FROBENIUS EXTENSIONS OF SUBALGEBRAS
OF HOPF ALGEBRAS

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ABSTRACT. We consider when extensions $S \subset R$ of subalgebras of a Hopf algebra are β-Frobenius, that is Frobenius of the second kind. Given a Hopf algebra H, we show that when $S \subset R$ are Hopf algebras in the Yetter-Drinfeld category for H, the extension is β-Frobenius provided R is finite over S and the extension of biproducts $S \star H \subset R \star H$ is cleft.

More generally we give conditions for an extension to be β-Frobenius; in particular we study extensions of integral type, and consider when the Frobenius property is inherited by the subalgebras of coinvariants.

We apply our results to extensions of enveloping algebras of Lie coloralgebras, thus extending a result of Bell and Farnsteiner for Lie superalgebras.

0. INTRODUCTION

In this paper we consider when various extensions $S \subset R$ of subalgebras of a Hopf algebra are β-Frobenius; such extensions generalize the usual notion of Frobenius extensions by having the module action on one side twisted by an automorphism β of S. It was already known that any extension of finite-dimensional Hopf algebras is β-Frobenius (a result of the third author [Sch 92]) as is any finite extension $U(K) \subset U(L)$ of enveloping algebras of Lie superalgebras (a result of Bell and Farnsteiner [BF]). Note that $U(L)$ is not an ordinary Hopf algebra, but rather a Hopf algebra in the category of $\mathbb{Z}_2$-graded modules. These results were an important motivation for this paper, and raised the question as to when an extension of Hopf algebras in a category was β-Frobenius.

One of the main results of this paper is that an extension $S \subset R$ of Hopf algebras of finite index in the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ for a given Hopf algebra H is β-Frobenius provided that the associated extension of Hopf algebras $S \star H \subset R \star H$ (the biproducts of S and R with H) has a normal basis (Theorem 5.6); this will happen whenever R and H are finite-dimensional, or when $R \star H$ is pointed. As an application we are able to generalize the [BF] result to Lie coloralgebras: if $U(K) \subset U(L)$ is a finite extension of enveloping algebras of Lie coloralgebras, then it is β-Frobenius. Moreover we give an explicit description of the automorphism β of $U(K)$, and of the Frobenius homomorphism $f : U(L) \to U(K)$ (Corollary 6.3).

Along the way we prove a number of other results about β-Frobenius extensions and conditions that ensure that an extension is β-Frobenius.

In Section 1 we give a short direct proof of the fact that an extension $B \subset A$ of finite-dimensional Hopf algebras is always β-Frobenius; moreover we give explicit

formulae for the automorphism β of B, the Frobenius homomorphism $f : A \to B$, and the dual bases of A over B with respect to f and β. We also discuss the notion of extensions of "integral type"; this definition, from [Sch 92], basically says that a finite extension $B \subset A$ of Hopf algebras has associated to it integral-like elements in A and in $\overline{A}^*$, where $\overline{A} = A/AB^+$. These elements behave as though $\overline{A}$ were a finite-dimensional Hopf algebra. The existence of such elements is crucial in showing that an extension is β-Frobenius.

In Section 2 we consider a more general situation to which our results apply: that of bi-Galois extensions. Our main example of this set-up is the following: assume that $B \subset A$ and H are Hopf algebras, and $\pi : A \to H$ is a surjective Hopf algebra map such that π restricted to B is also surjective. Then A is an H-comodule in the natural way, so we may define $R = A^{co\ H}$ and $S = B^{co\ H}$. This gives a bi-Galois extension for the pair (W, H), where W is the coalgebra A/AB^+; it is summarized in the diagram

$$
\begin{array}{ccccc}
S & \subset & B & \twoheadrightarrow & H \\
\cap & & \cap & & \| \\
R & \subset & A & \twoheadrightarrow & H.
\end{array}
$$

In the example of Hopf algebras $S \subset R$ in the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$, we have $B = S \star H$ and $A = R \star H$. In a more general bi-Galois extension, B and A are not themselves Hopf algebras, but rather bicomodule algebras with respect to another pair of Hopf algebras.

The main result of Section 3 is that under suitable conditions (including a certain Hopf extension being of integral type), the property of being a β-Frobenius extension is inherited by the subalgebras of coinvariants. That is, if the pair of algebras $B \subset A$ in a bi-Galois extension is β-Frobenius then the subalgebras of coinvariants $S \subset R$ will also be β-Frobenius. This will be applied to the case of Hopf algebras in categories in Section 5.

Section 4 is concerned with conditions under which extensions are of integral type. In fact we consider a more general situation: we look at extensions $K \subset H$, where H is a Hopf algebra but K is only a right coideal subalgebra. We prove (Theorem 4.8) that for $\overline{H} = H/HK^+$, the extension $K \subset H$ is of integral type provided $\dim \overline{H} < \infty$ and H has a (right) normal basis over K; moreover, $\overline{H}^*$ is a Frobenius algebra. As a consequence we show that an extension of Hopf algebras $K \subset H$ is β-Frobenius if the coradical of H is cocommutative and $\overline{H}$ is finite dimensional (Corollary 4.9). We also show (Theorem 4.10) that if $\dim K < \infty$ and H has a right normal basis over K, then K is a Frobenius algebra. This is applied to prove a Maschke-type theorem for (left) coideal subalgebras.

Section 5 studies Hopf algebras in the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$. Our main result, Theorem 5.6, has already been mentioned above; in addition we also give a description of the automorphism β of S, the Frobenius map $f : R \to S$, and the dual bases of R over S. We also show that any finite-dimensional Hopf algebra R in ${}^H_H\mathcal{YD}$ behaves like an ordinary Hopf algebra in that it is always a Frobenius algebra and satisfies a Maschke-type theorem: R is semisimple $\Leftrightarrow \varepsilon(t) \neq 0$, for t an integral in R (Corollary 5.8). An important example of Hopf algebras in ${}^H_H\mathcal{YD}$ is given by G-graded Hopf algebras. We discuss an old example of Radford in this context, and explicitly compute various data for it such as the integral and dual bases.

Finally in Section 6 we give our application to enveloping algebras of Lie coloralgebras. The results of Section 5 apply, since for a G-Lie coloralgebra L, where G is an abelian group with a given bicharacter, the enveloping algebra $U(L)$ is a G-graded Hopf algebra, and so is a Hopf algebra in the Yetter-Drinfeld category for $H = kG$.

We fix some notation. Throughout we work with algebras over a field k. For a Hopf algebra H, we denote the comultiplication $\triangle : H \rightarrow H \otimes H$ by $h \mapsto \sum h_1 \otimes h_2$. H has counit $\varepsilon : H \rightarrow k$ and antipode $S : H \rightarrow H$. When S is (composition) invertible, we denote its inverse by $\overline{S}$. Although sometimes S also denotes a subring, the meaning should be clear from the context. Let A be a right H-comodule, via $\delta : A \rightarrow A \otimes H$, $a \mapsto \sum a_0 \otimes a_1$. If $\pi : H \rightarrow \overline{H}$ is a coalgebra surjection, then A is also a right $\overline{H}$-comodule in the natural way, via $\overline{\delta} = (id \otimes \pi) \circ \delta$. In this situation we say that A has the *induced* $\overline{H}$-comodule structure.

Recall that for any algebra A, there is a left (right) action of A on its dual A^* given by the transpose of right (left) multiplication of A on itself. As in [Sw], we write $a \rightharpoonup f$ for this left action and $f \leftharpoonup a$ for the right action, for all $a \in A$, $f \in A^*$. Equivalently $(a \rightharpoonup f)(b) = f(ba)$ and $(f \leftharpoonup a)(b) = f(ab)$, for all $b \in A$. When f is in the coalgebra $A^\circ \subseteq A^*$, we have the usual formulas $a \rightharpoonup f = \sum f_2(a)f_1$ and $f \leftharpoonup a = \sum f_1(a)f_2$.

If A is any augmented algebra with augmentation $\varepsilon : A \rightarrow k$, one can define (left and right) integrals for A. That is, $\int_A^l = \{t \in A \mid at = \varepsilon(a)t, \text{ all } a \in A\}$ is the space of left integrals; similarly for $\int_A^r$, the right integrals. If A is a Frobenius algebra, then the spaces of left and right integrals are each one-dimensional. In this case, if $f \in A^*$ is the Frobenius homomorphism for A and we choose $t \in A$ such that $f \leftharpoonup t = \varepsilon$, then it is easy to see that $t \in \int_A^r$. If also A^* is augmented then $f \in \int_{A^*}^r$.

Finally, we call an extension of algebras $B \subset A$ *faithfully flat* if A is a faithfully flat left and right B-module. When A and B are Hopf algebras and the antipode is bijective, assuming faithful flatness on one side is sufficient, since S is then an antiautomorphism.

1. β-Frobenius extensions

We begin by reviewing some known results about β-Frobenius extensions, with some additional facts and characterizations which will be used later. β-Frobenius extensions generalize classical Frobenius extensions [K] and were introduced by Nakayama and Tsuzuku [NT 60], [NT 61]; they are also called *Frobenius extensions of the second kind.* Recall the definition:

1.1. Definition. Let $B \subset A$ be a ring extension and $\beta : B \rightarrow B$ a ring automorphism.

(a) If M is a left B-module then $_\beta M$ is defined to be the *left β-twisted B-module* with underlying set M and left action $b \cdot_\beta m = \beta(b) \cdot m$ for all $m \in M$, $b \in B$.

Similarly, one can define a *right β-twisted B-module.*

(b) $B \subset A$ is called a (left) *β-Frobenius extension* if

    (i) A is a finitely generated projective right B-module, and

    (ii) $A \cong {}_\beta \operatorname{Hom}_B(A_B, B_B)$ as (B, A)-bimodules, where $_\beta \operatorname{Hom}_B(A_B, B_B)$ is a left β-twisted (B, A)-bimodule via $(b \cdot \psi \cdot a)(x) = \beta(b)\psi(ax)$, for all $b \in B, a, x \in A$, and $\psi \in \operatorname{Hom}_B(A_B, B_B)$.

1.2. *Remark.* (a) In the case when $\beta = id$ we recover the classical notion of a Frobenius extension. If also $B = k$, then A is a Frobenius algebra since the isomorphism $A_A \to (A^*)_A$ of part (ii) of the definition is the classical Frobenius isomorphism.

(b) It will follow from Proposition 1.3 (c) that the choice of sides for a β-Frobenius extension does not matter; any (left) β-Frobenius extension is also a (right) β^{-1}-Frobenius extension. Note also that for any B-modules X_B and $_BY$, we have

$$X_\beta \otimes_B Y \cong X \otimes_B \,_{\beta^{-1}}Y$$

as vector spaces, via $x \otimes y \mapsto x \otimes y$.

As in the classical case, dual bases exist for a β-Frobenius extension. In fact, these concepts are equivalent, as we show next. A similar result is shown in [BF, Theorem 1.1]; see Remark 1.4 (a).

1.3. Proposition. *Let $B \subset A$ be a ring extension, $\beta : B \to B$ a ring automorphism, and $f : A \to \,_\beta B$ any (B, B)-bimodule map. Define*

$$F : A \to \operatorname{Hom}_B(A_B, B_B) \qquad \text{by } F(x) = fx, \text{ where } fx(a) = f(xa);$$
$$\tilde{F} : A_\beta \otimes_B A \to \operatorname{Hom}_B(A_B, A_B) \qquad \text{by } \tilde{F}(x \otimes y) = xfy = xF(y), \text{ where}$$
$$xF(y)(a) = xf(ya)$$

for all $x, y \in A$. Then the following are equivalent:
  (a) $B \subset A$ is a β-Frobenius extension via F.
  (b) F and $\tilde{F}$ are bijections.
  (c) There exist $r_i, l_i \in A, i = 1, ..., n$, such that $\forall a \in A,$

  (i) $a = \sum\limits_{i=1}^{n} r_i f(l_i a),$

  (ii) $a = \sum\limits_{i=1}^{n} (\beta^{-1} \circ f)(ar_i)l_i.$

Proof. $(a) \Rightarrow (b)$. First note that F is indeed a map of (B, A)-bimodules and $\tilde{F}$ is well defined. Next, F is bijective by the definition of a β-Frobenius extension; to see that $\tilde{F}$ is bijective, we note that $\tilde{F}$ is defined via F as follows:

$$A_\beta \otimes_B A \xrightarrow{\text{id} \otimes F} A_\beta \otimes_B \,_\beta \operatorname{Hom}_B(A_B, B_B)$$
$$\cong A \otimes_B \operatorname{Hom}_B(A_B, B_B) \qquad \text{by 1.2 (b)}$$
$$\cong \operatorname{Hom}_B(A_B, (A \otimes_B B)_B)$$
$$\qquad \qquad \text{(since } A \text{ is finitely generated projective over } B)$$
$$\cong \operatorname{Hom}_B(A_B, A_B).$$

For, following the isomorphisms, for any $x, y \in A$ we have $x \otimes y \mapsto x \otimes F(y) = x \otimes fy \mapsto x \otimes fy$. This now goes to the map $a \mapsto xf(ya)$, which is precisely $\tilde{F}(x \otimes y)$.

$(b) \Rightarrow (c)$. Since $\tilde{F} : A_\beta \otimes_B A \to \operatorname{Hom}_B(A_B, A_B)$ is bijective, we may choose $\sum\limits_{i=1}^{n} r_i \otimes l_i \in A_\beta \otimes_B A$ such that $\tilde{F}(\sum r_i \otimes l_i) = id_A$. But then

$$\sum r_i \, f(l_i a) = \tilde{F}(\sum r_i \otimes l_i)(a) = a,$$

so (i) holds.

As for (ii), let $x, y \in A$. Then

$$F\left(\sum (\beta^{-1} \circ f)(xr_i)l_i\right)(y) = f\left(\sum \beta^{-1}(f(xr_i))l_i y\right)$$
$$= \sum f(xr_i)f(l_i y) = f\left(x \sum r_i \, f(l_i y)\right) = f(xy) = F(x)(y).$$

Bijectivity of F now yields (ii).

$(c) \Rightarrow (a)$. A_B is finitely generated by (i), using the dual basis lemma. Moreover, $A \cong {}_\beta \operatorname{Hom}_B(A_B, B_B)$ via the inverse bijections:

$$x \overset{F}{\mapsto} fx \quad \text{and} \quad \sum \beta^{-1}(\psi(r_i))l_i \overset{G}{\leftarrow} \psi,$$

using (i) to see that $F \circ G = id$ and (ii) to see that $G \circ F = id$. It is easy to see that F is indeed a (B, A)-bimodule map. $\qquad \square$

1.4. *Remark.* (a) Any $f : A \to B$ satisfying part (c) is called a β-*Frobenius homomorphism*, and $\{r_i\}, \{l_i\}$ are a pair of *dual bases.* The Frobenius homomorphism f determines a B-semi-linear associative form $(\, , \,) : A \otimes A \to B$ via $(a, a') := f(aa')$, and it follows from 1.3 (c) that this form is nondegenerate. Here "semi-linear" means that for $b, b' \in B$, $(ba, a'b') = \beta(b)(a, a')b$. Conversely the existence of such a form determines an isomorphism $F : A \to \operatorname{Hom}(A, B)$ given by $Fx(a) := (x, a)$. This characterization of β-Frobenius is the one given in [BF, 1.1].

(b) In the classical case of a Frobenius algebra A, one may define the *Nakayama automorphism* η of A by $f(xy) = f(y\eta(x))$, for all $x, y \in A$. Equivalently $(x, y) = (y, \eta(x))$, where the bilinear form is as in (a).

(c) When $B \subset A$ is a β-Frobenius extension as above, A satisfies the separability condition

$$\sum ar_i \otimes l_i = \sum r_i \otimes l_i a$$

in the twisted tensor product $A_\beta \otimes_B A$, for all $a \in A$. For, given $a, x \in A$,

$$\tilde{F}\left(\sum ar_i \otimes l_i\right)(x) = \sum ar_i f(l_i x) = ax = \sum r_i f(l_i ax) = \tilde{F}\left(\sum r_i \otimes l_i a\right)(x);$$

the bijectivity of $\tilde{F}$ now yields the result.

(d) If $B \subset A$ are finite-dimensional and A_B is free, then (c) (ii) in the proposition is not needed; that is, $B \subset A$ is β-Frobenius if there exist a (B, B)-bimodule map $f : A \to {}_\beta B$ and $\{r_i, l_i\}$ in A such that $a = \sum r_i f(l_i a)$, for all a. For, by the comment in the proof of $(c) \Rightarrow (a)$, (i) implies $F \circ G = id$, and thus F is surjective. Since $A \cong B^{(n)}$ for some n, it follows that $\operatorname{Hom}_B(A, B) \cong \operatorname{Hom}_B(B, B)^{(n)} \cong B^{(n)}$ and so $\dim A = \dim \operatorname{Hom}_B(A, B)$. Thus F is bijective, and so $B \subset A$ is β-Frobenius.

A major example of β-Frobenius extensions is given by any pair $B \subset A$ of finite-dimensional Hopf algebras; this is essentially [Sch 92, 3.6 II]. We give here a shorter proof of this fact, and also obtain some new information about the form of the automorphism β, the map f, and the dual bases.

We fix the following notation. Let A be a finite-dimensional Hopf algebra, f_A a right integral in A^*, and $t \in A$ such that $f_A \leftharpoonup t = \varepsilon$ (that is, $f_A(ta) = \varepsilon(a)$, for all $a \in A$). We may choose t in this way since by the Larson-Sweedler theorem [LS] f_A is a generator for A^* as a (cyclic) right A-module. As noted in the introduction, it follows that $t \in \int_A^r$. Let $\alpha \in A^*$ be the *(right) modular function* for A; that is, $at = \alpha(a)t$, for all $a \in A$.

Parts of the next lemma are known: the fact that A is Frobenius is in [LS], and the form of the dual bases and the Nakayama automorphism η were shown in [OSch] under an additional hypothesis.

1.5. Lemma. *Let A be any finite-dimensional Hopf algebra with f_A, t, and α as above. Then A is a Frobenius algebra with Frobenius homomorphism f_A and dual*

bases $\{\overline{S}t_2, t_1\}$. *The Nakayama automorphism $\eta : A \to A$ has the following form:*

$$\begin{aligned}
\eta(a) &= \overline{S}^2(a \leftharpoonup \alpha) = (\overline{S}^2 a) \leftharpoonup \alpha, \\
\eta^{-1}(a) &= S^2(a \leftharpoonup \alpha^{-1}) = (S^2 a) \leftharpoonup \alpha^{-1}
\end{aligned}$$

for all $a \in A$. It follows that η has finite order dividing $2 \dim_k A$.

Proof. Since f_A is a right integral of A^*, $a \leftharpoonup f_A = f_A(a)1_A$, for all $a \in A$. It follows that

$$\begin{aligned}
\sum (\overline{S}t_2) f_A(t_1 a) &= \sum (\overline{S}t_3) f_A(t_1 a_1) t_2 a_2 \\
&= \sum f_A(ta_1) a_2 = \sum \varepsilon(a_1) a_2 = a,
\end{aligned}$$

and thus $\{\overline{S}t_2, t_1\}$ are dual bases of A and f_A is the Frobenius homomorphism.

