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POISSON TRANSFORMS ON VECTOR BUNDLES

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Abstract. Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G. Let (τ, V) be an irreducible unitary representation of K, and $G \times_K V$ the associated vector bundle. In the algebra of invariant differential operators on $G \times_K V$ the center of the universal enveloping algebra of Lie(G) induces a certain commutative subalgebra Z_τ. We are able to determine the characters of Z_τ. Given such a character we define a Poisson transform from certain principal series representations to the corresponding space of joint eigensections. We prove that for most of the characters this map is a bijection, generalizing a famous conjecture by Helgason which corresponds to τ the trivial representation.

Introduction

Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space of noncompact type. We fix an Iwasawa decomposition $G = KAN$. Let M be the centralizer of A in K. Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of G and A, respectively, and $\Sigma(\mathfrak{g}, \mathfrak{a})$ the root system for $\mathfrak{g}, \mathfrak{a}$. Let $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be the positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the ordering given by N. Let $D(G/K)$ be the algebra of invariant differential operators on G/K. It is well-known that the characters of $D(G/K)$ are parametrized by $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the complex dual space of $\mathfrak{a}$. Let $\mathcal{E}_\lambda(G/K)$ denote the space of joint eigenfunctions corresponding to λ. We write $g = \mathrm{k}(g) \exp \mathrm{H}(g) \mathrm{n}(g)$ for each $g \in G$ according to $G = KAN$. For each $\phi \in C^\infty(K/M)$ we define $P_\lambda \phi \in C^\infty(G/K)$ by

$$P_\lambda \phi(g) = \int_K \phi(k) e^{-(\lambda+\rho)\mathrm{H}(g^{-1}k)} dk.$$

Here ρ is the half sum of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (including multiplicities). It turns out that $P_\lambda \phi \in \mathcal{E}_\lambda(G/K)$. One can easily extend the definition of P_λ to the space $D'(K/M)$ (resp. $A'(K/M)$) of distributions (resp. analytic functionals) on K/M. In this paragraph we fix $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not in $-\mathbb{N} - \{0\}$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. It is proved by Helgason [Helg2] that P_λ defines a bijection from $C^\infty(K/M)_{K\text{-finite}}$ onto $\mathcal{E}_\lambda(G/K)_{K\text{-finite}}$. He also proves in the rank one case P_λ is a bijection from $A'(K/M)$ onto $\mathcal{E}_\lambda(G/K)$. He then conjectured this should be true for the higher rank case. The conjecture was eventually proved by six Japanese mathematicians in [KKMOOT]. It should be mentioned that a representation theoretic proof by Schmid, starting from the K-finite result, is indicated in [Sch]. Lewis, then a student of Helgason, made the following observation: Let $\mathcal{E}^*_\lambda(G/K)$ be the subspace of $\mathcal{E}_\lambda(G/K)$ where each element increases at most exponentially (see §2 for definition);

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then P_λ maps $D'(K/M)$ into $\mathcal{E}^*_\lambda(G/K)$. He was able to prove in the rank one case P_λ is a bijection from $D'(K/M)$ onto $\mathcal{E}^*_\lambda(G/K)$. See [Lew]. This result has been generalized to the higher rank case by Oshima and Sekiguchi [OS]. There is an alternative and independent proof by Wallach [Wall1]. By refining Wallach's idea van den Ban and Schlichtkrull have a third proof in [BS]. They define $\mathcal{E}^\infty_\lambda(G/K)$ as the subspace of $\mathcal{E}_\lambda(G/K)$ where each element and its derivatives increase at most exponentially (uniformly). Then they prove P_λ is a bijection from $C^\infty(K/M)$ onto $\mathcal{E}^\infty_\lambda(G/K)$. The bijectivity of P_λ from $D'(K/M)$ to $\mathcal{E}^*_\lambda(G/K)$ follows easily.

Let (τ, V) be an irreducible unitary representation of K. Let $G \times_K V$ be the associated vector bundle over G/K. The space of smooth sections of this vector bundle can be identified with

$$C^\infty \operatorname{Ind}_K^G(\tau) = \{f \in C^\infty(G, V) | f(gk) = \tau(k^{-1})f(g), \forall g \in G, \forall k \in K\}.$$

Let D_τ denote the algebra of invariant differential operators on $C^\infty \operatorname{Ind}_K^G(\tau)$. Notice when (τ, V) is the trivial representation we go back to the previous case. In the case where $\dim V = 1$, D_τ is commutative and its characters can be parameterized by $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. In [Shim] Shimeno is able to characterize the joint eigenspace of D_τ in terms of a Poisson transform for most of λ. Gaillard's results about the eigenforms of the Laplacian on hyperbolic spaces are illuminating. They show considerable variety even for a simple space. See [Ga] for details. van der Ven [Ven] considers vector-valued Poisson transforms in the rank one case, extending Gaillard's results. His emphasis, however, is on the singular eigenvalues. Minemura [Min] studies the properties of D_τ and obtains a result on the dimension of the spherical eigensections.

One of the difficulties people run into when trying to generalize the classical results is the complexity of D_τ, in particular its noncommutativity. The remedy used was either a condition on τ or a condition on (G/K). We put a mild condition on $\mathfrak{g}$ (see beginning of §4) but no restriction on τ. We replace D_τ with a subalgebra Z_τ coming from $\mathcal{Z}(\mathfrak{g})$, the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Then we are able to determine the characters of Z_τ. It turns out they are given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and Λ is given by the infinitesimal character of an irreducible representation of M contained in τ (see Proposition 1.11).

Let V be the representation space of τ, and

$$V = \bigoplus_{\sigma \in \widehat{M}} V(\sigma)$$

the isotypic decomposition of V into M-isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$. Write

$$V(\Lambda) = \bigoplus_{\sigma \in \tau, \Lambda_\sigma = \Lambda} V(\sigma).$$

Here Λ_σ is given by the infinitesimal character of σ. Let $\tau(\Lambda)$ be the restriction of τ to M with representation space $V(\Lambda)$. We define a Poisson transform (see §1 for definition)

$$P_\lambda \colon C^\infty \operatorname{Ind}_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \to \mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$$

by

$$P_\lambda \phi(g) = \int_K \tau(k)\phi(gk)dk.$$

Here $C^\infty \operatorname{Ind}_{MAN}^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$ is the space defined by

$$\{\phi \in C^\infty(G, V(\Lambda)) | \phi(gman) = a^{\lambda-\rho}\tau(m^{-1})\phi(g)\},$$

and $\mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ the subspace of the total eigenspace where each element and its derivatives increase at most exponentially (uniformly). Let $C(\lambda)$ be the generalized Harish-Chandra C-function corresponding to τ (Proposition 2.3), $C_0(\lambda)$ the restriction of $C(\lambda)$ to $V(\Lambda)$, and $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ as defined after Remark 1.5.

Theorem. *Let $\lambda - \Lambda \in \mathfrak{h}^*_\mathbb{C}$, satisfying the following conditions:*
(i) $2\langle\lambda-\Lambda,\alpha\rangle/\langle\alpha,\alpha\rangle \notin \mathbb{Z}$, $\forall\alpha \in \Sigma(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$, *with* $\alpha|\mathfrak{a} \neq 0$;
(ii) $2\langle\lambda,\beta\rangle/\langle\beta,\beta\rangle \notin -\mathbb{N}$, $\forall\beta\in\Sigma^+(\mathfrak{g},\mathfrak{a})$;
(iii) $\det C_0(\lambda) \neq 0$.
Then P_λ is a bijection.

This generalizes the result of van den and Ban and Schlichtkrull mentioned above which corresponds to τ the trivial representation.

We have similar results about distributions and K-finite sections, generalizing the above-mentioned results for τ trivial.

The main idea in the proof is to generalize the theory of asymptotic expansions developed in [Ban] and [BS]. By invoking Casselman's deep result [Ca] on globalization of Harish-Chandra modules, one might simplify our argument somehow. But we prefer a self-contained account. Besides, we think the theory of asymptotic expansions developed here is of interest on its own.

The paper is organized as follows: in Section 1 we study the invariant differential operators on $G \times_K V$. In Section 2 we introduce some function spaces on G. In Section 3 we state some results on the asymptotic expansion of an eigensection. In Section 4 we study the algebraic structure of a $(\mathfrak{g}, K)$-module. In Sections 5 and 6 we prove the results stated in Section 3. In Section 7 we study the leading terms of the asymptotic expansion. In Section 8 we give an inversion formula to the Poisson transform. In Sections 9 and 10 we extend the Poisson transform to vector-valued distributions.

1. Notations and preliminaries

Let G be a connected real semisimple Lie group with finite center and K a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space. We fix an Iwasawa decomposition $G = KAN$, and let M be the centralizer of A in K, M' the normalizer of A in K, $W = M'/M$ the Weyl group. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, and $\mathfrak{m}$ be the corresponding Lie algebras of G, K, A, N, and M, respectively, and $U(\mathfrak{g}), U(\mathfrak{k}), U(\mathfrak{a}), U(\mathfrak{n})$, and $U(\mathfrak{m})$ the corresponding universal enveloping algebras of the complexified Lie algebras. Let $\Sigma(\mathfrak{g},\mathfrak{a})$ be the restricted root system for $(\mathfrak{g},\mathfrak{a})$, and $\Delta = \{\alpha_1,\dots,\alpha_r\}$ the set of simple roots for the ordering of $\Sigma(\mathfrak{g},\mathfrak{a})$ given by N. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. If $g \in G$ we write $g = \mathrm{k}(g)\exp \mathrm{H}(g)\mathrm{n}(g)$ according to $G = KAN$.

Fix once and for all an irreducible unitary representation (τ, V) of K. Denote by $G \times_K V$ the associated vector bundle. Then the space of its smooth sections may be identified with the following space:

$$C^\infty \operatorname{Ind}_K^G(\tau) = \{f \in C^\infty(G,V) | f(gk) = \tau(k)^{-1} f(g), \forall g \in G, \forall k \in K\}.$$

Let D_τ denote the algebra of differential operators on $C^\infty \operatorname{Ind}_K^G(\tau)$ that commute with the left translations by elements of G. The remainder of this section will be devoted to the study of this algebra. First for each $f \in C^\infty(G,V)$ and $X \in \mathfrak{g}$ we define $L_X f$ and $R_X f$ as follows:

$$L_X f(g) = \left(\frac{d}{dt} f(\exp(-tX)g)\right)\Big|_{t=0},$$
$$R_X f(g) = \left(\frac{d}{dt} f(g \exp tX)\right)\Big|_{t=0}, \quad \forall g \in G.$$

Then L and R define two representations of $\mathfrak{g}$ which we extend to representations of $U(\mathfrak{g})$. Let $\mathrm{End}(V)$ denote the space of linear maps from V to itself. Then $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ is an associative algebra with the natural multiplication. Let $I(\tau)$ be the left ideal of $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ generated by $\{X \otimes 1 + 1 \otimes \tau(X) | X \in \mathfrak{k}.\}$.

Proposition 1.1. *With the above notations, we have*

$$U(\mathfrak{g}) \otimes \mathrm{End}(V) = (U(\mathfrak{a}) \otimes \mathrm{End}(V)) \oplus (\mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau)).$$

Proof. It suffices to show the left-hand side is contained in the right-hand side. Suppose $u \otimes T \in U(\mathfrak{g}) \otimes \mathrm{End}(V)$. By Poincaré-Birkhoff-Witt we can assume $u = u_1 u_2 u_3$, where $u_1 \in U(\mathfrak{n})$, $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. If $u_1 \in \mathfrak{n}U(\mathfrak{n})$ then $u \otimes T \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V)$. So we can assume $u = u_2 u_3$, where $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. Let $u_3 = X_1 \cdots X_j$, for $X_1, \ldots, X_j \in \mathfrak{k}$. It is easy to show $u_2 u_3 \otimes T \in (U(\mathfrak{a}) \otimes \mathrm{End}(V)) + I(\tau)$ by induction on j. $\square$

Define a K-action on $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ by

$$k(X \otimes T) = \mathrm{Ad}(k)X \otimes \tau(k)T\tau(k)^{-1},$$

for each $k \in K$. Let $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ be the fixed elements.

Proposition 1.2. *Let* $\Gamma_1 : U(\mathfrak{g}) \otimes \mathrm{End}(V) \to U(\mathfrak{a}) \otimes \mathrm{End}(V)$ *be the projection map according to the decomposition in Proposition 1.1. Then* Γ_1 *is a homomorphism from* $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ *into* $U(\mathfrak{a}) \otimes \mathrm{End}_M(V)$*, where*

$$\mathrm{End}_M(V) = \{T \in \mathrm{End}(V) | \tau(m)T = T\tau(m), \forall m \in M\}.$$

Proof. Since M preserves $\mathfrak{n}$, it is easy to see Γ_1 maps $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ into $U(\mathfrak{a}) \otimes \mathrm{End}_M(V)$. We now check Γ_1 is a homomorphism.

Suppose $D_1, D_2 \in (U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$. Then

$$D_1 - \Gamma_1(D_1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

Hence

$$D_1 D_2 - \Gamma_1(D_1) D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau) D_2.$$

Assume $D_2 = \sum u_i \otimes T_i$, for $u_i \in U(\mathfrak{g})$, and $T_i \in \mathrm{End}(V)$. Then for any $X \in \mathfrak{k}$,

$$\begin{aligned}(X \otimes 1 + 1 \otimes \tau(X)) D_2 &= \sum (X u_i \otimes T_i + u_i \otimes \tau(X) T_i) \\ &= \sum (\mathrm{ad}(X) u_i \otimes T_i + u_i \otimes [\tau(X), T_i]) \\ &\quad + \sum (u_i X \otimes T_i + u_i \otimes T_i \tau(X)).\end{aligned}$$

Then first summation is zero since $D_2 \in (U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$. The second one is just $D_2(X \otimes 1 + 1 \otimes \tau(X))$. So we have proved $I(\tau) D_2 \subset I(\tau)$. Hence

$$D_1 D_2 - \Gamma_1(D_1) D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

However,

$$D_2 - \Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau),$$

and

$$\Gamma_1(D_1)(\mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau)) \subset \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

Therefore

$$D_1 D_2 - \Gamma_1(D_1)\Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

This proves $\Gamma_1(D_1D_2) = \Gamma_1(D_1)\Gamma_1(D_2)$. □

For $D = \sum u_i \otimes T_i \in U(\mathfrak{g}) \otimes \mathrm{End}(V)$, and $f \in C^\infty(G, V)$, we define

$$\mu_1(D)f = \sum T_i R_{u_i} f.$$

It is not difficult to show for $D \in (U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ and $f \in C^\infty \mathrm{Ind}_K^G(\tau)$, $\mu_1(D)f$ remains in $C^\infty \mathrm{Ind}_K^G(\tau)$. So $\mu_1(D) \in D_\tau$. In fact μ_1 is a surjective homomorphism from $(U(\mathfrak{g}) \otimes \mathrm{End}(V))^K$ onto D_τ.

We define $\mu(D) = \mu_1(D \otimes 1)$, for each $D \in U(\mathfrak{g})^K$. By a theorem of Burnside which asserts that $\tau(U(\mathfrak{k})) = \mathrm{End}(V)$, one can prove μ is a surjective homomorphism from $U(\mathfrak{g})^K$ onto D_τ, using the surjectivity of μ_1.

