TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 3, March 1998, Pages 1045–1078
S 0002-9947(98)01886-8

ON THE ELLIPTIC EQUATION $\Delta u + ku - Ku^p = 0$ ON COMPLETE RIEMANNIAN MANIFOLDS AND THEIR GEOMETRIC APPLICATIONS

PETER LI, LUEN-FAI TAM, AND DAGANG YANG

Abstract. We study the elliptic equation $\Delta u + ku - Ku^p = 0$ on complete noncompact Riemannian manifolds with K nonnegative. Three fundamental theorems for this equation are proved in this paper. Complete analyses of this equation on the Euclidean space $\mathbf{R}^n$ and the hyperbolic space $\mathbf{H}^n$ are carried out when k is a constant. Its application to the problem of conformal deformation of nonpositive scalar curvature will be done in the second part of this paper.

0. Introduction

Let (M, g) be a complete n-dimensional Riemannian manifold and K a continuous function on M. The problem of the existence of positive C^2 solutions of the nonlinear elliptic partial differential equation

$$(*) \qquad \Delta u + ku - Ku^p = 0$$

originated from the study of pointwise conformal deformation of Riemannian metrics with prescribed scalar curvature, where $p > 1$ and Δ is the Laplace-Beltrami operator of the n-dimensional Riemannian manifold (M, g). Indeed, let $n \geq 3$, $p = \frac{n+2}{n-2}$, $k = -\frac{n-2}{4(n-1)} S_g$, where S_g is the scalar curvature of (M, g). If u is a positive C^2 solution of equation $(*)$ on (M, g), then the scalar curvature of the pointwise conformal metric $g_1 = u^{4/(n-2)} g$ is $-\frac{4(n-1)}{n-2} K$. Thus a function S on (M, g) is a pointwise conformal scalar curvature function if and only if equation $(*)$ has a positive C^2 solution for $K = -\frac{n-2}{4(n-1)} S$.

If M is compact and K is a constant, this is the well known Yamabe problem, and it has been completely solved (cf. [3], [4], [25], [34], [38], [39]). For the Yamabe problem on noncompact manifolds, there are counterexamples due to Z.R. Jin [16]. The reader is also referred to [2], [36], etc. For recent progress on equation $(*)$ on compact manifolds, we refer to [18], [35], and the references therein.

The problem of pointwise conformal deformation to metrics of nonpositive scalar curvature on 2-dimensional noncompact Riemannian manifolds has been studied

---

Received by the editors May 23, 1995.
1991 *Mathematics Subject Classification.* Primary 58G03; Secondary 53C21.
Key words and phrases. Conformal deformation, prescribing scalar curvature, complete Riemannian manifolds, semi-linear elliptic PDE, generalized maximum principle, analysis on manifolds.
The first two authors are partially supported by NSF grant DMS 9300422. The third author is partially supported by NSF grant DMS 9209330.

by many authors (cf. [5], [8], [15], [27], [30], [33], [37], etc). Recently, it has been analyzed fairly completely by M. Kalka and the third author in [21] and [22].

In this paper, we shall study equation $(*)$ on noncompact Riemannian manifolds. In general, we shall assume that K is a nonnegative function which will be continuous for the nonexistence results and Hölder continuous for the existence results. M will always be a complete noncompact n-dimensional Riemannian manifold. However, some of the main results remain true without the completeness assumption. All manifolds in this paper will be assumed to be smooth, connected, and noncompact without boundary.

The first interesting existence and nonexistence results on complete noncompact Riemannian manifolds were due to W. M. Ni [29], who studied equation $(*)$ on $\mathbf{R}^n$ with $k = 0$ and $p = \frac{n+2}{n-2}$. Subsequently, many authors have studied equation $(*)$ on $\mathbf{R}^n$ with $k = 0$. We refer the reader to [7], [8], [9], [12], [19], [23], [28], [29], etc. Equation $(*)$ is fairly well understood on $\mathbf{R}^n$ if $k = 0$ and $K \geq 0$. However, very little is known if $k \neq 0$.