Now replace a by $\eta(a)$ in the above formula:

$$\eta(a) = \sum (\overline{S}t_2) f_A(t_1 \eta(a)) = \sum (\overline{S}t_2) f_A(at_1),$$

since f_A is the Frobenius homomorphism. Thus

$$\begin{aligned}
S^2(\eta(a)) &= \sum f_A(at_1)(St_2) &= \sum f_A(a_1 t_1) a_2 t_2 (St_3) \\
&= \sum f_A(a_1 t) a_2 &= \sum \alpha(a_1) a_2 = a \leftharpoonup \alpha.
\end{aligned}$$

Hence $\eta(a) = \overline{S}^2(a \leftharpoonup \alpha)$. Since $\alpha \overline{S}^2 = \alpha$, it follows that $\eta(a) = (\overline{S}^2 a) \leftharpoonup \alpha$. Similarly, using $\alpha S = \alpha^{-1}$ and $\alpha S^2 = \alpha$, it follows that $\eta^{-1}(a) = S^2(a \leftharpoonup \alpha^{-1}) = (S^2 a) \leftharpoonup \alpha^{-1}$, for all $a \in A$.

We now consider the order of η. First, for $a \in A$,

$$\begin{aligned}
\eta^2(a) &= \overline{S}^2((\overline{S}^2 a) \leftharpoonup \alpha) \leftharpoonup \alpha \\
&= (\overline{S}^4 a) \leftharpoonup \alpha^2.
\end{aligned}$$

By induction it follows that $\eta^n(a) = (\overline{S}^{2n} a) \leftharpoonup \alpha^n$. Since $\alpha \in G(A^*)$, the order of α divides $n = \dim A$ [NZ] and so $\eta^n(a) = \overline{S}^{2n}(a)$; similarly the order of the modular function for A^*, $c \in A^{**} \cong A$, divides n. Since $S^4(a) = c(\alpha^{-1} \rightharpoonup a \leftharpoonup \alpha)c^{-1}$ by Radford's therorem [R 76], it follows that $S^{4n} = id$. Thus $\eta^{2n}(a) = \overline{S}^{4n}(a) = a$, and so the order of η divides $2n$. $\square$

We remark that as a consequence of Lemma 1.5, $f_A(\overline{S}t) = 1$. For, set $a = \overline{S}t$ in the dual basis formula in the proof. Since $\overline{S}t \in \int_A^l$, this gives $\overline{S}t f_A(\overline{S}t) = \overline{S}t$, and so $f_A(\overline{S}t) = 1$.

Now assume that B is a Hopf subalgebra of A; let t_B be a right integral in B and let $\alpha_B \in B^*$ be the right modular function for B. Since A is free over B, by [NZ], we may write

$$t = t_A = \tilde{\Lambda} t_B$$

for some $\tilde{\Lambda} \in A$.

1.6. Definition. Let $B \subset A$ be finite-dimensional Hopf algebras, with right modular functions α_A and α_B and Nakayama automorphisms η_A and η_B respectively. Let

$$\chi = \alpha_A * \alpha_B^{-1} \in Alg(B, k)$$

and

$$\beta = \eta_B^{-1} \circ \eta_A \in Aut(B),$$

where here α_A and η_A are the respective restrictions to B. Then χ is the *relative modular function* and β the *relative Nakayama automorphism.*

It is easy to see that the Nakayama automorphism of A restricts to an automorphism of B and that $\beta(b) = \Sigma \chi(b_1) b_2 = b \leftharpoonup \chi$, using the formulas in Lemma 1.5.

1.7. Theorem. *Let $B \subset A$ be finite-dimensional Hopf algebras, and consider A as a left $\overline{A} = A/AB^+$-comodule via the induced coaction. Choose $f_A \in \int_{A^*}^{r}$ and $t_A \in \int_{A}^{r}$ such that $f_A \leftharpoonup t_A = \varepsilon$. Let $t_B \in \int_{B}^{r}$, and write $t_A = \tilde{\Lambda} t_B$. Then $B \subset A$ is a β-Frobenius extension, with*

(a) *automorphism $\beta : B \to B$ given by $\beta = \eta_{B}^{-1} \circ \eta_{A}$, the relative Nakayama automorphism as in 1.6,*
(b) *Frobenius homomorphism $f : A \to B$, via $f(a) = \sum f_A(a_1 \overline{S} t_B) a_2$,*
(c) *dual bases $\{\overline{S} \Lambda_2, \Lambda_1\}$, where $\Lambda = (S\tilde{\Lambda}) \leftharpoonup \alpha^{-1} = \eta^{-1}(\overline{S}\tilde{\Lambda})$.*

Proof. We first note several consequences of the fact that A is free over B [NZ]. Setting $\overline{A} = A/AB^+$, we may consider A as a left $\overline{A}$-comodule in the natural way; it follows that $B = {}^{co\,\overline{A}}A$ (as in 2.4 (b)). Also by Remark 1.4 (d), it will suffice to show that the given dual bases satisfy 1.3 (c) (i).

The fact that $B = {}^{co\,\overline{A}}A$ implies that $Im f \subseteq B$. For, the comodule map $\rho : A \to \overline{A} \otimes A$ is given by $\rho = (\pi \otimes id) \circ \triangle$. Writing $u = \overline{S} t_B$, we have:

$$
\begin{aligned}
\rho(f(a)) &= (\pi \otimes id) \circ \triangle(\sum f_A(a_1 u) a_2) \\
&= \sum f_A(a_1 u) \overline{a}_2 \otimes a_3 \\
&= \sum f_A(a_1 u_1) \overline{a}_2 \varepsilon(u_2) \otimes a_3 \\
&= \sum f_A(a_1 u_1) \overline{a_2 u_2} \otimes a_3 \quad \text{since } u_2 \in B \\
&= \sum \overline{a_1 u \leftharpoonup f_A} \otimes a_2 \\
&= \sum f_A(a_1 u) \overline{1} \otimes a_2 \quad \text{since } f_A \in \int_{A^*}^{r} \\
&= \overline{1} \otimes f(a).
\end{aligned}
$$

Thus $f(a) \in B$.

Since $u \in \int_{B}^{l}$, for all $b \in B$ we have

$$
f(ab) = \sum f_A(a_1 b_1 u) a_2 b_2 = \sum f_A(a_1 \varepsilon(b_1) u) a_2 b_2 = f(a) b
$$

and so f is a right B-map. On the left, using the Nakayama automorphism η_A,

$$
\begin{aligned}
f(ba) &= \sum f_A(b_1 a_1 u) b_2 a_2 = \sum f_A(a_1 u \eta_A(b_1)) b_2 a_2 \\
&= \sum f_A(a_1 u \, \alpha_B^{-1}(\eta_A(b_1))) b_2 a_2 \\
&= \sum \alpha_B^{-1}(\overline{S}^2(b_1 \leftharpoonup \alpha_A)) b_2 f_A(a_1 u) a_2 \quad \text{by Lemma 1.5} \\
&= \sum \alpha_A(b_1) \alpha_B^{-1}(b_2) b_3 f(a) \\
&= \sum \chi(b_1) b_2 f(a) = \beta(b) f(a).
\end{aligned}
$$

Thus f is a left β-twisted B-map.

It remains only to check 1.3 (c) (i) for the dual bases $\{\overline{S}\Lambda_2, \Lambda_1\}$, where $\eta(\Lambda) = \overline{S}\tilde{\Lambda}$ and $t_A = \tilde{\Lambda}t_B$. Now for $a \in A$,

$$
\begin{aligned}
\sum(\overline{S}\Lambda_2)f(\Lambda_1 a) &= \sum(\overline{S}\Lambda_3)f_A(\Lambda_1 a_1 \overline{S}t_B)\Lambda_2 a_2 \\
&= \sum f_A(\Lambda_1 a_1 \overline{S}t_B)\varepsilon(\Lambda_2)a_2 \\
&= \sum f_A(\Lambda a_1 \overline{S}t_B)a_2 \\
&= \sum f_A(a_1 \overline{S}t_B \eta(\Lambda))a_2 \\
&= \sum f_A(a_1 \overline{S}(\tilde{\Lambda}t_B))a_2 \quad \text{by the form of } \eta(\Lambda) \\
&= \sum f_A(a_1 \overline{S}t_A)a_2 \\
&= \sum f_A(\varepsilon(a_1)\overline{S}t_A)a_2 \quad \text{since } \overline{S}t_A \in \int_A^l \\
&= f_A(\overline{S}t_A)a = a
\end{aligned}
$$

since $f_A(\overline{S}t_A) = 1$ by choice of t_A with respect to f_A, as noted after Lemma 1.5. Thus $B \subset A$ is β-Frobenius. $\qquad\square$

1.8. Corollary. *Let $B \subset A$ be finite-dimensional Hopf algebras. Then the following are equivalent:*

(a) *$B \subset A$ is a classical Frobenius extension;*

(b) *$\beta = id$, where β is the relative Nakayama automorphism as in 1.6;*

(c) *$\alpha_A|_B = \alpha_B$, where α_A and α_B are the right modular functions for A and B, respectively.*

Proof. $(c) \Rightarrow (b)$ by the form of χ in Definition 1.6, and $(b) \Rightarrow (a)$ is trivial. Thus it remains to prove $(a) \Rightarrow (c)$. The argument is the same as that in [OSch, 4.8, $(2) \Rightarrow (3)$]: since A is both Frobenius and β-Frobenius, there exists an isomorphism $\phi : A \to \ _{\beta^{-1}}A$ from A as an ordinary (B, A)-module to A as a left β^{-1}-twisted (B, A)-module. Let $u = \phi(1)$; then u is a unit in A, with inverse $w = \phi^{-1}(1)$. It follows that for all $b \in B$,

$$ub = \phi(1)b = \phi(b) = \beta^{-1}(b)\phi(1) = \beta^{-1}(b)u.$$

Applying ε, we see that $\varepsilon(b) = \varepsilon(\beta^{-1}(b))$, for all b, since $\varepsilon(u) \neq 0$. Thus $\varepsilon\beta = \varepsilon$. But since $\beta(b) = \sum \chi(b_1)b_2$, it follows that $\varepsilon\beta = \chi$. Thus $\chi = \varepsilon$. Now by 1.6 again, the restriction of the modular function of A to B is the modular function of B: $\alpha|_B = \alpha_B$. $\qquad\square$

We remark that this result is reminiscent of a classical result on locally compact groups [We, Ch. II, Sec. 9]: if G is such a group and L a closed subgroup of G, consider the locally compact topological space $X = G/L$ of left cosets of L in G; G acts on X by left translation. Then X has a non-zero G-invariant Radon measure $\Leftrightarrow$ the modular function for G restricted to L is the modular function for L.

1.9. Example. Let H be any finite-dimensional Hopf algebra and $D(H)$ its Drinfeld double [Dr 86]. Following the relations for $D(H)$ given in [R 93], we may write $D(H) = H^{*cop} \bowtie H$; thus $D(H) = H^{*cop} \otimes H$ as coalgebras, and the multiplication is given by H^{*cop} and H acting on each other via the right and left coadjoint actions. See also [M, 10.3.5] for details. In particular we may consider $H \subset D(H)$, via $H \cong \varepsilon \bowtie H$, and apply the above results to this extension, which is always β-Frobenius by Theorem 1.7.

By [R 93], $D(H)$ is always unimodular, and thus $\alpha = \varepsilon$ in $A = D(H)$. Consequently, for χ as in Definition 1.6 and Theorem 1.7, $\chi = (\alpha_H)^{-1}$, where α_H is the

right modular function for $B = H$. It follows that the automorphism β is given by

$$\beta(h) = \sum \alpha_H^{-1}(h_1)h_2$$

for all $h \in H$. Thus, as in Corollary 1.8, the extension $H \subset D(H)$ is Frobenius $\Leftrightarrow \alpha_H = \varepsilon \Leftrightarrow H$ is unimodular. This last fact is also noted in [CMZ, Cor.4.5]. We may also describe the Frobenius homomorphism f as follows. Using Theorem 1.7(b) and Example 1.12, we see that $f(a) = \sum \lambda(\overline{a_1})a_2$, where $\lambda \in \int_{\overline{D(H)^*}}^r$. Now, $\overline{D(H)} = D(H)/D(H)H^+ \cong H^{*cop}$ as coalgebras, and thus a right integral λ in $\overline{D(H)}^* \cong H^{op}$ corresponds to a left integral in H. That is, we may write $\lambda = u \in \int_H^l$. Then for an element $\gamma \bowtie h \in D(H)$, where $\gamma \in H^{*cop}, h \in H$, we have $\lambda(\overline{\gamma \bowtie h}) = \lambda(\overline{\gamma \bowtie \varepsilon(h)1}) = \varepsilon(h)\gamma(u)$, and thus

$$
\begin{aligned}
f(\gamma \bowtie h) \quad &:= \quad \sum \lambda(\overline{\gamma_2 \bowtie h_1})(\gamma_1 \bowtie h_2) \\
&= \quad \sum \gamma_2(u)\gamma_1 \bowtie h \\
&= \quad \sum (u \rightharpoonup \gamma) \bowtie h = \gamma(u)\varepsilon \bowtie h.
\end{aligned}
$$

Thus in fact $f : D(H) \to \varepsilon \bowtie H \cong H$; one may check that f is a left β-twisted map.

We may also find a dual basis for $D(H)$ over H. For, let $\tilde{\Lambda}$ be a left integral in $H^{* \, cop}$ and t_H a right integral in H. Then $t = \tilde{\Lambda} \bowtie t_H$ is a (left and right) integral in $D(H)$ [R 93]. Since $\alpha = \varepsilon$ in $D(H)$, Theorem 1.7 (c) gives that $\Lambda = S\tilde{\Lambda}$. Thus the dual bases are $\{\overline{S}\Lambda_2, \Lambda_1\} = \{\tilde{\Lambda}_1, S\tilde{\Lambda}_2\}$.

We next turn to studying more general conditions which will guarantee that a given extension $B \subset A$ is β-Frobenius. Although we may no longer have actual integrals in A and B, the hypotheses we use involve the existence of integral-like elements, as used in [Sch 92]. If A is any augmented algebra with augmentation $\varepsilon : A \to k$, let $\int_A^r$ (respectively $\int_A^l$) denote the space of right (left) integrals of A.

1.10. Definition. Let W be a Hopf algebra, $U \subset W$ a Hopf subalgebra, and $\overline{W} = W/WU^+$ the (left) quotient coalgebra; note that $\overline{W}^*$ is an augmented algebra by evaluation at $\overline{1}$. Then the extension $U \subset W$ is of *(right) integral type* if
  (a)   there exists $0 \neq \lambda \in \int_{\overline{W}^*}^r$,
  (b)   there exists $\Lambda \in W$ such that $\lambda \leftharpoonup \Lambda = \varepsilon$ on $\overline{W}$,
  (c)   there exists $\chi \in Alg(U, k)$ such that $\lambda \leftharpoonup u = \chi(u)\lambda$, for all $u \in U$.

Note that (b) means that $\lambda(\Lambda \cdot \overline{w}) = \lambda(\overline{\Lambda w}) = \varepsilon(\overline{w}) = \varepsilon(w)$, for all $w \in W$ (in particular $\lambda(\overline{\Lambda}) = 1$); alternatively, $\int_{\overline{W}^*}^r \leftharpoonup W = \overline{W}^*$. Similarly (c) means that $\lambda(\overline{uw}) = \chi(u)\lambda(\overline{w})$, for all $u \in U, w \in W$; or alternatively $\lambda \leftharpoonup U = k\lambda$.

Similarly we may define an extension $U \subset W$ to be of *left integral type* by using $\overline{W} = W/U^+W$ and assuming there exist $0 \neq \lambda \in \int_{\overline{W}^*}^l, \Lambda$, and χ with the appropriate properties. In either case the hypothesis generalizes properties of normal Hopf subalgebras.

1.11. Example. Let U be a normal Hopf subalgebra of W of finite index (that is, $\dim \overline{W}$ is finite). Then $U \subset W$ is of right and left integral type. For, since U is normal, $U^+W = WU^+$ and $\overline{W} = W/WU^+$ is a finite-dimensional Hopf algebra. Thus $\overline{W}^*$ is also a finite-dimensional Hopf algebra, and so by the Larson-Sweedler theorem contains a right (left) integral $\lambda \neq 0$; moreover $\lambda \leftharpoonup \overline{W} \cong \overline{W}^*$. Thus there exists $\overline{\Lambda} \in \overline{W}$ such that $\lambda \leftharpoonup \overline{\Lambda} = \varepsilon$; in fact $\overline{\Lambda}$ is a right integral in $\overline{W}$. Choose

$\Lambda \in W$ to be any pre-image of $\overline{\Lambda}$; then (b) holds. For (c), we may choose $\chi = \varepsilon$. For, if $u \in U$ and $w \in W$, then $\overline{uw} = \varepsilon(u)\overline{w}$ since $\overline{U^+W} = \overline{WU^+} = \overline{0}$. Thus $\lambda \leftharpoonup u = \varepsilon(u)\lambda$, for all $u \in U$.

In fact $U \subset W$ is always of (right and left) integral type if W is finite-dimensional.

1.12. Example. We reconsider the case of finite-dimensional Hopf algebras of Theorem 1.7. It is not difficult to check that we may use $\lambda \in \overline{A}^*$ defined by $\lambda(\overline{a}) := f_A(a\overline{S}t_B)$, and that Λ and χ are exactly as given in 1.7. Thus any such extension is of right (left) integral type. This fact was shown in [Sch 92], though without the explicit formulas for λ, Λ, and χ.

1.13. Example. We show in Corollary 4.9 that any extension $B \subset A$ of pointed Hopf algebras of finite index is always of integral type.

The importance of integral-type extensions comes from the following result of Schneider. Recall that if A is a left W-comodule via $\sigma : A \to W \otimes A$ and $\pi : W \to \overline{W}$ is a coalgebra morphism, then A has an induced left $\overline{W}$-comodule structure via $(\pi \otimes id) \circ \sigma$.

1.14. Theorem ([Sch 92, 3.3]). *Let W be a Hopf algebra with bijective antipode, let U be a Hopf subalgebra such that $U \subset W$ is a faithfully flat extension of right integral type, and let $\overline{W} = W/WU^+$. Let A be a left W-comodule algebra such that $^{coW}A \subset A$ is W-Galois, and let $B = {}^{co\overline{W}}A$, where A has the induced $\overline{W}$-comodule structure. Then $B \subset A$ is a β-Frobenius extension.*

In particular the Frobenius homomorphism $f : A \to B$ is given by $f(a) = \sum \lambda(\overline{a}_{-1})a_0$, for all $a \in A$, and the automorphism $\beta : B \to B$ is given by $\beta(b) = \sum \chi(\overline{b}_{-1})b_0$, for all $b \in B$, where λ and χ are as in Definition 1.10.

In fact the result in [Sch 92] is stated for extensions of left integral type; the present version follows from that one by using $H := W^{op\,cop}$, $H' := U^{op\,cop}$, and the right H-comodule algebra $A^{op} \to A^{op} \otimes W^{op\,cop}$ determined by $\sigma : A \to W \otimes A$.

2. BICOMODULE ALGEBRAS AND BI-GALOIS EXTENSIONS

Since we wish to show that in a Frobenius extension $B \subset A$, certain well-behaved subalgebras $S \subset R$ are also Frobenius, we introduce in this section the kind of extensions we will be looking at. We first consider a more general situation.

2.1. Definition. Let C and D be k-coalgebras and A a k-vector space.
(1) A is a left (C, D)-*bicomodule* if
 (a) A is a left C-comodule, with coaction $\sigma : A \to C \otimes A$,
 (b) A is a right D-comodule, with coaction $\rho : A \to A \otimes D$, and
 (c) ρ is a left C-comodule map (equivalently, σ is a right D-comodule map).
That is, the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \sigma\ } & C \otimes A \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle id\otimes\rho} \\
A \otimes D & \xrightarrow[\sigma\otimes id]{} & C \otimes A \otimes D
\end{array}
$$

(2) Now assume that C and D have grouplike elements c_0 and d_0 respectively. Then we may define the coinvariants of A with respect to these grouplikes to be

$$^{coC}A = \{a \in A \mid \sigma(a) = c_0 \otimes a\} \text{ and } A^{coD} = \{a \in A \mid \sigma(a) = a \otimes d_0\}.$$

Then the set $S = (^{co\,C}A) \cap (A^{co\,D})$ is the set of *bicoinvariant* elements.