For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we introduce an important function Ψ_λ on G with values in $\mathrm{End}(V)$ as follows:

$$\Psi_\lambda(nak) = a^{\lambda+\rho}\tau(k)^{-1},$$

for $n \in N$, $a \in A$, and $k \in K$. Here ρ is the half sum of the positive roots for $(\mathfrak{g}, \mathfrak{a})$. Notice that for each $v \in V$, the function: $g \to \Psi_\lambda(g) \cdot v$ belongs to $C^\infty \mathrm{Ind}_K^G(\tau)$.

Proposition 1.3. *For each $D \in U(\mathfrak{g})^K$, and $v \in V$,*

$$\mu(D)(\Psi_\lambda \cdot v) = \Psi_\lambda \cdot (\Gamma_1(D \otimes 1)(\lambda + \rho)v).$$

Proof. Since both sides are left N-invariant and behave in the same way under the right K-action, it is sufficient to show they are equal when restricted to A. By definition

$$D \otimes 1 = D_1 + \Gamma_1(D \otimes 1) + D_2,$$

where $D_1 \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V)$, and $D_2 \in I(\tau)$. It is easy to see that

$$\mu_1(D_1)(\Psi_\lambda \cdot v)|A = 0,$$

and

$$\mu_1(D_2)(\Psi_\lambda \cdot v) = 0.$$

So

$$\mu(D)(\Psi_\lambda \cdot v)|A = a^{\lambda+\rho}\Gamma_1(D \otimes 1)(\lambda + \rho)v. \quad \square$$

Corollary 1.4. *There exists a homomorphism $\Gamma' \colon D_\tau \to U(\mathfrak{a}) \otimes \mathrm{End}_M(V)$. Moreover, for each $D \in U(\mathfrak{g})^K$, $\Gamma'(\mu(D)) = \Gamma_1(D \otimes 1)$.*

Remark 1.5. It has been proved in Section 3 in [Min] that Γ' is injective, using results from [Lep].

In general D_τ is very complicated. For instance it is not abelian in most of the cases. For this reason we replace it by $\mu(\mathcal{Z}(\mathfrak{g}))$ which we denote by Z_τ.

Choose $\mathfrak{t}$ a maximal abelian subalgebra in $\mathfrak{m}$. Then $\mathfrak{h}_\mathbb{C} = (\mathfrak{t}+\mathfrak{a})_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the root system for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ be the set of positive roots for some ordering, and $\mathfrak{g}_\mathbb{C}^+$ (resp. $\mathfrak{g}_\mathbb{C}^-$) the sum of positive (resp. negative) root spaces. Choose an ordering such that $\mathfrak{n} \subset \mathfrak{g}_\mathbb{C}^+$. We consider each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ (resp. $\mathfrak{t}_\mathbb{C}^*$) an element of $\mathfrak{h}_\mathbb{C}^*$ by the requirement that λ be zero in $\mathfrak{t}$ (resp. $\mathfrak{a}$). Let

$$P = \{\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) | \alpha|\mathfrak{a} \neq 0\}, \quad P_0 = \{\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) | \alpha|\mathfrak{a} = 0\}.$$

Write

$$\rho = \frac{1}{2}\sum_{\alpha\in P}\alpha, \quad \rho_0 = \frac{1}{2}\sum_{\alpha\in P_0}\alpha.$$

Let Θ be the Cartan involution of $\mathfrak{g}$ with fixed point set $\mathfrak{k}$ and extend it to an automorphism of $\mathfrak{g}_\mathbb{C}$. Then $\alpha \to -\Theta\alpha$ is a permutation of P, so $\rho|\mathfrak{t} = 0$. Hence ρ can be viewed as the half sum of positive roots for $(\mathfrak{g}, \mathfrak{a})$.

Let $\gamma' \colon \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h}_\mathbb{C})$ be defined by

$$Z - \gamma'(Z) \in \mathfrak{g}_\mathbb{C}^- U(\mathfrak{g}),$$

for $Z \in \mathcal{Z}(\mathfrak{g})$. We define $\gamma(Z)(\mu) = \gamma'(Z)(\mu - \rho - \rho_0)$, for each $\mu \in \mathfrak{h}_\mathbb{C}^*$. This is the usual Harish-Chandra homomorphism.

Let $V = \bigoplus_{\sigma\in\widehat{M}} V(\sigma)$ be the decomposition into the M-isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$.

For each irreducible representation (σ, V_σ) of M, we get a Lie algebra representation of $\mathfrak{m}$ by differentiation. We denote the representation by $d\sigma$. In general this is not irreducible. Fortunately it is a multiple of an irreducible representation of $\mathfrak{m}$. This fact can be seen in the following way. Let M_0 be the identity component of M. By structure theory (see 1.1.3.8 in [War]) one can find $Z(A)$, a finite subgroup of M where each element commutes with every element of M_0. Choose an irreducible representation (σ, V_1) of M_0 in (σ, V_σ). For each $z \in Z(A)$, $(\sigma, \sigma(z)V_1)$ gives an irreducible representation of M_0 in (σ, V_σ), which is equivalent to (σ, V_1). Since σ is irreducible, $V_\sigma = \sum_{z\in Z(A)} \sigma(z)V_1$. So by Schur's lemma the center $\mathcal{Z}(\mathfrak{m})$ of $U(\mathfrak{m})$ acts on V_σ by scalars. The action is determined by $\Lambda_\sigma \in \sqrt{-1}\mathfrak{t}^*$ as follows: For each Z in $\mathcal{Z}(\mathfrak{m}), d\sigma(Z) = \gamma(Z)(\Lambda_\sigma)\mathrm{I}_{V_\sigma}$, where γ is the Harish-Chandra homomorphism for $(\mathfrak{m}, \mathfrak{t})$, and I_{V_σ} the identity map of V_σ. We choose Λ_σ the highest weight of σ plus ρ_0.

Let $\Gamma \colon D_\tau \to U(\mathfrak{a}) \otimes \mathrm{End}_M(V)$ be defined by

$$\Gamma(D)(\lambda) = \Gamma'(D)(\lambda + \rho).$$

Theorem 1.6. *For each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}_\mathbb{C}^*$,*

$$\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)\mathrm{I}_{V(\sigma)}.$$

We give a proof below using a well-known proposition about $\mathcal{Z}(\mathfrak{g})$. A more self-contained proof is in [Wall]. First for the proof and later use we recall the definition of Poisson transforms.

Let (δ, V_δ) be a finite dimensional representation of $B = MAN$, the minimal parabolic subgroup of G. Let $C^\infty \operatorname{Ind}_B^G(\delta)$ be the space defined by

$$\{\phi \in C^\infty(G, V_\delta) | \phi(gman) = a^{-\rho}\delta^{-1}(man)\phi(g), \forall g \in G, \forall man \in B\}.$$

Let $C^\infty \operatorname{Ind}_B^G(\delta)$ be endowed with the topology from $C^\infty(G, V_\delta)$. We will specify the topology on $C^\infty \operatorname{Ind}_K^G(\tau)$ in the next section.

Definition 1.7. A Poisson transform is a continuous, linear, G-equivariant map from $C^\infty \operatorname{Ind}_B^G(\delta)$ into $C^\infty \operatorname{Ind}_K^G(\tau)$.

Given $T \in \operatorname{Hom}_M(V_\delta, V)$, and $\phi \in C^\infty \operatorname{Ind}_B^G(\delta)$, we write

$$P_T\phi(g) = \int_K \tau(k)T(\phi(gk))dk.$$

One can easily check P_T is a Poisson transform.

Proposition 1.8. *The map $T \to P_T$ is a bijection from* $\operatorname{Hom}_M(V_\delta, V)$ *onto the space of Poisson transforms.*

This result appears in [Ven]. We include a proof for completeness. Suppose P is a Poisson transform from $C^\infty \operatorname{Ind}_B^G(\delta)$ into $C^\infty \operatorname{Ind}_K^G(\tau)$. Define the Poisson kernel $p \in [C^\infty \operatorname{Ind}_B^G(\delta)]' \otimes V$, the strong topological dual of $C^\infty \operatorname{Ind}_B^G(\delta)$ tensored by V, by

$$\langle p, \phi\rangle = (P\phi)(e),$$

for each $\phi \in C^\infty \operatorname{Ind}_B^G(\delta)$. By the G-equivariance of P the Poisson kernel completely determines P by

$$P\phi(x) = \langle p, L_{x^{-1}}\phi\rangle,$$

for any $\phi \in C^\infty \operatorname{Ind}_B^G(\delta)$. Here $L_{x^{-1}}\phi(g) = \phi(xg)$.

By Section 9 there is a K-equivariant isomorphism between $(C^\infty \operatorname{Ind}_B^G(\delta))'$ and $C^{-\infty} \operatorname{Ind}_M^K(\check{\delta}|M)$, where $C^{-\infty} \operatorname{Ind}_M^K(\check{\delta}|M)$ denotes the space of vector-valued distributions $f\colon C^\infty(K, \mathbb{C}) \to V_\delta^*$, such that

$$R_m f = \check{\delta}(m)^{-1} f,$$

for any $m \in M$. Here $\check{\delta}$ is the dual representation of $\delta|M$. And $R_m f(\phi) = f(R_{m^{-1}}\phi)$, with $(R_{m^{-1}}\phi)(k) = \phi(km^{-1})$. So

$$p \in C^{-\infty} \operatorname{Ind}_M^K(\check{\delta}|M) \otimes V.$$

However, for $\phi \in C^\infty \operatorname{Ind}_B^G(\delta)$,

$$\langle p, L_k\phi\rangle = P(L_k\phi)(e) = P\phi(k^{-1}) = \tau(k)(P\phi(e)) = \tau(k)(\langle p, \phi\rangle).$$

Hence $p \in (C^{-\infty} \operatorname{Ind}_M^K(\check{\delta}|M) \otimes V)^K$. Let π be the representation of K in $V_\delta^* \otimes V$ defined by $\pi(k)(v \otimes w) = v \otimes \tau(k)w$, for $v \in V_\delta^*$, and $w \in V$. Then $p \in C^{-\infty}(K, V_\delta^* \otimes V)$, and $L_k p = \pi(k^{-1})p$. By Lemma 9.3 p must be smooth. Its transformation properties imply that p is determined by $p(e)$, which belongs to $(V_\delta^* \otimes V)_M \cong \operatorname{Hom}_M(V_\delta, V)$.

Proof of Proposition 1.8. From the definition of P_T, it is immediate that the Poisson kernel of P_T evaluated at the identity is T. This shows the map $T \to P_T$ is injective. On the other hand, let P be a Poisson transform, and let p be its Poisson kernel. Then

$$\begin{aligned} P\phi(x) &= \langle p, L_{x^{-1}}\phi\rangle \\ &= \int_K \langle p(k), \phi(xk)\rangle dk \\ &= \int_K \tau(k)p(e)\phi(xk)dk. \end{aligned}$$

This proves $P = P_{p(e)}$, whence the surjectivity. □

Lemma 1.9.

$$\int_K F(\mathrm{k}(g^{-1}k))dk = \int_K F(k)e^{-2\rho \mathrm{H}(gk)}dk.$$

Let σ be a finite dimensional representation of M and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Then $\sigma\otimes(-\lambda)\otimes 1$ defines a representation of B by $man \to a^{-\lambda}\sigma(m)$.

Corollary 1.10. *For each* $\phi \in C^\infty \operatorname{Ind}_B^G(\sigma\otimes(-\lambda)\otimes 1)$*,* $T \in \operatorname{Hom}_M(V_\sigma, V)$*,*

$$P_T\phi(g) = \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk.$$

Proof.

$$\begin{aligned} P_T\phi(g) &= \int_K \tau(k)T\phi(gk)dk \\ &= \int_K \tau(k)T\phi(\mathrm{k}(gk)\exp \mathrm{H}(gk)\mathrm{n}(gk))dk \\ &= \int_K e^{(\lambda-\rho)\mathrm{H}(gk)}\tau(k)T\phi(\mathrm{k}(gk))dk. \end{aligned}$$

By Lemma 1.9,

$$\begin{aligned} &\int_K e^{(\lambda-\rho)\mathrm{H}(gk)}\tau(k)T\phi(\mathrm{k}(gk))dk \\ &\quad= \int_K e^{(\lambda+\rho)\mathrm{H}(g\mathrm{k}(g^{-1}k))}\tau(\mathrm{k}(g^{-1}k))T\phi(\mathrm{k}(g\mathrm{k}(g^{-1}k)))dk \\ &\quad= \int_K e^{(-\lambda+\rho)\mathrm{H}(g^{-1}k)}\tau(\mathrm{k}(g^{-1}k))T\phi(k)dk \\ &\quad= \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk. \quad \square \end{aligned}$$

Proof of Theorem 1.6. Let δ be the restriction of τ to M with $V(\sigma)$ as the representation space. It is well-known that $L_Z\phi = \gamma(Z)(\Lambda_\sigma - \lambda)\phi$ for each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\phi \in C^\infty \operatorname{Ind}_B^G(\delta\otimes(-\lambda)\otimes 1)$. See [Vogan]. Let $*$ denote adjoint on $U(\mathfrak{g})$. By Corollary 5.31 on p. 324 in [Helg1],

$$\begin{aligned} R_Z P_T\phi &= L_{Z^*}P_T\phi = P_T L_{Z^*}\phi \\ &= P_T(\gamma(Z^*)(\Lambda_\sigma - \lambda)\phi) = P_{\gamma(Z^*)(\Lambda_\sigma-\lambda)T}\phi = P_{\gamma(Z)(-\Lambda_\sigma+\lambda)T}\phi. \end{aligned}$$

On the other hand, by Proposition 1.3 and Corollary 1.10,

$$R_Z P_T\phi = P_{\Gamma(\mu(Z))(\lambda)T}\phi.$$

So

$$P_{\gamma(Z)(-\Lambda_\sigma+\lambda)T} = P_{\Gamma(\mu(Z))(\lambda)T}.$$

By Proposition 1.8 we conclude

$$\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)\mathrm{I}_{V(\sigma)}. \quad \square$$

By definition a character of Z_τ is a homomorphism from Z_τ to $\mathbb{C}$.

