition, then $K = C^r$ is affine for A_{N_r} (via B_{N_1}). $\square$

Acknowledgment. I am pleased to thank Marina Ratner, for encouraging me to extend my work, and César Silva, for several helpful conversations. This research was partially supported by a grant from the National Science Foundation.

References

[JR] A. del Junco and D. Rudolph, On ergodic actions whose self-joinings are graphs, *Ergodic Theory Dynamical Systems* **7** (1987) 531–557. MR **89e:**28029

[MT] G. A. Margulis and G. M. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* **116** (1994) 347–392. MR **95k:**22013

[R] M. Ratner, Raghunathan's conjectures for cartesian products of real and p-adic Lie groups, *Duke Math. J.* **77** (1995) 275–382. MR **96d:**22015

[W] D. Witte, Measurable quotients of unipotent translations on homogeneous spaces, *Trans. Amer. Math. Soc.* **345** (1994) 577–594. MR **95a:**22005

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078
E-mail address: `dwitte@math.okstate.edu`