TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 12, December 1997, Pages 4775–4786
S 0002-9947(97)02072-2

RESTRICTION OF STABLE BUNDLES IN CHARACTERISTIC p

TOHRU NAKASHIMA

Abstract. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a nonsingular projective variety defined over k and H an ample line bundle on X. We shall prove that there exists an explicit number m_0 such that if E is a μ-stable vector bundle of rank at most three, then the restriction $E_{|D}$ is μ-stable for all $m \geq m_0$ and all smooth irreducible divisors $D \in |mH|$. This result has implications to the geometry of the moduli space of μ-stable bundles on a surface or a projective space.

Introduction

Let X be a nonsingular projective variety of dimension n defined over an algebraically closed field k, and let E be a vector bundle of rank r on X. The problem of whether the μ-(semi)stability of E with respect to an ample divisor H is preserved under restriction to a divisor $D \in |mH|$ has been studied intensively in recent years. When the characteristic of k is arbitrary, a fundamental theorem of Mehta-Ramanathan states that the answer is positive for a general member D and sufficiently large m ([M-R1], [M-R2]). However, most of the important results have been proved under the assumption that the characteristic is zero. For example, H. Flenner's theorem on the effective estimate of m has no known analogue in positive characteristic ([F]).

In this paper we would like to contribute to the restriction problem when $\operatorname{char} k = p > 0$. Under the assumption that $r = 2$ or 3, we shall give an explicit number m_0 such that for all $m \geq m_0$, every μ-stable bundle E with fixed Chern classes remains μ-stable when restricted to arbitrary smooth irreducible divisors $D \in |mH|$. As a consequence, we obtain the restriction morphism between moduli spaces of μ-stable bundles

$$j_D : M_X(r, c_i) \to M_D(r, c_{i|D}),$$

where $M_X(r, c_i)$ denotes the moduli space of μ-stable bundles E of rank r ($r = 2$, 3) with Chern classes $c_i(E) = c_i$. We remark that a restriction theorem with no genericity assumption on D has been obtained by F. Bogomolov if $n = 2$ and $\operatorname{char} k = 0$ ([B1], [B2]). Recently, A. Moriwaki has generalized the result to higher dimensional varieties ([Mo3]).

The lower bound m_0 mentioned above can be expressed as a function of $\delta_H(E)$ and $\mu_H(X)$, where $\delta_H(E) = \{2rc_2(E) - (r-1)c_1(E)^2\} \cdot H^{n-2}$ and $\mu_H(X)$ denotes the minimal slope of the Harder-Narasimhan filtration of the tangent bundle of X. The appearance of the number $\mu_H(X)$ comes from our use of an analogue of

Received by the editors April 30, 1996.
1991 *Mathematics Subject Classification.* Primary 14D20, 14F05.

Bogomolov-Gieseker inequality, which has been proved for rank two bundles in [N1]. The inequality is described as follows. If E is a μ-semistable torsion-free sheaf of rank $r = 2$ or 3, we have $\delta_H(E) \geq 0$ if $\mu_H(X) \geq 0$, and otherwise

$$\delta_H(E) \geq -\alpha \frac{\mu_H(X)^2}{p^2 H^n},$$

where $\alpha = 1$ if $r = 2$, $\alpha = 9$ if $r = 3$. Inequalities of different types have been obtained for rank two bundles on surfaces ([S-B]) and non-uniruled varieties ([Mo2]).

It is natural to ask whether the results in the present paper generalize to bundles of rank greater than three. This problem is closely related to the boundedness of the family of μ-semistable sheaves in positive characteristic, which is at present unknown except when $r \leq 3$ or $n = 2$. Roughly speaking, the restriction theorem, the Bogomolov-Gieseker inequality and the boundedness may be considered as almost equivalent properties of μ-semistable sheaves.

In the first section we will prove analogues of the Bogomolov-Gieseker inequality. The restriction theorem will be proved in section two. This result will be applied in the last section to the study of moduli spaces of μ-stable bundles on surfaces and projective spaces.

The author is very grateful to the referee for pointing out several inaccuracies in the original version of this paper.

1. The Bogomolov-Gieseker inequality

Throughout this paper all varieties are defined over an algebraically closed field k of characteristic $p > 0$. Let X be a nonsingular projective variety of dimension $n \geq 2$ over k and E a rank r torsion-free sheaf on X. For an ample line bundle H on X, we define the slope of E by

$$\mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{r}$$

and let

$$\delta_H(E) := \{2rc_2(E) - (r-1)c_1(E)^2\} \cdot H^{n-2}.$$

Even if E is μ-semistable, the Bogomolov-Gieseker inequality $\delta_H(E) \geq 0$ does not hold in general. Nevertheless, one expects that some lower bound for $\delta_H(E)$, which depends on X and H, may exist. In fact, the boundedness of semistable sheaves implies the existence of a lower bound. For rank two μ-semistable bundles, an effective lower bound has been given in [N1]. The purpose of this section is to extend the result to μ-semistable torsion-free sheaves of rank at most three.