Proposition 1.11. *A character χ of Z_τ is given by $\lambda - \Lambda_\sigma$, where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and $\sigma \in \tau$. More specifically, $\chi(\mu(Z)) = \gamma(Z)(\lambda - \Lambda_\sigma)$, for each $Z \in \mathcal{Z}(\mathfrak{g})$.*

Lemma 1.12. *Let S be the common zeros of $p_1, \dots, p_m$ in $S(\mathfrak{h}_{\mathbb{C}})$. Assume in addition S is $\widetilde{W}$ invariant, $\widetilde{W}$ denoting the Weyl group for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then one can find $q_1, \dots, q_n$ in $I(\mathfrak{h}_{\mathbb{C}})$, the $\widetilde{W}$-invariants in $S(\mathfrak{h}_{\mathbb{C}})$, such that S is the common zeros of $q_1, \dots, q_n$.*

Proof. Write $R_i(X) = \prod_{s \in \widetilde{W}}(X - p_i^s)$, and $w = |\widetilde{W}|$. Then

$$R_i(X) = X^w + p_{i1}X^{w-1} + \cdots + p_{iw}.$$

It is easy to see we can use p_{ij} as our $q_1, \dots, q_n$. $\square$

Proof of Proposition 1.11. Let $A = \mu \circ \gamma^{-1} \colon I(\mathfrak{h}_{\mathbb{C}}) \to Z_\tau$. By Theorem 1.6 $\ker(A) = \{\rho \in I(\mathfrak{h}_{\mathbb{C}}) | p | (-\Lambda_\sigma + \mathfrak{a}_{\mathbb{C}}^*) = 0$, for all $\sigma \in \tau\}$. Here we use Remark 1.5 which asserts that Γ is injective. Suppose $\chi \colon Z_\tau \to \mathbb{C}$ is a character of Z_τ. Then there exists $\mu \in \mathfrak{h}_{\mathbb{C}}^*$, such that $\chi \circ A = \chi_\mu$, where χ_μ is the homomorphism defined by evaluation at μ. Obviously $p(\mu) = 0$, for all $p \in \ker(A)$. Let

$$S = \bigcup_{\sigma \in \tau, w \in \widetilde{W}} w(-\Lambda_\sigma + \mathfrak{a}_{\mathbb{C}}^*) \subset \mathfrak{h}_{\mathbb{C}}^*.$$

Obviously one can find $p_1, \dots, p_m$ in $S(\mathfrak{h}_{\mathbb{C}})$ such that S is the common zeros of $p_1, \dots, p_m$. Then by Lemma 1.12 we can find $q_1, \dots, q_n$ in $I(\mathfrak{h}_{\mathbb{C}})$ such that S is the common zeros of $q_1, \dots, q_n$. This shows $q_1, \dots, q_n$ are in $\ker(A)$. So $q_1(\mu) = \cdots = q_n(\mu)$. Therefore $\mu \in S$, i.e. $\mu = w(\lambda - \Lambda_\sigma)$ for some $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\sigma \in \tau$, and $w \in \widetilde{W}$. $\square$

For $s \in M'$, define $s.(X \otimes T) = \mathrm{Ad}(s)X \otimes \tau(s)T\tau(s^{-1})$, for $X \in U(\mathfrak{a})$, and $T \in \mathrm{End}(V)$. The next proposition is about a property of the generalized Harish-Chandra homomorphism. It is a weak version of a conjecture by Lepowsky.

Proposition 1.13. *For each $s \in M'/M$, $s.\Gamma(D) = \Gamma(D)$, for each $D \in Z_\tau$.*

For the proof of this result we need more facts about Weyl groups. Let $\widetilde{W}_1 \subset \widetilde{W}$ be the subgroup where every element stabilizes $\mathfrak{a}$. It is well-known there is a surjective homomorphism $\widetilde{W}_1 \to M'/M$. The kernel $\widetilde{W}_0$ is the Weyl group for $(\mathfrak{m}, \mathfrak{t})$.

Lemma 1.14. *For each $s \in M'/M$, choose $w(s)$ in $\widetilde{W}_1$ in the preimage of s under the homomorphism above. Then $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$.*

Proof (by Vogan). Take a maximal torus T of M_0. sTs^{-1} is another maximal torus. So there is $m \in M_0$, such that $msTs^{-1}m^{-1} = T$. To avoid cumbersome notations we assume $sTs^{-1} = T$. It is easy to see that $\mathrm{Ad}(s)^*$, the transpose of $\mathrm{Ad}(s)$, preserves $\Sigma(\mathfrak{m}, \mathfrak{t})$. We can also assume $\mathrm{Ad}(s)^*$ preserves $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. For $Z \in \mathcal{Z}(\mathfrak{m})$,

$$Z - \gamma'(Z) \in \mathfrak{m}^- U(\mathfrak{m}).$$

Hence

$$\mathrm{Ad}(s)Z - \mathrm{Ad}(s)\gamma'(Z) \in \mathfrak{m}^- U(\mathfrak{m}).$$

So

$$\begin{aligned}\sigma^s(Z) &= \sigma(\mathrm{Ad}(s)Z) = \mathrm{Ad}(s)\gamma'(Z)(\Lambda_\sigma - \rho_0)\\ &= \gamma'(Z)(\mathrm{Ad}(s)^*\Lambda_\sigma - \rho_0) = \gamma(Z)(\mathrm{Ad}(s)^*\Lambda_\sigma).\end{aligned}$$

Hence

$$\Lambda_{\sigma^s} = \mathrm{Ad}(s)^*\Lambda_\sigma = w(s)\Lambda_\sigma. \quad \square$$

Proof of Proposition 1.13. Take $Z \in \mathcal{Z}(\mathfrak{g})$ such that $D = \mu(Z)$. Then for each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and $s \in M'$,

$$s.\Gamma(D)(\lambda)|V(\sigma) = s.\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\mathrm{Ad}(s)^*\lambda - \Lambda_{\sigma^s})\mathrm{I}_{V(\sigma)}.$$

By Lemma 1.14, $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$. So

$$\begin{aligned}s.\Gamma(D)(\lambda)|V(\sigma) &= \gamma(Z)(\mathrm{Ad}(s)^*\lambda - w(s)\Lambda_\sigma)\mathrm{I}_{V(\sigma)}\\ &= \gamma(Z)(\lambda - \Lambda_\sigma)\mathrm{I}_{V(\sigma)}\\ &= \Gamma(\mu(Z))(\lambda)|V(\sigma)\\ &= \Gamma(D)(\lambda)|V(\sigma). \quad \square\end{aligned}$$

Now let $\overline{\mathfrak{n}} = \theta\mathfrak{n}$. Similarly as in Proposition 1.1 we get

$$U(\mathfrak{g}) \otimes \mathrm{End}(V) = U(\mathfrak{a}) \otimes \mathrm{End}(V) \oplus (\overline{\mathfrak{n}}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau)).$$

Then we define $\widetilde{\Gamma}_1 \colon U(\mathfrak{g}) \otimes \mathrm{End}(V) \to U(\mathfrak{a}) \otimes \mathrm{End}(V)$ as the projection according to this decomposition.

Corollary 1.15. *For each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$,*

$$\widetilde{\Gamma}_1(Z \otimes 1)(\lambda) = \Gamma(\mu(Z))(\lambda + \rho).$$

Proof. Take $s \in M'$ such that $\mathrm{Ad}(s)^*\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^-(\mathfrak{g}, \mathfrak{a})$. By definition

$$Z \otimes 1 - \Gamma_1(Z \otimes 1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

Hence

$$s.(Z \otimes 1) - s.\Gamma_1(Z \otimes 1) \in \overline{\mathfrak{n}}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

So

$$\widetilde{\Gamma}_1(Z \otimes 1) = s.\Gamma_1(Z \otimes 1).$$

Hence

$$\begin{aligned}\widetilde{\Gamma}_1(Z \otimes 1)(\lambda) &= \tau(s)\Gamma_1(Z \otimes 1)(\mathrm{Ad}(s)^*\lambda)\tau(s^{-1})\\ &= \tau(s)\Gamma(\mu(Z))(\mathrm{Ad}(s)^*\lambda - \rho)\tau(s^{-1})\\ &= \tau(s)\Gamma(\mu(Z))(\mathrm{Ad}(s)^*(\lambda + \rho))\tau(s^{-1})\\ &= \Gamma(\mu(Z))(\lambda + \rho). \quad \square\end{aligned}$$

2. Some function spaces on G

In this section we introduce a certain growth condition on a function on G with values in V. It turns out the condition is satisfied by $P_T\phi$ for any $\phi \in C^\infty \operatorname{Ind}_B^G(\delta)$, where δ is a certain finite dimensional representation of B.

For each $g \in G$, we denote by $\|g\|$ the operator norm of $\operatorname{Ad}(g)$ on $\mathfrak{g}$, which is equipped with the inner product $\langle X, Y\rangle_\Theta = -K(X, \Theta Y)$. Here K is the Killing form on $\mathfrak{g}$.

Lemma 2.1. (i) $\|g\| = \|\Theta g\| = \|g^{-1}\| \geq 1$;
(ii) $\|g_1 g_2\| \leq \|g_1\| \, \|g_2\|$;
(iii) *if* $g = k_1 a k_2$ *with* $k_1, k_2 \in K$, $a \in A$, *then*

$$\|g\| = \exp\left(\max_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} |\alpha(\log a)|\right);$$

(iv) *there are constants* $C_1, C_2 > 0$, *such that if* $x = \exp X$ *with* $X \in \mathfrak{p}$, *then* $e^{C_1|X|} \leq \|x\| \leq e^{C_2|X|}$. *Here* $\mathfrak{p}$ *is the* -1 *eigenspace of* Θ, *and* $|X| = \sqrt{\langle X, X\rangle_\Theta}$;
(v) $\|a\| \leq \|an\|$, *for* $a \in A$, *and* $n \in N$.

Proof. See [BS]. □

For any function $f : G \to V$ and $r \in \mathbb{R}$, we write

$$\|f\|_r = \sup_{g \in G} \|g\|^{-r} |f(g)|.$$

We say f increases at most exponentially if $\|f\|_r < \infty$, for some $r \in \mathbb{R}$. Let $C_r(G, V)$ denote the Banach space of continuous functions f on G with values in V with $\|f\|_r \leq \infty$.

Example 2.2. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and σ a finite dimensional representation of M. Let $r(\lambda) = C_1^{-1} |\operatorname{Re}\lambda - \rho|$, where C_1 is the constant in Lemma 2.1 (iv). Then for each $\phi \in C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$, and $T \in \operatorname{Hom}_M(V_\sigma, V)$, $P_T\phi$ belongs to $C_{r(\lambda)}(G, V)$. This is in [BS] when τ is trivial and τ in general does not offer additional difficulties.

Write

$$C_r^\infty(G, V) = \{f \in C^\infty(G, V) | L_u f \in C_r(G, V), \forall u \in U(\mathfrak{g})\}.$$

We endow $C_r(G, V)$ with its standard topology: Let $X_1, \ldots, X_p$ be a basis of $\mathfrak{g}$, and $X^I = X^{i_1} \cdots X^{i_p} \in U(\mathfrak{g})$ for $I = (i_1, \ldots, i_p) \in \mathbb{N}^p$. For $q \in \mathbb{N}$ and $f \in C^q(G, V)$, a q times continuously differentiable function from G to V, we define

$$\|f\|_{q,r} = \sum_{|I| \leq q} \|L_{X^I} f\|_r.$$

Endowed with this norm the space

$$C_r^q(G, V) = \{f \in C^q(G, V) | \|f\|_{q,r} < \infty\}$$

is a Banach space. Obviously $C_r^q \subset C_r^{q'}$ if $q' \leq q$, $C_r^\infty(G, V) = \bigcap_q C_r^q(G, V)$. The topology on $C_r^\infty(G, V)$ is given by the family of norms $\| \cdot \|_{q,r}$, $q \in \mathbb{N}$, on $C_r^\infty(G, V)$. We now consider for each $q \in \mathbb{N}$ the action of L and R on $C_r^\infty(G, V)$. Recall for $g, x \in G$, and $f \in C^q(G, V)$, $L_x f(g) = f(x^{-1}g)$, and $R_x f(g) = f(gx)$. Obviously L_x leaves $C_r^q(G, V)$ invariant. In fact $\|L_x f\|_{q,r} \leq C\|x\|^{r+s}\|f\|_{q,r}$, for each $f \in C_r^q(G, V)$, and $x \in G$. Here C and s are constants. On the other hand,

$\|R_x f\|_{q,r} \leq \|x\|^r \|f\|_{q,r}$. From Example 2.2, we see P_T maps $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$ into $C^\infty_{r(\lambda)}(G, V)$ continuously.

Recall from Proposition 1.11 a character of Z_τ is given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and Λ is the infinitesimal character of an irreducible representation of M in τ. Let $\mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ denote the corresponding eigenspace of Z_τ. Let

$$\mathcal{E}^\infty_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau) = \mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) \cap C^\infty_r(G, V),$$
$$\mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) = \bigcup_{r \in \mathbb{R}} \mathcal{E}^\infty_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau).$$

Our goal is to describe $\mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ in terms of the Poisson transform, at least for "generic" $\lambda - \Lambda$. The following result due to Harish-Chandra is very important to us. See [Wall2].

Proposition 2.3. *Let $\overline{N} = \Theta N$. Then $C(\lambda)$ defined by*

$$C(\lambda) = \int_{\overline{N}} \tau(\mathrm{k}(\overline{n})) e^{-(\lambda+\rho)\mathrm{H}(\overline{n})} d\overline{n}$$

is holomorphic on $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} | \operatorname{Re}\langle \lambda, \alpha \rangle > 0$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$. Moreover there exists a meromorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$.

Proposition 2.4. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that* $\operatorname{Re}\langle \lambda, \alpha \rangle > 0$*, for $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then*

$$\lim_{t \to \infty} e^{(-\lambda+\rho)(H)} P_T \phi(g \exp tH) = C(\lambda) T\phi,$$

for each $H \in \mathfrak{a}^+$, $T \in \operatorname{Hom}_M(V_\sigma, V)$, and $\phi \in C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$. Here $\mathfrak{a}^+ = \{X \in \mathfrak{a} | \alpha(X) > 0, \forall \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$.