On Riemannian manifolds, some interesting results were obtained by P. Aviles and R. McOwen [1], and more recently by Z.R. Jin [17] and C.F. Gui and X.F. Wang [14]. However, they all assumed that M is simply connected with sectional curvature bounded from both sides by negative constants. Note that these are both very strong topological and geometrical conditions, and they imply, in particular, that M is diffeomorphic to $\mathbf{R}^n$ and the exponential map at any point is a diffeomorphism.

The aim of our program is to analyze equation $(*)$ on general complete noncompact Riemannian manifolds assuming as few and as weak geometrical and topological conditions as possible. While the techniques developed in this paper can be equally well applied to more general nonlinear equations of the form

$$(**) \qquad \Delta u + k(x)u - f(x, u) = 0$$

for a large class of nonnegative functions f, we restrict ourselves to equation $(*)$, mainly for its simplicity and its clear geometrical significance.

It is well-known that by the monotone iteration scheme (cf. [2], [15], [29], [31]), if there exist positive supersolution u_+ and subsolution u_- of $(*)$ such that $u_- \leq u_+$, then $(*)$ has a positive solution u satisfying $u_- \leq u \leq u_+$. It is often not too difficult to construct a supersolution or a subsolution. However, in many cases, it is much more difficult to construct a supersolution and a subsolution at the same time, especially when we require that $u_+ \geq u_-$. Under the assumption that $K \geq 0$, we will prove that the solvability of equation $(*)$ is reduced to the existence of positive supersolutions on compact domains and the existence of a positive global subsolution. More precisely, we will prove the following theorem:

Theorem 1.2. *Let K be a permissible function on M. Suppose that $(*)$ has a positive subsolution u_-. Then equation $(*)$ has a positive C^2 maximal solution u on Ω, and $u \geq u_-$.*

Here K is said to be *permissible* if M can be exhausted by compact domains Ω_i such that $K > 0$ on $\partial\Omega_i$, and $(*)$ admits a positive supersolution on any compact domain X of M. A *maximal solution* of $(*)$ is a positive solution which dominates any other solution. This reduction theorem greatly simplifies the problem, and there is now no need to construct a global positive supersolution so that the supersolution and subsolution satisfy the relation $u_- \leq u_+$. We should mention that in Theorem

1.2, the special case when $k = 0$ and $M = \Omega$ is a domain in the Euclidean space $\mathbf{R}^n$ was proved by K.S. Cheng and W.M. Ni in [9].

The condition that $(*)$ has a positive supersolution on a compact domain X can be expressed in terms of the sign of the first eigenvalue $\lambda_{\mathfrak{K}_0 \cap X}$ (see Definition 2.2) of the operator $\mathbf{L} = -\Delta - k$ defined on the zero set $\mathfrak{K}_0$ of K. The results are contained in the following:

Theorem 2.2. *Let $K \geq 0$ be a Hölder continuous nonnegative function on M. Let $\mathfrak{K}_0 = \{x \in M | K(x) = 0\}$. If the equation $(*)$ has a positive supersolution on a compact domain X, then $\lambda_{\mathfrak{K}_0 \cap X} \geq 0$. Conversely, if $\lambda_{\mathfrak{K}_0 \cap X} > 0$, then $(*)$ has a positive supersolution on X.*

Theorem 2.2 shows the significance of the sign of the first eigenvalue of the associated linear operator $\mathbf{L}u = -\Delta u - ku$ of equation $(*)$. It explains the seemingly mysterious condition that $A^2 B^{-2} \leq (n-1)^2/n(n-2)$ in [1], theorem B, due to P. Aviles and R. McOwen. In particular, it implies that theorems 3 and 4 in [17] cannot be true without additional assumptions (cf. Examples 2.1 and 2.2 for more details).

For nonexistence results, we will use the method of proof of the generalized maximum principle of Cheng and Yau ([41], [11]), and obtain:

Theorem 3.2. *Let M be a complete noncompact manifold with Ricci curvature bounded from below by $-a^2$ for some constant $a \geq 0$. Let w be a positive C^2 function such that $|\nabla \log w|$ is bounded and such that*

$$\Delta w + kw \leq 0.$$

Let $p > 1$, and let $K \geq 0$ be a continuous function. Suppose

$$K(x) w^{p-1}(x) \geq C \left(\prod_{i=0}^{m} \log^{(i)} (r(x)) \right)^{-1}$$

near infinity for some constant $C > 0$ and positive integer m, where $r(x)$ is the distance from x to some fixed point $o \in M$. Then equation $(*)$ does not admit any positive solution.