A torsion-free sheaf E is said to be *p-semistable* with respect to H if, for all $m \geq 0$, the m-th iterated Frobenius pull-back $(F^m)^*E$ is μ-semistable with respect to H. The following result has been proved for vector bundles in [Mo1, Theorem 1], and its generalization to torsion-free sheaves poses no significant problems.

Proposition (1.1). *Let X be a nonsingular projective variety of dimension $n \geq 2$. Let E be a torsion-free sheaf of rank r on X which is p-semistable with respect to H. Then $\delta_H(E) \geq 0$ if $r \leq 3$.*

Let T_X denote the tangent bundle of X and let

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{s-1} \subset T_s = T_X$$

be the Harder-Narasimhan filtration (H-N filtration for short) with respect to H. We define

$$\mu_H(X) := \mu_H(T_X/T_{s-1}).$$

Proposition (1.2). *Assume that $\mu_H(X) \geq 0$, and let E be a μ-semistable torsion-free sheaf of rank r. Then $\delta_H(E) \geq 0$ if $r \leq 3$.*

Proof. It suffices to show that under the assumption $\mu_H(X) \geq 0$, every μ-semistable sheaf E is p-semistable. Assume that E is not p-semistable, and let m be the smallest integer such that $(F^m)^*E$ is unstable. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = (F^m)^*E$$

be the H-N filtration of $(F^m)^*E$ and let $G_i = E_i/E_{i-1}$. By arguing as in the proof of [N1, Lemma 1], for $0 < i < s$ we obtain a non-trivial $\mathcal{O}_X$-homomorphism

$$f_i : T_X \to \mathcal{H}om_{\mathcal{O}_X}(E_1, (F^m)^*E/E_1).$$

It follows that we can find $0 < i < j \leq s$ and a non-zero map

$$f : T_X \to \mathcal{H}om_{\mathcal{O}_X}(G_i, G_j).$$

Since at least one of G_i and G_j is of rank one, the latter sheaf is μ-semistable. Hence we obtain $\mu_H(X) \leq \mu_H(G_j) - \mu_H(G_i) < 0$, which is a contradiction. This completes the proof. ☐

Remark. The condition $\mu_H(X) \geq 0$ is satisfied for varieties with nef tangent bundle T_X (that is, the tautological bundle $\mathcal{O}(1)$ on $\mathbb{P}(T_X)$ is nef). Another example is a Fano n-fold X such that T_X is μ-semistable with respect to $H = -K_X$: we have $\mu_H(X) = (-K_X)^n/n > 0$ in this case. If X is a del Pezzo surface, the μ-semistability of T_X has been proved ([Fa]).

Corollary (1.3). *Let X be a Fano threefold. If the tangent bundle of X is μ-semistable with respect to $-K_X$, then $c_1(X)^3 \leq 72$.*

Proof. Since we have $\mu_H(X) = \mu_H(T_X) = (-K_X)^3/3 > 0$, Proposition (1.2) yields

$$\left\{3c_2(X) - c_1(X)^2\right\} \cdot c_1(X) \geq 0.$$

We have $H^3(X, \mathcal{O}_X) \cong H^0(X, K_X) = 0$. Further, by [N1, Corollary 3]

$$H^2(X, \mathcal{O}_X) \cong H^1(X, K_X) = 0.$$

Hence $\chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) \leq 1$ and, by the Riemann-Roch formula,

$$1 \geq \frac{1}{24} c_1(X) c_2(X).$$

Therefore we obtain $c_1(X)^3 \leq 72$, as desired. ☐

In view of Proposition (1.1), the following can be proved as in [N1, Theorem 1].

Proposition (1.4). *Let X be a nonsingular projective variety of dimension $n \geq 2$. Let E be a rank two torsion-free sheaf which is μ-semistable with respect to H. Then*

1. *If $\mu_H(X) \geq 0$, then $\delta_H(E) \geq 0$.*
2. *If $\mu_H(X) < 0$, then*

$$\delta_H(E) \geq -\frac{\mu_H(X)^2}{p^2 H^n}.$$

The rest of this section is devoted to proving an inequality of the above type for rank three μ-semistable sheaves. The following lemma is elementary but very useful.