Proof. First we observe $k \to \tau(k) T\phi(g \exp tHk)$ is a function on K/M. By Theorem 5.20 in Chapter I in [Helg1],

$$\begin{aligned} P_T\phi(g \exp tH) &= \int_{\overline{N}} \tau(\mathrm{k}(\overline{n})) T\phi(g \exp tH \mathrm{k}(\overline{n})) e^{-2\rho \mathrm{H}(\overline{n})} d\overline{n} \\ &= \int_{\overline{N}} e^{-(\lambda+\rho)\mathrm{H}(\overline{n})} \tau(\mathrm{k}(\overline{n})) T\phi(g \exp tH\overline{n}) d\overline{n} \\ &= e^{(\lambda-\rho)tH} \int_{\overline{N}} e^{-(\lambda+\rho)\mathrm{H}(\overline{n})} \tau(\mathrm{k}(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n}. \end{aligned}$$

Here $a_t = \exp tH$. So

$$e^{-(\lambda-\rho)tH} P_T\phi(g \exp tH) = \int_{\overline{N}} e^{-(\lambda+\rho)\mathrm{H}(\overline{n})} \tau(\mathrm{k}(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n}$$

since $a_t \overline{n} a_t^{-1} \to e$, as $t \to \infty$. Formally we have

$$P_T\phi(g \exp tH) \to C(\lambda) T\phi(g),$$

as $t \to \infty$. To justify the exchange of two limits we use an argument due to Helgason. Let $\lambda = \xi + \sqrt{-1}\eta$, for $\xi, \eta \in \mathfrak{a}^*$. Our assumption on λ amounts to $A_\xi \in \mathfrak{a}^+$, where A_ξ is given by $\langle \mu, A_\xi \rangle = K(\xi, \mu)$, for each $\mu \in \mathfrak{a}^*$. It was proved by Harish-Chandra that

$$B(H, \mathrm{H}(\overline{n})) \geq 0, \quad B(H, \mathrm{H}(\overline{n}) - \mathrm{H}(a_t \overline{n} a_t^{-1})) \geq 0,$$

for each $H \in \mathfrak{a}^+$. Thus if we choose ε such that $0 < \varepsilon < 1$, $A_\rho - \varepsilon A_\xi \in \mathfrak{a}^+$, and put

$$C = \sup_{\overline{n}, t} |\tau(\mathrm{k}(\overline{n})) T\phi(g \mathrm{k}(a_t \overline{n} a_t^{-1}))| < \infty,$$

then

$$
\begin{aligned}
&|e^{-(\lambda+\rho)\mathrm{H}(\overline{n})}\tau(\mathrm{k}(\overline{n}))T\phi(ga_t\overline{n}a_t^{-1})| \\
&\quad = |e^{-(\lambda+\rho)\mathrm{H}(\overline{n})}e^{(\lambda-\rho)\mathrm{H}(a_t\overline{n}a_t^{-1})}\tau(\mathrm{k}(\overline{n}))T\phi(g\mathrm{k}(a_t\overline{n}a_t^{-1}))| \\
&\quad \leq Ce^{-(\xi+\rho)\mathrm{H}(\overline{n})}e^{(\xi-\rho)\mathrm{H}(a_t\overline{n}a_t^{-1})} \\
&\quad \leq Ce^{-(\xi+\rho)\mathrm{H}(\overline{n})}e^{(\xi-\varepsilon\xi)\mathrm{H}(a_t\overline{n}a_t^{-1})} \\
&\quad \leq Ce^{-(\xi+\rho)\mathrm{H}(\overline{n})}e^{(\xi-\varepsilon\xi)\mathrm{H}(\overline{n})} \\
&\quad \leq Ce^{(-\varepsilon\xi-\rho)\mathrm{H}(\overline{n})}.
\end{aligned}
$$

This being integrable over $\overline{N}$ justifies letting $t \to \infty$ under the integral sign and proves Proposition 2.4. □

3. Asymptotics

By a formal expansion at a point $H_0 \in \mathfrak{a}^+$, we mean a formal sum

$$\sum_{\xi \in X} p_\xi(H,t)e^{t\xi(H)},$$

where X is a subset of $\mathfrak{a}_{\mathbb{C}}^*$ such that the subset $X(N)$ given by

$$X(N) = \{\xi \in X \,|\, \mathrm{Re}\,\xi(H_0) \geq N\}$$

is a finite set for each $N \in \mathbb{R}$, where p_ξ is a continuous function defined in a neighborhood of $\{H_0\} \times \mathbb{R}$ and polynomial in the last variable.

Let f be a function $\mathfrak{a}^+ \to V$. If $N \in \mathbb{R}$ we say the formal sum is asymptotic to f of order N at H_0, if there exist a neighborhood of H_0 in $\mathfrak{a}^+$, say U, and constants $\varepsilon \geq 0$, $C \geq 0$, such that

$$\left| f(tH) - \sum_{\xi \in X(N)} p_\xi(H,t)e^{t\xi(H)} \right| \leq Ce^{(N-\varepsilon)t},$$

for each $H \in U$, $t \geq 0$. Moreover, we say the formal expansion is an asymptotic expansion for f at H_0 if for every $N \in \mathbb{R}$ it is asymptotic to f of order N at H_0. We write this as

$$f(tH) \sim \sum_{\xi \in X} p_\xi(H,t)e^{t\xi(H)} \qquad (t \to \infty).$$

The following result shows that the p_ξ's are essentially unique.

Proposition 3.1. *Let $X \subset \mathfrak{a}_{\mathbb{C}}^*$, $\sum_{\xi \in X} p_\xi(H,t)e^{t\xi(H)}$ and $\sum_{\xi \in X} q_\xi(H,t)e^{t\xi(H)}$ be formal expansions at H_0, both assumed to be asymptotic to $f\colon \mathfrak{a}^+ \to V$. Then for each $\xi \in X$, there is a neighborhood U of H_0, such that $p_\xi = q_\xi$ on $U \times \mathbb{R}$.*

Proof. See Proposition 3.1 in [BS]. □

Let $\lambda - \Lambda$ be a character of Z_τ in the sense of Proposition 1.11, where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and Λ is given by the infinitesimal character of an irreducible representation of M. Let $X(\lambda, \Lambda)$ be the subset of $\mathfrak{a}_{\mathbb{C}}^*$ defined by

$$X(\lambda,\Lambda) = \{w(\lambda-\Lambda)+\Lambda_\sigma-\rho-\mathbb{N}\cdot\Delta | w \in \widetilde{W}, \sigma \in \tau, (w(\lambda-\Lambda)+\Lambda_\sigma)|\mathfrak{t}=0\}.$$

Then we have the following results.

Theorem 3.2. (i) *For each* $f \in \mathcal{E}^{\infty}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$, $x \in G$, *and* $\xi \in X(\lambda, \Lambda)$, *there exists a unique polynomial* $p_{\lambda,\xi}(f, x, \cdot)$ *on* $\mathfrak{a}$ *with values in* V, *such that*

$$f(tH) \sim \sum_{\xi \in X(\lambda,\Lambda)} p_{\lambda,\xi}(f, x, tH) e^{t\xi(H)} \qquad (t \to \infty),$$

at every $H_0 \in \mathfrak{a}^+$, *and the polynomials have degree* $\leq d$, *where* d *is the number of elements in* $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

(ii) *Let* $r \in \mathbb{R}$ *and* $\xi \in X(\lambda, \Lambda)$; *there exists* $r' \in \mathbb{R}$ *such that* $f \to p_{\lambda,\xi}(f, \cdot, \cdot)$ *is a continuous map of* $\mathcal{E}^{\infty}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$ *into* $C^{\infty}_{r'}(G, V) \otimes P_d(\mathfrak{a})$, *equivariant for the left action of* G *on* $\mathcal{E}^{\infty}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$ *to* $C^{\infty}_{r'}(G, V) \otimes P_d(\mathfrak{a})$.

Theorem 3.3. *Let* Ω *be an open set in* $\mathfrak{a}^*_{\mathbb{C}}$. *Let* $\{f_\lambda\}_{\lambda \in \Omega}$ *be a holomorphic family in* $C^{\infty}_r \operatorname{Ind}_K^G(\tau)$ *such that* $f_\lambda \in \mathcal{E}^{\infty}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$ *for each* $\lambda \in \Omega$. *Fix* $\lambda_0 \in \Omega$ *and* $\xi_0 \in X(\lambda_0, \Lambda)$. *Let*

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0\}.$$

There exist an open neighborhood $\Omega_0 \subset \Omega$ *of* λ_0 *and a constant* $r' \in \mathbb{R}$ *such that the map* $(\lambda, H) \to \sum_{\xi \in \Xi(\lambda)} p_{\lambda,\xi}(f_\lambda, \cdot, H) e^{\xi(H)}$ *is continuous from* $\Omega \times \mathfrak{a}^+$ *into* $C^{\infty}_{r'}(G, V)$ *and in addition holomorphic in* λ.

We shall prove these results in Sections 5 and 6.

4. Some algebraic results

This section is a necessary preparation for the proof of the theorems stated in last section. It is strongly influenced by [Ban] and [BS].

Let E be the set of W-harmonic polynomials in $\mathfrak{a}^*$. It is well-known that $j \colon E \otimes I(\mathfrak{a}) \to S(\mathfrak{a})$ is bijective, where $j(e \otimes h) = eh$.

Now let $r \colon I(\mathfrak{h}_{\mathbb{C}}) \to I(\mathfrak{a})$ be the restriction map. **We assume r is surjective for the rest of the thesis**. According to [Helg3] if G/K is irreducible there are just four exceptions, and they only occur among symmetric spaces of exceptional groups. Pick a set of algebraically independent homogeneous generators of $I(\mathfrak{a})$, say, $p_1, \ldots, p_m$. Choose homogeneous elements $q_1, \ldots, q_m$ in $I(\mathfrak{h}_{\mathbb{C}})$, such that $r(q_i) = p_i$, for $i = 1, \ldots, m$. Let $I_1(\mathfrak{h}_{\mathbb{C}})$ be the polynomial ring of $q_1, \ldots, q_m$.

For any $\mu \in \mathfrak{h}^*_{\mathbb{C}}$, let

$$I^-_{1,\mu} = \{(T_\mu p)^- | p \in I_1(\mathfrak{h}_{\mathbb{C}})\}.$$

Here $(T_\mu p)(\nu) = p(\mu + \nu)$, for each $\nu \in \mathfrak{h}^*_{\mathbb{C}}$, and $(T_\mu p)^-(\lambda) = p(\mu + \lambda)$, for each $\lambda \in \mathfrak{a}^*$.

Proposition 4.1. *Let* $j_\mu \colon E \otimes I^-_{1,\mu} \to S(\mathfrak{a})$ *be defined by*

$$j_\mu(e \otimes h) = eh.$$

Then j_μ *is bijective.*

Proof. Observe $(T_\mu q_i)^- = p_i + r_i$, with $\deg r_i < \deg p_i$. Using the fact that j is bijective and by induction we are done. □

Let $\mathcal{Z}_1(\mathfrak{g}) = \gamma^{-1}(\mathrm{I}_1(\mathfrak{h}))$. Here γ is the Harish-Chandra homomorphism. For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $\Lambda = \Lambda_\sigma$ for some $\sigma \in \tau$, let

$$I(\lambda, \Lambda) = \{Z \in \mathcal{Z}_1(\mathfrak{g}) | \gamma(Z)(\lambda - \Lambda) = 0\}.$$

Recall $I(\tau)$ is the left ideal of $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ generated by $X \otimes 1 + 1 \otimes \tau(X)$, for all $X \in \mathfrak{k}$. Let $J(\lambda, \Lambda)$ be the left ideal generated by $I(\lambda, \Lambda)$ and $I(\tau)$. Let

$$\mathfrak{Y}_{\lambda,\Lambda} = U(\mathfrak{g}) \otimes \mathrm{End}(V)/J(\lambda, \Lambda).$$

Our interest in $\mathfrak{Y}_{\lambda,\Lambda}$ comes from the fact that for $f \in \mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$, the map $u \otimes T \to TR_u f$ factors through $\mathfrak{Y}_{\lambda,\Lambda}$ since f is killed by $J(\lambda, \Lambda)$. We shall find below an underlying vector space for $\mathfrak{Y}_{\lambda,\Lambda}$ independent of λ.

Write $\mathfrak{Y} = U(\overline{\mathfrak{n}}) \otimes E \otimes \mathrm{End}(V)$. We shall construct a linear bijection of $\mathfrak{Y}$ with $\mathfrak{Y}_{\lambda,\Lambda}$. First we identify $\mathfrak{Y}$ with a subspace of $U(\mathfrak{g}) \otimes \mathrm{End}(V)$ as follows: $u \otimes e \otimes T \to (u \cdot e) \otimes T$, for $u \in U(\overline{\mathfrak{n}})$, $e \in E$, and $T \in \mathrm{End}(V)$. Here $\cdot$ denotes the multiplication in $U(\mathfrak{a} + \overline{\mathfrak{n}})$. Let $\Psi\colon \mathfrak{Y} \otimes \mathcal{Z}_1(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathrm{End}(V)/I(\tau)$ be the map defined by

$$\Psi(y \otimes Z) = y \cdot (Z \otimes 1) + I(\tau),$$

for $y \in \mathfrak{Y}$, $Z \in \mathcal{Z}_1(\mathfrak{g})$. Here $\cdot$ means the multiplication in $U(\mathfrak{g}) \otimes \mathrm{End}(V)$.

Proposition 4.2. *In the setting above, Ψ is bijective.*

Proof. By the Iwasawa decomposition we have

$$U(\mathfrak{g}) \otimes \mathrm{End}(V)/I(\tau) \cong U(\overline{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes \mathrm{End}(V).$$

Via this isomorphism the degree on $U(\mathfrak{a})$ induces a degree (denoted by $\deg_{\mathfrak{a}}$) on $U(\mathfrak{g}) \otimes \mathrm{End}(V)/I(\tau)$. Let $\mathfrak{Y} \otimes \mathcal{Z}_1(\mathfrak{g})$ be filtered by the total degree on $E \otimes \mathcal{Z}_1(\mathfrak{g})$. Notice

$$\deg_{\mathfrak{a}}(Z \otimes 1 - (T_{\rho-\Lambda_\sigma}\gamma(Z))^{-1} \otimes 1 + I(\tau)) < \deg(Z \otimes 1),$$

for $Z \in \mathcal{Z}_1(\mathfrak{g})$, and each $\sigma \in \tau$. So Ψ preserves the filtrations. It also follows that the graded map

$$gr\Psi\colon U(\overline{\mathfrak{n}}) \otimes gr(E \otimes \mathcal{Z}_1(\mathfrak{g})) \otimes \mathrm{End}(V) \to U(\overline{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes \mathrm{End}(V)$$

associated to Ψ, is given by

$$u \otimes e \otimes Z \otimes T \to u \cdot e \cdot (T_{\rho-\Lambda_\sigma}\gamma(Z))^{-} \otimes T,$$

for $u \in U(\overline{\mathfrak{n}})$, $e \in E$, $Z \in \mathcal{Z}_1(\mathfrak{g})$, and $T \in \mathrm{Hom}(V(\sigma), V)$ (here we use Proposition 1.15). This is bijective because of Proposition 4.1. ☐

Corollary 4.3. (i) *Ψ maps $\mathfrak{Y} \otimes I(\lambda, \Lambda)$ onto $J(\lambda, \Lambda)$ modulo $I(\tau)$;* (ii) *for each $u \in U(\mathfrak{g}) \otimes \mathrm{End}(V)$ there exists a unique $y \in \mathfrak{Y}$, such that $u - y \in J(\lambda, \Lambda)$.*

Proof. See Corollary 5.2 in [BS]. ☐

From the corollary we obtain a linear bijection b_λ of $\mathfrak{Y}_{\lambda,\Lambda}$ onto $\mathfrak{Y}$, defined by $u - b_\lambda(u + J(\lambda, \Lambda)) \subset J(\lambda, \Lambda)$. Through this bijection $\mathfrak{Y}$ is equipped with a $(\mathfrak{g}, K)$-module structure from $\mathfrak{Y}_{\lambda,\Lambda}$, by making b_λ a morphism of modules. Recall the $\mathfrak{g}$ action on $\mathfrak{Y}_{\lambda,\Lambda}$ is induced from left multiplication in $U(\mathfrak{g})$, and the K action is induced from the following K action on $U(\mathfrak{g}) \otimes \mathrm{End}(V)$,

$$k.(u \otimes T) = \mathrm{Ad}(k)u \otimes T\tau(k^{-1}),$$

for each $k \in K$, $u \in U(\mathfrak{g})$, and $T \in \mathrm{End}(V)$. Notice the difference from the action we use to define $U(\mathfrak{g})^K$.