(3) If also C and D are bialgebras and A is an algebra, then A is a (C, D)-*bicomodule algebra* if σ and ρ are algebra maps.

2.2. *Remark.* (a) An easy example of a bicomodule is given as follows: Let A be any coalgebra, and C and D two quotient coalgebras of A. Then A becomes a right D-comodule and a left C-comodule via the induced coactions, and it is easy to see that A is a (C, D)-bicomodule.

(b) If $a \in A$, then we may write $\sigma(a) = \sum a_{-1} \otimes a_0 \in C \otimes A$ and $\rho(a) = \sum a_0 \otimes a_1 \in A \otimes D$. This double usage of a_0 is justified since the compatibility condition 2.1 (c) allows us to write

$$(id \otimes \rho)\sigma(a) = (\sigma \otimes id)\rho(a) = \sum a_{-1} \otimes a_0 \otimes a_1.$$

In the next two lemmas we relate the comodule structures of various subcomodules of coinvariants of a bicomodule A.

2.3. Lemma. *Suppose C and D are coalgebras with grouplikes c_o and d_0 (as in Definition 2.1(2)). Let A be a (C, D)-bicomodule and set $B = {}^{co\,C}A$ and $E = A^{co\,D}$. Then:*

(a) *B is a right D-comodule via ρ,*

(b) *E is a left C-comodule via σ;*

(c) *set $S = B \cap E$; then $S = B^{co\,D} = {}^{co\,C}E$.*

Proof. (a) First note that $A \otimes D$ is a left C-comodule via $\sigma \otimes id$. Moreover $B = {}^{co\,C}A$ implies that ${}^{co\,C}(A \otimes D) = B \otimes D$. Now if $b \in B$, then $\sigma(b) = c_0 \otimes b$. By the bicomodule property,

$$\begin{aligned}(\sigma \otimes id)\rho(b) &= (id \otimes \rho)\sigma(b) \\ &= (id \otimes \rho)(c_0 \otimes b) = c_0 \otimes \rho(b).\end{aligned}$$

Thus $\rho(b) \in {}^{co\,C}(A \otimes D) = B \otimes D$.

(b) This is similar, using $(C \otimes A)^{co\,D} = C \otimes E$.

(c) Since E is a left C-comodule by (b), $B \cap E = ({}^{co\,C}A) \cap E = {}^{co\,C}E$. Also B is a right D-comodule by (a), and so $B \cap E = B \cap (A^{co\,D}) = B^{co\,D}$. □

We next require a known lemma [Sch 92, 1.3] although we sketch a proof for completeness, since we are working on the other side. Recall that if W is a Hopf algebra, $U \subset W$ is a *right coideal subalgebra* if U is a subalgebra and $\triangle U \subseteq U \otimes W$. It follows that WU^+ is a coideal and a left ideal of W, and we have the canonical quotient map

$$\mu : W \to \overline{W} := W/WU^+.$$

$\overline{W}$ is a coalgebra and a left W-module, and W is a left $\overline{W}$-comodule via the induced coaction.

2.4. Lemma. *Let W be a Hopf algebra and $U \subset W$ a right coideal subalgebra such that $U \subset W$ is faithfully flat. Then*

(a) *the Galois map $\gamma : W \otimes_U W \to \overline{W} \otimes W$ given by $x \otimes y \mapsto \sum \overline{x}_1 \otimes x_2 y$ is bijective, and*

(b) *$U = {}^{co\,\overline{W}}W$.*

Proof. We follow the argument in [Sch 92, 1.3], switching from left to right.

(a) The map γ is a bijection since it has the inverse $\overline{\gamma} : \overline{v} \otimes w \mapsto \sum v_1 \otimes (Sv_2)w$, for all $v, w \in W$, where here S is the antipode in W.

(b) This will follow from the commutativity of the diagram

$$
\begin{array}{ccccc}
U & \subset & W & \overset{i_1}{\underset{i_2}{\rightrightarrows}} & W \otimes_U W \\
\downarrow & & \| & & \| \wr \gamma \\
{}^{co\,\overline{W}}W & \subset & W & \overset{\overline{\triangle}_W}{\underset{i_2}{\rightrightarrows}} & \overline{W} \otimes W
\end{array}
$$

$x, y \in W$. The top row of the diagram is exact since $U \subset W$ is faithfully flat [Wa, Th.13.1], and the bottom row is also exact. But then $U = \mathrm{Ker}\,(\overline{\triangle}_W, i_2) = {}^{co\,\overline{W}}W$. □

We now specialize to the following situation. Let W and H be Hopf algebras, U a Hopf subalgebra of W, and let $\overline{W}$ be as above. If A is a (W, H)-bicomodule algebra, then A is a $(\overline{W}, H)$-bicomodule in the natural way, via $\overline{\sigma} = (\mu \otimes id) \circ \sigma : A \to \overline{W} \otimes A$, and the $\overline{W}$-coinvariants ${}^{co\overline{W}}A$ are a subring of A since $\overline{W}$ is a left W-module. From now on we use the notation

$$R = A^{co\,H},\ B = {}^{co\,\overline{W}}A,\ S = R \cap B.$$

By Lemma 2.3, $S = {}^{co\,\overline{W}}R = B^{co\,H}$. We next show that these subrings are U-comodules.

2.5. Lemma. *Assume $U \subset W$ is a faithfully flat extension of Hopf algebras, and let $\overline{W}, A, B, R$, and S be as above. Then:*
   (a) *B is a left U-comodule via σ,*
   (b) *S is a left U-comodule via σ.*

Proof. (a) We first show that $\sigma^{-1}(U \otimes A) = B$. Choose $a \in \sigma^{-1}(U \otimes A)$. Then $(\mu \otimes id)\sigma(a) = \overline{1} \otimes a$, and thus $a \in {}^{co\overline{W}}A = B$. Hence $\sigma^{-1}(U \otimes A) \subseteq B$.

On the other hand, choose $b \in B$. By Lemma 2.4, $U = {}^{co\,\overline{W}}W$, and thus ${}^{co\,\overline{W}}(W \otimes A) = U \otimes A$, where $W \otimes A$ is a $\overline{W}$-comodule via $\overline{\triangle}_W \otimes id$. Now

$$
\begin{aligned}
(\overline{\triangle}_W \otimes id)\sigma(b) &= (\mu \otimes id^2)(\triangle_W \otimes id)\sigma(b) \\
&= (\mu \otimes id^2)(id \otimes \sigma)\sigma(b) \quad \text{since } \sigma \text{ is a comodule map} \\
&= \sum \overline{b_{-1}} \otimes (b_0)_{-1} \otimes (b_0)_0 \\
&= \overline{1} \otimes \sigma(b) \quad \text{since } b \in {}^{co\,\overline{W}}A.
\end{aligned}
$$

Thus $\sigma(b) \in {}^{co\,\overline{W}}(W \otimes A) = U \otimes A$, and so $b \in \sigma^{-1}(U \otimes A)$.

Now note that if $U \subset W$ is any inclusion of coalgebras and A is a left W-comodule via $\sigma : A \to W \otimes A$, then $\sigma^{-1}(U \otimes A)$ is a left U-comodule. In our case this means that B is a left U-comodule.

(b) Recall that $S = R \cap B$ by Lemma 2.3. The result now follows since $\sigma(S) \subset \sigma(R) \subset W \otimes R$ by 2.3 (b) and since $\sigma(S) \subset \sigma(B) \subset U \otimes B$ by (a) above, for then $\sigma(S) \subset (W \otimes R) \cap (U \otimes B) = U \otimes (R \cap B) = U \otimes S$. □

We now come to the situation we wish to study. Note that ${}^{coW}A \subset A$ is left W-Galois if the map $A \otimes_{coW\,A} A \to W \otimes A$ given by $x \otimes y \mapsto \sum x_{-1} \otimes x_0 y$ is bijective; when S_W is bijective this is equivalent to $x \otimes y \mapsto \sum x_{-1} \otimes y x_0$ being bijective. Thus A is left W-Galois $\iff A^{cop}$ is right $W^{op\,cop}$-Galois. The second map will be used later on.

2.6. Definition. Let W and H be Hopf algebras with bijective antipodes, $U \subset W$ a Hopf subalgebra such that $U \subset W$ is a faithfully flat extension of algebras, and A a

(W, H)-bicomodule algebra. As above, set $\overline{W} = W/WU^+$, $B = {}^{co\,\overline{W}}A$, $R = A^{co\,H}$, and $S = R \cap B$. Assume:

(a) $R \subset A$ and $S \subset B$ are faithfully flat H-Galois extensions, and

(b) ${}^{co\,W}A \subset A$ is a W-Galois extension.

Then (U, W, H, A) is called a *faithfully flat bi-Galois extension.*

We represent this situation in the following commutative diagram:

$$
\begin{array}{ccccc}
S & \subset & B & \overset{\rho}{\underset{i_1}{\rightrightarrows}} & B \otimes H \\
\cap & & \cap & & \cap \\
R & \subset & A & \overset{\rho}{\underset{i_1}{\rightrightarrows}} & A \otimes H \quad \cdot \\
& & \sigma \downarrow\downarrow i_2 & & \\
& & \overline{W} \otimes A & &
\end{array}
$$

We note that bi-Galois extensions have also been studied recently by Schauenburg [Sb] in the following special case: $U = k1$ (so $\overline{W} = W$) and $B = R = k$; that is, the coinvariants in A for both W and H are trivial.

In all of our applications, we consider the following case:

2.7. Main Example. Let H and A be Hopf algebras with bijective antipodes and a surjective Hopf algebra map $\pi : A \to H$. Suppose B is a Hopf subalgebra of A such that $B \subset A$ is faithfully flat and such that $\pi : B \to H$ is also surjective. Consider A as a left A-comodule via $\sigma = \triangle_A$ and as a right H-comodule via $\rho = (id \otimes \pi) \circ \triangle_A$. Now set $R = A^{co\,H}$ and $S = B^{co\,H}$. Using $W = A$ and $U = B$, we have $\overline{W} = \overline{A} = A/AB^+$, and so $B = {}^{co\,\overline{A}}A$ by Lemma 2.4 (a). Because of these simplifying assumptions, (B, A, H, A) will be a faithfully flat bi-Galois extension provided $R \subset A$ and $S \subset B$ are faithfully flat H-Galois extensions.

We express this in the diagram

$$
\begin{array}{ccccc}
S & \subset & B & \overset{\pi}{\twoheadrightarrow} & H \\
\cap & & \cap & & \| \\
R & \subset & A & \overset{\pi}{\twoheadrightarrow} & H
\end{array}
$$

One property which will guarantee that the H-Galois extensions $R \subset A$ and $S \subset B$ are faithfully flat is the existence of a total integral, that is, a right H-comodule map $\Gamma : H \to B$ such that $\Gamma(1) = 1$; see [KT] or [Sch 90]. We consider two important special cases in which this property holds:

(a) The extension $S \subset B$ is *cleft*; that is, such a $\Gamma : H \to B$ exists which is (convolution) invertible. This is equivalent to assuming that A and B are crossed products; that is, $A = R\#_\sigma H$ and $B = S\#_\sigma H$, for some invertible cocycle $\sigma : H \otimes H \to S$. In fact this will happen whenever A is finite-dimensional or pointed, by [Sch 92]. Cleft extensions will be considered further in Section 4.

(b) Assume in addition that $\Gamma : H \to B$ is a Hopf algebra map such that $\pi\Gamma = id$; that is, B is a Hopf algebra with a projection. It now follows by Radford's theorem [R 85] that $B = S \star H$, a biproduct; similarly $A = R \star H$. The above diagram then becomes

(2.8)
$$
\begin{array}{ccccc}
S & \subset & S \star H & \overset{\varepsilon \otimes id}{\twoheadrightarrow} & H \\
\cap & & \cap & & \| \\
R & \subset & R \star H & \overset{\varepsilon \otimes id}{\twoheadrightarrow} & H
\end{array}
$$

This is the situation discussed in detail in Section 5: that is, $S \subset R$ are Hopf algebras in the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$.

In fact the situation in (b) will always occur in the set-up of Example 2.7 whenever $S = k$. For, $R \subset A$ being H-Galois means that the Galois map $\gamma : A \otimes_R A \to A \otimes H$ given by $a \otimes b \mapsto \sum ab_1 \otimes \pi(b_2)$, for all $a, b \in A$, is bijective. Tensoring on the left by $k \cong A/A^+$ as a right A-module, we see that $k \otimes_R A \cong H$, given by $1 \otimes b \mapsto \pi(b)$. Restricting this map to the H-Galois extension $k \subset B$, we see that π restricted to B is an isomorphism. Thus $B \cong H$, and there exists a Hopf algebra map $\Gamma : H \to B \subset A$ such that $\pi\Gamma = id$.

We record some general facts about the situation of our main example.

2.8. Proposition. *Let (B, A, H, A) be a faithfully flat bi-Galois extension as in Example 2.7, with $R = A^{co\,H}$ and $S = B^{co\,H}$. Then:*
  (a) *$R \otimes_S B \cong A$ via multiplication,*
  (b) *$B \otimes_S R \cong A$ via multiplication,*
  (c) *Let $\overline{R} = R/RS^+$ and $\overline{A} = A/AB^+$. Then $\overline{R} \cong \overline{A}$, as left R-modules and right S-modules, via the map induced by the inclusion $R \subset A$.*

Proof. (a) Let $\mathcal{M}^H_B$ be the category of right (B, H)-Hopf modules. Since $S \subset B$ is H-Galois and faithfully flat, by [Sch 90] there is a category equivalence $\mathcal{M}_S \leftrightarrows \mathcal{M}^H_B$, given by $M \mapsto M \otimes_S B$ and $V \mapsto V^{co\,H}$, for $M \in \mathcal{M}_S, V \in \mathcal{M}^H_B$. Now $A \in \mathcal{M}^H_B$ using the given H-comodule structure $\rho : A \to A \otimes H$, and $A^{co\,H} = R$; hence using $M = R$, the map $R \otimes_S B \to A$, given by $r \otimes b \mapsto rb$, is a bijection.

(b) This is similar to (a), using the equivalence ${}_S\mathcal{M} \leftrightarrows {}_B\mathcal{M}^H$.

(c) First note that given any S-module M and ideal I of S, $M \otimes_S S/I \cong M/MI$. Thus using $I = S^+$, we have $M \otimes_S k \cong M/MS^+$. We apply this fact for $M = R$ and use $R \otimes_S B \cong A$ from part (a): $R/RS^+ \cong R \otimes_S k \cong R \otimes_S B \otimes_B k \cong A \otimes_B k \cong A/AB^+$. □

3. Frobenius extensions of subalgebras

In this section we prove our main result about when the coinvariants in a bi-Galois extension are β-Frobenius.

We recall the notion of bi-Galois extensions from Definition 2.6: $U \subset W$ and H are Hopf algebras with bijective antipodes, A is a (W, H)-bicomodule algebra, $\overline{W} = W/WU^+, R = A^{co\,H}, B = {}^{co\,\overline{W}}A$, and $S = R \cap B$, with various Galois and faithful flatness assumptions. The diagram in 2.6 may be helpful. Part (a) of the theorem is due to Schneider, as noted in Theorem 1.14.

3.1. Theorem. *Let (U, W, H, A) be a faithfully flat bi-Galois extension, and assume that $U \subset W$ is of right integral type, with λ and χ as in Definition 1.10. Define*

$$f : A \to B \quad by \quad f(a) := \sum \lambda(\overline{a_{-1}})a_0,$$
$$\beta : B \to B \quad by \quad \beta(b) := \sum \chi(b_{-1})b_0$$

for all $a \in A, b \in B$. Then:

(a) $B \subset A$ is a β-Frobenius extension with Frobenius map f.

(b) $S \subset R$ is a β_R-Frobenius extension with Frobenius map f_R, where
$f_R : R \to S$ and $\beta_R : S \to S$ are the restrictions of f and β to R and S.

(c)   We may choose dual bases as follows:
Let $\gamma : A \otimes_{co\ W_A} A \to W \otimes A$, via $x \otimes y \mapsto \sum y_{-1} \otimes xy_0$, be the Galois iso-
morphism, and let $r_i, l_i \in A$, $1 \leq i \leq n$, be such that $\gamma(\sum r_i \otimes l_i) = \Lambda \otimes 1$.
Then $\{r_i, l_i\}$ are dual bases of $B \subset A$.

Moreover, there exist $x_j, y_j \in R$, $1 \leq j \leq m$, such that

$$\sum_j x_j \otimes y_j = \sum_i r_i \otimes l_i \ \ in \ A_\beta \otimes_B A,$$

and all elements $\{x_j, y_j\}$ having this property are dual bases of the β_R-Frobenius
extension $S \subset R$ with respect to f_R (and of the β-Frobenius extension $B \subset A$).

Proof. (a) This is Theorem 1.14, which was shown by constructing dual bases and
using part (c) of Proposition 1.3.

(b) We make three preliminary observations.

(1) f and β restrict to f_R and β_R:

First, if $a \in R$, then $f(a) = \sum \lambda(\bar{a}_{-1})a_0 \in R$ since $\sigma(R) \subset W \otimes R$ by Lemma 2.3
(b), with $C = W, D = H$, and $E = R$. Similarly, if $b \in S$, then $\beta(b) = \sum \chi(b_{-1})b_0 \in$
S since $\sigma(S) \subset U \otimes S$ by Lemma 2.5 (b).

Note that $\chi : U \to k$ is an algebra map by 1.10 (c), and hence both β and β_R are
algebra automorphisms of B and S, respectively, with inverse $b \mapsto \sum \chi(Sb_{-1})b_0$.

(2) R_S is flat:

For $X \in {}_S\mathcal{M}$, the functor

$$X \mapsto A \otimes_S X \cong A \otimes_B B \otimes_S X \cong A \otimes_R R \otimes_S X$$

is exact since B_S is flat by the bi-Galois extension hypothesis and A_B is flat since
$B \subset A$ is β-Frobenius, by (a). Since also $R \subset A$ is faithfully flat by the bi-Galois
hypothesis, it follows that $X \mapsto R \otimes_S X$ is exact.

(3) $R_{\beta_R} \otimes_S A \to A_\beta \otimes_B A$, via $r \otimes a \mapsto r \otimes a$, is bijective:

Note that

$$R_{\beta_R} \otimes_S A \cong R \otimes_S {}_{\beta_R^{-1}}A, \ \ via \ \ r \otimes a \mapsto r \otimes a, \ by \ 1.2(b), \ and$$
$$B_\beta \otimes_B A \cong {}_{\beta^{-1}}A, \ \ via \ \ b \otimes a \mapsto \beta^{-1}(b)a.$$

Thus

$$R_{\beta_R} \otimes_S A \cong R \otimes_S {}_{\beta_R^{-1}}A \cong R \otimes_S B_\beta \otimes_B A \cong A_\beta \otimes_B A,$$

where the last isomorphism is Proposition 2.8 (a).

We now prove (b) by applying Proposition 1.3(b). Define F and $\tilde{F}$ in terms of
f and β and also $F_R, \tilde{F}_R$ in terms of f_R and β_R; that is,

$$F_R : R \to \ \mathrm{Hom}_S(R_S, S_S) \ \ via \ \ F_R(x) = (f_R)x, \ \ and$$
$$\tilde{F}_R : R_{\beta_R} \otimes_S R \to \ \mathrm{Hom}_S(R_S, R_S) \ \ via \ \ \tilde{F}_R(x \otimes y)(a) = xf(y\,a).$$

By 1.3 and part (a) above, F and $\tilde{F}$ are bijective, and we must show that F_R
and $\tilde{F}_R$ are bijective.