Lemma (1.5). *Let E be a torsion-free sheaf of rank r on X. Assume that E has a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that $G_i = E_i/E_{i-1}$ is torsion-free of rank r_i for each i. We put

$$\alpha_i = \frac{r(\mu_H(G_i) - \mu_H(E))}{H^n}.$$

Then

$$\delta_H(E) \geq \sum_{i=1}^{s} \left\{ \frac{r}{r_i}\delta_H(G_i) - \frac{r_i}{r}\alpha_i^2 H^n \right\}.$$

Proof. We have

$$\begin{aligned} 2c_2(E)\cdot H^{n-2} &= \left\{ 2\sum_{i=1}^{s} c_2(G_i) + 2\sum_{i<j} c_1(G_i)c_1(G_j) \right\} \cdot H^{n-2} \\ &= \left\{ 2\sum_{i=1}^{s} c_2(G_i) + c_1(E)^2 - \sum_{i=1}^{s} c_1(G_i)^2 \right\} \cdot H^{n-2} \\ &\geq \sum_{i=1}^{s} \left(\frac{\delta_H(G_i)}{r_i} - \frac{c_1(G_i)^2\cdot H^{n-2}}{r_i} \right) + c_1(E)^2\cdot H^{n-2}. \end{aligned}$$

On the other hand, since for each i we have

$$\left\{ \frac{r}{r_i}c_1(G_i) - c_1(E) - \alpha_i H \right\} \cdot H^{n-1} = 0,$$

the Hodge index theorem yields

$$\left\{ \frac{r}{r_i}c_1(G_i) - c_1(E) - \alpha_i H \right\}^2 \cdot H^{n-2} \leq 0.$$

Hence

$$-\frac{c_1(G_i)^2\cdot H^{n-2}}{r_i} \geq \left\{ -\frac{2}{r}c_1(G_i)\cdot c_1(E) + \frac{r_i}{r^2}c_1(E)^2 \right\} \cdot H^{n-2} - \frac{r_i}{r^2}\alpha_i^2 H^n.$$

Since $\sum_{i=1}^{s} c_1(G_i) = c_1(E)$ and $\sum_{i=1}^{s} r_i = r$, after summing up these inequalities for $1 \leq i \leq s$, we obtain

$$\sum_{i=1}^{s} -\frac{c_1(G_i)^2\cdot H^{n-2}}{r_i} \geq -\frac{c_1(E)^2\cdot H^{n-2}}{r} - \sum_{i=1}^{s} \frac{r_i}{r^2}\alpha_i^2 H^n.$$

Combining this with the inequality for $2c_2(E)\cdot H^{n-2}$, we obtain the claimed bound for $\delta_H(E)$. ∎

Assume that $\mu_H(X) < 0$. Let E be a μ-semistable torsion-free sheaf of rank three such that F^*E is not μ-semistable. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = F^*E$$

be the H-N filtration ($s = 2$ or 3). Let $G_i = E_i/E_{i-1}$ and $r_i = \operatorname{rk} G_i$. There are three types of (r_i): (2, 1), (1, 2), (1, 1, 1). We have the following estimates for α_i.

Lemma (1.6). *$\alpha_1 > 0$, and*

$$\frac{2\mu_H(X)}{H^n} \le \alpha_2 < \alpha_1 \le -\frac{\mu_H(X)}{H^n} \quad \text{in case of type } (2, 1),$$
$$\frac{\mu_H(X)}{H^n} \le \alpha_2 < \alpha_1 \le -\frac{2\mu_H(X)}{H^n} \quad \text{in case of type } (1, 2),$$
$$\frac{3\mu_H(X)}{H^n} \le \alpha_3 < \alpha_2 < \alpha_1 \le -\frac{3\mu_H(X)}{H^n} \quad \text{in case of type } (1, 1, 1).$$

Proof. The inequality $\alpha_1 > 0$ is clear. As in the proof of Proposition (1.2), for each $i = 1, 2$, we have a non-trivial $\mathcal{O}_X$-homomorphism

$$f_i : T_X \to \mathcal{H}om_{\mathcal{O}_X}(E_i, F^*E/E_i) =: H_i.$$

Since the slope of the maximal destabilizing subsheaf of H_i is $\mu_H(G_{i+1}) - \mu_H(G_i)$, we see that $\mu_H(X) \le \mu_H(G_{i+1}) - \mu_H(G_i)$ for $i = 1, 2$. Then it is easy to deduce the following estimates for $\mu_H(E_1)$:

$$\mu_H(E_1) \le \mu_H(F^*E) - \frac{2}{3}\mu_H(X) \quad \text{in case of type } (2, 1),$$
$$\mu_H(E_1) \le \mu_H(F^*E) - \frac{1}{3}\mu_H(X) \quad \text{in case of type } (1, 2),$$
$$\mu_H(E_1) \le \mu_H(F^*E) - \mu_H(X) \quad \text{in case of type } (1, 1, 1).$$

The claim follows immediately from the above inequalities. □

Theorem (1.7). *Let X be a nonsingular projective variety of dimension $n \ge 2$ with an ample line bundle H. Let E be a μ-semistable torsion-free sheaf of rank three on X.*