Let τ_λ denote the resulting $\mathfrak{g}$ action on $\mathfrak{Y}$. Notice the action of $\overline{\mathfrak{n}}$ on $\mathfrak{Y}$ is just the left multiplication. The action of $\mathfrak{a}$ can be determined as follows: Let

$y \in \mathfrak{Y} \subset U(\mathfrak{g}) \otimes \mathrm{End}(V)$, $H \in \mathfrak{a}$; then $H \cdot y$ can be written (modulo $I(\tau)$) as $\Psi(\sum y_i \otimes Z_i)$ according to Proposition 4.2. Then by the definition of τ_λ we have

$$(*) \qquad \tau_\lambda(H)y = \sum \gamma(Z_i)(\lambda - \Lambda)y_i.$$

For each $k \in \mathbb{N}$, let $\overline{\mathfrak{n}}^k$ be the linear span of k times product of $\overline{\mathfrak{n}}$ in $U(\overline{\mathfrak{n}})$. Then τ_λ induces a representation τ_λ^k of $\mathfrak{a} + \mathfrak{m}$ on the finite dimensional space $\mathfrak{Y}/\overline{\mathfrak{n}}^k\mathfrak{Y}$. In particular τ_λ^1 is a representation of $\mathfrak{a} + \mathfrak{m}$ on $\mathfrak{Y}/\overline{\mathfrak{n}}\mathfrak{Y} \cong E \otimes \mathrm{End}(V)$. By $(*)$ we know τ_λ and τ_λ^k are holomorphic in λ.

Let $\{\lambda_1, \ldots, \lambda_l\}$ be the set of weights of τ_λ^1 restricted to $\mathfrak{a}$, and $\Lambda_k \subset -\mathbb{N} \cdot \Delta$ an enumeration of the weights of the $\mathfrak{a}$-module $U(\overline{\mathfrak{n}})/\overline{\mathfrak{n}}^k U(\overline{\mathfrak{n}})$.

Proposition 4.4. *For each $k \in \mathbb{N}$, $k \geq 1$, the set of weights of $(\tau_\lambda^k, \mathfrak{a})$ is*

$$\{\lambda_i + \mu | i = 1, \ldots, l, \mu \in \Lambda_k\}.$$

Proof. By induction on k. It is trivial for $k = 1$. For $k > 1$, the induction step is a consequence of the following two exact sequences of $\mathfrak{a}$-modules.

$$0 \to \overline{\mathfrak{n}}^{k-1}U(\overline{\mathfrak{n}})/\overline{\mathfrak{n}}^k U(\overline{\mathfrak{n}}) \otimes \mathfrak{Y}_{\lambda,\Lambda} \to \mathfrak{Y}_{\lambda,\Lambda}/\overline{\mathfrak{n}}^k \mathfrak{Y}_{\lambda,\Lambda} \to \mathfrak{Y}_{\lambda,\Lambda}/\overline{\mathfrak{n}}^{k-1}\mathfrak{Y}_{\lambda,\Lambda} \to 0,$$
$$0 \to \overline{\mathfrak{n}}^{k-1}U(\overline{\mathfrak{n}})/\overline{\mathfrak{n}}^k U(\overline{\mathfrak{n}}) \to U(\overline{\mathfrak{n}})/\overline{\mathfrak{n}}^k U(\overline{\mathfrak{n}}) \to U(\overline{\mathfrak{n}})/\overline{\mathfrak{n}}^{k-1} U(\overline{\mathfrak{n}}) \to 0. \qquad \square$$

Let $\overline{V}_k = \mathfrak{Y}/\overline{\mathfrak{n}}^k\mathfrak{Y}$, and $\widetilde{V}_k$ be a finite dimensional subspace of $\mathfrak{Y}$ mapped bijectively onto $\overline{V}_k$ by the canonical projection. Let $\pi\colon \widetilde{V}_k \to \overline{V}_k$ be the restriction of the canonical projection. Define $m\colon \mathfrak{Y} \to U(\mathfrak{g}) \otimes \mathrm{End}(V)$ by

$$m(u \otimes e \otimes T) = (u \cdot e) \otimes T,$$

for $u \in U(\overline{\mathfrak{n}})$, $e \in E$, and $T \in \mathrm{End}(V)$.

Let V_k be the image of $\widetilde{V}_k$ under m, and $\eta\colon V_k \to \widetilde{V}_k$ be the inverse of $m|\widetilde{V}_k$. Let $\mathcal{Z}(\mathfrak{a} + \mathfrak{m})$ be the center of $U(\mathfrak{a} + \mathfrak{m})$.

Proposition 4.5. *For $k \in \mathbb{N}$, $k \geq 1$, there exist*

(i) *an algebra homomorphism $b_k(\lambda, \cdot)\colon \mathcal{Z}(\mathfrak{a} + \mathfrak{m}) \to \mathrm{End}(V_k)$,*

(ii) *a linear map $y_\lambda\colon \mathcal{Z}(\mathfrak{a} + \mathfrak{m}) \otimes V_k \to \overline{\mathfrak{n}}^k U(\mathfrak{a} + \overline{\mathfrak{n}}) \otimes \mathrm{End}(V)$, both depending polynomially on λ, such that for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, and $v \in V_k$,*

$$Dv - b_k(\lambda, D)v - y_\lambda(D, v) \in J(\lambda, \Lambda).$$

Proof. Let $p_\lambda\colon U(\mathfrak{g}) \otimes \mathrm{End}(V) \to \mathfrak{Y}$ be the map defined by

$$p_\lambda(u \otimes T) = \tau_\lambda(u)(1 \otimes 1 \otimes T),$$

for $u \in U(\mathfrak{g})$, and $T \in \mathrm{End}(V)$. For $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, $\tilde{v} \in \widetilde{V}_k$ we define the maps

$$\tilde{b}_k(\lambda, D) = \pi^{-1} \circ \tau_\lambda^k \circ \pi \in \mathrm{End}\, \widetilde{V}_k,$$
$$\tilde{y}_\lambda(D, \tilde{v}) = p_\lambda((D \otimes 1) \cdot m(\tilde{v})) - m(\tilde{b}_k(\lambda, D)\tilde{v}) \in \mathfrak{Y}.$$

Then $b_k(\lambda, \cdot)$ and y_λ are defined by

$$b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta, \quad y_\lambda(D, v) = m(\tilde{y}_\lambda(D, \eta(v))),$$

for $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, $v \in V_k$. $\square$

Corollary 4.6. *As a representation of $\mathfrak{a}$, $b_k(\lambda, \cdot)$ has the same weights as $(\tau_\lambda^k, \mathfrak{a})$, i.e. $\{\lambda_i + \mu | i = 1, \ldots, l, \mu \in \Lambda_k\}$.*

Proof. By definition $b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta$, and $\eta = (m|\widetilde{V}_k)^{-1}$. So $b_k(\lambda, \cdot)$ has the same weights as $\tilde{b}_k(\lambda, \cdot)$. Since $\tilde{b}_k(\lambda, \cdot) = \pi^{-1} \circ \tau_\lambda^k \circ \pi$, the proof is complete. □

Let V_k^* be the dual space of V_k, and $b_k^*(\lambda, \cdot)$ be the transpose of $b_k(\lambda, \cdot)$. For each weight ξ of $b_k^*(\lambda, \cdot)$ we denote by $P_{\lambda,\xi}$ the projection map from V_k^* onto the generalized weight space of ξ, along the remaining generalized weight spaces. We now consider the holomorphic dependence of $P_{\lambda,\xi}$ on λ.

Proposition 4.7. *There exists for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and each weight ξ a unique polynomial $q_{\lambda,\xi}$ on $\mathfrak{a}$ with values in $\mathrm{End}(V_k^*)$, such that*

$$P_{\lambda,\xi} q_{\lambda,\xi}(H) P_{\lambda,\xi} = q_{\lambda,\xi}(H),$$
$$\exp b_k^*(\lambda, H) = \sum_{\xi} e^{\xi(H)} q_{\lambda,\xi}(H),$$

for $H \in \mathfrak{a}$.

Proof. Let $V_k^*(\xi)$ be the generalized weight space of ξ. Then the restriction of $b_k^*(\lambda, \cdot)$ to $V_k^*(\xi)$ gives a representation of $\mathfrak{a}$. $\mathfrak{a}$ is abelian so in particular solvable. Hence by Lie's theorem one can find a basis such that $b_k^*(\lambda, H)|V_k^*(\xi)$ corresponds to an upper triangular matrix, for each $H \in \mathfrak{a}$. The diagonal entries are $\xi(H)$. So there exists a unique polynomial $q_{\lambda,\xi}(H)$ on $\mathfrak{a}$ with values in $\mathrm{End}(V_k^*)$, such that

$$\exp b_k^*(\lambda, H)|V_k^*(\xi) = e^{\xi(H)} q_{\lambda,\xi}(H). \quad \square$$

Let F be an N-dimensional complex vector space, and τ_z a family of representations of $\mathfrak{a}$ in F, depending on a parameter $z \in \mathbb{C}^n$. For each weight ξ of τ_z let $P_{z,\xi}$ be the projection map from F onto the generalized weight space $V(\xi)$, along the remaining generalized weight spaces. Fix $z_0 \in \mathbb{C}^n$, and ξ_0 a weight of τ_{z_0}.

Lemma 4.8. *Given any neighborhood $N(\xi_0)$ of ξ_0 there exist a neighborhood $V(\xi_0)$ of ξ_0 in $N(\xi_0)$, and a neighborhood $\Omega(z_0)$ of z_0, such that*

$$P(z) = \sum_{\xi \in V(\xi_0)} P_{z,\xi} \in \mathrm{End}(F)$$

is holomorphic in z in $\Omega(z_0)$.

Proof. We use the argument in Chapter II in [Kato]. First let us consider the case where $\dim \mathfrak{a} = 1$. Pick a nonzero element $H_0 \in \mathfrak{a}$. Let

$$T(z) = \tau_z(H_0) \in \mathrm{End}(F).$$

Then $\lambda_0 = \xi_0(H_0)$ is an eigenvalue of $T(z_0) = \tau_{z_0}(H_0)$. Define

$$R(z, \lambda) = (T(z) - \lambda)^{-1},$$

for $z \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$. By Theorem 1.5 in Section 3 of Chapter II in [Kato], $R(z, \lambda)$ is holomorphic in the two variables z and λ in each domain where λ is not an eigenvalue of $T(z)$. Moreover, for each (z_1, λ) in such a domain,

$$R(z, \lambda) = R(z_1, \lambda) + \sum_{I \in \mathbb{N}^n} R_I(\lambda)(z - z_1)^I,$$

where $R_I(\lambda)$ are determined by $R(z_1, \lambda)$, and they are holomorphic in λ. This is called the second Neumann series for the resolvent. It is uniformly convergent for sufficiently small $z - z_1$ and $\lambda \in \Gamma$ if Γ is a compact subset of the resolvent set of $T(z_1)$.

Let Γ be a closed positively oriented curve in the resolvent set of $T(z_0)$ enclosing λ_0 but no other eigenvalues of $T(z_0)$. Then

$$P(z) = -\frac{1}{2i\pi}\int_\Gamma R(z,\lambda)d\lambda$$

is holomorphic in z, for $z - z_0$ sufficiently small.

It is easy to see $P(z)$ is equal to the sum of the eigenprojections for all eigenvalues of $T(z)$ lying inside Γ. This basically takes care of the case $\dim \mathfrak{a} = 1$. In general we choose a basis $e_1, \dots, e_m$ for $\mathfrak{a}$. We can duplicate the above process to $T_i(z) = \tau_z(e_i)$, for $i = 1, \dots, m$. Thus we get $P_i(z)$, $i = 1, \dots, m$. Then the composition of P_i is our $P(z)$. ☐

Fix $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$, and ξ_0 a weight of $b^*_k(\lambda_0, \cdot)$. For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0\}.$$

Proposition 4.9. *There exist a neighborhood $\Omega_0(\lambda_0)$ of λ_0 and a neighborhood $V(\xi_0)$ of ξ_0, such that*

$$P(\lambda) = \sum_{\xi \in V(\xi_0)} P_{\lambda,\xi} \in \operatorname{End}(V^*_k)$$

is holomorphic in $\Omega_0(\lambda_0)$, and

$$\{\xi \in V(\xi_0) | \xi \text{ is a weight of } b^*_k(\lambda, \cdot)\} \cap X(\lambda, \Lambda) \subset \Xi(\lambda).$$

Proof. It follows at once from Lemma 4.8. ☐

5. Existence of asymptotic expansion

The methods we use in this section are similar to those used in [Ban], Section 12. Also see [BS], Section 6.

Fix $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $H_0 \in \mathfrak{a}^+$ and $r \in \mathbb{R}$. If A_1, A_2 are Banach spaces we denote by $B(A_1, A_2)$ the Banach space of bounded linear operators from A_1 to A_2.

Proposition 5.1. *There exist, for each $N \in \mathbb{R}$,*

(a) *open neighborhoods Ω of $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ and U of $H_0 \in \mathfrak{a}^+$,*
(b) *constants $k, q \in \mathbb{N}, r' \geq r$, and $C, \varepsilon > 0$,*
(c) *a continuous map*

$$\Phi \colon \Omega \times U \to B(C^q_r(G, V), V^*_k \otimes C_{r'}(G, V)),$$

holomorphic in the first variable, and

(d) *a linear form $\eta \in (V^*_k)^*$, such that*

(i) *$\Phi(\lambda, H)$ intertwines the left actions of G on $C^q_r(G, V)$ and $C_{r'}(G, V)$, for all $(\lambda, H) \in \Omega \times U$, and* (ii)

$$\|R_{\exp tH} f - (\eta \circ \exp b^*_k(\lambda, tH) \otimes 1)\Phi(\lambda, H) f\|_{r'} \leq C\|f\|_{q,r} e^{(N-\varepsilon)t},$$

for $f \in \mathcal{E}_{\lambda - \Lambda} \operatorname{Ind}^G_K(\tau) \cap C^q_r(G, V)$, $\lambda \in \Omega$, $H \in U$, $t \geq 0$.

Proof. In the same way as for Proposition 12.6 in [Ban]. ☐

We now begin the proof of Theorem 3.2. Using Proposition 4.7 we can write

$$(\eta \circ \exp b^*_k(\lambda, tH) \otimes 1)\Phi(\lambda, H) = \sum_\xi p_{\lambda,\xi}(H, t) e^{t\xi(H)},$$

for $\lambda \in \Omega$, $H \in U$, $t \geq 0$, where the summation extends to the weights ξ of $b_k^*(\lambda, \cdot)$ which by Corollary 4.6 is the set

$$\{\lambda_i + \mu | i = 1, \dots, l, \mu \in \Lambda_k\},$$

and where $p_{\lambda,\xi}(H,t) = (\eta \circ q_{\lambda,\xi}(tH) \otimes 1)\Phi(\lambda, H) \in B(C_r^q, C_{r'})$, which is continuous in H and polynomial in t. From (d) (ii) of Proposition 5.1 we have

$$\|R_{\exp tH} f - \sum_{\xi} e^{t\xi(H)} p_{\lambda,\xi}(H,t) f\|_{r'} \leq C \|f\|_{q,r} e^{t(N-\varepsilon)},$$

for $f \in \mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) \cap C_r^q(G,V)$. Since N is arbitrary we have for each $g \in G$,

$$f(g \exp tH) \sim \sum_{\xi \in \widetilde{X}(\lambda,\Lambda)} (p_{\lambda,\xi}(H,t)f)(g) e^{t\xi(H)} \qquad (t \to \infty).$$

Here $\widetilde{X}(\lambda, \Lambda) = \{\lambda_i + \mu | i = 1, \dots, l, \mu \in -\mathbb{N} \cdot \Delta\}$.