To check that F_R is bijective, we claim that the following diagram commutes:

$$
\begin{array}{ccccccc}
R & \subset & A & \xrightarrow[\;i_1\;]{\;\rho\;} & A \otimes H \\[2pt]
F_R \downarrow & & \downarrow \phi & & \downarrow \psi \\[4pt]
\operatorname{Hom}_S(R_S, S_S) & \hookrightarrow & \operatorname{Hom}_S(R_S, B_S) & \xrightarrow[\operatorname{Hom}(id, i_1)]{\operatorname{Hom}(id, \rho)} & \operatorname{Hom}_S(R_S, B_S \otimes H)
\end{array}
$$

where $\phi(a)(r) := f(ar)$ and $\psi(a \otimes h)(r) := f(ar) \otimes h$. Clearly the left square commutes since both directions give $r \mapsto (x \mapsto f(rx))$, and similarly the right lower square commutes, with $a \mapsto (x \mapsto f(ax) \otimes 1)$. It remains to consider the upper right square. If $a \in A$, then $\psi\rho : a \mapsto \psi(\sum a_0 \otimes a_1) = (x \mapsto \sum f(a_0 x) \otimes a_1)$ and $\operatorname{Hom}(id, \rho)\phi : a \mapsto (x \mapsto f(ax)) \mapsto (x \mapsto \rho(f(ax)))$. But now

$$
\begin{aligned}
\rho(f(ax)) &= \rho(\sum \lambda(\overline{a_{-1} x_{-1}}) a_0 x_0) \\
&= \sum \lambda(\overline{a_{-1} x_{-1}})(a_0)_0 (x_0)_0 \otimes (a_0)_1 (x_0)_1 \\
&= \sum \lambda(\overline{a_{-1} x_{-1}})(a_0)_0 x_0 \otimes (a_0)_1 \\
&= \sum \lambda(\overline{(a_0)_{-1} x_{-1}})(a_0)_0 x_0 \otimes a_1 \\
&= \sum f(a_0 x) \otimes a_1,
\end{aligned}
$$

where the third equality follows since $x \in R$ implies $x_0 \in R$ (since $\sigma(R) \subset W \otimes R$ by Lemma 2.3 (b)) and thus $\rho(x_0) = x_0 \otimes 1$ since $R = A^{co\,H}$, and the fourth equality follows from the bicomodule condition. Thus the diagram commutes.

Now, the top row is exact since $R = A^{co\,H}$; the bottom row is exact since $S = B^{co\,H}$ and since $\operatorname{Hom}_S(R_S, -)$ is left exact.

ϕ is bijective, since $R \otimes_S B \to A$ is bijective by Proposition 2.8, and hence

$$
\phi : A \xrightarrow{F} \operatorname{Hom}_B(A_B, B_B) \cong \operatorname{Hom}_B(R \otimes_S B, B_B) \cong \operatorname{Hom}_S(R_S, B_S)
$$

is bijective since F is bijective.

Finally, ψ is bijective. For,

$$
\psi : A \otimes H \xrightarrow{F \otimes id} \operatorname{Hom}_B(A_B, B_B) \otimes H \xrightarrow{\cong} \operatorname{Hom}_B(A_B, B_B \otimes H) \cong \operatorname{Hom}_S(R_S, B_S \otimes H)
$$

where the last isomorphism follows from $R \otimes_S B \cong A$ as above and the middle mapping is given by $\varphi \otimes h \mapsto (a \mapsto \varphi(a) \otimes h)$; it is an isomorphism since A_B is finitely-generated projective by part (a). Thus ψ is bijective, and so F_R is bijective using the diagram.

Now we show that $\tilde{F}_R$ is bijective. We claim that the following diagram commutes:

$$
\begin{array}{ccccccc}
R_{\beta_R} \otimes_S R & \subset & R_{\beta_R} \otimes_S A & \xrightarrow[id \otimes i_1]{id \otimes \rho} & R_{\beta_R} \otimes_S A \otimes H \\[2pt]
\tilde{F}_R \downarrow & & \downarrow \tilde{\phi} & & \downarrow \tilde{\psi} \\[4pt]
\operatorname{Hom}_S(R_S, R_S) & \overset{\operatorname{Hom}(id, i)}{\hookrightarrow} & \operatorname{Hom}_S(R_S, A_S) & \xrightarrow[\operatorname{Hom}(id, i_1)]{\operatorname{Hom}(id, \rho)} & \operatorname{Hom}_S(R_S, A_S \otimes H)
\end{array}
$$

where $i : R \subset A$ is the inclusion map,

$$
\tilde{\phi}(x \otimes y)(r) := xf(yr), \text{ and } \tilde{\psi}(x \otimes y \otimes h)(r) := xf(yr) \otimes h.
$$

The left square commutes since both possibilities give $x \otimes y \mapsto (r \mapsto xf(yr))$, and similarly the right lower square commutes, with $x \otimes y \mapsto (r \mapsto xf(yr) \otimes 1)$. It

remains to consider the right upper square. For $x \in R_{\beta_R}, y \in A$, we have

$$\tilde{\psi} \circ (id \otimes \rho) : x \otimes y \mapsto \tilde{\psi}(\sum x \otimes y_0 \otimes y_1) = (r \mapsto \sum xf(y_0 r) \otimes y_1).$$

The fact that $\mathrm{Hom}(id, \rho) \circ \tilde{\phi}$ gives the same result follows using $\rho(f(yr)) = \sum f(y_0 r) \otimes y_1$, which we showed above in the argument for F_R. Thus the diagram commutes.

The upper row is exact since, first, $R = A^{co\ H}$ implies $R \subset A \rightrightarrows A \otimes H$ is exact, and second, R_{β_R} is flat as a right S-module (since $X_\beta \otimes_S Y \cong X \otimes_{S\ \beta^{-1}}Y$ for all S-modules X and Y by 1.2 (b), and hence X flat as an S-module implies X_β is also flat over S).

The lower row is also exact, since in fact $R \subset A \rightrightarrows A \otimes H$ is exact as (R, R), and hence (S, S)-bimodules, and because $\mathrm{Hom}_S(R_S, -)$ is left exact.

Next, $\tilde{\phi}$ and $\tilde{\psi}$ are bijective. $\tilde{\phi}$ is bijective since it is the composition of the isomorphisms

$$R_{\beta_R} \otimes_S A \quad \overset{\cong}{\to} \quad A_\beta \otimes_B A \quad \text{by (3)}$$

$$\overset{\tilde{F}}{\to} \quad \mathrm{Hom}_B(A_B, A_B)$$

$$\cong \quad \mathrm{Hom}_S(R_S, A_S)$$

where the last isomorphism is induced by $R \otimes_S B \cong A$, which is 2.8(a). $\tilde{\psi}$ is the composition of the isomorphisms

$$R_{\beta_R} \otimes_S A \otimes H \quad \overset{\cong}{\to} \quad A_\beta \otimes_B A \otimes H \quad \text{by (3) again}$$
$$\overset{\tilde{F} \otimes id}{\to} \quad \mathrm{Hom}_B(A_B, A_B) \otimes H$$
$$\cong \quad \mathrm{Hom}_B(A_B, A_B \otimes H) \quad \text{since } A_A \text{ is fin. gen. projective}$$
$$\cong \quad \mathrm{Hom}_S(R_S, A_S \otimes H) \quad \text{by 2.8(a) as above.}$$

Using the diagram, it now follows that $\tilde{F}_R$ is bijective, proving (b).

(c) The following diagram commutes:

$$
\begin{array}{ccc}
R_{\beta_R} \otimes_S R & \xrightarrow{\quad \tilde{F}_R \quad} & \mathrm{Hom}_S(R_S, R_S) \\
{\scriptstyle g} \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(id, i)} \\
A_\beta \otimes_B A & \xrightarrow{\ \tilde{F}\ } \mathrm{Hom}_B(A_B, A_B) & \cong \mathrm{Hom}_S(R_S, A_S)
\end{array}
$$

where the lower right isomorphism is induced by $R \otimes_S B \cong A$ and the map g is the isomorphism of (b) (3) restricted to $R_{\beta_R} \otimes_S R$.

Now by Proposition 1.3 and the proof there of $(b) \Rightarrow (c)$, it follows that if $B \subset A$ is a β-Frobenius extension with Frobenius homomorphism f, and $\tilde{F}$ is as above, then $\{r_i, l_i\}$ are dual bases $\Leftrightarrow \tilde{F}(\sum_i r_i \otimes l_i) = id_A \Leftrightarrow \sum_i r_i f(l_i x) = x$, for all $x \in A$, where we consider $\sum r_i \otimes l_i \in A_\beta \otimes_B A$.

It follows that if $\{r_i, l_i\}$ are chosen in A such that $\gamma(\sum r_i \otimes l_i) = \Lambda \otimes 1$, then $\{r_i, l_i\}$ are dual bases of $B \subset A$. Indeed, for all $x \in A$,

$$
\begin{aligned}
\sum r_i f(l_i x) &= \sum r_i \lambda(\overline{(l_i)_{-1} x_{-1}})(l_i)_0 x_0 \\
&= \sum \lambda(\overline{\Lambda x_{-1}}) x_0 \quad \text{since} \quad \sum (l_i)_{-1\beta} \otimes r_i (l_i)_0 = \Lambda \otimes 1 \\
&= \sum \varepsilon(x_{-1}) x_0 = x.
\end{aligned}
$$

Now by (b), since $S \subset R$ is β-Frobenius, there exist dual bases $\{x_j, y_j\}$ of $S \subset R$; thus $\tilde{F}_R(\sum x_j \otimes_S y_j) = id_R$ by the above. By commutativity of the diagram, it must be that $g(\sum x_j \otimes_S y_j) = \sum x_j \otimes_B y_j = \sum r_i \otimes l_i$ in $A_\beta \otimes_B A$.

Finally, given $\{r_i, l_i\}$ dual bases for $B \subset A$, assume that $\{x_j, y_j\}$ are elements of R such that $\sum x_j \otimes y_j = \sum r_i \otimes l_i$ in $A_\beta \otimes_B A$. Then

$$\tilde{F}(\sum x_j \otimes_B y_j) = \tilde{F}(\sum r_i \otimes l_i) = id_A.$$

Since the above diagram commutes, it follows that $\mathrm{Hom}(id, i)\tilde{F}_R(\sum x_j \otimes_S y_j) = i$ and thus that $\tilde{F}_R(\sum x_j \otimes_S y_j) = id_R$, since $\mathrm{Hom}(id, i)$ is injective and $\mathrm{Hom}(id, i)(id_R) = i$. Thus $\{x_j, y_j\}$ are dual bases of $S \subset R$. □

3.2. Remark. In the theorem, β^{-1} can be described as the restriction of a function which is defined on A and not just on B. For

$$\beta^{-1}(a) := \sum \lambda(\overline{S(a_{-1})\Lambda})a_0, \quad \text{for all } a \in A,$$
$$\beta^{-1}(ab) = \beta^{-1}(a)\beta^{-1}(b), \quad \text{for all } a \in A, b \in B.$$

Consequently $\sum_j \beta^{-1}(x_j)y_j = \varepsilon(\Lambda)$, where $\{x_j, y_j\}$ are as in Theorem 3.1(c).

Proof. (a) First note that for $b \in B, \beta^{-1}(b) = \sum \chi(\overline{Sb_{-1}})\lambda(\overline{\Lambda})b_0 = \sum \chi(\overline{Sb_{-1}})b_0$ using the form of β in Theorem 3.1 (and 1.14) and the fact that $\lambda(\overline{\Lambda}) = 1$, as noted after Definition 1.10. Consequently $\beta^{-1}(\beta(b)) = b$ since χ is multiplicative. Now for all $a \in A, b \in B$,

$$\begin{aligned}
\beta^{-1}(ab) &= \sum \lambda(S(a_1 b_{-1})\overline{\Lambda})a_0 b_0 \\
&= \sum \chi(Sb_{-1})\lambda(Sa_{-1}\overline{\Lambda})a_0 b_0 \quad \text{by 1.10 (c)} \\
&= \beta^{-1}(a)\sum \chi(Sb_{-1})b_0 \\
&= \beta^{-1}(a)\beta^{-1}(b).
\end{aligned}$$

(b) Now by (a), the map $A_\beta \otimes_B A \to A$, via $x \otimes y \mapsto \beta^{-1}(x)y$, is well-defined, and thus $\sum_j \beta^{-1}(x_j)y_j = \sum_i \beta^{-1}(r_i)l_i$.

Recall we have chosen $\{r_i, l_i\}$ so that $\gamma(\sum r_i \otimes l_i) = \sum(l_i)_{-1} \otimes r_i(l_i)_0 = \Lambda \otimes 1$. Under the map $W \otimes A \to W \otimes A$, via $w \otimes a \mapsto \sum a_{-1}\overline{S}w \otimes a_0$, we see that $\Lambda \otimes 1 \mapsto \overline{S}\Lambda \otimes 1$. It follows that

$$\gamma(\sum l_i \otimes r_i) = \sum(r_i)_{-1} \otimes (r_i)_0 l_i = \overline{S}\Lambda \otimes 1.$$

Now

$$\begin{aligned}
\sum \beta^{-1}(r_i)l_i &= \sum \lambda(S((r_i)_{-1})\overline{\Lambda})(r_i)_0 l_i \\
&= \lambda(S(\overline{S}\Lambda)\overline{\Lambda}) \\
&= \lambda(\Lambda \cdot \overline{\Lambda}) = \varepsilon(\Lambda) \quad \text{by 1.10 (b).} \quad □
\end{aligned}$$

We consider the case of our main example, 2.7, and show that more can be said. The reader should compare this result with Theorem 1.7.

3.3. Corollary. *Assume we are in the situation of Example 2.7, that is, $B \subset A$ and H are Hopf algebras, $B \subset A$ is a faithfully flat extension, $\pi : A \to H$ is a Hopf surjection which is also surjective when restricted to B. Let $R = A^{coH}$ and $S = B^{coH}$ using the H-comodule structure induced on A and B by π. Assume that $R \subset A$ and $S \subset B$ are faithfully flat H-Galois extensions, and that $B \subset A$ is of right integral type. Then $S \subset R$ is β-Frobenius, with β and f as in Theorem 3.1, and:*

(a) *The element $\Lambda \in A(= W)$ in 1.10 (b) can be chosen in R.*

(b) *Dual bases of $S \subset R$ may be found as follows:*

There exist $x_i, y_i \in R, 1 \le i \le m$, such that in $A_\beta \otimes_B A$,

$$\sum x_i \otimes y_i = \sum \overline{S}\Lambda_2 \otimes \Lambda_1.$$

More generally, for any $r \in R$, there exist $x_i, y_i \in R$ such that in $A_\beta \otimes_B A$,

$$\sum x_i \otimes y_i = \sum \overline{S}r_2 \otimes r_1.$$

In particular $\{\overline{S}\Lambda_2, \Lambda_1\}$ are dual bases for $B \subset A$, and $\{x_i, y_i\}$ are dual bases for $S \subset R$.

Proof. Using $W = A$ and $U = B$, we see that the assumptions imply that (B, A, H, A) is a faithfully flat bi-Galois extension. Thus Theorem 3.1 applies.

(a) By 2.8 (b), $A = BR$; hence there exist $b_i \in B, u_i \in R, 1 \le i \le n$, such that $\Lambda = \sum_i b_i u_i$. Then, using 1.10 (b) and (c),

$$\varepsilon = \lambda \cdot \Lambda = \sum_i (\lambda \cdot b_i)u_i = \sum_i \chi(b_i)\lambda \cdot u_i = \lambda \cdot \sum_i \chi(b_i)u_i = \lambda \cdot \Lambda',$$

where $\Lambda' = \sum_i \chi(b_i)u_i \in R$.

(b) Given $r \in R$, and any $z \in R$,

$$
\begin{aligned}
\tilde{F}(\sum \overline{S}r_2 \otimes r_1)(z) &= \sum (\overline{S}r_2)f(r_1 z) \\
&= \sum \overline{S}r_3 \lambda(\overline{r_1 z_1})r_2 z_2 \quad \text{by the definition of } f \\
&= \sum (\overline{S}r_3)r_2 \lambda(\overline{r_1 z_1})z_2 \\
&= \sum \lambda(\overline{r z_1})z_2 \in R.
\end{aligned}
$$

Thus $\tilde{F}(\sum \overline{S}r_2 \otimes r_1) \in \mathrm{Hom}_S(R, R)$.

By the diagram used in the proof of Theorem 3.1 (c), it follows that $\sum \overline{S}r_2 \otimes r_1$ lies in the image of $R_{\beta_R} \otimes_S R$ under g; that is, $\sum \overline{S}r_2 \otimes r_1 = \sum x_i \otimes y_i$, for some $x_i, y_i \in R$.

The first statement now follows, choosing $\Lambda \in R$ as in (a), or using Theorem 3.1(c) since under the Galois map $\gamma : A \otimes A \to A \otimes A$ given by $x \otimes y \mapsto \sum y_1 \otimes xy_2$, we have $\gamma(\sum \overline{S}\Lambda_2 \otimes \Lambda_1) = \Lambda \otimes 1$. $\qquad\square$

4. Extensions of integral type

It is clear from Theorem 1.14 (from [Sch 92]) and from our Theorem 3.1 that it is important to find conditions which guarantee that an extension of Hopf algebras is of integral type. We do this here by generalizing the methods of Larson and Sweedler [LS] to cleft extensions of finite index. In fact we consider a more general situation than that of Hopf subalgebras; for most of what we do, a right coideal subalgebra will suffice.

As in Section 2, let K be a right coideal subalgebra of the Hopf algebra H and let $\overline{H} = H/HK^+$; $\overline{H}$ is a coalgebra and a left H-module.

4.1. Definition. Let $K \subset H$ be a right coideal subalgebra of H and let $\overline{H} = H/HK^+$. Then H has a *right normal basis over* K if $H \cong \overline{H} \otimes K$ as right K-modules and left $\overline{H}$-comodules. Here H has the induced left $\overline{H}$-comodule structure.

4.2. *Remark.* If H has a right normal basis over K, then H is free over K, and so $K \subset H$ is right faithfully flat. Thus, by Lemma 2.4 we know that $K = {}^{co\,\overline{H}}H$ and that the Galois map $\gamma : H \otimes_K H \to \overline{H} \otimes H$, via $x \otimes y \mapsto \sum \overline{x_1} \otimes x_2 y$, is a bijection.

In addition, the Galois map γ is also a bijection, when applied to $H \otimes_k K$:

$$\gamma \colon H \otimes K \to H \square_{\overline{H}} H.$$

This last fact is dual to Lemma 2.4(a) and requires $K = {}^{co\overline{H}}H$.

Similarly, one could define a normal basis property given a quotient left module and coalgebra of H. We next require a known lemma [T 79, Theorem 1], although we sketch a direct proof for completeness.