1. *If $\mu_H(X) \ge 0$, then $\delta_H(E) \ge 0$.*
2. *If $\mu_H(X) < 0$, then*

$$\delta_H(E) > -\frac{9\mu_H(X)^2}{p^2 H^n}.$$

Proof. (1) follows from Proposition (1.2). Assume $\mu_H(X) < 0$. Let m denote the smallest integer such that $(F^m)^*E$ is not μ-semistable. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = (F^m)^*E$$

be the H-N filtration of $(F^m)^*E$ and put $G_i = E_i/E_{i-1}$. We will give the lower bounds for $\delta_H(E)$ according to the three types of $r_i = \operatorname{rk} G_i$. In case of type $(2, 1)$, we have $\delta_H(G_1) \ge -\mu_H(X)^2/(p^2H^n)$ by Proposition (1.4) and $\delta_H(G_2) \ge 0$. Applying Lemmas (1.5) and (1.6) to $(F^{m-1})^*E$, we obtain

$$\delta_H((F^m)^*E) = p^{2m}\delta_H(E) > -\left(2 + \frac{3}{2p^2}\right)\frac{\mu_H(X)^2}{H^n}.$$

Hence

$$\delta_H(E) > -\left(2 + \frac{3}{2p^2}\right)\frac{\mu_H(X)^2}{p^2H^n}.$$

Similarly, in case of type $(1,2)$

$$\delta_H(E) > -\left(4 + \frac{3}{2p^2}\right)\frac{\mu_H(X)^2}{p^2H^n}.$$

Finally, in case of type $(1,1,1)$ we obtain

$$\delta_H(E) > -\frac{9\mu_H(X)^2}{p^2H^n}.$$

Comparing the above bounds for $\delta_H(E)$, we get the theorem. □

Corollary (1.8). *Assume that X is a threefold with the ample canonical bundle K_X. If the tangent bundle of X is μ-semistable with respect to K_X, then the following inequality holds:*

$$\{3c_2(X) - c_1(X)^2\} \cdot (-c_1(X)) \geq 0.$$

Proof. Applying Theorem (1.7) to $E = T_X$ and $H = K_X$, we obtain

$$\{3c_2(X) - c_1(X)^2\} \cdot (-c_1(X)) > -\frac{3}{2p^2} > -1.$$

The claim is clear, since the left-hand side is an integer. □

2. The restriction theorem

Let X be a nonsingular projective variety of dimension $n \geq 2$ and let H be an ample line bundle on X. In this section we will prove the restriction theorem mentioned in the introduction. We will treat the two cases $r = 2$ and $r = 3$ separately. Let $\delta = \delta_H(E)$, $\mu = \mu_H(X)$ and $h = H^n$.

Theorem (2.1). *Let E be a rank two vector bundle on X which is μ-stable with respect to H. Let $D \subset X$ be a smooth irreducible divisor with $D \in |mH|$. Then $E_{|D}$ is μ-stable with respect to $H_{|D}$ under either one of the following conditions:*

1. $\mu \geq 0$ *and* $m > \frac{1}{2}\left(\delta + \frac{1}{h}\right)$*;*
2. $\mu < 0$ *and*

$$m > \max\left\{\sqrt{\frac{1}{h}\left(\delta + \frac{\mu^2}{p^2h}\right)}, \frac{1}{2}\left(\delta + \frac{1}{h}\right)\right\}.$$

Proof. Assume that $E_{|D}$ is not μ-stable with respect to $H_{|D}$. Then there exist a rank one torsion-free sheaf Q on D satisfying $\mu_{H_{|D}}(Q) \leq \mu_{H_{|D}}(E_{|D})$ and a surjection $E_{|D} \to Q$. Let E' be the rank two torsion-free sheaf on X defined by the exact sequence

$$0 \to E' \to E \to Q \to 0.$$

Then we have $c_1(E') = c_1(E) - D$ and $c_2(E') = c_2(E) - c_1(E) \cdot D + c_1(Q) \cdot H_{|D}$. Hence

$$\delta_H(E') \leq \delta_H(E) - m^2H^n.$$

By Proposition (1.4), it follows that under assumption (1) or (2), E' is unstable with respect to H. Consider the H-N filtration of E':

$$0 \to G_1 \to E' \to G_2 \to 0,$$

where the G_i are rank one torsion-free sheaves. For $i = 1, 2$, let

$$\alpha_i = \frac{2(\mu_H(G_i) - \mu_H(E'))}{H^n}.$$

Since G_1 is a subsheaf of E which is μ-stable, we have

$$0 < \alpha_1 \leq m - \frac{1}{H^n},$$

and we have $\alpha_2 = -\alpha_1$. Applying Lemma (1.5) to E', we have $\delta_H(E') \geq -\alpha_1^2 H^n \geq -(m - 1/(H^n))^2 H^n$. It follows that

$$m \leq \frac{1}{2}\left(\delta_H(E) + \frac{1}{H^n}\right).$$

This contradicts the assumption on m. Hence the theorem is proved. □

Theorem (2.2). *Let X, H be as before and let E be a rank three vector bundle on X which is μ-stable with respect to H. Then for every smooth irreducible $D \in |mH|$, $E_{|D}$ is μ-stable under one of the following conditions:*