The function $\log^{(i)} t$ is defined inductively by $\log^{(0)} t = t$, and $\log^{(m+1)} t = \log \log^{(m)} t$. Theorem 3.2 and its refined version give us a sharp condition on K for nonexistence results of the equation $(*)$ by constructing a positive supersolution of the equation $\Delta u + ku = 0$.

It turns out that, in many cases, a sharp subsolution of $(*)$ can be obtained by constructing a positive subsolution of the same linear equation $\Delta u + ku = 0$. In fact, we will prove:

Lemma 4.1. *Let u be a positive function on M satisfying $\Delta u + ku \geq \phi u$ for some positive function ϕ. Suppose K is a function on M such that $Ku^{p-1} \leq C\phi$ for some constant C. Then for sufficiently small $\epsilon > 0$ the function ϵu is a subsolution of $(*)$.*

Using the method of proof of Theorem 3.2, one may also obtain an upper estimate of the maximal solution of $(*)$ in terms of K and supersolutions of $\Delta u + ku = 0$. Together with Lemma 4.1, one may express the asymptotic behavior of the maximal solution of $(*)$ in terms of the asymptotic behavior of K and k.

Using the general theory mentioned above, we will analyze the existence and nonexistence results obtained for equation $(*)$ for $\mathbf{R}^n$ and $\mathbf{H}^n$. These results are extremely sharp. Assuming that k is constant, we present the precise exponential growth rates on the function K near infinity such that beyond these growth rates equation $(*)$ has no C^2 positive solution and below them it always has a maximal C^2 positive solution as long as K is a permissible function for $(*)$ on M. The results are somewhat technical and lengthy. However, if one is only concerned with the *critical exponential growth rate* of the function K, then they can be easily summarized as follows:

Theorem A. *Let M be either the n-dimensional Euclidean space $\mathbf{R}^n$ $(n \geq 1)$ or the n-dimensional simply connected hyperbolic space $\mathbf{H}^n$ $(n \geq 2)$ of constant negative sectional curvature -1. Let ν be the first eigenvalue for the Laplacian operator on M. For a constant k, let λ be the first eigenvalue of the associated linear operator $\mathbf{L} = -\Delta - k$ on M. If $\lambda < 0$, then the upper critical exponential growth rate of the function K for equation $(*)$ on M is ∞. If $\lambda \geq 0$, then the upper and lower critical exponential growth rate are the same and are both equal to $(p-1)(\sqrt{\nu} - \sqrt{\lambda})$.*

Here the *upper (lower) critical exponential growth rate* of the function K for equation $(*)$ on M is defined to be the extended real number

$$\sup\{\alpha\} \qquad (\inf\{\beta\}),$$

where the sup is taken for all α such that the equation $\Delta u + ku - e^{\alpha r(x)}u^p = 0$ has a positive C^2 solution on M and the inf is taken for all β such that the equation $\Delta u + ku - e^{\beta r(x)}u^p = 0$ has no positive C^2 solution on M. We would like to indicate here that many of the results for the special case that $k = 0$ on $\mathbf{R}^n$ in section 4 are more or less known (cf. [7], [9], [12], [19], [23], [28], [29], etc). The reason for their inclusion in this paper is mainly for the sake of a unified treatment and the completeness of the paper. Moreover, our approach is different; more information is made available and it enables us to generalize these results on $\mathbf{R}^n$ to manifolds of zero or nonnegative Ricci curvature, which will be carried out in detail in the second part of the paper.

After the examination of some other manageable examples, notably, the scaled product of the hyperbolic space $\mathbf{H}^n$ with $\mathbf{H}^m$, there seem to be reasons to expect that the following two conjectures might be true.

Conjecture 1. *Theorem* A *remains valid if M is a complete noncompact Riemannian manifold with Ricci curvature bounded from below by a constant and k is a uniformly bounded smooth function.*

Conjecture 2. *Let M be a complete noncompact Riemannian manifold with Ricci curvature bounded from below by a constant, and k a uniformly bounded smooth function.*

- **(a)** *If the first eigenvalue of the associated linear operator $\mathbf{L}u = -\Delta u - ku$ on M is negative, then the critical exponential growth rate of the function K for equation $(*)$ on M is ∞.*
- **(b)** *If the critical exponential growth rate of the function K for equation $(*)$ on M is ∞, then equation $(*)$ has a positive C^2 solution on M for every strictly positive smooth function K on M.*

Notice that Theorem 2.2 implies that the strict positivity condition on the function K in conjecture 2 (b) is necessary.