4.3. Lemma. *Let $\overline{H} = H/I$ be a quotient coalgebra and left H-module. Define $K := {}^{co\,\overline{H}}H$, a right coideal subalgebra. Assume $H \cong \overline{H} \otimes K$ as right K-modules and left $\overline{H}$-comodules. Then $I = HK^+$, and so H has a right normal basis over K as in Definition 4.1.*

Proof. This is dual to Lemma 2.4(b), since H is left $\overline{H}$-faithfully coflat. For, consider the diagram

$$
\begin{array}{ccccc}
H \otimes K & \overset{\longrightarrow}{\longrightarrow} & H & \longrightarrow & H/HK^+ \\[2ex]
\shortparallel\wr & & \shortparallel & & \downarrow \\[2ex]
H\square_{\overline{H}}H & \overset{\varepsilon\otimes 1}{\underset{1\otimes\varepsilon}{\longrightarrow}} & H & \longrightarrow & H/I
\end{array}
$$

Note that $HK^+ \subset I$, and hence the diagram is commutative. $\square$

4.4. *Remark.* In the situation of Definition 4.1, the following are all well-defined categories of Hopf modules:

$$\overline{H}_H\mathcal{M}, \quad {}_H\mathcal{M}^{\overline{H}}, \quad \mathcal{M}_K^H, \quad {}_K\mathcal{M}^H.$$

For, a k-space M is in $\overline{H}_H\mathcal{M}$ if it is a left H-module, a left $\overline{H}$-comodule via $\rho : M \to \overline{H} \otimes M$, and ρ is a left H-module map. This is well-defined since $\overline{H}$ is a left H-module; similarly for ${}_H\mathcal{M}^{\overline{H}}$. The second two are well-defined because K is a right coideal subalgebra.

We require some known facts about Hopf modules.

4.5. Proposition ([Sch 90], [MD])**.** *Assume that H is $\overline{H}$-cleft, with $K = {}^{co\,\overline{H}}H$. Then the following are inverse equivalences:*
(a)

$$
\begin{array}{ccc}
{}_K\mathcal{M} & \leftrightarrows & \overline{H}_H\mathcal{M}, \\[1ex]
M & \mapsto & H \otimes_K M, \\[1ex]
{}^{co\,\overline{H}}V & \leftarrowtail & V.
\end{array}
$$

In particular, for any $V \in \overline{H}_H\mathcal{M}$, $H \otimes_K^{co\,\overline{H}} V \overset{\cdot}{\to} V$ is an isomorphism, where $\cdot$ is the action of H on V.

(b)

$$\mathcal{M}^{\overline{H}} \quad \leftrightarrows \quad \mathcal{M}_K^H,$$
$$M \quad \mapsto \quad M\square_{\overline{H}} H,$$
$$\overline{V} := V/VK^+ \quad \hookleftarrow \quad V.$$

In particular, for any $V \in \mathcal{M}_K^H$, $V \to \overline{V}\square_{\overline{H}}H$ is an isomorphism, given by the comodule structure of V.

Proof. (a) is a special case of [Sch 90, 3.7], since $K \subset H$ is faithfully flat, and (b) is a special case of [Sch 90, 4.7].

(a) and (b) also follow from [MD] using γ and its dual in 4.2. $\qquad\square$

We consider various structures on duals of (co)modules.

4.6. Definition. (a) **Transposed (co)actions:** If A is an algebra and V a left A-module, then $V^* = \operatorname{Hom}(V, k)$ becomes a *right* A-module via $(f \cdot a)(v) := f(av)$, for all $v \in V$. V^* is called the *transposed* A-module; similarly for right A-modules.

Note that if A acts on $V = A$ by left multiplication, then the transposed action of A on A^* is simply $f \cdot a = f \leftharpoonup a$, as discussed in Section 0.

Dually, let C be a coalgebra and $V \to V \otimes C$, $v \mapsto \sum v_0 \otimes v_1$, a *finite-dimensional* right C-module. Then V^* is a *left* C-comodule via $V^* \to C \otimes V^*$, $f \mapsto \sum f_{-1} \otimes f_0$, where for all $v \in V$,

$$\sum f_0(v)f_{-1} := \sum f(v_0)v_1.$$

V^* is called the *transposed* C-comodule. Similarly we may begin with a finite-dimensional left C-comodule.

(b) **Contragredient (co)actions:** Let H be a Hopf algebra. If V is a left H-module, then the *contragredient left* H-module structure on V^* is defined by $(h \cdot f)(v) := f((Sh) \cdot v)$, for all $v \in V$. Similarly we may begin with a right H-module.

Note that if H acts on $V = H$ by left multiplication, then the contragredient left H-action on H^* is $h \cdot f = f \leftharpoonup Sh$.

If V is a finite-dimensional right H-comodule, then the *contragredient right* H-*comodule* structure on V^* is defined by $f \mapsto \sum f_0 \otimes Sf_{-1}$, where $f \mapsto \sum f_{-1} \otimes f_0$ is the transposed left comodule structure above. Similarly we may begin with a finite-dimensional left H-comodule.

4.7. Lemma. *Let H be a Hopf algebra, $K \subset H$ a right coideal subalgebra, and $\overline{H} = H/I$ a left H-module coalgebra quotient.*

(a) Let $V \in {}_H\mathcal{M}^{\overline{H}}$ with $\dim V < \infty$. Then $V^* \in {}_H^{\overline{H}}\mathcal{M}$, where V^* is the transposed $\overline{H}$-comodule and the contragredient H-module.

(b) Let $V \in {}_K\mathcal{M}^H$ with $\dim V < \infty$. Then $V^* \in \mathcal{M}_K^H$, where V^* is the contragredient H-comodule and the transposed K-module.

Proof. (a) This is similar to the Larson-Sweedler argument [LS]. Let $\delta_V : V \to V \otimes \overline{H}$, $v \mapsto \sum v_0 \otimes v_1$, and $\delta_{V^*} : V^* \to \overline{H} \otimes V^*$, via $f \mapsto \sum f_{-1} \otimes f_0$, be the $\overline{H}$-comodule structure and its transpose, as in 4.6. We must show $\delta_{V^*}(h \cdot f) = \sum h_1 f_{-1} \otimes h_2 \cdot f_0$, for all $f \in V^*$, $h \in H$, where $h \cdot f$ is the contragredient H-module action. That is, for all $v \in V$, we need

$$\sum (h_2 \cdot f_0)(v)(h_1 \cdot f_{-1}) = \sum (h \cdot f)(v_0)v_1$$

or equivalently $\sum f_0((Sh_2) \cdot v)(h_1 \cdot f_{-1}) = \sum f(Sh \cdot v_0)v_1$. Now

$$
\begin{aligned}
\sum f_0((Sh_2) \cdot v)(h_1 \cdot f_{-1}) &= \sum h_1 \cdot (f((Sh_2) \cdot v)_0)(Sh_2 \cdot v)_1) \quad \text{by 4.6(a)} \\
&= \sum h_1 \cdot (f((Sh_3) \cdot v_0)(Sh_2 \cdot v_1)) \quad \text{since } V \in {}_H\mathcal{M}^{\overline{H}} \\
&= \sum f((Sh) \cdot v_0)v_1.
\end{aligned}
$$

(b) This is dual to (a). Let $\triangle_V : V \to V \otimes H$, $v \mapsto \sum v_0 \otimes v_1$, and $V^* \to H \otimes V^*$, $f \mapsto \sum f_{-1} \otimes f_0$, be the H-comodule structure and its transpose. Then $\triangle_{V^*} : V^* \to V^* \otimes H$, $f \mapsto \sum f_0 \otimes Sf_{-1}$, is the H-comodule structure of $V^* \in \mathcal{M}_K^H$.

We must show that $\triangle_{V^*}(f \cdot k) = \sum f_0 \cdot k_1 \otimes (Sf_{-1})k_2$, for all $k \in K$, $f \in V^*$; that is, for all $v \in V$,

$$
\sum (f_0 \cdot k_1)(v)(Sf_{-1})k_2 = \sum (f \cdot k)(v_0)Sv_1,
$$

or equivalently

$$
\sum f_0(k_1 \cdot v)(Sf_{-1})k_2 = \sum f(k \cdot v_0)Sv_1.
$$

Now

$$
\begin{aligned}
\sum f_0(k_1 \cdot v)(Sf_{-1})k_2 &= \sum f((k_1 \cdot v)_0)S((k_1 \cdot v)_1)k_2 \quad \text{by 4.6(1)} \\
&= \sum f(k_1 \cdot v_0)S(k_2 \cdot v_1)k_3 \quad \text{since } V \in {}_K\mathcal{M}^H \\
&= \sum f(k \cdot v_0)Sv_1.
\end{aligned}
$$

$\square$

We return now to the situation of Definition 4.1. That is, K is a right coideal subalgebra of the Hopf algebra H, and $\overline{H} = H/HK^+$. Then $\overline{H}^*$ is an algebra, since $\overline{H}$ is a coalgebra; moreover $\overline{H}^*$ is augmented via $f \mapsto f(\overline{1})$, and thus we may discuss the existence of integrals in $\overline{H}^*$. Note also that $\overline{H}^*$ is a right H-module, considered as the transposed right H-module of $\overline{H}$ considered as a left H-module. That is, given $f \in \overline{H}^*$, $h \in H$, and $\overline{g} \in \overline{H}$, we have $(f \cdot h)(\overline{g}) = f(\overline{hg})$.

The following theorem lays the foundation to give cases of extensions that are of (right) integral type.

4.8. Theorem. *Let H be a Hopf algebra, K a right coideal subalgebra, and $\overline{H} = H/HK^+$ as above. Assume H has a right normal basis over K, and that $\dim \overline{H} < \infty$. Then:*
(a) *$\int_{\overline{H}^*}^r$ is one-dimensional, with basis $\lambda \neq 0$.*
(b) *There exists $\Lambda \in H$ such that $\lambda \leftharpoonup \Lambda = \varepsilon$ on $\overline{H}$.*
(c) *There exists $\tilde{\chi} \in Alg(K, k)$ such that $\lambda \leftharpoonup Sk = \tilde{\chi}(k)\lambda$ for all $k \in K$.*
(d) *$\overline{H}^*$ is a Frobenius algebra.*

Proof. First, $\overline{H} \in {}_H\mathcal{M}^{\overline{H}}$, where the right $\overline{H}$-comodule structure is given by $\triangle_{\overline{H}}$, that is, $\overline{H} \to \overline{H} \otimes \overline{H}$ via $\overline{h} \mapsto \sum \overline{h}_1 \otimes \overline{h}_2$, and $\overline{H}$ is a left H-module as usual. Thus $\overline{H}^* \in {}_H^{\overline{H}}\mathcal{M}$ by Lemma 4.7(a), and so, by Proposition 4.5,

$$(*) \qquad\qquad H \otimes_K {}^{co\,\overline{H}}\overline{H}^* \xrightarrow{\ \sim\ } \overline{H}^*$$

is an isomorphism. $\overline{H}^*$ is the transposed $\overline{H}$-comodule; hence

$$
\begin{aligned}
{}^{co\,\overline{H}}\overline{H}^* &= \{\varphi \in \overline{H}^* \mid \forall \overline{h} \in \overline{H}, \ \textstyle\sum \varphi_0(\overline{h})\varphi_{-1} = \varphi(\overline{h})\overline{1}\} \\
&= \{\varphi \in \overline{H}^* \mid \forall \overline{h} \in \overline{H}, \ \varphi(\overline{h})\overline{1} = \textstyle\sum \varphi(\overline{h}_1)\overline{h}_2\} \quad \text{by 4.6(1)} \\
&= \textstyle\int_{\overline{H}^*}^r,
\end{aligned}
$$

where the last equality follows as noted in Section 0.

Also $H \cong \overline{H} \otimes K$ as right K-modules and left $\overline{H}$-comodules by assumption. Thus using $(*)$, we obtain

$$(**) \qquad \overline{H}^* \cong H \otimes_K \int_{\overline{H}^*}^r \cong (\overline{H} \otimes K) \otimes_K \int_{\overline{H}^*}^r \cong \overline{H} \otimes \int_{\overline{H}^*}^r,$$

where the isomorphism is as left $\overline{H}$-comodules. The result now follows:

(a) Clearly $\int_{\overline{H}^*}^r$ is one-dimensional (hence in particular non-zero), since $\dim \overline{H} = \dim \overline{H}^* < \infty$. Pick $0 \neq \lambda \in \int_{\overline{H}^*}^r$

(b) Since $H \otimes_K \int_{\overline{H}^*}^r \overset{\cdot}{\longrightarrow} \overline{H}^*$ is surjective, given by $h \otimes \lambda \mapsto h \cdot \lambda = \lambda \leftharpoonup Sh$, it follows that there exists $\tilde{\Lambda} \in H$ such that $\lambda \leftharpoonup S\tilde{\Lambda} = \varepsilon$ on $\overline{H}$. Pick $\Lambda = S\tilde{\Lambda}$; then $\lambda \leftharpoonup \Lambda = \varepsilon$ on $\overline{H}$.

(c) Since $\int_{\overline{H}^*}^r = {}^{co\,\overline{H}}\overline{H}^* \in {}_K\mathcal{M}$, it follows that $\int_{\overline{H}^*}^r \leftharpoonup SK = \int_{\overline{H}^*}^r$. and so there exists $\tilde{\chi} \in K^*$ such that $\lambda \leftharpoonup Sl = \tilde{\chi}(l)\lambda \;\; \forall l \in K$. Note that by its definition, $\tilde{\chi}$ must be in $Alg(K, k)$.

(d) First, we see that $(**)$ determines an isomorphism $\phi : \overline{H} \to \overline{H}^*$ of left $\overline{H}$-comodules.

Taking transposes, $\phi^* : (\overline{H}^*)^* \to \overline{H}^*$ is an isomorphism of left $\overline{H}^*$-modules (for since $C = \overline{H}$ is finite-dimensional, any finite-dimensional left C-comodule V dualizes to a left C^*-module; that is, $V \to C \otimes V$ gives $C^* \otimes V^* \to V^*$). Now since $\overline{H}^* \cong \overline{H}^{**}$ as $\overline{H}^*$-modules, $\overline{H}^*$ is Frobenius. $\qquad \square$

It may be useful to describe these isomorphisms more explicitly. Let λ be a basis of $\int_{\overline{H}^*}^r$. For $h \in H$, the action of h on λ in the isomorphism $(*)$ is given by $h \cdot \lambda = \lambda \leftharpoonup Sh$. In $(**)$ we have identified $\overline{H}$ as a subspace of H via the cleft map. That is, let $\gamma : \overline{H} \to \overline{H} \otimes K \cong H$ via $\overline{h} \mapsto \overline{h} \otimes 1$. Then ϕ is given by $\overline{h} \mapsto \lambda \leftharpoonup (S\gamma(\overline{h}))$.

Now consider the isomorphism $\phi^* : \overline{H}^{**} \to \overline{H}^*$, where $\overline{H}^*$ is a left $\overline{H}^*$-module by left multiplication. The left $\overline{H}^*$-module structure of $\overline{H}^{**}$ is transposed to the right $\overline{H}^*$-module structure defined by multiplication on $\overline{H}^*$ as noted above; explicitly, the transposed $\overline{H}$-comodule structure $\overline{H}^* \to \overline{H} \otimes \overline{H}^*$, $g \mapsto \sum g_{-1} \otimes g_0$, is defined by $\sum g_0(\overline{h})g_{-1} = \sum g(\overline{h}_1)\overline{h}_2$, for all $\overline{h} \in \overline{H}$. Thus, the dual $\overline{H}^*$-module structure on $\overline{H}^{**}$ is given by

$$\begin{array}{ccccl} \overline{H}^* & \otimes & \overline{H}^{**} & \to & \overline{H}^{**}, \\ f & \otimes & \varphi & \mapsto & (g \mapsto \sum f(g_{-1})\varphi(g_0)). \end{array}$$

This action is in fact $f \rightharpoonup \varphi$. For, note that for all $\overline{h} \in \overline{H}$, $\sum f(g_{-1})g_0(\overline{h}) = \sum g(\overline{h}_1)f(\overline{h}_2) = gf(\overline{h})$, and so $\sum f(g_{-1})g_0 = gf$. Thus

$$\begin{aligned} (f \cdot \varphi)(g) &:= \sum f(g_{-1})\varphi(g_0) = \varphi(\sum f(g_{-1})g_0) \\ &= \varphi(gf) = (f \rightharpoonup \varphi)(g), \end{aligned}$$

which is what we would expect.

Part (b) of the next result has already been shown in Example 1.12. Part (c) extends work of [Sch 92], where it was shown that when H is pointed then $K \subset H$ is $\overline{H}$-cleft.

4.9. Corollary. *Let H be a Hopf algebra, $K \subset H$ a Hopf subalgebra, and $\overline{H} = H/HK^+$ as above. Assume that $\overline{H}$ is finite dimensional, and that either*

(a) *H has a right normal basis over K and the antipode of K is bijective, or*

(b) *H is finite-dimensional, or*

(c) *the coradical of H is cocommutative.*

Then $K \subset H$ is of right integral type and is a β-Frobenius extension. Moreover, $\overline{H}^*$ is a Frobenius algebra.

Proof. (a) Since H has a right normal basis over K, Theorem 4.8 applies. Noting that K is a Hopf subalgebra, we see that $SK \subset K$, and so we may take $\chi = \tilde{\chi} \circ \overline{S}$ to get 1.5 (c). Thus $K \subset H$ is of right integral type.

(b) This is a special case of (a), since by [Sch 92, 2.4(2b)], $K \subset H$ is $\overline{H}$-cleft whenever H is finite-dimensional. But cleftness implies the normal basis isomorphism.

For case (c), let $\overline{k}$ denote the algebraic closure of our base field k and consider $K' = \overline{k} \otimes K \subset H' = \overline{k} \otimes H$. Now H' is pointed, and thus $K' \subset H'$ is $\overline{H'}$-cleft by [Sch 92, 4.3(2)]. Thus Theorem 4.8 applies to give that $K' \subset H'$ is of right integral type, and that $\overline{H'}^*$ is a Frobenius algebra.

However $\overline{H'} = H'/H'(K')^+ \cong \overline{k} \otimes \overline{H}$ as coalgebras, and so $\overline{k} \otimes \overline{H}^* \cong \overline{H'}^*$ as algebras. Thus $\overline{H}^*$ is Frobenius, and $\int_{\overline{H}^*}^r$ is one-dimensional. The other properties also clearly descend from $K' \subset H'$ to $K \subset H$. Thus $K \subset H$ is of right integral type.

Finally, when H_0 is cocommutative $K \subset H$ is faithfully flat by [T 72]. It now follows by Theorem 1.14 with $A = W = H$ and $B = U = K$ that in all these cases $K \subset H$ is β-Frobenius. $\qquad\square$

The next result can be considered a "dual version" of Theorem 4.8.

4.10. Theorem. *Let H be a Hopf algebra, K a right coideal subalgebra, and $\overline{H} = H/HK^+$. Assume H has a right normal basis over K and that $\dim K < \infty$. Then*

(a) $\overline{K^*} := K^*/K^* \cdot K^+$ *is one-dimensional, and*

(b) K *is a Frobenius algebra.*

Proof. First, note that $K \in {}_K\mathcal{M}^H$, where the H-comodule structure is given by $\triangle : K \to K \otimes H$. Thus by Lemma 4.7(b), $K^* \in \mathcal{M}_K^H$, where K acts on K^* via $\leftharpoonup$. Now

$$
\begin{aligned}
K^* &\cong \overline{K^*}\square_{\overline{H}}H & \text{by 4.5(b) with } M = K^* \\
&\cong \overline{K^*}\square_{\overline{H}}(\overline{H} \otimes K) & \text{by the normal basis isomorphism} \\
&\cong \overline{K^*} \otimes K,
\end{aligned}
$$

where the isomorphisms are as right K-modules. The isomorphism $K^* \to \overline{K^*} \otimes K$ is given explicitly by $f \mapsto \sum \overline{f_0} \otimes \varphi(f_1)$, where $\varphi : H \to K$ defines the normal basis isomorphism $H \xrightarrow{\cong} \overline{H} \otimes K$, via $\overline{h} \mapsto \sum \overline{h_1} \otimes \varphi(h_2)$.