1. $\mu \geq 0$ *and*
$$m > \max\left\{\sqrt{\frac{\delta}{2h}}, \sqrt{\frac{2}{3h}\left(\delta + \frac{2}{h}\right)}, \frac{1}{12}\left(\delta + \frac{1}{2h}\right), \frac{1}{2}\left(\frac{\delta}{2} + \frac{1}{h}\right)\right\};$$
2. $\mu < 0$ *and*
$$m > \max\left\{\sqrt{\frac{1}{2h}\left(\delta + \frac{9\mu^2}{p^2 h}\right)}, \sqrt{\frac{2}{3h}\left(\delta + \frac{2}{h} + \frac{\mu^2}{p^2 h}\right)}, \frac{1}{4}\left(\frac{\delta}{3} + \frac{\mu^2}{2p^2 h}\right), \frac{1}{2}\left(\frac{\delta}{2} + \frac{1}{2h}\right)\right\}.$$

Proof. The proof is similar to the rank two case. Assume that $E_{|D}$ is not μ-stable and let Q be its destabilizing torsion-free quotient. First we treat the case when Q is of rank one. Let E' be the kernel of the surjection $E_{|D} \to Q$. As before, we have

$$\delta_H(E') \leq \delta_H(E) - 2m^2 H^n.$$

By Theorem (1.7), E' is unstable if m satisfies the inequality

$$m > \max\left\{\sqrt{\frac{\delta_H(E)}{2H^n}}, \sqrt{\frac{1}{2H^n}\left(\delta_H(E) + \frac{9\mu_H(X)^2}{p^2 H^n}\right)}\right\}.$$

We shall give upper bounds for m according to the types of the H-N filtration of E'. For $1 \leq i \leq 3$, define

$$\alpha_i = \frac{3(\mu_H(G_i) - \mu_H(E'))}{H^n}.$$

Assume that we are in case of type $(2, 1)$. Then we have

$$0 < \alpha_1 \leq m - \frac{1}{2H^n}$$

since G_1 is a subsheaf of E. Using Proposition (1.4) and Lemma (1.5), we obtain

$$m \leq \begin{cases} \frac{1}{12}\left(\delta_H(E)+\frac{1}{2H^n}\right) & \text{if } \mu_H(X)\geq 0, \\ \frac{1}{4}\left(\delta_H(E)+\frac{\mu_H(X)^2}{2p^2H^n}\right) & \text{if } \mu_H(X)<0. \end{cases}$$

Similarly, in case of type $(1,2)$ we have $0<\alpha_1\leq m-1/(H^n)$. Hence

$$m \leq \begin{cases} \sqrt{\frac{2}{3H^n}\left(\delta_H(E)+\frac{2}{H^n}\right)}-\frac{1}{H^n} & \text{if } \mu_H(X)\geq 0, \\ \sqrt{\frac{2}{3H^n}\left(\delta_H(E)+\frac{2}{H^n}+\frac{\mu_H(X)^2}{p^2H^n}\right)}-\frac{1}{H^n} & \text{if } \mu_H(X)<0. \end{cases}$$

Assume that we are in case of type $(1,1,1)$. We have $0<\alpha_1\leq m-1/(H^n)$, since

$$\mu_H(G_1)\leq \mu_H(E')+\frac{mH^n-1}{3}.$$

Hence

$$\begin{aligned} \mu_H(G_2) &= 2\mu_H(E_2)-\mu_H(G_1) \\ &> 2\mu_H(E')-\left(\mu_H(E')+\frac{mH^n-1}{3}\right) \\ &= \mu_H(E')-\frac{mH^n-1}{3}. \end{aligned}$$

It follows that $\alpha_2>-(m-1/(H^n))$. Finally, we have

$$\begin{aligned} \mu_H(G_3) &= 3\mu_H(E')-\mu_H(G_2)-\mu_H(G_3) \\ &> 3\mu_H(E')-2\left(\mu_H(E')+\frac{mH^n-1}{3}\right) \\ &= \mu_H(E')-\frac{2(mH^n-1)}{3}. \end{aligned}$$