Lemma 5.2. *Let $X \subset \mathfrak{a}_{\mathbb{C}}^*$ and $f \colon \mathfrak{a}^+ \to V$. Assume that for each $H_0 \in \mathfrak{a}^+$ there is a given formal expansion*

$$\sum_{\xi \in X} p_{\xi, H_0}(H,t) e^{t\xi(H)}$$

which is an asymptotic expansion for f at H_0. Then for each $\xi \in X$ there exists a unique continuous function $p_\xi \colon \mathfrak{a}^+ \to V$ such that for each $H_0 \in \mathfrak{a}^+$ there is a neighborhood U with

$$p_{\xi,H_0}(H,t) = p_\xi(tH),$$

for $H \in U$, and $t > 0$.

Proof. See Corollary 3.4 in [BS]. □

As can be seen in the proof of Proposition 12.6 in [Ban], for $t > 0$, $H \in U$ with $tH \in U$, $\Phi(\lambda, tH) = \Phi(\lambda, H)$. Thus for $t > 0$, $H \in U$ with $tH \in U$, $(p_{\lambda,\xi}(H,t)f)(g) = (p_{\lambda,\xi}(tH,1)f)(g)$. By Lemma 5.2, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $r \in \mathbb{R}$, and $\xi \in \widetilde{X}(\lambda, \Lambda)$, there exist constants $r' \in \mathbb{R}$, $q \in \mathbb{N}$, and a unique continuous map $p_{\lambda,\xi}(\cdot,\cdot,\cdot) \colon \mathfrak{a}^+ \to B(\mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) \cap C_r^q(G,V), C_{r'}(G,V))$, such that

$$f(g \exp tH) \sim \sum_{\xi \in \widetilde{X}(\lambda,\Lambda)} p_{\lambda,\xi}(f,g,tH) e^{t\xi(H)} \qquad (t \to \infty),$$

at every $H_0 \in \mathfrak{a}$, for $f \in \mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) \cap C_r^q(G,V)$.

To complete the proof of Theorem 3.2 it remains to show (1) we can replace $\widetilde{X}(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$, (2) $p_{\lambda,\xi}(f,g,H)$ is a polynomial in H with order $\leq d$. We shall finish the proof in the next section. We now consider the holomorphic dependence in λ in order to prove Theorem 3.3.

Let $r \in \mathbb{R}$ and Ω be an open set in $\mathfrak{a}_{\mathbb{C}}^*$. Let $\{f_\lambda\}_{\lambda \in \Omega}$ be a holomorphic family in $C_r^\infty(G,V)$, and $f_\lambda \in \mathcal{E}_{\lambda-\Lambda}^\infty \operatorname{Ind}_K^G(\tau)$, for each $\lambda \in \Omega$. We now study the asymptotic expansion of f_λ. Fix $\lambda_0 \in \Omega$, and $\xi_0 \in \widetilde{X}(\lambda_0, \Lambda)$.

Proposition 5.3. *There exist a neighborhood $\Omega(\lambda_0)$ of λ_0 in Ω and a neighborhood $V(\xi_0)$ of ξ_0 in $\mathfrak{a}_{\mathbb{C}}^*$, such that*

$$(\lambda, H) \to \sum_{\xi \in V(\xi_0)} p_{\lambda,\xi}(f_\lambda, \cdot, H) e^{\xi(H)}$$

is continuous from $\Omega(\lambda_0)\times U$ *to* $C^{q'}_{r'}(G,V)$ *for some* $q'\in\mathbb{N}$, $r'\in\mathbb{R}$, *and in addition holomorphic in* λ. *Moreover, we can choose* $V(\xi_0)$ *such that* $V(\xi_0)\cap X(\lambda,\Lambda)\subset\Xi(\lambda)$.

Proof. It follows from Proposition 4.9. □

6. Differential equations for the coefficients

In this section we derive certain differential equations for the vector-valued functions $p_{\lambda,\xi}(f,g,\cdot)$ on $\mathfrak{a}^+$, where $f\in\mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}^G_K(\tau)$, and $g\in G$.

Fix $Z\in\mathcal{Z}(\mathfrak{g})$, and $D=\mu(Z)\in Z_\tau$. We can choose finitely many x_i in $\overline{\mathfrak{n}}U(\overline{\mathfrak{n}})$, and $v_i\in U(\mathfrak{a})\otimes\operatorname{End}(V)$, such that

$$Z-\widetilde{\Gamma}_1(Z\otimes 1)-\sum x_iv_i\in I(\tau),$$

and $\operatorname{ad}(\mathfrak{a})$ acts on x_i by a weight $-\eta_i\neq 0$, where $\eta_i\in\mathbb{N}\cdot\Delta$, and v_i, $\widetilde{\Gamma}_1(Z\otimes 1)\in U(\mathfrak{a})\otimes\operatorname{End}(V)$ which can be interpreted as differential operators with constant coefficients on $C^\infty(\mathfrak{a},V)$.

Proposition 6.1. *Let* $f\in\mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}^G_K(\tau)$. *Then the functions* $p_{\lambda,\xi}(f,\cdot,\cdot)e^\xi$ *from* $G\times\mathfrak{a}^+$ *to* V *satisfy the following recursive equations:*

$$\begin{aligned}1\otimes\partial(\widetilde{\Gamma}_1(Z\otimes 1)&-\gamma(Z)(\lambda-\Lambda))(p_{\lambda,\xi}(f,\cdot,\cdot)e^\xi)\\&=-\sum_{i,\xi+\eta_i\in\widetilde{X}(\lambda,\Lambda)}R_{x^i}\otimes e^{-\eta_i}\partial(v^i)(p_{\lambda,\xi+\eta_i}(f,\cdot,\cdot)e^{\xi+\eta_i}),\end{aligned}$$

for all $\xi\in\widetilde{X}(\lambda,\Lambda)$.

The proof is the same as for Proposition 7.1 in [BS].

Proof of Theorem 3.2. Let

$$V=\bigoplus_{\Lambda_1\in\mathfrak{t}^*}V(\Lambda_1),$$

where $V(\Lambda_1)=\bigoplus_{\sigma\in\tau,\Lambda_\sigma=\Lambda_1}V(\sigma)$. Let $P(\Lambda_1)$ be the projection from V to $V(\Lambda_1)$. By Corollary 1.15 $\Gamma_1(Z\otimes 1)|V(\Lambda_1)=(T_{\rho-\Lambda_1}\gamma(Z))^-\otimes\mathrm{I}_{V(\Lambda_1)}$. For $\xi_1,\xi_2\in\mathfrak{a}^*_{\mathbb{C}}$, we say $\xi_1\prec\xi_2$ if there exists $\eta\in\mathbb{N}\cdot\Delta$ such that $\xi_2=\xi_1+\eta$. This defines a partial order on $\mathfrak{a}^*_{\mathbb{C}}$. For each $f\in\mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}^G_K(\tau)$, define $E(\lambda,\Lambda,f)$ by

$$E(\lambda,\Lambda,f)=\{\xi\in\widetilde{X}(\lambda,\Lambda)|p_{\lambda,\xi}(f,\cdot,\cdot)\not\equiv 0\}.$$

Let $E_L(\lambda,\Lambda,f)$ denote the set of maximal elements in $E(\lambda,\Lambda,f)$. Suppose $\xi\in E_L(\lambda,\Lambda,f)$. Then $p_{\lambda,\xi}(f,\cdot,\cdot)\not\equiv 0$. So one can find $g\in G$, $\Lambda_1\in\mathfrak{t}^*$, such that $P(\Lambda_1)p_{\lambda,\xi}(f,g,\cdot)\not\equiv 0$.

Since the right-hand side of the equation in Proposition 6.1 is zero because ξ is maximal in $E(\lambda,\Lambda,f)$,

$$\partial(\widetilde{\Gamma}_1(Z\otimes 1)-\gamma(Z)(\lambda-\Lambda))(p_{\lambda,\xi}(f,g,\cdot)e^\xi)=0.$$

So

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z))^- -\gamma(Z)(\lambda-\Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f,g,\cdot)e^\xi)=0.$$

We extend $p_{\lambda,\xi}(f,g,\cdot)e^\xi$ to a function on $\mathfrak{a}^++\sqrt{-1}\mathfrak{t}\subset\mathfrak{h}=\mathfrak{a}+\sqrt{-1}\mathfrak{t}$, by abuse of notation still denoted by $p_{\lambda,\xi}(f,g,\cdot)e^\xi$, by the requirement that it be constant in the $\mathfrak{t}$ direction. Hence

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z))-\gamma(Z)(\lambda-\Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f,g,\cdot)e^\xi)=0.$$

So

$$\partial((\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi-\Lambda_1+\rho}) = 0.$$

By Theorem 3.13, Chapter III in [Helg1], $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi-\Lambda_1+\rho} = \sum q_i e^{\mu_i}$, where q_i are polynomials on $\mathfrak{h}, \mu_i \in \mathfrak{h}^*_{\mathbb{C}}$. Recall that $p_{\lambda,\xi}(f, g, tH)$ is a polynomial in t. We conclude $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial on $\mathfrak{h}$, and

$$\xi - \Lambda_1 + \rho = w(\lambda - \Lambda),$$

for some $w \in \widetilde{W}$. Also $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a $\widetilde{W}(w(\lambda-\Lambda))$-harmonic, where $\widetilde{W}(\mu) = \{w \in \widetilde{W} | w\mu = \mu\}$, for each $\mu \in \mathfrak{h}^*_{\mathbb{C}}$. So

$$\deg(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)) \leq d.$$

Here d is the number of elements in $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. It follows that we can replace $\widetilde{X}(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$ since $E_L(\lambda, \Lambda, f) \subset X(\lambda, \Lambda)$.

By induction on ξ using Proposition 6.1 one can easily show $p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial with degree $\leq d$. Note we only need to show it for $g = e$. So this completes the proof of Theorem 3.2.

The proof of Theorem 3.3 follows from Proposition 5.3. □

7. Leading exponents

We further consider the properties of a leading term in the asymptotic expansion of $f \in \mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}^G_K(\tau)$.

Proposition 7.1. *For each $\xi \in E_L(\lambda, \Lambda, f)$, $man \in B$, $H \in \mathfrak{a}$, and $g \in G$,*

$$p_{\lambda,\xi}(f, gman, H) = e^{\xi(\log a)}\tau(m)^{-1}p_{\lambda,\xi}(f, g, H + \log a).$$

Proof. The same as for Theorem 8.4 in [BS]. □

Let $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$. We introduce conditions on $\lambda - \Lambda$ and λ as follows:

$$\mathfrak{A}_1 = \{\lambda - \Lambda | \lambda \in \mathfrak{a}^*_{\mathbb{C}}, \Lambda \in \mathfrak{t}^*_{\mathbb{C}}, \langle\lambda - \Lambda, \alpha^\vee\rangle \notin \mathbb{Z}, \forall\alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}), \alpha|\mathfrak{a} \neq 0\},$$

$$\mathfrak{A}_2 = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} | \langle\lambda, \beta^\vee\rangle \notin -\mathbb{N}, \forall\beta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}.$$

Let

$$\widetilde{W}_0 = \{w \in \widetilde{W} | w|\mathfrak{a} = \mathrm{id}\}, \quad \widetilde{W}_1 = \{w \in \widetilde{W} | w\mathfrak{a} = \mathfrak{a}\}.$$

Proposition 7.2. *Suppose $\lambda - \Lambda \in \mathfrak{A}_1$. We have*

(i) *if $w(\lambda - \Lambda) = \lambda - \Lambda$ for some $w \in \widetilde{W}$, then $w \in \widetilde{W}_0$;*

(ii) *if there exist $w \in \widetilde{W}$, $\sigma \in \tau$ such that*

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0,$$

then $w \in \widetilde{W}_1$, and $\Lambda_\sigma = w\Lambda$.

Proof. (i) Since $w(\lambda - \Lambda) = \lambda - \Lambda$, $w = w_{\alpha_1} \cdots w_{\alpha_s}$, where $\alpha_j \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and $\langle\lambda - \Lambda, \alpha_j\rangle = 0$. Then we conclude $\alpha_j|\mathfrak{a} = 0$ from $\mathfrak{A}_1$. So $w \in \widetilde{W}_0$. (ii) For any $\beta \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with $\beta|\mathfrak{a} = 0$, we have $\langle w(\lambda - \Lambda) + \Lambda_\sigma, \beta\rangle = 0$ since $(w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0$. Hence

$$\frac{2\langle\lambda - \Lambda, w^{-1}\beta\rangle}{\langle\beta, \beta\rangle} = -\frac{2\langle\Lambda_\sigma, \beta\rangle}{\langle\beta, \beta\rangle},$$
$$\frac{2\langle\lambda - \Lambda, w^{-1}\beta\rangle}{\langle w^{-1}\beta, w^{-1}\beta\rangle} = -\frac{2\langle\Lambda_\sigma, \beta\rangle}{\langle\beta, \beta\rangle}.$$

The right-hand side being integral forces $w^{-1}\beta|\mathfrak{a} = 0$. This shows w preserves $\mathfrak{t}$. Therefore w preserves $\mathfrak{a}$. Trivially $\Lambda_\sigma = w\Lambda$. □

Proposition 7.3. *Let $f \in \mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$. Suppose $\lambda-\Lambda \in \mathfrak{A}_1$, and ξ in $E_L(\lambda, \Lambda, f)$. Then $\xi \in W\lambda - \rho$, and $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ for each $g \in G$.*

Proof. In the last section we showed if $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot) \not\equiv 0$, then there exists $w \in \widetilde{W}$, such that $\xi - \Lambda_\sigma + \rho = w(\lambda - \Lambda)$. So

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0.$$

By Proposition 7.2 (ii) $w \in \widetilde{W}_1$. So $\xi + \rho = w\lambda$. Hence $\xi \in W\lambda - \rho$. □

We also showed that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is $\widetilde{W}(w(\lambda - \Lambda))$-harmonic. Since $w \in \widetilde{W}_1$, $w(\lambda - \Lambda) \in \mathfrak{A}_1$. By Proposition 7.2 (i) $\widetilde{W}(w(\lambda - \Lambda)) \subset \widetilde{W}_0$. We conclude $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$. This shows $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ since $\sigma \in \tau$ is arbitrary. In this case we denote it by $p_{\lambda,\xi}(f, g)$.

Corollary 7.4. *If $\lambda - \rho \in E_L(\lambda, \Lambda, f)$, and in addition λ is regular, i.e., $W(\lambda) = \{w \in W|w\lambda = \lambda\} = e$, then*

$$p_{\lambda,\lambda-\rho}(f, g) = P(\Lambda)p_{\lambda,\lambda-\rho}(f, g).$$

Proof. If for some $\sigma \in \tau$, such that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g) \not\equiv 0$, then there exists $w \in \widetilde{W}_1$, with

$$w\lambda = (\lambda - \rho) + \rho, w\Lambda_\sigma = \Lambda.$$

λ being regular implies $w \in \widetilde{W}_0$. But then $P(\Lambda) = P(\Lambda_\sigma)$ by definition. □

By Appendix II in [KKMOOT] if $\lambda \in \mathfrak{A}_2$, then $\lambda-\rho$ is always maximal in $W\lambda-\rho$. So we have the following definition.