We intend to continue our study of equation $(*)$ on complete noncompact Riemannian manifolds in the second part of the paper [26], using the results proved for the two model cases $\mathbf{R}^n$ and $\mathbf{H}^n$.

Acknowledgment

The authors would like to thank Professor S.Y. Cheng for providing the reference [10], and for his comments. The third author would like to thank Professor J. Cheeger for his constant moral support and for his kind help. He would also like to thank his colleagues and the Chairman, Professor T. Lawson, at the Department of Mathematics at Tulane University for their great support.

1. Reduction to the existence of a positive subsolution

The aim of this section is to prove some basic properties of the equation

$$(*) \qquad \Delta u + ku - Ku^p = 0,$$

where k and K are Hölder continuous functions on M. The function K is assumed to be non-negative, and the constant p is assumed to be bigger than 1. It is well-known that, by the monotone iteration scheme (cf. [2], [15], [29], [31]), if there exist positive continuous functions u_+ and u_- in $H^2_{loc}(M)$ such that

$$\begin{cases} \Delta u_+ + ku_+ - Ku_+^p \leq 0, \\ \Delta u_- + ku_- - Ku_-^p \geq 0, \\ u_+ \geq u_-, \end{cases}$$

then equation $(*)$ has a positive C^2 solution u on M such that $u_- \leq u \leq u_+$.

It is often not too difficult to construct either a supersolution u_+ or a subsolution u_- of $(*)$ on M. But it is rather difficult to construct both at the same time so that they satisfy $u_+ \geq u_-$.

The main result of this section is to show that if $K \geq 0$ satisfies a fairly weak positivity condition, then the existence of a positive solution of equation $(*)$ on M is equivalent to the existence of a positive subsolution on M together with the existence of positive supersolutions on compact subdomains in M. This greatly simplifies the problem and makes it possible to study the existence theory of solutions of equation $(*)$ on general complete Riemannian manifolds. Let us first introduce two definitions.

Definition 1.1. A Hölder continuous function K on a connected noncompact manifold Ω is said to be *essentially positive* if it is nonnegative and there exists an exhaustion by compact domains $\{\Omega_i\}_{i=1}^\infty$ in Ω such that $\Omega = \bigcup_{i=1}^\infty \Omega_i$ and $K|_{\partial\Omega_i} > 0$ for all $i \geq 1$. K is said to be *permissible* if it is essentially positive on Ω and $(*)$ has a positive supersolution on every compact subdomain X of Ω.

In §2, we will discuss the conditions for a function K to be permissible.

Definition 1.2. A positive C^2 solution u of equation $(*)$ on a connected noncompact manifold Ω is said to be a *maximal* solution if for every solution v of equation $(*)$, we have $v(x) \leq u(x)$ for all $x \in \Omega$.

It is easy to see that a maximal solution is unique. The following is an immediate consequence of the uniqueness of maximal solutions.

Proposition 1.1. *Let G be a group which acts on the Riemannian manifold Ω by isometry. Assume that both k and K are G-invariant functions. If equation $(*)$ has a positive C^2 maximal solution u on Ω, then u is a G-invariant function.*

Proof. Let $g \in G$ be an isometry of the Riemannian manifold Ω. Since the Laplacian Δ is invariant by an isometry and because both k and K are G-invariant functions, it follows that the pullback g^*v of a positive C^2 solution v under the map g is a positive C^2 solution and $v \to g^*v$ defines a one-to-one and onto map from the collection of all positive C^2 solutions to itself. Thus, if u is a positive C^2 maximal solution of equation $(*)$, then so is the pullback g^*u. By the uniqueness of a maximal solution, u is G-invariant. $\square$

Theorem 1.2. *Let K be permissible. Suppose that $(*)$ has a positive subsolution u_-. Then equation $(*)$ has a positive C^2 maximal solution u on Ω, and $u \geq u_-$.*

Before we prove the theorem, let us prove four lemmas.