Hence we obtain $\alpha_3>-2(m-1/(H^n))$. From these bounds for α_i, we obtain $\alpha_i^2\leq(m-1/(H^n))^2$ for $i=1$, 2, and $\alpha_3^2<4(m-1/(H^n))^2$. Thus, by Lemma (1.5),

$$\delta(E')>-2\left(m-\frac{1}{H^n}\right)^2 H^n.$$

Therefore we obtain

$$m\leq\frac{1}{2}\left(\frac{\delta_H(E)}{2}+\frac{1}{H^n}\right).$$

Putting these altogether, we conclude that $E_{|D}$ has no destabilizing rank one quotient sheaves. If Q is a sheaf of rank two, then let $\widetilde{Q}:=\bigwedge^2 Q/tors$ denote the quotient of the exterior product $\bigwedge^2 Q$ by its torsion subsheaf. Then $\widetilde{Q}$ is a rank one torsion-free sheaf, and there is a surjection

$$\bigwedge^2 E_{|D}\to\widetilde{Q}.$$

Since we have $\bigwedge^2 E\cong E^\vee(c_1(E))$, $\bigwedge^2 E$ is μ-stable and $\delta_H(\bigwedge^2 E)=\delta_H(E)$. Further, it is easy to see that $\widetilde{Q}$ is a destabilizing quotient of $\bigwedge^2 E_{|D}$. Thus we are reduced to the case Q is of rank one. This completes the proof of the theorem. □

3. Application to the moduli space

In this section we consider some applications of the restriction theorem to the study of moduli spaces of μ-stable bundles.

Let S be a nonsingular projective surface with an ample line bundle H. For $L \in \operatorname{Pic}(S)$, we denote by $M = M_S(r, L, c_2)$ the moduli space of vector bundles of rank r on S with $\det(E) \cong L$, $c_2(E) = c_2$ which are μ-stable with respect to H. $M_S(r, L, c_2)$ has a natural compactification $\overline{M}_S(r, L, c_2)$, the moduli of Gieseker semistable torsion-free sheaves. We recall that $\overline{M}_S(r, L, c_2)$ is constructed as the geometric invariant theory quotient of a certain Quot scheme Quot^{ss} ([G]). Let $\pi : \mathrm{Quot}^{ss} \to \overline{M}_S(r, L, c_2)$ be the quotient morphism.

We fix a tautological sheaf $\mathcal{F}$ on $S \times \mathrm{Quot}^{ss}$. For a smooth curve $C \subset S$ of genus $g(C)$, we define $\mathcal{F}^C = \mathcal{F}_{|C \times \mathrm{Quot}^{ss}}$. Let $p_Q : C \times \mathrm{Quot}^{ss} \to \mathrm{Quot}^{ss}$ and $p_C : C \times \mathrm{Quot}^{ss} \to C$ be the natural projections. The proof of the following lemma is exactly the same as that of [N2, Proposition 1.3], where the result is stated in the case $\operatorname{char} k = 0$.

Lemma (3.1). *Assume that $c_1(L) \cdot C = rd$ for some integer d, and let A be a line bundle of degree $-d + g(C) - 1$ on C. Then there exists a line bundle $\operatorname{Det}_{\mathcal{F}}(C)$ on $\overline{M}_S(r, L, c_2)$ such that*

$$\pi^* \operatorname{Det}_{\mathcal{F}}(C) \cong \det(p_Q)_!(\mathcal{F}^C \otimes p_C^* A)^\vee.$$

Let $\overline{M}_C(r, L_{|C})$ denote the moduli of semistable rank r vector bundles F on C with $\det F \cong L_{|C}$. Then we have

Proposition (3.2). *If $r = 2$ or 3, then there exists an explicit number m_0 such that for all $m \geq m_0$ and every smooth curve $C \in |rmH|$, we obtain the restriction morphism of moduli spaces*

$$j_C : M_S(r, L, c_2) \to \overline{M}_C(r, L_{|C})$$

which satisfies $\operatorname{Det}_{\mathcal{F}}(C)_{|M} \cong j_C^* \mathcal{L}_C$ for an ample line bundle $\mathcal{L}_C$ on $\overline{M}_C(r, L_{|C})$. Furthermore, if m is sufficiently large, j_C is an injective immersion (namely, an injective morphism with injective differential map).

Proof. We obtain the number m_0 immediately from Theorems (2.1) and (2.2). The existence of an ample line bundle $\mathcal{L}_C$ with the required property follows from the description of $\operatorname{Pic} \overline{M}_C(r, L_{|C})$ given in [D-N]. Thus the first claim is proved.

For the second claim, note that since $M_S(r, \mathcal{L}, c_2)$ is of finite type, if m is sufficiently large we have $H^q(S, \mathcal{H}om(E, F)(-C)) = 0$ for $q \leq 1$ and every E, $F \in M$ by Serre's vanishing theorem. It follows that

$$H^0(S, \mathcal{H}om(E, F)) \cong H^0(C, \mathcal{H}om(E_{|C}, F_{|C}))$$

and the natural map $H^1(S, \mathcal{H}om(E, F)) \to H^1(C, \mathcal{H}om(E_{|C}, F_{|C}))$ is injective. Therefore we deduce that j_C is an injective immersion. □

Next we study μ-stable bundles on the projective space $\mathbb{P}^n$ by restricting them to smooth complete intersections. Restrictions to hyperplanes have already been considered in [E] in the case $\operatorname{char} k > 0$. The following approach to moduli spaces is similar to [P], which treats the case $\operatorname{char} k = 0$.