Definition 7.5. Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$. For $f \in \mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$, $\beta_\lambda(f)$ is defined by

$$\beta_\lambda(f) = p_{\lambda,\lambda-\rho}(f, \cdot).$$

We call β_λ the boundary value map.

Theorem 7.6. *Let $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$. Then*

(i) β_λ maps $\mathcal{E}^\infty_{\lambda-\Lambda,r}\operatorname{Ind}_K^G(\tau)$ linearly, continuously, and G-equivariantly into $C^\infty\operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$ for each $r \in \mathbb{R}$, where $\tau(\Lambda)$ is the restriction of τ to M with representation space $V(\Lambda)$.

(ii) Let $\Omega \subset \mathfrak{a}^*_{\mathbb{C}}$ be open, $\{f_\lambda\}_{\lambda\in\Omega}$ a holomorphic family in $\mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$; then $\lambda \to \beta_\lambda(f_\lambda)$ is holomorphic in $\Omega \cap \mathfrak{A}_2$.

Proof. (i) comes from Theorem 3.2; (ii) is a result of Theorem 3.3. □

Finally we notice for certain λ we can obtain the boundary value map by a simple limit procedure.

Lemma 7.7. *Let $\lambda - \Lambda \in \mathfrak{A}_1$. If $\operatorname{Re}\langle\lambda, \alpha\rangle > 0$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, then*

$$\beta_\lambda f(g) = \lim_{t\to\infty} e^{(-\lambda+\rho)(tH)} f(g \exp tH),$$

for $f \in \mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$, and $H \in \mathfrak{a}^+$.

Proof. The condition on λ implies that $\operatorname{Re}\xi(H) < \operatorname{Re}(\lambda-\rho)(H)$ for all $\xi \in X(\lambda,\Lambda)$ with $\xi \neq \lambda-\rho$. Then the result follows from Theorem 3.2 and the very definition of asymptotic expansion. □

For each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$, we define $P_\lambda\phi$ by

$$P_\lambda\phi(g) = \int_K \tau(k)\phi(gk)dk.$$

From the proof of Theorem 1.6 we conclude $P_\lambda\phi \in \mathcal{E}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$. By Example 2.2 $P_\lambda\phi \in \mathcal{E}^\infty_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$.

Corollary 7.8. *Under the same conditions as in Lemma 7.7,*

$$\beta_\lambda P_\lambda\phi = C(\lambda)\phi,$$

for each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$.

Proof. By Proposition 2.4 and Lemma 7.7. □

8. The inversion of the Poisson transform

Let $C(\lambda)$ be the generalized Harish-Chandra C-function given by

$$C(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)\mathrm{H}(\overline{n})}\tau(\mathrm{k}(\overline{n}))d\overline{n}.$$

Recall $P_\lambda\colon C^\infty \operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1) \to \mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ is defined by

$$P_\lambda\phi(g) = \int_K \tau(k)\phi(gk)dk.$$

Theorem 8.1. *Let* $\lambda-\Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, *and* $C_0(\lambda)$ *the restriction of* $C(\lambda)$ *to* $V(\Lambda)$. *Then*

$$\beta_\lambda P_\lambda\phi = C_0(\lambda)\phi,$$

for each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$.

Proof. If $\operatorname{Re}\langle\lambda,\alpha\rangle > 0$, for all $\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})$, then by Corollary 7.8,

$$\beta_\lambda P_\lambda\phi = C_0(\lambda)\phi.$$

Since $P_\lambda\phi$ is a holomorphic family in $\mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$, by Theorem 7.6 the left-hand side is holomorphic. The right-hand side is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$. Hence two sides must coincide. □

Corollary 8.2. *If in addition we assume* $\det C_0(\lambda) \neq 0$, *then* β_λ *is surjective. Hence* P_λ *is injective.*

Theorem 8.3. *Let* $\lambda-\Lambda \in \mathfrak{A}_1$, *and* $\lambda \in \mathfrak{A}_2$, *and* $\det C_0(\lambda) \neq 0$. *Then* P_λ *is bijective, and the inverse of* P_λ *is given by* $C_0(\lambda)^{-1}\beta_\lambda$.

For the proof we recall a definition which can be found in [Wall], Section 11.6. Let $\mathfrak{V}$ be a finitely generated $(\mathfrak{g},K)$-module.

Definition 8.4. $\mathfrak{V}^*_{mod}$ denotes the set of all $\mu \in \mathfrak{V}^*$, such that there exists $d_\mu \in \mathbb{R}$ and for each $\nu \in \mathfrak{V}$ there exist an analytic function $f_{\mu,\nu}$ and a constant $C_{\mu,\nu} > 0$ with the following properties:
(i) $L_u f_{\mu,\nu}(k) = \mu(k^{-1}.(u.\nu))$, for $u \in U(\mathfrak{g})$, $k \in K$,
(ii) $|f_{\mu,\nu}(g)| \leq C_{\mu,\nu}\|g\|^{d_\mu}$, for each $g \in G$.

Recall that $(C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))'$ is the strong topological dual of $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$. The following result can also be found in [Wall], Section 11.7.

Proposition 8.5. *Let* $(C^\infty \operatorname{Ind}_B^G(\sigma\otimes(-\lambda)\otimes 1))_{K\text{-finite}}$ *denote the space of* K*-finite elements in* $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$*. Then*

$$(C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}})^*_{mod} = (C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))'.$$

Before we go ahead with the proof of Theorem 8.3, we mention the following result about the irreducibility of the principal series representations. Let $\sigma \in \widehat{M}$.

Lemma 8.6. *As a* $(\mathfrak{g}, K)$*-module* $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$ *is irreducible if* $\lambda - \Lambda \in \mathfrak{A}_1$.

Proof. This is a direct consequence of Theorem 1.1 in [SV]. □

Proof of Theorem 8.3. It suffices to show β_λ is injective. Assume the opposite. Then there exists $f_0 \in \mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$, such that $\beta_\lambda f_0 = 0$, and $f_0 \not\equiv 0$. We can assume $f_0(e) \neq 0$ since β is G-equivariant. Define f_K by

$$f_K(g) = \int_K \operatorname{tr} \tau(k) f_0(kg) dk.$$

Then f_K is K-finite, and $f_K(e) = \frac{1}{\dim(\tau)} f_0(e) \neq 0$. Let

$$\mathfrak{W} = L_{U(\mathfrak{g})} L_K f_K.$$

Then $\mathfrak{W}$ is a finitely generated $(\mathfrak{g}, K)$-module. Let $\mathfrak{W}_1$ be an irreducible submodule of $\mathfrak{W}$. By the subrepresentation theorem and Lemma 8.4 there exists $\sigma \in \widehat{M}$, such that $\mathfrak{W}_1 \cong C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$. So there is a $(\mathfrak{g}, K)$ map

$$P_\sigma \colon C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}} \to \mathfrak{W}.$$

It is easy to see $\Lambda = \Lambda_\sigma$. Define $\mu \in \mathfrak{W}^* \otimes V$ by

$$\mu(\nu) = \nu(e),$$

for each $\nu \in \mathfrak{W}$.

Taking $f_{\mu,\nu} = \nu \in \mathcal{E}^\infty_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ in Definition 8.4, we can verify that (i) and (ii) are satisfied. So $\mu \in \mathfrak{W}^*_{mod} \otimes V$. Hence

$$\mu^\sharp = \mu \circ P_\sigma \in (C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}})^*_{mod} \otimes V.$$

Then by Proposition 8.5,

$$\mu^\sharp \in (C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))' \otimes V.$$

Now define $P_\sigma^\sharp \colon C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1) \to C^\infty \operatorname{Ind}_K^G(\tau)$ by

$$P_\sigma^\sharp \phi(g) = \mu^\sharp(L_{g^{-1}} \phi).$$

Since P_σ is a $\mathfrak{g}$ map and eigensections are analytic we can show that for ϕ in $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$,

$$P_\sigma \phi = P_\sigma^\sharp \phi,$$

by showing they are identical at e along with their derivatives.

We observe that $P_\sigma^\sharp$ is a linear, continuous, and G-equivariant map from $C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$ to $C^\infty \operatorname{Ind}_K^G(\tau)$. By Proposition 1.8 we conclude $\sigma \in \tau$, and there exists $T \in \operatorname{Hom}_M(V_\sigma, V)$ such that $P_\sigma^\sharp = P_T$. Hence

$$P_\sigma = P_T \colon C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}} \to \mathfrak{W}.$$

Taking $\sigma \in C^\infty \operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$ such that $0 \neq f = P_T\phi$, then $f = P_\lambda(T\phi)$. Notice $T\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$. So

$$B_\lambda f = \beta_\lambda P_\lambda(T\phi) = C(\lambda)T\pi \neq 0.$$

This contradicts the assumption $f \in \mathfrak{W} \subset \ker(\beta_\lambda)$. □

9. Vector-valued distributions

Suppose K is a Lie group and V a finite dimensional space over $\mathbb{C}$. Let $C^{-\infty}(K, V)$ denote all continuous $\mathbb{C}$-linear maps from $C_c^\infty(K, \mathbb{C})$ to V. Let M be a compact subgroup of K, and (π, V) a finite dimensional representation of M. Let $C^{-\infty} \operatorname{Ind}_M^K(\pi)$ be the space defined by

$$\{f \in C^{-\infty}(K, V) | R_m f(\phi) = \pi(m^{-1}) f(\phi), \forall \phi \in C_c^\infty(K, \mathbb{C}), \forall m \in M\}.$$

Here $R_m f(\phi) = f(R_{m^{-1}}\phi)$, with $R_{m^{-1}}\phi(k) = \phi(km^{-1})$.

Let $(\check{\pi}, V^*)$ be the dual representation of (π, V), and $\langle , \rangle$ the nondegenerate bilinear form on $V \times V^*$. Let $(C_c^\infty \operatorname{Ind}_M^K(\pi))'$ be the strong dual of $C_c^\infty \operatorname{Ind}_M^K(\pi)$. For each $T \in (C_c^\infty \operatorname{Ind}_M^K(\pi))'$, $\phi \in C_c^\infty(K, \mathbb{C})$, and $v \in V$, we define $\xi_1(T)(\phi) \in V^*$ by

$$\langle v, \xi_1(T)(\phi)\rangle = T(\xi_1(\phi, v)),$$

where $\xi_1(\phi, v)(k) = \int_M \phi(km)\pi(m) v dm$. It is easy to show that

$$\xi_1(T) \in C^{-\infty} \operatorname{Ind}_M^K(\check{\pi}).$$

Proposition 9.1. *The map* $\xi_1 \colon (C_c^\infty \operatorname{Ind}_M^K(\pi))' \to C^{-\infty} \operatorname{Ind}_M^K(\check{\pi})$ *is bijective.*

Proof. Define

$$\eta_1 \colon C^{-\infty} \operatorname{Ind}_M^K(\check{\pi}) \to (C_c^\infty \operatorname{Ind}_M^K(\pi))'$$

as follows: for each $f \in C^{-\infty} \operatorname{Ind}_M^K(\check{\pi})$, and $\phi \in C_c^\infty \operatorname{Ind}_M^K(\pi)$, the map

$$f_\phi \colon u \to f(\langle \phi, u\rangle)$$

is a linear map from V^* to V^*. Then we define

$$\eta_1(f) = \operatorname{tr}(f_\phi).$$

It is a long but rather straightforward calculation to show ξ_1 and η_1 are inverses to each other. □

Now let $G = KAN$, and (δ, V_δ) be a finite dimensional representation of $B = MAN$. Let

$$C^\infty \operatorname{Ind}_B^G(\delta) = \{f \in C^\infty(G, V_\delta) | R_{man} f = a^{-\rho}\delta^{-1}(man) f, \forall man \in B\},$$

$$C^{-\infty} \operatorname{Ind}_B^G(\delta) = \{f \in C^{-\infty}(G, V_\delta) | R_{man} f = a^{-\rho}\delta^{-1}(man) f, \forall man \in B\}.$$

For $T \in (C^\infty \operatorname{Ind}_B^G(\delta))'$, $\xi(T)$ is defined by

$$\langle v, \xi(T)(\phi)\rangle = T(\xi(\phi, v)),$$

for each $v \in V_\delta$, and $\phi \in C_c^\infty(G, \mathbb{C})$. Here $\xi(\phi, v)$ is defined as follows: for each $g \in G$,

$$\xi(\phi, v)(g) = \int_{MAN} \phi(gman) a^\rho \delta(man) v dm da dn.$$

Now we show $\xi(T) \in C^{-\infty} \operatorname{Ind}_B^G(\check{\delta})$. By definition,

$$\langle v, \xi(T)(R_{(man)^{-1}}\phi)\rangle = T(\xi(R_{(man)^{-1}}\phi, v)).$$

However, it is a simple calculation to see

$$\xi(R_{(man)^{-1}}\phi, v) = \xi(\phi, a^{-\rho}\delta(man)v).$$

Hence

$$\begin{aligned} \langle v, R_{man}\xi(T)(\phi)\rangle &= \langle v, \xi(T)(R_{(man)^{-1}}\phi)\rangle \\ &= T(\xi(\phi, a^{-\rho}\delta(man)v)) \\ &= \langle a^{-\rho}\delta(man)v, \xi(T)(\phi)\rangle \\ &= \langle v, a^{-\rho}\check{\delta}((man)^{-1})T(\phi)\rangle. \end{aligned}$$

This proves $\xi(T) \in C^{-\infty} \operatorname{Ind}_B^G(\check{\delta})$.

Theorem 9.2. *Let ξ be defined as above. Then ξ is a G-equivariant bijection from $(C^\infty \operatorname{Ind}_B^G(\delta))'$ to $C^{-\infty} \operatorname{Ind}_B^G(\check{\delta})$.*

Lemma 9.3. *Let L be a Lie group and (π, V) a finite dimensional representation of L on V. Suppose $f \in C^{-\infty}(L, V)$, satisfying*

$$R_l f = \pi(l^{-1}) f,$$

for each $l \in L$. Let dl be the right invariant Haar measure on L. Then there exists a unique vector $v \in V$, such that

$$f(\phi) = \int_L \phi(l)\pi(l^{-1}) v dl,$$

for each $\phi \in C_c^\infty(L, \mathbb{C})$.