Lemma 1.3. *Let Ω be either a compact subdomain of M or $\Omega = M$. Suppose u_+ and u_- are a positive C^2 supersolution and subsolution for $(*)$ on Ω. When Ω is compact, we assume that $(u_+ - u_-)|_{\partial\Omega} \geq 0$. On the other hand, if $\Omega = M$; then we assume that $\lim_{r\to\infty} \inf\{\frac{u_+(x)}{u_-(x)} | x \in M\backslash B_o(r)\} \geq 1$, where $o \in M$ is a fixed point and $B_o(r)$ is the geodesic ball of radius r centered at o. Then $u_+ \geq u_-$ on Ω.*

Proof. Let $\phi = u_-/u_+$, then $\phi \leq 1$ on the boundary of Ω or at ∞. Since $u_- = u_+\phi$, we have

$$\Delta\phi + 2\langle \nabla \log u_+, \nabla\phi\rangle + K u_+^{p-1}\phi(1 - \phi^{p-1}) \geq 0.$$

It follows from the strong maximum principle (cf. [13]) that ϕ cannot have a maximum value > 1 in the interior of Ω. $\square$

Lemma 1.4. *Let $\mathbf{H}^n$ be the simply connected hyperbolic space form of constant sectional curvature -1. Given any constants $\epsilon > 0$, $\beta > 0$, and λ, there is a positive increasing function $\phi_\epsilon(r)$ such that the function defined on an ϵ geodesic ball $B(\epsilon)$ of $\mathbf{H}^n$ given by $V_\epsilon(x) = \phi_\epsilon(r(x))$ satisfies*

$$\Delta V_\epsilon + \lambda V_\epsilon - \beta V_\epsilon^p \leq 0,$$

and

$$V_\epsilon|_{\partial B(\epsilon)} = \infty.$$

Here $r(x)$ is the distance from x to the center of $B(\epsilon)$. In particular, the value $V_\epsilon(0)$ is a constant depending only on the constants $p > 1$, ϵ, β, and λ.

Proof. In normal polar geodesic coordinates, the metric on the n-dimensional hyperbolic space $\mathbf{H}^n$ is of the form

$$dr^2 + \sinh^2 r\, \omega^2,$$

where ω^2 is the standard metric on the $(n-1)$-dimensional sphere S^{n-1}. Let $\sigma = \frac{2}{p-1}$, $a(r) = \sinh^2 \frac{\epsilon}{2} - \sinh^2 \frac{r}{2}$, and $\phi(r) = a^{-\sigma}(r)$. Then $\phi(r)$ is a positive increasing function in $0 \leq r < \epsilon$. Letting $V(x) = \phi(r(x))$, we clearly have $V|_{\partial B(\epsilon)} = \infty$. Direct computation shows that

$$\Delta V + \lambda V = \frac{1}{4}a^{-\sigma-2}\{\sigma((1+\sigma)\sinh^2 r + 2na\cosh r) + 4a^2\lambda\} \leq C^{p-1}V^p$$

where $C > 0$ is a constant depending only on the constants p, ϵ, and λ. Use the non-linearity of the inequality, one can define

$$\phi_\epsilon = C\beta^{\frac{-1}{p-1}}\phi,$$

which has the desired properties. □

Lemma 1.5. *Let $\Omega \subset M$ be an open bounded domain. If there is a compact subdomain $X \subset \Omega$ such that $K \geq 0$ on X and $K > 0$ on ∂X, then there exists a positive constant C such that every C^2 positive solution u of equation $(*)$ on Ω must satisfy*