Since $\dim K = \dim K^* < \infty$, clearly $\dim \overline{K^*} = 1$, proving (a). Since $K^* \cong K$ as right K-modules, K is Frobenius, proving (b). $\qquad\square$

All of this section could have been proved on the other side as well. The next corollary will be used for Corollary 5.8, where it is needed on the other side, so we will state and prove it on the other side here.

4.11. Corollary. *Let A and H be Hopf algebras and $\pi : A \to H$ a surjective Hopf algebra map. Consider A as a right H-comodule via π and define $R := A^{co\, H}$. Assume that $\dim R < \infty$ and that there exists a right H-colinear and invertible map $\gamma : H \to A$. Then*

(a) R *is a Frobenius algebra and has non-zero left integrals and right integrals.*
(b) *Choose* $0 \neq t \in \int_R^r$. *Then* $\varepsilon(t) \neq 0 \Leftrightarrow R$ *is separable.*

Proof. (a) More generally, let $A \to \overline{A} = A/I$ be a quotient right A-module and coalgebra and assume there exists $\gamma : \overline{A} \to A$ which is right $\overline{A}$-colinear and invertible. Let $R := A^{co\,\overline{A}}$, a left coideal subalgebra of A, and assume that $\dim R < \infty$. Then by [MD, (1.4),(1.5)],

$$R \otimes \overline{A} \to A, \quad \text{given by } r \otimes \overline{a} \mapsto r\gamma(\overline{a}),$$

is bijective, left R-linear, and right $\overline{A}$-colinear. By Lemma 4.3, $I = R^+ A$, and so Theorem 4.10 (on the other side) applies. Thus R is Frobenius. The algebra R is augmented via the restriction of ε_A, so $\int_R^r$ is one-dimensional since it is the right annihilator of R^+; similarly for $\int_R^l$.

(b) If R is separable, then $R \xrightarrow{\varepsilon} k$ splits, as right R-modules, and thus $\varepsilon(t) \neq 0$ for $0 \neq t \in \int_R^r$.

Conversely, we first show that $S_A^2(R) \subset R$. To see this, first note that π is a Hopf algebra map, so in particular $\pi \circ S_A = S_H \circ \pi$, where S_H is the antipode for H. Thus, $\forall r \in R$,

$$(\pi \otimes id)\triangle S_A^2 r = (\pi \otimes id)\sum S_A^2 r_1 \otimes S_A^2 r_2 = \sum (S_H)^2 \pi(r_1) \otimes S_A^2 r_2 = 1 \otimes S_A^2 r,$$

and so $S_A^2(R) \subset R$

Now part (b) follows from the next lemma, which is [K, 5.2] on the other side. We give the argument for completeness.

Lemma [K]. *Let* $R \subset A$ *be a finite-dimensional left coideal subalgebra of* A *such that* $S^2(R) \subset R$. *Let* $0 \neq t \in \int_R^r$ *such that* $\varepsilon(t) = 1$. *Then*
(a) $\sum rSt_1 \otimes t_2 = \sum St_1 \otimes t_2 r$, *for all* $r \in R$,
(b) $\sum St_1 \otimes t_2 \in R \otimes R$.
In particular, R *is separable, since* $\sum (St_1)t_2 = \varepsilon(t) = 1$.

Proof. (a) Indeed,

$$
\begin{aligned}
\sum St_1 \otimes t_2 r &= \sum r_1 S(t_1 r_2) \otimes t_2 r_3 \\
&= \sum r_1 S((tr_2)_1) \otimes (tr_2)_2 \\
&= \sum r(St_1) \otimes t_2,
\end{aligned}
$$

since t is a right integral.

(b) Write $\triangle t = \sum_i t_{1,i} \otimes t_{2,i}$, where $\{t_{2,i}\}$ is a k-basis of R, and consider $X = \sum_i k(St_{1,i})$. Then $RX \subset X$ by (a), and also $1 \in X$, since $1 = \varepsilon(t) = \sum S(t_1(St_2)) = \sum S^2(t_2)St_1 \in RX$, because $t_2 \in R$. Hence $R \subset RX \subset X$, and so $R = X$, since $\dim X \leq \dim R$ by construction. Thus $\sum St_1 \otimes t_2 \in R \otimes R$. $\square$

5. Hopf algebras in Yetter-Drinfeld categories

Let H be a Hopf algebra with a bijective antipode. The Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is the braided monoidal category whose objects M are both left H-modules and left H-comodules and satisfy the compatibility condition

(5.1) $$\sum h_1 m_{-1} \otimes h_2 \cdot m_0 = \sum (h_1 \cdot m)_{-1} h_2 \otimes (h_1 \cdot m)_0$$

for all $h \in H$, $m \in M$. The braiding $\tau : V \otimes W \mapsto W \otimes V$ in this category is given by

$$(5.2) \qquad \tau(v \otimes w) = \sum v_{-1} \cdot w \otimes v_0.$$

For details, see [Y]. Now, $R \in {}_H^H\mathcal{YD}$ is a bialgebra in this category if it is both an algebra and a coalgebra, and the bialgebra structure maps are all category morphisms. Moreover, $\triangle_R$ must be multiplicative in ${}_H^H\mathcal{YD}$, using τ on $R \otimes R$; that is

$$(5.3) \qquad \triangle_R(rs) = \sum r^1((r^2)_{-1} \cdot s^1) \otimes (r^2)_0 s^2,$$

where $\triangle_R(r) = \sum r^1 \otimes r^2$ denotes the comultiplication of R. For a given H, an R which was an algebra and a coalgebra in ${}_H\mathcal{M}$ and in ${}^H\mathcal{M}$, and satisfied the compatibility conditions (5.1) and (5.3), was called *admissible* in [R 85]. The fact that this is the same as R being a bialgebra in ${}_H^H\mathcal{YD}$ is noted in [Mj]. Given such an R, we can form Radford's biproduct $A = R \star H$ [R 85, Theorem 1], which is a usual bialgebra. As an algebra, $A = R \star H$ is the smash product $R\#H$, and as a coalgebra it is the smash coproduct, that is

$$\triangle_A(r \star h) = \sum (r^1 \star (r^2)_{-1}h_1) \otimes ((r^2)_0 \star h_2).$$

R is a Hopf algebra in ${}_H^H\mathcal{YD}$ if it has an antipode S_R, that is, a convolution inverse to the identity. $R \star H$ is then a Hopf algebra with antipode given by $S(r \star h) = \sum(1 \star S_H(r_{-1}h))(S_R r_0 \star 1)$ [R 85, Theorem 2]. S_R is a map in ${}_H^H\mathcal{YD}$, so it is both H-linear and H-colinear. For details of these constructions, see also [M, Section 10.6]. Note that under $\triangle_A$, R is a left coideal subalgebra of A, but not a subcoalgebra.

Now let $S \subset R$ be an extension of Hopf algebras in the Yetter-Drinfeld category ${}_H^H\mathcal{YD}$. Set $A = W = R \star H$ and $B = U = S \star H$, $\sigma = \triangle_A$ and $\rho = id \otimes \triangle_H$, and assume that A is left or right faithfully flat over B. Then $R \cong A^{co\,H}$ and $B = {}^{co\,\overline{A}}A$, where $\overline{A} = \overline{W} = (R \star H)/(R \star H)(S \star H)^+$. In fact $\overline{A} \cong \overline{R} = R/RS^+$; see Lemma 5.5 below.

Note that we now have a Hopf algebra surjection $\pi : R \star H \twoheadrightarrow H$, $r \star h \mapsto \varepsilon_R(r)h$, and so also have the commutative diagram as in Example 2.7 (b),

$$(5.4) \qquad \begin{array}{ccccc} S & \subset & S \star H & \overset{\pi}{\twoheadrightarrow} & H \\ \cap & & \cap & & \| \\ R & \subset & R \star H & \overset{\pi}{\twoheadrightarrow} & H \end{array}$$

This diagram will be used frequently in the following sections.

Now $S \subset S \star H$ and $R \subset R \star H$ are faithfully flat H-Galois extensions, and $k = {}^{co\,A}A \subset A$ is A-Galois; thus $(S \star H, R \star H, H, R \star H)$ is a faithfully flat bi-Galois extension.

5.5. Lemma. *Let $S \subset R$ be an extension of Hopf algebras in the category ${}_H^H\mathcal{YD}$. Then RS^+ is a coideal of R. If also $B \subset A$ is left or right faithfully flat, then the inclusion $R \subset A$ induces an isomorphism of coalgebras $\overline{R} \cong \overline{A}$.*

Proof. We first check that RS^+ is a coideal of R; note that there is something to prove here since $\triangle_R$ is not a usual algebra map. Thus, choose $r \in R$, $s \in S^+$. Since S^+ is a coideal of S, and S is a subcoalgebra of R, we have $\triangle_R s \in S \otimes S^+ + S^+ \otimes S$.

Since S is in $^H_H\mathcal{YD}$ and ε_S is an H-module map, if follows that $H \cdot S^+ \subseteq S^+$. Then

$$
\begin{aligned}
\triangle_R(rs) &= \sum r^1((r^2)_{-1} \cdot s^1) \otimes (r^2)_0 s^2 \\
&\in \sum r^1((r^2)_{-1} \cdot S) \otimes (r^2)_0 S^+ + \sum r^1((r^2)_{-1} \cdot S^+) \otimes (r^2)_0 S \\
&\in R \otimes RS^+ + RS^+ \otimes R.
\end{aligned}
$$

Thus RS^+ is a coideal and $\overline{R} = R/RS^+$ is a coalgebra.

Now assume the faithfully flat hypothesis. Then by the above remarks, we are in the situation of 2.7, and thus Proposition 2.8 applies. Consider $R \subset A$ via $r \mapsto r\star 1$. Then

$$
\triangle_A(r \star 1) = \sum ((r^1 \star 1)(1 \star (r^2)_{-1})) \otimes ((r^2)_0 \star 1).
$$

Under the map $A \to \overline{A}$, $r \star h \mapsto (\overline{r \star 1})(\overline{1 \star h}) = \varepsilon(h)(\overline{r \star 1})$. Thus

$$
\begin{aligned}
\triangle_{\overline{A}}(\overline{r \star 1}) &= \sum \varepsilon((r^2)_{-1})(\overline{r^1 \star 1}) \otimes ((\overline{r^2})_0 \star 1) \\
&= \sum (\overline{r^1 \star 1}) \otimes (\overline{r^2 \star 1}).
\end{aligned}
$$

Thus the inclusion $\overline{R} \subseteq \overline{A}$ is a coalgebra isomorphism.   □

We write the H-comodule structure of R as $\delta_R : R \to H \otimes R$, $r \mapsto \sum r_{-1} \otimes r_0$. We say $S \subset R$ has *finite index* if $\overline{R} = R/RS^+$ is finite dimensional.

5.6. Theorem. *Let $S \subset R$ be an extension of Hopf algebras of finite index in $^H_H\mathcal{YD}$, and let $A = R \star H$ and $B = S \star H$. Assume that A has a right normal basis over B and that R, S and H have bijective antipodes. Then $\int^r_{\overline{R}^*} = k\lambda$ for some $\lambda \neq 0$, we may choose $\Lambda \in R$ such that $\lambda \leftharpoonup \Lambda = \varepsilon$, and there exist $\chi_H \in Alg\,(H,k)$ and $\chi \in Alg\,(S,k)$ such that*

$$
\lambda(\overline{h \cdot r}) = \chi_H(h)\lambda(\overline{r}) \quad and \quad \lambda(\overline{sr}) = \chi(s)\lambda(\overline{r}),
$$

for all $r \in R$, $s \in S$, and $h \in H$.

Consequently $S \subset R$ is β-Frobenius, where $\beta : S \to S$ and the Frobenius homomorphism $f : R \to S$ are given by

$$
\begin{aligned}
\beta(s) &:= \sum \chi(s^1)\chi_H((s^2)_{-1})(s^2)_0, \\
f(r) &:= \sum \lambda(\overline{r^1})r^2,
\end{aligned}
$$

for all $s \in S$, $r \in R$. Moreover, dual bases (with respect to f) are given by

$$
\{\overline{S}_R((\Lambda^2)_0), \; \chi_H((\Lambda^2)_{-1})\overline{S}_H((\Lambda^2)_{-2}) \cdot \Lambda^1\}
$$

Proof. Since A has a right normal basis over B, $B \subset A$ is of right integral type, by Corollary 4.9. Since $R \subset A = R \star H$ and $S \subset B$ are faithfully flat, Theorem 3.1 and Corollary 3.3 apply to give $\int^r_{\overline{A}^*} = \int^r_{\overline{R}^*} = k\lambda$, for some $\lambda \neq 0$, and $\Lambda \in R$, $\chi \in Alg\,(B,k)$, such that $\lambda \leftharpoonup \Lambda = \varepsilon$ and $\lambda \leftharpoonup b = \chi(b)\lambda$, for all $b \in B$.

Moreover $S \subset R$ is β-Frobenius, with $\beta(s) := \sum \chi(s_1)s_2$, for all $s \in S$, and the Frobenius homomorphism $f(r) := \sum \lambda(\overline{r_1})r_2$, for all $r \in R$. Dual bases of $B \subset A$ are given by $\{\overline{S}\Lambda_2, \Lambda_1\}$.

The difficulty is that these descriptions of β, f, and the dual bases use $\triangle_A$ and not $\triangle_R$. To express them in terms of $\triangle_R$, we use [R 85, Theorem 3]. That is, $A = R \star H$ has a projection π onto H, so that there exists $\gamma : H \to A$, a Hopf algebra map, satisfying $\pi\gamma = id_H$. In our case, clearly $\gamma(h) = 1 \star h$; in particular, $\gamma(H) \subset B$. Then $A^{co\,H} = R$ is a coalgebra, via

$$
\triangle_R(r) = \sum r^1 \otimes r^2 := \sum r_1 \gamma\pi(S_A r_2) \otimes r_3,
$$

an H-module algebra via

$$h \cdot r := \sum \gamma(h_1) r \gamma(S_H h_2),$$

and an H-comodule algebra via

$$\sum r_{-1} \otimes r_0 := \sum \pi(r_1) \otimes r_2.$$

The antipode on R is $S_R r := \sum \gamma\pi(r_1) S_A r_2$, with inverse $\overline{S}_R r = \sum (\overline{S}_A r_2) \gamma\pi(r_1)$, where the antipode S_A of A is bijective. See [R 85, p. 337].

First note that $\chi_H = \chi_A \circ \gamma$. Using $\overline{R} \cong \overline{A} = A/AB^+$, we get

$$
\begin{aligned}
\lambda(\overline{h \cdot r}) &= \lambda(\overline{\sum \gamma(h_1) r \gamma(S_H h_2)}) \\
&= \lambda(\overline{\sum \gamma(h_1) r \varepsilon(\gamma(S_H h_2))}) \quad \text{since } \gamma(H) \subseteq B \\
&= \lambda(\overline{\gamma(h) r}) \\
&= \chi(\gamma(h)) \lambda(\overline{r}).
\end{aligned}
$$

We can now check the formulas for β and f. Using $\triangle_R$ as above, for all $s \in S$ we have

$$
\begin{aligned}
\sum \chi(s^1) \chi_H((s^2)_{-1})(s^2)_0 &= \sum \chi(s_1 \gamma\pi(S_A s_2)) \chi_H(\pi s_3) r_4 \\
&= \sum \chi(s_1) \chi(\gamma\pi(S_A s_2)) \chi\gamma(\pi s_3) s_4 \\
&= \sum \chi(s_1) s_2 = \beta(s),
\end{aligned}
$$

and for all $r \in R$,

$$
\begin{aligned}
\sum \lambda(\overline{r^1}) r^2 &= \sum \lambda(\overline{r_1 \gamma\pi(S_A r_2)}) r_3 \\
&= \sum \lambda(\overline{r_1 \varepsilon(\gamma\pi(S_A r_2))}) r_3 \quad \text{as } \gamma(H) \subseteq B \\
&= \sum \lambda(\overline{r_1}) r_2 = f(r).
\end{aligned}
$$

Finally, the fact that the dual bases $\{\overline{S}_A \Lambda_2, \Lambda_1\}$ can be rewritten as the desired elements in the statement of the theorem follows from the next lemma. We would like to thank N. Andruskiewitsch for help in the computations in the lemma.

5.7. Lemma. *In $A_\beta \otimes_B A$, we have, for all $r \in R$,*
  (a) $\sum \overline{S}_A r_2 \otimes r_1 = \sum (\overline{S}_A r_5) \gamma\pi(r_4) \otimes \chi(\gamma\pi(r_3))(\gamma\pi(\overline{S}_A r_2)) r_1$.
  (b) $\sum \overline{S}_A r_2 \otimes r_1 = \sum \overline{S}_R(r^2)_0 \otimes \chi_H((r^2)_{-1}) \overline{S}_H((r^2)_{-2}) \cdot r^1$.

Proof. For (a), since $\gamma(H) \subseteq B$,

$$
\begin{aligned}
\sum \overline{S}_A r_2 \otimes r_1 &= \sum (\overline{S}_A r_4) \gamma\pi(r_3) \gamma\pi(\overline{S}_A r_2) \otimes r_1 \\
&= \sum (\overline{S}_A r_4) \gamma\pi(r_3) \otimes \beta^{-1}(\gamma\pi(\overline{S}_A r_2)) r_1.
\end{aligned}
$$

Since $\beta^{-1}(\gamma\pi(\overline{S}_A r_2)) = \sum \chi^{-1}(\gamma\pi(\overline{S}_A r_3)) \gamma\pi(\overline{S}_A r_2)$, it follows that

$$\sum \overline{S}_A r_2 \otimes r_1 = \sum (\overline{S}_A r_5) \gamma\pi(r_4) \otimes \chi\gamma\pi(r_3) \gamma(\pi)(\overline{S}_A r_2) r_1.$$

  (b) Since $\triangle_R r = \sum r^1 \otimes r^2 = \sum r_1 \gamma\pi(S_A r_2) \otimes r_3$, it follows that

$$\sum r^1 \otimes (r^2)_{-2} \otimes (r^2)_{-1} \otimes (r^2)_0 = \sum r_1 \gamma\pi(S_A r_2) \otimes \pi(r_3) \otimes \pi(r_4) \otimes r_5.$$

Thus

$$
\begin{aligned}
&\sum \overline{S}_R(r^2)_0 \otimes \chi_H((r^2)_{-1}) S_H((r^2)_{-2})) \cdot r^1 \\
&= \sum (\overline{S}_A r_6) \gamma\pi(r_5) \otimes \chi\gamma(\pi(r_4)) \overline{S}_H(\pi(r_3)) \cdot (r_1 \gamma\pi(S_A r_2)) \\
&= \sum (\overline{S}_A r_7) \gamma\pi(r_6) \otimes \chi\gamma(\pi r_5) \gamma(\overline{S}_H(\pi r_4)) r_1 \gamma\pi(S_A r_2) \gamma(S_H \overline{S}_H(\pi r_3)) \\
&= \sum (\overline{S}_A r_5) \gamma\pi(r_4) \otimes \chi(\gamma\pi(r_3)) \gamma\pi(\overline{S} r_2) r_1.
\end{aligned}
$$

This finishes the proof of the lemma. $\qquad \square$

Note that the hypotheses of Theorem 5.6 are always satisfied if either $A = R \star H$ is finite-dimensional or A is pointed. In either case, A is $\overline{A}$-cleft by [Sch 92] and thus has the normal basis property (see also Corollary 4.9).

As a corollary, we obtain analogues of the Larson-Sweedler results [LS] for ordinary finite-dimensional Hopf algebras.