For given integers $r \geq 2$ and $c_i\,(2 \leq i \leq r)$, let $M_{\mathbb{P}^n}(r, 0, c_i)$ be the moduli space of rank r μ-stable vector bundle E on $\mathbb{P}^n$ with $c_1(E) = 0$ and $c_i(E) = c_i \in A^i(\mathbb{P}^n) \cong \mathbb{Z}$, where A^i denotes the Chow group of codimension i cycles. Similarly, for a smooth subvariety $X \subset \mathbb{P}^n$, we denote by $M_X(r, 0, c_i)$ the moduli space of rank r bundles E on X which are μ-stable with respect to $H = \mathcal{O}_X(1)$, with $c_1(E) = 0$, $c_i(E) = c_i H^i \in A^i(X)$. We notice that if $r \leq 3$, these moduli spaces are quasi-projective schemes ([Ma1], [Ma2]).

Lemma (3.3). *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of type $(d_1, \dots, d_m)$ with $\sum_{j=1}^m d_j \geq n+1$. If $\dim(X) \geq 3$ or $\dim(X) = 2$ and $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$, then T_X is μ-stable with respect to $\mathcal{O}_X(1)$.*

Proof. It suffices to prove the μ-stability of the cotangent bundle Ω_X^1. Assume that Ω_X^1 is not μ-stable, and let $F \subset \Omega_X^1$ be a destabilizing subsheaf of rank r with $0 < r < n-m$. We may assume F to be reflexive by considering its double dual; hence we obtain an invertible subsheaf $\det F \subset \Omega_X^r$. Since $\mathrm{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)]$ if $\dim(X) \geq 3$, we have $\det F = \mathcal{O}_X(l)$ for some integer l. Thus $H^0(X, \Omega_X^r(-l)) \neq 0$ and, moreover, the destabilizing condition yields

$$l \geq \frac{r(\sum_{j=1}^m d_j - n - 1)}{n-m} \geq 0.$$

However, this is a contradiction since we have $H^0(X, \Omega_X^r(-l)) = 0$ for $l \geq 0$ by [D, Théorème 1.5]. This completes the proof. □

Theorem (3.4). *Let $n \geq 3$, let c_i be integers and let $r = 2$ or 3. Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of type $(d_1, \dots, d_m)$ with $\dim(X) \geq 2$. There exists an integer d_0 depending only on n, m, c_2 and p such that if $d_j > d_0$, then for all $E \in M_{\mathbb{P}^n}(r, 0, c_i)$, $E_{|X}$ is μ-stable with respect to $\mathcal{O}_X(1)$. Furthermore, for sufficiently large d_j and $d = \prod_{j=1}^m d_j$, the restriction morphism*

$$j_X : M_{\mathbb{P}^n}(r, 0, c_i) \to M_X(r, 0, dc_i)$$

defines an open immersion.

Proof. We shall consider only the rank two case, since the proof in the rank three case is entirely similar. First we choose an integer d_1 such that

$$d_1 > \max\{2c_2, n+1\}$$

and let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d_1. By Theorem (2.1), every $E \in M_{\mathbb{P}^n}(2, 0, c_2)$ restricts to a μ-stable bundle $E_{|X}$ with $c_1(E_{|X}) = 0$, $c_2(E_{|X}) = c_2 d_1 H^2$. Since the tangent bundle of X is μ-stable by Lemma (3.3), we have

$$\mu_H(X) = \mu_H(T_X) = \frac{(n+1-d_1)d_1}{n-1} < 0.$$

Let d_2 an integer such that

$$d_2 > \max\left\{\sqrt{4c_2 d_1 + \frac{(n+1-d_1)^2}{p^2(n-1)^2}},\ 2c_2 d_1^2 + \frac{1}{2d_1}\right\}.$$

Applying again Theorem (2.1) to $E_{|X}$, we see that for every smooth $Y \in |d_2 H|$, $E_{|Y}$ is μ-stable. Repeating this process, we get the first assertion.

Let $\mathcal{I}_X$ denote the ideal sheaf of X. We observe that if $\dim(X) \geq 2$ and d_j are sufficiently large, then we have $H^q(\mathbb{P}^n, \mathcal{H}om(E, F) \otimes \mathcal{I}_X) = 0$ for $q \leq 2$ and for all E, $F \in M_{\mathbb{P}^n}(2, 0, c_2)$. Indeed, since $M_{\mathbb{P}^n}(2, 0, c_2)$ is of finite type, we can use Serre's vanishing theorem and the Koszul resolution of $\mathcal{I}_X$ to prove the claim. Therefore the cohomology sequences induced by the exact sequence

$$0 \to \mathcal{H}om(E,F) \otimes \mathcal{I}_X \to \mathcal{H}om(E,F) \to \mathcal{H}om(E_{|X}, F_{|X}) \to 0$$

yield the isomorphisms $H^q(\mathbb{P}^n, \mathcal{H}om(E,F)) \cong H^q(X, \mathcal{H}om(E_{|X}, F_{|X}))$ for $q \leq 1$. It follows that the restriction morphism defines an open immersion. □

For the restriction to complete intersection curves, we have the following result.