Proof. We use an argument due to Helgason. For ϕ and ψ in $C_c^\infty(L, \mathbb{C})$, we define $\phi * \psi$ in $C_c^\infty(L, \mathbb{C})$ by

$$\phi * \psi(x) = \int_L \phi(l)\psi(xl^{-1}) dl.$$

Then

$$f(\phi * \psi) = \int_L \phi(l) f(R_{l^{-1}}\psi) dl = \int_L \phi(l)\pi(l^{-1}) f(\psi) dl.$$

Choose a sequence ψ_n such that $\check{\psi}_n \to \delta$, the delta function, as $n \to +\infty$. Here $\check{\psi}_n(l) = \psi_n(l^{-1})$. Let $v_n = f(\psi_n)$. Then

$$(*) \qquad f(\phi * \psi_n) = \int_L \phi(l)\pi(l^{-1}) v_n dl.$$

We can choose an appropriate ϕ (e.g. close to δ), such that $\int_L \phi(l)\pi(l^{-1}) dl$ is invertible. Since $\phi * \psi_n \to \phi$, by letting $n \to +\infty$ in $(*)$, we conclude there exists $v \in V$, such that $v_n \to v$, and

$$f(\phi) = \int_L \phi(l)\pi(l^{-1}) v dl.$$

The uniqueness follows from the fact that there is ϕ such that $\int_L \phi(l)\pi(l^{-1})dl$ is invertible. □

Proof of Theorem 9.2. First we construct the inverse η of ξ as follows: Take $f \in C^{-\infty}\operatorname{Ind}_B^G(\check{\delta})$, and $\psi \in C^\infty(K,\mathbb{C})$.Then $\phi \to f(\psi\otimes\phi)$ defines a continuous linear map from $C_c^\infty(A\times N,\mathbb{C})$ to V_δ^*, where

$$(\psi\otimes\phi)(kan) = \psi(k)\phi(an).$$

It is easy to check this map satisfies all the conditions as in Lemma 9.3 if we take $L = AN$, $\pi(an) = a^\rho\check{\delta}(an)$. So there exists a unique element in V_δ^*, which we denote by $f^-(\psi)$, such that

$$f(\psi\otimes\phi) = \int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)f^-(\psi)dadn.$$

Notice $a^{2\rho}dadn$ gives a right invariant Haar measure on AN. It is fairly easy to see $f^- \in C^{-\infty}\operatorname{Ind}_M^K(\check{\delta}|M)$. Then by Proposition 9.1 $\eta_1(f^-)$ gives an element in $(C^\infty\operatorname{Ind}_M^K(\delta|M))'$. Since $C^\infty\operatorname{Ind}_M^K(\delta|M)\cong C^\infty\operatorname{Ind}_B^G(\delta)$, one can view $\eta_1(f^-)$ as an element in $(C^\infty\operatorname{Ind}_B^G(\delta))'$. Finally we define $\eta(f)$ by

$$\eta(f) = \eta_1(f^-).$$

The final step of the proof is to show $\eta\circ\xi = \mathrm{id}$, and $\eta\circ\xi = \mathrm{id}$. For each $T\in(C^\infty\operatorname{Ind}_B^G(\delta))'$, $\psi\in C^\infty(K,\mathbb{C})$, and $\phi\in C_c^\infty(A\times N,\mathbb{C})$,

$$\xi(T)(\psi\otimes\phi) = \int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)(\xi(T))^-dadn.$$

So for each $v\in V$,

$$(**)\qquad \langle v,\xi(T)(\psi\otimes\phi)\rangle = \langle v,\int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)(\xi(T))^-(\psi)dadn\rangle.$$

By definition

$$\begin{aligned}\xi(\psi\otimes\phi,v)(k) &= \int_{MAN}(\psi\otimes\phi)(kman)a^\rho\delta(man)vdmdadn\\ &= \int_{MAN}\psi(km)\delta(m)\phi(an)a^\rho\delta(an)vdmdadn\\ &= \xi_1(\psi,v_1),\end{aligned}$$

where $v_1 = \int_{A\times N}a^\rho\phi(an)\delta(an)vdadn$. So by $(**)$

$$\begin{aligned}\langle v,\xi(T)(\psi\otimes\phi)\rangle &= T(\xi_1(\psi,v_1))\\ &= \langle v_1,\xi_1(T)(\psi)\rangle\\ &= \langle v,\int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)\xi_1(T)(\psi)dadn\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)(\xi(T))^-(\psi)dadn\\ &\qquad= \int_{A\times N}\phi(an)a^\rho\check{\delta}^{-1}(an)\xi_1(T)(\psi)dadn\rangle.\end{aligned}$$

By comparing both sides we have $\xi_1(T) = (\xi(T))^-$. Hence

$$T = \xi_1^{-1}((\xi(T))^-) = \eta_1((\xi(T))^-) = \eta(\xi(T)).$$

Similarly we can verify $\xi \circ \eta = \mathrm{id}$. Note it is enough to check on functions of the form $\psi \otimes \phi$. □

Now suppose V_δ is a Hilbert space. Let δ^* be the representation defined as follows: for each $g \in G, w, v \in V_\delta$, we have $\langle \delta(g)v, w\rangle = \langle v, \delta(g)^t w\rangle$; then $\delta^*(g) = \delta(g^{-1})^t$. Let $C^{-\infty}\operatorname{Ind}_B^G(\delta^*)$ be the space of conjugate linear maps f from $C_c^\infty(G, \mathbb{C})$ to V_δ, such that

$$R_{man} f = a^{-\rho}\delta^*((man)^{-1})f.$$

For each $T \in (C^\infty \operatorname{Ind}_B^G(\delta))'$, and $\phi \in C_c^\infty(G, \mathbb{R}), \xi(T)(\phi)$ is defined by

$$\langle v, \xi(T)(\phi)\rangle = T(\xi(\phi, v)),$$

for each $v \in V_\delta$. Here

$$\xi(\phi, v)(g) = \int_{MAN} \phi(gman)a^\rho \delta(man)v dm da dn.$$

Corollary 9.4. *ξ is a bijection from $(C^\infty \operatorname{Ind}_B^G(\delta))'$ onto $C^{-\infty}\operatorname{Ind}_B^G(\delta^*)$.*

Let σ be a unitary representation of M and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. $\sigma \otimes \overline{\lambda} \otimes 1$ is the representation of B defined by $man \to a^{\overline{\lambda}}\sigma(m)$. Then

$$(\sigma \otimes \overline{\lambda} \otimes 1)^* = \sigma \otimes (-\lambda) \otimes 1.$$

Corollary 9.5. *The map*

$$\xi\colon (C^\infty \operatorname{Ind}_B^G(\sigma \otimes \overline{\lambda} \otimes 1))' \to C^{-\infty}\operatorname{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$$

is a bijection.

10. Distribution boundary values

We now introduce a weak growth condition in the eigenspace $\mathcal{E}_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$. Recall in Section 2 we have

$$C_r^q(G, V) = \{f \in C^q(G, V) | \|f\|_{q,r} < \infty\},$$

$q \in \mathbb{N}$ and $r \in \mathbb{R}$. $C_r^\infty(G, V) = \bigcap_q C_r^q(G, V)$. We define $\mathfrak{F}$ to be the space

$$\mathfrak{F} = \bigcap_r C_r^\infty(G, V) = \bigcap_{q,r} C_r^q(G, V),$$

endowed with the projective limit topology for the intersection over q and r (i.e., the topology given by the family of forms $\|\cdot\|_{q,r}$). Using the same argument as on p. 142 in [BS] we conclude $\mathfrak{F}$ is a Fréchet space. It follows from Section 2 that L and R act smoothly on $\mathfrak{F}$.

Let $\mathfrak{F}'$ be the space dual to $\mathfrak{F}$, equipped with the strong dual topology. For each $T \in \mathfrak{F}', q \in \mathbb{N}$, and $r \in \mathbb{R}$, we define

$$\|T\|'_{q,r} = \sup\{T(\varphi) | \varphi \in \mathfrak{F}, \|\varphi\|_{q,r} \le 1\}.$$

The space $C_r^q(G, V)' = \{T \in \mathfrak{F}' | \|T\|'_{q,r} < \infty\}$ with this norm is the dual space of $C_r^q(G, V)$. Moreover, we have $\mathfrak{F}' = \bigcup_{q,r} C_r^q(G, V)'$. By duality $\mathfrak{F}'$ is the inductive limit of these spaces. Using Lemma 2.1 we can prove that for some $b \in \mathbb{R}$, $\int_G \|g\|^b dg < \infty$. It follows that there is a continuous injection of $C_r^0(G, V)$ into $C_{b-r}^0(G, V)'$ defined by integration over G. Hence there is a continuous injection of $C_r^0(G, V)$ into $\mathfrak{F}'$.

Let $q' \geq q$, and $r \in \mathbb{R}$. For each $T \in C_r^q(G,V)'$, and $\varphi \in C_r^{q'}(G,\mathbb{R})$, we define an element $L^\vee(\varphi)T$ in $C_r^{q'-q}(G,V)$ by

$$\langle v, L^\vee(\varphi)T(x)\rangle = T(R_{x^{-1}}\varphi \cdot v).$$

Note if $f \in C_r^0(G,V)$, and $\varphi \in C_{b-r}^0(G,\mathbb{C})$, then

$$L^\vee(\varphi)f(x) = \int_G \varphi(g)f(gx)dg.$$

Lemma 10.1. *Let $q, q' \in \mathbb{N}$ with $q \leq q'$. There exist $s \geq 0$ and $C \geq 0$ such that*

$$\|L^\vee(\varphi)T\|_{q'-q,r} \leq C\|T\|'_{q',r}\|\varphi\|_{q',r-s},$$

for all $r \in \mathbb{R}$, $T \in C_r^q(G,V)'$, and $\varphi \in C_{r-s}^{q'}(G,\mathbb{R})$.

Proof. See Lemma 11.1 in [BS]. $\square$

Let $\mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$ denote the closed subspace $\mathcal{E}_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau) \cap \mathfrak{F}'$. We call the elements of $\mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$ eigensections of weak moderate growth. Notice if $f \in \mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$, and $\varphi \in C_c^\infty(C,\mathbb{R})$, then $L^\vee(\varphi)f \in \mathcal{E}^\infty_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$ by Lemma 10.1. For $\lambda - Ł \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, and $f \in \mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$, we define a vector-valued distribution $\overline{\beta}_\lambda f$ on G by

$$\overline{\beta}_\lambda f(\varphi) = \beta_\lambda(L^\vee(\varphi)f)(e),$$

for each $\varphi \in C_c^\infty(G,\mathbb{R})$.

Proposition 10.2. *$\overline{\beta}_\lambda$ is a linear, continuous, and G-equivariant map from $\mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$ to $C^{-\infty}\operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$.*

Proof. It suffices to show $\overline{\beta}_\lambda f \in C^{-\infty}\operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$. By definition,

$$\begin{aligned} L^\vee(R_{(man)^{-1}}\varphi)f(x) &= f(R_{x^{-1}}R_{(man)^{-1}}\varphi) \\ &= f(R_{(manx)^{-1}}\varphi) \\ &= L^\vee(\varphi)f(manx). \end{aligned}$$

However, β_λ is G-equivariant. Hence

$$\begin{aligned} B_\lambda(L^\vee(R_{(man)^{-1}}\varphi)f)(e) &= \beta_\lambda(L^\vee(\varphi)f)(man) \\ &= \tau(\Lambda)(m^{-1})a^{\lambda-\rho}\beta_\lambda(L^\vee(\varphi)f)(e). \end{aligned}$$

This proves $\overline{\beta}_\lambda f \in C^{-\infty}\operatorname{Ind}_B^G(\tau(\Lambda)\otimes(-\lambda)\otimes 1)$. $\square$

For each $T \in (C^\infty\operatorname{Ind}_B^G(\tau(\Lambda)\otimes\overline{\lambda}\otimes 1))'$, we define $\overline{P}_\lambda T$ as follows:

$$\langle v, \overline{P}_\lambda T(g)\rangle = T(P(\Lambda)L_g\Phi_\lambda \cdot v)\rangle,$$

for each $v \in V$. Here $\Phi_\lambda(x)$ is the transpose of $\Psi_\lambda(x^{-1})$, and $P(\Lambda)$ the projection from V to $V(\Lambda)$. The motivation of this definition is from Corollary 1.10.

Proposition 10.3. *$\overline{P}_\lambda T \in \mathcal{E}^*_{\lambda-\Lambda}\operatorname{Ind}_K^G(\tau)$, for $T \in (C^\infty\operatorname{Ind}_B^G(\tau(\Lambda)\otimes\overline{\lambda}\otimes 1))'$. And $\overline{P}_\lambda$ is linear, continuous, and G-equivariant.*

Proof. Similar to the proof for Corollary 11.3 in [BS]. $\square$

Lemma 10.4. *Let $T \in (C^\infty\operatorname{Ind}_B^G(\tau(\Lambda)\otimes\overline{\lambda}\otimes 1))'$, and $\varphi \in C_c^\infty(G,\mathbb{R})$. Then $L^\vee(\varphi)\overline{P}_\lambda T = P_\lambda(L^\vee(\varphi)\xi(T))$. Here ξ is the isomorphism in Corollary 9.5, and $L^\vee(\varphi)\xi(T)(x) = \xi(T)(R_{x^{-1}}\varphi)$.*

Proof. $L^\vee(\varphi)$, $\overline{P}_\lambda$, and P_λ are continuous. So it is enough to check for $T \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. The proof follows from the G-equivariance of P_λ. □

By a similar argument we get

Lemma 10.5. *Let $f \in \mathcal{E}^*_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$, and $\varphi \in C_c^\infty(G, \mathbb{R})$. Then*

$$L^\vee(\varphi)\overline{\beta}_\lambda f = \beta_\lambda(L^\vee(\varphi)f).$$

Theorem 10.6. *Under the same conditions as in Theorem 8.3, $\overline{P}_\lambda$ defines a G-equivariant topological isomorphism from $(C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$ onto $\mathcal{E}^*_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$. And $\eta \circ C_0(\lambda)^{-1} \circ \overline{\beta}_\lambda$ gives the inverse of $\overline{P}_\lambda$.*

Proof. By Theorem 8.1 and Lemma 10.4, 10.5,

$$L^\vee(\varphi)\overline{\beta}_\lambda \overline{P}_\lambda T = \beta_\lambda P_\lambda L^\vee(\varphi)\xi(T) = C_0(\lambda)L^\vee(\varphi)\xi(T),$$

for $T \in (C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$. Similarly, for each $f \in \mathcal{E}^*_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$,

$$L^\vee(\varphi)\overline{P}_\lambda \eta(C_0(\lambda)^{-1}\overline{\beta}_\lambda f) = P_\lambda C_0(\lambda)^{-1}\beta_\lambda L^\vee(\varphi)f = L^\vee(\varphi)f.$$

So we have

$$\overline{\beta}_\lambda \circ \overline{P}_\lambda = C_0(\lambda) \circ \xi, \quad \overline{P}_\lambda \circ \eta \circ C_0(\lambda)^{-1}\overline{\beta}_\lambda = \mathrm{id}. \quad \square$$

Remark 10.7. Let $\mathcal{E}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau) = \mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) \cap C_r(G,V)$ be equipped with the Banach space topology inherited from $C_r(G,V)$. Then $\mathcal{E}^*_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ is identical with the inductive limit topology for the union $\mathcal{E}^*_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau) = \bigcup_r \mathcal{E}_{\lambda-\Lambda,r} \operatorname{Ind}_K^G(\tau)$. See p. 146 in [BS].

A classical result asserts that the left K-finite elements in $\mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)$ increase at most exponentially. So by the remark above we easily get

Corollary 10.8. *Under the same conditions as in Theorem 8.3, P_λ is a bijection from $C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (_\lambda) \otimes 1)_{K\text{-finite}}$ to $\mathcal{E}_{\lambda-\Lambda} \operatorname{Ind}_K^G(\tau)_{K\text{-finite}}$.*

Remark 10.9. I think by Schmid's method indicated in [Sch] one should be able to get a bijection on the level of hyperfunctions from Corollary 10.8.

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