5.8. Corollary. *Let R be a finite-dimensional Hopf algebra in $^H_H \mathcal{YD}$. Then R is a Frobenius algebra, and $\int^r_R$ is one-dimensional. If $0 \neq t \in \int^r_R$, then $\varepsilon(t) \neq 0 \Leftrightarrow R$ is a separable k-algebra.*

Proof. Let $A = R \star H$, the biproduct as above; A is a Hopf algebra. Then $\pi : A \to H$, via $r \star h \mapsto \varepsilon(r)h$, is a surjective Hopf algebra map with $R = A^{co\,H}$. Thus π is split, via $\gamma : H \to A$, $h \mapsto 1 \star h$. The result is now a special case of Corollary 4.11. $\square$

In fact Theorem 5.6 gives more information in this case: the Frobenius homomorphism $f : R \to k$ is given by $f(r) = \sum \lambda(r^1)r^2 = \lambda(r)$ (since $f(r) \in k$), and dual bases for R (with respect to f) are given by

$$\{\overline{S}_R((t^2)_0), \; \chi_H((t^2)_{-1})\overline{S}_H((t^2)_{-2}) \cdot t^1\}.$$

We remark that it is quite complicated to prove directly the equations showing that the given elements in Theorem 5.6 are dual bases. The direct proof, entirely inside $^H_H \mathcal{YD}$, is difficult even for the special case above when R is finite-dimensional and $S = k$. If also H is finite-dimensional, then the computations simplify slightly and we have an explicit formula for χ_H, as we see next.

5.9. *Remark.* Let H be finite-dimensional; we give an elementary proof of Corollary 5.8 as well as an explicit formula for χ_H. For then $A = R \star H$ is a finite-dimensional Hopf algebra, so it is a Frobenius algebra. Since A is free over R on both sides, it follows that R is also Frobenius. If $0 \neq t \in \int^r_R$ and $u \in \int^r_H$, then one can check directly that $w = t \star u \in \int^r_A$, using that ε_R is an H-module map.

Now for any $h \in H$,

$$\alpha_A(h)w = hw = \sum (h_1 \cdot t) \star h_2 u = \sum (h_1 \cdot t)\alpha_H(h_2) \star u,$$

and thus $\alpha_A(h)t = \sum (h_1 \cdot t)\alpha_H(h_2)$, where α_A and α_H are the right modular functions for A and H, respectively. It follows that $h \cdot t = \chi_H(h)t$, where

$$\chi_H = \alpha_A|_H * \alpha_H^{-1} \in H^*.$$

Since ε_R is an H-module map, it follows that $\varepsilon_R(h \cdot t) = \varepsilon_H(h)\varepsilon_R(t) = \chi(h)\varepsilon_R(t)$. Thus if $\varepsilon_R(t) \neq 0$, it follows that $\chi_H = \varepsilon_H$ and so $t \in R^H$. Using this fact, a twisted version of the usual Maschke argument shows that $\sum t_1 \otimes St_2$ centralizes R. To see this, first note that since R is Frobenius, $\int^r_R$ is one-dimensional, and thus (as for usual Hopf algebras) t is also in $\int^l_R$ since $\varepsilon(t) \neq 0$. Using (5.3), it then follows that for all $r \in R$,

$$(*) \qquad \begin{aligned} \triangle_R t \otimes r &= \sum \triangle_R(\varepsilon(r^1)t) \otimes r^2 = \sum \triangle_R(r^1 t) \otimes r^2 \\ &= \sum r^1((r^2)_{-1} \cdot t^1) \otimes (r^2)_0 t^2 \otimes r^3. \end{aligned}$$

Also

$$(**) \qquad \varepsilon(h)\triangle(t) = \triangle(\varepsilon(h)t) = \triangle_R(h \cdot t) = h \cdot \triangle(t) = \sum h_1 \cdot t^1 \otimes h_2 \cdot t^2,$$

for all $h \in H$, since $t \in R^H$. Now for any $r \in R$, we have

$$
\begin{aligned}
\sum t^1 \otimes (St^2)r &= \sum r^1((r^2)_{-1} \cdot t^1) \otimes S((r^2)_0 t^2) r^3 \text{ using } (*) \\
&= \sum r^1((r^2)_{-2} \cdot t^1) \otimes S((r^2)_{-1} \cdot t^2) S((r^2)_0) r^3 \\
&= \sum r^1 t^1 \otimes (St^2)\varepsilon((r^2)_{-1}) S((r^2)_0) r^3 \text{ using } (**) \\
&= \sum r^1 t^1 \otimes (St^2)(Sr^2) r^3 \\
&= \sum rt^1 \otimes St^2.
\end{aligned}
$$

Since $\sum t_1 St_2 = \varepsilon(t) \neq 0$, it follows that R is separable.

We will see in Example 5.12 that if R is not separable, then χ_H can be non-trivial and $t \notin R^H$.

5.10. Example: Graded Hopf algebras. Let $H = kG$, where G is an abelian group with a bicharacter $\langle \mid \rangle : G \times G \to k^*$, and consider the category of kG-comodules ${}^{kG}\mathcal{M}$. Recall that ${}^{kG}\mathcal{M}$ is just the category of G-graded k-modules; moreover it is contained in the Yetter-Drinfeld category ${}^H_H \mathcal{YD}$, for $H = kG$, since any $M \in {}^{kG}\mathcal{M}$ is also a left kG-module via $g \cdot m_h = \langle g|h \rangle m_h$, for all $g, h \in G$ and $m_h \in M_h$, the h-component of M. Then for G-graded modules V and W, and homogeneous elements $v \in V_g$ and $w \in W_h$, the twist is given by

$$
\tau : V \otimes W \to W \otimes V, \text{ via } v \otimes w \mapsto \langle g|h \rangle w \otimes v.
$$

For A and B G-graded algebras, $A \otimes B$ becomes an associative algebra via

$$
(m_A \otimes m_B) \circ (id \otimes \tau \otimes id) : A \otimes B \otimes A \otimes B \to A \otimes B.
$$

R is a G-graded Hopf algebra if it is both a G-graded algebra and a G-graded coalgebra, if $\triangle_R$ is multiplicative in ${}^{kG}\mathcal{M}$ using the above multiplication in $A \otimes A$, and if the antipode S is a map in ${}^{kG}\mathcal{M}$. It will then follow that the biproduct $A = R \star kG$ is an ordinary Hopf algebra. If $S \subset R$ are G-graded Hopf algebras, diagram (5.4) becomes

$$
\begin{array}{ccccc}
S & \subset & S \star kG & \overset{\pi}{\twoheadrightarrow} & kG \\
\cap & & \cap & & \| \\
R & \subset & R \star kG & \overset{\pi}{\twoheadrightarrow} & kG
\end{array}
$$

As a special case, we will consider extensions $U(K) \subset U(L)$ of G-Lie coloralgebras in Section 6. We also consider an old example of Radford:

5.11. Radford's example revisited. We reconsider the example of Radford [R 85, Section 4]. We first rewrite it in the language of graded Hopf algebras, which we believe simplifies the exposition, and then compute for it the data in Theorem 5.6. Let $G = \mathbb{Z}_n$, and assume that the base field k contains a primitive n^{th} root of 1, say ω. Define

$$
\langle \mid \rangle : \mathbb{Z}_n \times \mathbb{Z}_n \to k^* \quad \text{by} \quad \langle i \mid j \rangle = \omega^{ij},
$$

for all $i, j \in \mathbb{Z}_n$. Then $\langle \mid \rangle$ is a bicharacter on $\mathbb{Z}_n$, so as above it determines a braiding on the category of $\mathbb{Z}_n$-graded modules: if $M = \bigoplus_{i \in \mathbb{Z}_n} M_i$ and $N = \bigoplus_{j \in \mathbb{Z}_n} N_j$, and $m_i \in M_i$, $n_j \in N_j$, then

$$
\tau : M \otimes N \to N \otimes M \quad \text{is given by} \quad m_i \otimes n_j \mapsto \omega^{ij}(n_j \otimes m_i).
$$

Radford's example (in the case $m = 1$) is defined to be the k-algebra

$$R = k\langle u_0, \cdots, u_{n-1} \mid u_i u_j = 0,\ i \neq j, \quad \text{and} \quad u_i^{n+1} = \alpha_i u_i,\ i = 0, \cdots, n-1 \rangle$$

where the scalars $\alpha_i \in k^*$ will be determined below. Note that for all i, $e_i = \alpha_i^{-1} u_i^n$ is an idempotent and $1_R = \sum_{i=0}^{n-1} e_i$. Thus as an algebra, $R = \bigoplus_{i=0}^{n-1} S_i$, where $S_i = k\text{-span}\{u_i^j \mid j = 1, \cdots, n\}$, and so $\dim R = n^2$. If also $\alpha_i^{-1/n} = \beta_i \in k$, then $(\beta_i u_i)^n = e_i$ and so $S_i \cong k\mathbb{Z}_n$, for all i. R becomes a $\mathbb{Z}_n$-graded algebra by defining $\text{degree}(u_i) := i$. That is, $R = \bigoplus_{i \in \mathbb{Z}_n} R_i$, with $R_i R_j \subseteq R_{i+j(\text{mod } n)}$, where $R_i = k\text{-span}\{u_j^k \mid kj \equiv i \ (\text{mod } n)\}$.

We next consider R as a graded coalgebra. Define

$$\triangle u_i = \sum_{l=0}^{n-1} u_l \otimes u_{i-l} \quad \text{and} \quad \varepsilon(u_i) = \delta_{0,i},$$

for all $i = 0, \cdots, n-1$. Thus $\triangle u_i \in \sum_l R_l \otimes R_{i-l}$. We define $\triangle(u_i^k)$ so that $\triangle$ becomes a graded-multiplicative map on R. Using induction, this means that

$$\triangle(u_i^k) := (\triangle u_i)^k \quad \text{(where we use the graded product on } R \otimes R\text{)}$$

$$(*) \qquad = \sum_{l=0}^{n-1} \omega^{l(i-l)(1+2+\cdots+(k-1))} u_l^k \otimes u_{i-l}^k$$

$$= \sum_{l=0}^{n-1} \omega^{l(i-l)\frac{k(k-1)}{2}} u_l^k \otimes u_{i-l}^k.$$

For $k = n+1$, $(*)$ must be compatible with the relations $u_i^{n+1} = \alpha_i u_i$ in R. Solving, we see that a necessary and sufficient condition on the α_i for $\triangle$ to be graded-multiplicative is that

$$(**) \qquad \alpha_i = \left(\omega^{\frac{n(n+1)}{2}} \right)^{l(i-l)} \alpha_l \alpha_{i-l},$$

for all $i, l = 0, \cdots,\ n-1$. Now if n is odd, $\omega^{\frac{n(n+1)}{2}} = 1$ and if n is even, $\omega^{\frac{n(n+1)}{2}} = -1$. Summarizing, we have:

Proposition. *With generators and relations as above, R is a $\mathbb{Z}_n$-graded bialgebra provided we can find $\alpha_i \in k^*$ such that*
(1) $\alpha_i = \alpha_l \alpha_{i-l}$, all i, l, if n is odd, or
(2) $\alpha_i = (-1)^{l(i-l)} \alpha_l \alpha_{i-l}$, all i, l if n is even.
Moreover $R \cong (k\mathbb{Z}_n)^{(n)}$ as an algebra if $\alpha_i^{1/n} \in k$, for all i.

Example. If n is odd, condition (1) is satisfied by setting $\alpha_i = \omega^i$. If n is even and k contains a primitive 4th root of 1, condition (2) is satisfied by setting $\alpha := (-1)^{\frac{i(n-i)}{2}} \omega^i$. We continue with these assumptions on k and the α_i.

Now R becomes a $\mathbb{Z}_n$-graded Hopf algebra by defining $S_R u_i = \alpha_{n-i}^{-1} u_{n-i}^{n-1}$; note that $S_R u_i \subseteq R_i$ since $(n-1)(n-i) \equiv i \ (\text{mod } n)$. S_R is extended to R as a graded (anti-) homomorphism; then for $1 \leq k \leq n$

$$S_R(u_i^k) = (\alpha_{n-1})^{-1} \omega^{i^2 \binom{k}{2}} u_{n-i}^{n-k}.$$

We check that $S_R * id = \varepsilon$ on u_i:

$$(S_R * id)(u_i) = \sum_{l=0}^{n-1}(Su_l)u_{i-l} = \sum_{l=0}^{n-1}\alpha_{n-l}^{-1}u_{n-l}^{n-1}u_{i-l} = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ 1 & \text{if} \quad i = 0 \end{cases}$$
$$= \delta_{i,0} = \varepsilon(u_i).$$

Radford shows that S_R has order $2n$. However in the biproduct $A = R \star H = R \# k\mathbb{Z}_n$ we have $S_A^2 = id$, and thus A is involutory.

Thus we have constructed a semisimple graded Hopf algebra which is not a usual Hopf algebra, since $\triangle_R$ is not an algebra map in the usual sense. Such an example is not possible if we use $R = u(L)$, the restricted enveloping algebra of a restricted Lie superalgebra $L = L_0 \oplus L_1$, which is a $\mathbb{Z}_2$-graded Hopf algebra. For in that case, it is shown in [Be] that $u(L)$ is semisimple only if $L_1 = 0$; that is, $u(L)$ is an ordinary Hopf algebra.

We now turn to the data in Corollary 5.4 and Theorem 5.6. First, it is easy to see that $t = \sum_{k=1}^{n} u_0^k$ is a two-sided integral for R, since $tu_j = 0 = \varepsilon(u_j)$ if $j \neq 0$, and $tu_0 = t = \varepsilon(u_0)t$ (since $\alpha_0 = 1$ implies $u_0^{n+1} = u_0$). Now $\varepsilon(t) = \sum_{k=1}^{n} \varepsilon(u_0)^k = n \neq 0$ in k, as predicted since R is semisimple.

Next we compute the dual bases for R. In this case, the formula in Theorem 5.6 simplifies considerably. For, we may choose Λ to be any (right) integral in R; thus let $\Lambda = t$ as above. Since R is semisimple, $\chi_H = \varepsilon$ from Remark 5.9; thus the dual bases after Corollary 5.8 become

$$\{\overline{S_R}((t^2)_0), \overline{S_H}((t^2)_{-1}) \cdot t'\}.$$

More specifically,

$$\triangle_R t = \sum_{k=1}^{n} \triangle_R(u_0^k) = \sum_{k} \sum_{l=0}^{n-1} \omega^{-l^2\binom{k}{2}}u_l^k \otimes u_{n-l}^k.$$

Since $\overline{S}$ is the composition inverse of S, it is easy to see that

$$\overline{S}(u_{n-i}^k) = \alpha_{n-i}\omega^{-i^2\binom{k}{2}}u_i^{n-k}.$$

For $t^2 = u_{n-l}^k \in R_{-lk}$, we have $(t^2)_0 = t^2$ and $(t^2)_{-1} = -lk$. Then the dual bases for R are

$$\left\{\alpha_{n-i}\omega^{-i^2\binom{n}{2}} + u_i^{n-k}, \ u_i^k\right\}.$$

5.12. Example. Again we consider a $\mathbb{Z}_n$-graded Hopf algebra, with the same bicharacter on $\mathbb{Z}_n$ as in Example 5.11, and thus the same braiding. Let

$$R = k[\, x \mid x^n = 0\,].$$

R is a $\mathbb{Z}_n$-graded algebra by setting $R_i = kx^i$, and it is also a $\mathbb{Z}_n$-graded coalgebra by setting $\triangle_R x = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$ and extending multiplicatively using the twist map. Thus

$$\triangle_R x^k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q x^i \otimes x^{k-i},$$

where $\begin{bmatrix} k \\ i \end{bmatrix}_q$ is the q-binomial coefficient using $q = \omega$. R is a $\mathbb{Z}_n$-graded Hopf algebra, with

$$S_R(x^k) = (-1)^k \omega^{\binom{k}{2}}x^k.$$

It is easy to see that $t = x^{n-1}$ is a left and right integral for R. As in Example 5.10, $\mathbb{Z}_n$ acts on $r_{\bar{j}} \in R_{\bar{j}}$ via $\bar{i} \cdot r_{\bar{j}} = \langle i \mid j \rangle r_{\bar{j}} = \omega^{ij} r_{\bar{j}}$. Then $\bar{i} \cdot t = \omega^{i(n-1)} t = \omega^{-i} t$. Thus using Remark 5.9, we see that $\chi_H(\bar{i}) = \omega^{-i}$. In particular, $\chi_H \neq \varepsilon_H$ and $t \notin R^G$.

Using the comment after Corollary 5.8, we can find dual bases for R. Since

$$\triangle_R t = \triangle_R x^{n-1} = \sum_{i=0}^{n-1} \begin{bmatrix} n-1 \\ i \end{bmatrix}_q x^i \otimes x^{n-1-i}$$

and since $x^i \in R_{\bar{i}}$, the dual bases are

$$\left\{ \overline{S}_R(x^{n-1-i}), \; \chi_H(\overline{n-1-i}) \overline{S}_H(\overline{n-1-i}) \cdot \begin{bmatrix} n-1 \\ i \end{bmatrix}_\omega x^i \right\}$$

for all i. Simplifying, the right-hand terms become

$$\begin{bmatrix} n-1 \\ i \end{bmatrix}_\omega \omega^{i+1} \overline{i+1} \cdot x^i = \begin{bmatrix} n-1 \\ i \end{bmatrix}_\omega \omega^{i+1} \omega^{-(i+1)i} x^i = \begin{bmatrix} n-1 \\ i \end{bmatrix}_\omega \omega^{-i^2+1} x^i.$$

Together with the above formulas for $\overline{S}$ and the action, this gives the dual bases:

$$\left\{ (-1)^{n-i-1} \omega^{-\binom{n-i-1}{2}} x^{n-i-1}, \; \begin{bmatrix} n-1 \\ i \end{bmatrix}_\omega \omega^{-i^2+1} x^i \right\}.$$

We remark that in this case, $A = R \star H$ is just the Taft algebra of dimension n^2 [Tf]. For, writing $\mathbb{Z}_n = \langle g \rangle$ multiplicatively, we have $g \cdot x = gxg^{-1} = \omega^{-1} x$, or $xg = \omega gx$. In the Hopf algebra A, $g \in G(A)$ and $\triangle_A x = x \otimes g + 1 \otimes x$.

6. Lie superalgebras and Lie coloralgebras

We now apply our main theorem to recover the result of Bell and Farnsteiner [BF] concerning when an extension $U(K) \subset U(L)$ is β-Frobenius, where $K \subset L$ are Lie superalgebras. Moreover, with very little extra work, we generalize their result to enveloping algebras of Lie coloralgebras. For basic facts on Lie coloralgebras, see [Sche] or [BMPZ].

Thus as in Example 5.10, let $H = kG$, where G is an abelian group (written multiplicatively) and $\langle \mid \rangle : G \times G \to k^*$ is a symmetric bicharacter; that is, $\langle \mid \rangle$ is a bi-homomorphism and satisfies $\langle g|h \rangle^{-1} = \langle h|g \rangle$, for all $g, h \in G$. Then a G-*Lie coloralgebra* with respect to $\langle \mid \rangle$ is a G-graded vector space $L = \bigoplus_{g \in G} L_g$ with a G-graded k-linear binary operation $[\ , \ ] : L \otimes L \to L$ such that for any $x \in L_g, y \in L_h, z \in L_l$,

(1) $[x, y] = -\langle g|h \rangle [y, x]$, and

(2) $\langle l|g \rangle [x, [y, z]] + \langle h|l \rangle [z, [x, y]] + \langle g|h \rangle [y, [z, x]] = 0$. These are the graded versions of anti-symmetry and the Jacobi identity, respectively. A universal enveloping algebra $U(L)$ of L exists [Sche], and a PBW theorem holds for $U(L)$. However, to describe it, we need more notation.