Theorem (3.5). *Let $n \geq 3$, c_i be integers and $r = 2$ or 3. For sufficiently large d_j and a general smooth complete intersection curve $C \subset \mathbb{P}^n$ of type $(d_1, \ldots, d_{n-1})$, every $E \in M_{\mathbb{P}^n}(r, 0, c_i)$ restricts to a μ-stable bundle on C and the restriction morphism defines an injective immersion of moduli spaces*

$$j_C : M_{\mathbb{P}^n}(r, 0, c_i) \to M_C(r, 0).$$

Proof. For sufficiently large d_j, a general complete intersection surface $S \subset \mathbb{P}^n$ of type $(d_1, \ldots, d_{n-2})$ satisfies $\mathrm{Pic}(S) \cong \mathbb{Z}[\mathcal{O}_S(1)]$ (cf. [M-R1, 2.2. Proposition]), and we have $H^q(\mathbb{P}^n, \mathcal{H}om(E,F) \otimes \mathcal{I}_C) = 0$ for $q \leq 1$. Hence the theorem follows as before. □

References

[B1] F.Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. of the USSR, Izvestija **13** (1979), 499–555. MR **80j:**14014

[B2] F.Bogomolov, *Stability of vector bundles on surfaces and curves*, Einstein metrics and Yang-Mills connections, Lecture Notes Pure Appl. Math., vol. 145 Marcel Dekker, New York, 1993, 35–49. MR **94i:**14021

[D] P.Deligne, *Cohomologie des intersections complètes*, SGA7, exp.XI., Lecture Notes in Math., 340, Springer, 1973, 39–61. MR **50:**7135

[D-N] J.-M.Drezet, M.S.Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbe algébriques*, Invent.Math. **97** (1989), 53–94. MR **90d:**14008

[E] L.Ein, *Stable vector bundles on projective spaces in char $p > 0$*, Math.Ann. **254** (1980), 53–72. MR **81d:**14010

[F] H.Flenner, *Restriction of semistable bundles on projective varieties*, Comment.Math.Helvetici **59** (1984), 635–650. MR **86m:**14014

[Fa] R.Fahlaoui, *Stabilité du fibré tangent des surfaces de del Pezzo*, Math.Ann. **283** (1989), 171–176. MR **89k:**14063

[G] D.Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann.Math. **106** (1977), 45–60. MR **81h:**14014

[Ma1] M.Maruyama, *Moduli of stable sheaves I*, J.Math. Kyoto Univ. **17** (1977), 91–126. MR **56:**8567

[Ma2] M.Maruyama, *On boundedness of families of torsion free sheaves*, J.Math. Kyoto Univ. **21** (1981), 673–701. MR **83a:**14019

[Mo1] A.Moriwaki, *A note on Bogomolov-Gieseker's inequality in positive characteristic*, Duke Math.J **64** (1991), 361–375. MR **92m:**14054

[Mo2] A.Moriwaki, *Frobenius pull-back of vector bundles of rank 2 over non-uniruled varieties*, Math.Ann. **296** (1993), 441–451. MR **94j:**14039

[Mo3] A.Moriwaki, *Arithmetic Bogomolov-Gieseker's inequality*, Amer.J.Math. **117** (1995), 1325–1347. MR **96i:**14022

[M-R1] V.B.Mehta, A.Ramanathan, *Semistable sheaves on projective varieties and their restriction to curves*, Math.Ann. **258** (1982), 213–224. MR **83f:**14013

[M-R2] V.B.Mehta, A.Ramanathan, *Restriction of stable sheaves and representations of the fundamental group*, Invent.Math. **77** (1984), 163–172. MR **85m:**14026

[N1] T.Nakashima, *Bogomolov-Gieseker inequality and cohomology vanishing in characteristic p*, Proc.Amer.Math.Soc. **123** (1995), 3609–3613. MR **96b:**14058

[N2] T.Nakashima, *Singularity of the moduli space of stable bundles on surfaces*, Compositio Math. **100** (1996), 125–130. MR **97d:**14018

[P] R.Paoletti, *Seshadri constants, gonality of space curves, and restriction of stable bundles*, J.Diff.Geom. **40** (1994), 475–504. MR **95k:**14046

[S-B] N.I.Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in positive characteristic*, Invent.Math. **106** (1991), 243–262. MR **92h:**14027

Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo, 192-03 Japan

E-mail address: `nakasima@math.metro-u.ac.jp`