First, by the symmetry of $\langle \mid \rangle$, we have $\langle g|g \rangle = \pm 1$ for each $g \in G$, and thus $G = G_+ \cup G_-$, where

$$G_+ = \{ g \in G \mid \langle g|g \rangle = 1 \} \quad \text{and} \quad G_- = \{ g \in G \mid \langle g|g \rangle = -1 \}.$$

Then $L = L_+ \oplus L_-$, where $L_+ = \bigoplus_{g \in G_+} L_g$ and $L_- = \bigoplus_{g \in G_-} L_-$; note that $[L_+, L_+] \subseteq L_+$ and that $[L_-, L_-] \subseteq L_+$.

For the special case of Lie superalgebras, in which $G = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$, the bicharacter is given by $\langle g|h \rangle = (-1)^{gh}$; note that here we write G under addition. Thus $G_+ = \{\overline{0}\}$ and $G_- = \{\overline{1}\}$, and so $L_+ = L_0$, which is an ordinary Lie algebra, and $L_- = L_1$.

In general a G-homogeneous basis of L can be written as $X = X_+ \cup X_-$, where X_+ is a basis of L_+ and X_- is a basis of L_-; in a total ordering of X, we assume that $y < x$ for all $y \in X_+$ and $x \in X_-$.

6.1. PBW Theorem ([BMPZ, p.85]). *Assume that char $k \neq 2, 3$ and let $X = X_+ \cup X_-$ be a G-homogeneous basis of the G-Lie coloralgebra L. Then the universal enveloping algebra $U(L)$ has a G-homogeneous basis over k consisting of 1 and all monomials of the form*

$$y_{i_1}^{n_1} y_{i_2}^{n_2} \cdots y_{i_k}^{n_k} x_{j_1} x_{j_2} \cdots x_{j_l}$$

where $y_{i_1} < y_{i_2} < \cdots < y_{i_k}$ in X_+ and $x_{j_1} < x_{j_2} < \cdots < x_{j_l}$ in X_-, and $n_i \geq 0$.

For the rest of this section, assume that char $k \neq 2, 3$ and that $K \subset L$ are G-Lie coloralgebras. Since we wish to consider the situation when $[U(L) : U(K)] < \infty$, it follows from the PBW theorem that *we must assume that $L_+ \subset K \subset L$*, or equivalently that $K_+ = L_+$, and that $[L : K] < \infty$. We now follow the notation in [BF], used there for Lie superalgebras.

Fix homogeneous elements $x_1, \cdots, x_n \in L_-$ whose cosets $\{x_i + K\}$ form a basis of L/K; note that $[L : K] = n$. Let $\mathcal{F} = \{0, 1\}^n$ be the set of multi-indices of length n; in particular let $\mathbf{1} = (1, 1, \cdots, 1)$ and $\mathbf{0} = (0, \cdots, 0)$. For any $I = (i(1), \cdots, i(n)) \in \mathcal{F}$, we define

$$\mathbf{x}^I = x_1^{i(1)} \cdots x_n^{i(n)};$$

thus $\mathbf{x}^0 = 1$ and $\mathbf{x}^1 = x_1 x_2 \cdots x_n$. Also define $|I| := \sum_{l=1}^{n} i(l)$, the *weight* of I; $J \leq I$ if $j(l) \leq i(l)$ for all $l = 1, \cdots, n$. By the PBW theorem, every element u of $U(L)$ can be written uniquely as

$$u = \sum_{I \in \mathcal{F}} u_I \mathbf{x}^I$$

where $u_I \in U(K)$. $U(L)$ has a filtration via $V(m) := \sum_{i=0}^{m} U(K)(L_-)^i$; note that $V(n) = U(L)$. $V(m)$ is also a right $U(K)$-module, and again by PBW,

$$V(m) = \bigoplus_{|I| \leq m} U(K)\mathbf{x}^I = \bigoplus_{|I| \leq m} \mathbf{x}^I U(K).$$

We now put this setup into the format of Theorem 5.6. The Hopf algebra in question is $H = kG$. H acts on $U(L)$ as in Example 5.10: for any $g \in G$ and homogeneous element $z \in U(L)_h$, $g \cdot z = \langle g|h \rangle z$. Thus in the smash product $U(L) \star H$, $(1 \star g)(z \star 1) = (g \cdot z) \star g = \langle g|h \rangle z \star g$. We abbreviate this by $gz = \langle g|h \rangle zg$. The diagram of 5.10 becomes

$$
\begin{array}{ccccc}
S = U(K) & \hookrightarrow & B = U(K) \star kG & \overset{\pi}{\to} & kG \\
\cap & & \cap & & \| \\
R = U(L) & \hookrightarrow & A = U(L) \star kG & \overset{\pi}{\to} & kG.
\end{array}
$$

By Lemma 5.5, we know that $\overline{A} = A/AB^+ \cong \overline{R} = R/RS^+$ as coalgebras. Moreover $\dim_k \overline{A} = \dim_k \overline{R} = 2^n$, and $\overline{A} \cong \overline{R}$ has k-basis $\{\overline{\mathbf{x}^I} | I \in \mathcal{F}\}$. Since the coalgebra

structure of $\overline{R}$ is induced from that of R, $\triangle_R z = z \otimes 1 + 1 \otimes z$ for each $z \in L$, and $\triangle_R$ is graded-multiplicative, it follows that

$$\triangle \overline{\mathbf{x}^I} = \sum_{J \leq I} \alpha_{I,J} \overline{\mathbf{x}^J} \otimes \overline{\mathbf{x}^{I-J}},$$

for some scalars $\alpha_{I,J} \in k$. Now $\overline{A}^*$ has a basis $\{f_I | I \in \mathcal{F}\}$ dual to the $\{\overline{\mathbf{x}^I}\}$; that is, $f_I(\overline{\mathbf{x}^J}) = \delta_{I,J}$.

6.2. Proposition. *The extension $U(K) \star kG \subset U(L) \star kG$ is of (right) integral type. The data λ and χ of Definition 1.10 may be constructed explicitly as follows:*
   (a) *$\lambda \in I_r(\overline{A}^\star)$ is given by $\lambda = f_{\mathbf{1}}$.*
   (b) *To define $\chi \in B^\star$, we proceed as follows:*
   (i) *Define $\overline{ad} : K \to gl(L/K) \cong gl_n(k)$ by $\overline{ad}(b)(y + K) := ad(b)(y) + K = [b, y] + K$, for $b \in K, y \in L$. Now let*

$$\chi(b) := \begin{cases} \mathrm{tr}\,(\overline{ad}(b)) & \text{for } b \in K_1, \\ \\ 0 & \text{for } b \in K_g,\ g \neq 1. \end{cases}$$

χ is a morphism of Lie coloralgebras, and so extends to a homomorphism $U(K) \to k$.
   (ii) *For $g \in G$, define*

$$\chi(g) := \langle g | g_1 g_2 \cdots g_n \rangle,$$

where the $x_i \in L_{g_i}$, for $g_i \in L_-$, are the homogeneous basis of L/K fixed above.

Proof. (a) First, since $U(L) \star kG$ is pointed, the extension is of right integral type by Corollary 4.9. The algebra structure of $\overline{A}^*$ is given by convolution, determined by $\triangle$ (given above) on $\overline{A} = \overline{R}$. Define $\tilde{f}_i \in \overline{R}^*$ by $\tilde{f}_i = f_I$, for $I = (0, \cdots, 1, 0, \cdots, 0)$; that is, with a 1 only in the ith position. One may check that $\tilde{f}_i^2 = 0$, for all i, and that $\tilde{f}_i \tilde{f}_j = \langle g_j | g_i \rangle \tilde{f}_j \tilde{f}_i = f_{(i+j)}$, for $i < j$, where $I = (i + j)$ has a 1 in the ith and j^{th} position. It follows that

$$\overline{R}^* = \overline{A}^* = k\langle \tilde{f}_1, \cdots, \tilde{f}_n \mid \tilde{f}_i^2 = 0 \text{ for all } i,\ \tilde{f}_i \tilde{f}_j = \langle g_j | g_i \rangle \tilde{f}_j \tilde{f}_i \text{ for all } i \neq j \rangle$$

with identity element $\overline{\varepsilon}$. $\overline{A}^*$ is augmented via evaluation at $\overline{1}$; thus $\langle \tilde{f}_i, \overline{1} \rangle = 0$, for all i. It is now straightforward to see that $f_{\mathbf{1}}$ is a right integral in $\overline{A}^*$.
   (b) We must check that for all $b \in B$ and $w \in W = A, \lambda(\overline{bw}) = \chi(b)\lambda(\overline{w})$. First, since $\overline{A}$ is a left A-module, it suffices to check this for w such that $\overline{w}$ is a basis element of $\overline{A}$, that is, $w = \mathbf{x}^I$, for some I. Moreover, we may assume that $b \in U(K)$, for if $b = z \star g$, where $z \in U(K)$ and $g \in G$, then $gw \in kwg$ and thus $\overline{bw} = \overline{(z \star g)w} \in k\overline{zw}$.
   If $|I| < n$, then $bw \in V(m) = \bigoplus_{|J| \leq m} \mathbf{x}^J U(K)$ for $m < n$, and so $\lambda(\overline{bw}) \in \sum_{|J| \leq m} \lambda(\mathbf{x}^J U(K))$. However if $c \in U(K)$, then $\overline{\mathbf{x}^J c} = \varepsilon(c) \overline{\mathbf{x}^J}$ and thus $\lambda(\overline{\mathbf{x}^J}) = 0$ by (1), since $\lambda = f_{\mathbf{1}}$ and $|J| < n$. Thus $\lambda(\overline{bw}) = 0 = \chi(b)\lambda(\overline{w})$ and there is nothing to prove.
   We may therefore assume that $|I| = n$ and that $w = x_1 x_2 \cdots x_n = \mathbf{x}^{\mathbf{1}}$. First consider the case that $a = g \in G$. Then, assuming $x_i \in L_{g_i}$,

$$\begin{aligned} gx_1 \cdots x_n &= \langle g | g_1 \rangle x_1 g x_2 \cdots x_n = \cdots \\ &= \langle g | g_1 \rangle \langle g | g_2 \rangle \cdots \langle g | g_n \rangle x_1 x_2 \cdots x_n g \\ &= \langle g | g_1 g_2 \cdots g_n \rangle x_1 \cdots x_n g. \end{aligned}$$

Thus

$$\begin{array}{rl} \lambda(\overline{gw}) & = \langle g|g_1\cdots g_n\rangle\lambda(\overline{x_1\cdots x_n g}) \\ & = \langle g|g_1\cdots g_n\rangle\lambda(\overline{w}) = \chi(g)1, \end{array}$$

since $g \in kG \subset U(K) \star kG$ and $\varepsilon(g) = 1$. This shows (ii).

Now assume that $b \in K$; in fact we may assume that b is homogeneous, say $b \in K_g$. It follows that

$(*)$
$$\begin{array}{rl} bx_1x_2\cdots x_n & = [b,x_1]x_2\cdots x_n + \langle g|g_1\rangle x_1[b,x_2]\cdots x_n \\ & \quad + \cdots + \langle g|g_1\cdots g_{n-1}\rangle x_1\cdots x_{n-1}[b,x_n] + \langle g|g_1\cdots g_n\rangle x_1\cdots x_n b. \end{array}$$

This relation is easy to check using $[b,x_i] = bx_i - \langle g|g_i\rangle x_i b$. Note that $\overline{x_1\cdots x_n b} = \varepsilon(b)\overline{x_1\cdots x_n} = \overline{0}$ and thus we may ignore the last term. If $b \in K_-$, then $[b,x_i] \in [L_-,L_-] \subseteq L_+ = K_+$. Thus each term

$$x_1\cdots x_i[b,x_{i+1}]\cdots x_n \in V(n-1)$$

and so we see immediately that $\lambda(\overline{bx_1\cdots x_n}) = 0$. This agrees with our definition of $\chi(b) = 0$ for $b \notin K_1$.

We may therefore assume that $b \in K_+$. Now consider the map $\phi : K \to gl(L/K)$ given by $\phi(b)(x_i + K) = [b,x_i] + K$, and write $[b,x_i] = \sum_j \alpha_{ij}x_j + K$; that is, $\phi(b)$ has matrix $[\alpha_{ij}]$. Now for any $c \in K, \overline{x_1\cdots x_i c x_{i+2}\cdots x_n} \in \overline{V(n-1)}$, and we know $\lambda(\overline{V(n-1)}) = 0$ by the above. Similarly, since $x_i^2 \in K$ for all i, and $[x_i,x_j] \in [L_-,L_-] \subseteq L_+ = K_+$, any monomial m obtained from w by replacing some x_k by $x_i, k \neq i$, satisfies $m \equiv 0 \bmod V(n-2) \subset V(n-1)$, and thus $\lambda(\overline{m}) = 0$. Using these observations in $(*)$, we see that

$$\lambda(\overline{bx_1\cdots x_n}) = (\alpha_{11} + \langle g|g_1\rangle\alpha_{22} + \cdots + \langle g|g_1\cdots g_{n-1}\rangle\alpha_{nn})\lambda(\overline{x_1\cdots x_n}).$$

Now if $g \neq 1$, then $[b,x_i] \in L_{g_i} \neq L_{g_i}$, and so $\alpha_{ii} = 0$, for all $i = 1,\cdots,n$. It follows that $\lambda(\overline{bx_1\cdots x_n}) = 0 = \chi(b)1$. Thus we may assume $g = 1$. But now $\langle g|h\rangle = 1$ for all $h \in G$, by properties of the bicharacter, and so

$$\lambda(\overline{bx_1\cdots x_n}) = (\alpha_{11} + \alpha_{22} + \cdots + \alpha_{nn})\lambda(\overline{w}) = \chi(b)1.$$

Thus $\chi(b) = \text{trace}\,(\overline{ad}(b))$, proving (i). $\qquad\square$

We note that the remaining piece of data in Definition 1.10, namely Λ, does not seem to be easy to compute explicitly. This is analogous to the difficulty of computing the integral in a restricted enveloping algebra of a (usual) restricted Lie algebra.

6.3. Corollary. *Assume $K \subset L$ are Lie coloralgebras such that $K_+ = L_+$ and $[L : K] < \infty$. Then $U(K) \subset U(L)$ is a β-Frobenius extension. The Frobenius homomorphism $f : U(L) \to U(K)$ is given by $f(r) = \sum f_1(\overline{r^1})r^2$, for all $r \in U(L)$, where $\triangle_{U(L)}(r) = \sum r^1 \otimes r^2$ and f_1 is the integral in $\overline{U(L)}^*$, as in Proposition 6.2. The automorphism $\beta : U(K) \to U(K)$ is given as follows:*

$$\beta(b) = \begin{cases} b + tr(\overline{ad}(b)) & \text{if } b \in K_1, \text{ for } \overline{ad}(b) \in gl(L/K) \\ \langle g|g_1\cdots g_n\rangle b & \text{if } b \in K_g, \text{ for } g \neq 1, \end{cases}$$

where $\{x_i + K\}, i = 1,\cdots,n, x_i \in L_{g_i}$, form a basis for L over K.

Proof. The first part of the corollary follows from Proposition 6.2 and our main theorem, 3.1, since $U(K) \star kG \subset U(L) \star kG$ is a (B, A, H, A) faithfully-flat bi-Galois extension, with $H = kG, A = U(L) \star kG, B = U(K) \star kG, R = U(L)$, and $S = U(K)$. Thus $U(K) \subset U(L)$ is a β-Frobenius extension, with $\beta(b) = \sum \chi(b_{-1})b_0$, where χ is described as in 6.2. The Frobenius homomorphism is $f = f_{\mathbf{1}}$, the integral in $\overline{U(L)}^*$, since $\overline{A} \cong \overline{R}$ by Lemma 5.5. Note also that we may use $\triangle_{U(L)}(r)$ in f rather that $\triangle_A(r)$ by Theorem 5.6. Thus we need only check what β does to elements of K.

For any $b \in K$, $\triangle_R b = b \otimes 1 + 1 \otimes b$. Passing to $B = U(K) \star H$, we have $\triangle_A b = \sum (b_1 \star (b_2)_{-1}) \otimes ((b_2)_0 \star 1)$. The kG-comodule structure on K gives $b \mapsto g \otimes b$ if $b \in K_g$, and thus $\triangle_A b = b \otimes 1 + g \otimes b$. Thus:

(a) if $b \in K_1$, $\beta(b) = \chi(b) \cdot 1 + \chi(1)b = b + \chi(b) \cdot 1 = b + tr(\overline{ad}(b)) \cdot 1$, and
(b) if $b \in K_g$, for $g \neq 1$, then

$$\beta(b) = \chi(b) \cdot 1 + \chi(g) \cdot b = 0 + \langle g | g_1 \cdots g_n \rangle b.$$

Thus β is as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.4. *Remark.* (1) When $K \subset L$ are Lie superalgebras, we recover Bell and Farnsteiner's result [BF]. For then $G = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$, and we see that our χ is exactly their λ on K itself. For the automorphism β, note that when $g = \overline{1}$, $\langle g | g_1 \cdots g_n \rangle = \langle \overline{1} | \overline{1}^n \rangle = (-1)^n$, where $n = \dim(L/K)$, and thus $\beta(b) = (-1)^n b$ if $b \in K_g$. When $g = \overline{0}, \beta(b) = b + \chi(b)1$ for $b \in K_g$, as before. This agrees with the automorphism α in [BF].

(2) If K is a Lie ideal of L, that is $[K, L] \subset K$, then one might expect as in Example 1.11 that the extension $U(K) \subset U(L)$ is actually Frobenius, and not just β-Frobenius (for when K and L are ordinary Lie algebras, K a Lie ideal of L implies that $U(L)$ is a normal Hopf subalgebra of $U(L)$). However, this is not the case. It is true that $\overline{ad}(b) = 0$, for $b \in K_1$, and thus $\beta(b) = b$, as in Corollary 6.3. But if $b \in K_g$, for $g \neq 1$, then $\beta(b) = \langle g | g_1 \cdots g_n \rangle b$ as before; thus $\beta(b) = b$ only if $\langle g | g_1 \cdots g_n \rangle = 1$. For the case of superalgebras, this happens $\Leftrightarrow n$ is even.

The difficulty here is that $U(K) \star kG$ is not, in general, normal in $U(L) \star kG$, for $U(K)$ is not a Hopf subalgebra of $U(L) \star kG$ under the new comultiplication.

Note added in proof

M. Takeuchi has pointed out to us a more conceptual approach to Theorem 3.1. He defines a β-*H-Frobenius* extension to be an extension $B \subset A$ of right H-comodule algebras, which is β-Frobenius with Frobenius map f, such that β and f are H-colinear. For example $B \subset A$ in our 3.1 is β-H-Frobenius, and $U \subset W$ in our 1.14 is β-W-Frobenius. In the other direction he shows that if $U \subset W$ is a right coideal subalgebra of the Hopf algebra W and if the extension is β-W-Frobenius, then it is an extension of right integral type.

Takeuchi proves the following: let $B \subset A$ be β-H-Frobenius, with $R = A^{co\,H}$ and $S = B^{co\,H}$, and assume the following functor is a category equivalence:

$$\mathcal{M}_S \to \mathcal{M}_B^H$$
$$M \mapsto M \otimes_S B.$$

Then $S \subset R$ is a β-Frobenius extension.

Theorem 3.1 is a consequence of this result. Takeuchi's approach has been used by Y. Doi in "A note on Frobenius extensions in Hopf algebras", to appear in Communications in Algebra.

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