$$u|_{\partial X} \leq C.$$

Proof. Since $K > 0$ on ∂X and $X \subset \Omega$ is a compact subdomain, there exists a positive constant ϵ, strictly less than the injectivity radius of X, such that the ϵ-neighborhood $U_\epsilon(\partial X)$ of ∂X in M is contained in Ω and $K \geq \beta > 0$ on $U_\epsilon(\partial X)$ for some positive constant β. Since Ω is bounded, by scaling the metric if necessary, we may assume that the Ricci curvature on Ω is bounded from below by $-(n-1)$. Let $V_\epsilon(r)$ be the function defined in Lemma 1.4 with $\lambda = \sup\{k(x) \mid x \in \Omega\}$, and let $C = V_\epsilon(0)$. For any point $x_0 \in \partial X$, let r_0 be the distance function to the point x_0 in the geodesic ball $B_{x_0}(\epsilon) \subset \Omega$. Since $Ric|_\Omega \geq -(n-1)$, a comparison argument shows that $V_\epsilon(r_0(x))$ is a smooth supersolution of equation $(*)$ on $B_{x_0}(\epsilon)$. Since $V_\epsilon(r_0)|_{\partial B_{x_0}(\epsilon)} = \infty$, it follows from Lemma 1.3 that

$$u(x) \leq V_\epsilon(r_0(x))$$

on $B_{x_0}(\epsilon)$ for all C^2 solution u of equation $(*)$ on Ω. In particular, $u(x_0) \leq V_\epsilon(0) = C$. Since C is independent of u and $x_0 \in \partial X$, the proof is completed. □

Lemma 1.6. *Let Ω be a bounded domain in M. Assume that $K \geq 0$ is Hölder continuous on Ω, continuous on $\overline{\Omega}$, and $K > 0$ on $\partial\Omega$. If equation $(*)$ has a bounded C^2 solution v on Ω such that v is bounded from below by a positive constant on Ω, then the boundary value problem*

$$\begin{cases} \Delta u + ku - Ku^p = 0, \\ u|_{\partial\Omega} = \infty \end{cases}$$

has a positive C^2 solution u on Ω and $u \geq v$.

Proof. For any constant $N \geq 1$, Nv is a positive supersolution on Ω since

$$\begin{aligned} \Delta(Nv) + kNv - K(Nv)^p &= N(\Delta v + kv) - K(Nv)^p \\ &= NKv^p(1 - N^{p-1}) \\ &\leq 0. \end{aligned}$$

Let us define

$$C = \inf_\Omega v,$$

which is positive by the assumption on v. Since v is also bounded from above in Ω, there is a positive integer N_0 such that $N_0 C \geq \sup_\Omega v$. By the method of supersolution-subsolution, the boundary value problem

$$\begin{cases} \Delta u + ku - Ku^p = 0, \\ u|_{\partial\Omega} = NC \end{cases}$$

has a positive C^2 solution v_N on Ω such that $v \leq v_N \leq Nv$ for all integers $N \geq N_0$.

Now Lemma 1.3 implies that $\{v_N\}_{N\geq N_0}$ is an increasing sequence of solutions of equation $(*)$ on Ω. Lemma 1.5 implies that for any $\epsilon > 0$ sufficiently small, there is a constant $C(\epsilon) > 0$ such that

$$\sup_N \left(\sup_{\partial\Omega_\epsilon} v_N \right) \leq C(\epsilon),$$

where $\Omega_\epsilon = \{x \in \Omega \mid dist(x, \partial\Omega) > \epsilon\}$. Consider the function $C(\epsilon)C^{-1}v$, where $C = \inf_\Omega v$. We may choose $C(\epsilon)$ sufficiently large so that $C(\epsilon)C^{-1} > 1$; then $C(\epsilon)C^{-1}v$ is a supersolution of $(*)$. By Lemma 1.3, we have $v_N \leq C(\epsilon)C^{-1}v$ in Ω_ϵ for all N. Hence v_N is uniformly bounded on compact subsets of Ω. Thus $\{v_N\}_{N\geq N_0}$ converges pointwise to a solution u on Ω in the distribution sense. Standard regularity theory (cf. [4]) then implies that u is a C^2 solution of equation $(*)$ on Ω. It is also clear that $u \geq v$ and $u|_{\partial\Omega} = \infty$. □

Proof of Theorem 1.2. Let u_- be a positive subsolution of $(*)$ on Ω. The assumption that K is permissible implies that there exists a compact exhaustion $\{\Omega_i\}_{i\geq 1}$ of Ω such that $K > 0$ on $\partial\Omega_i$ for all $i \geq 1$. Since Ω_i is compact and K is permissible, equation $(*)$ has a positive supersolution on Ω_i. By observing that multiplying a supersolution by a constant bigger than 1 again produces a supersolution, we may assume that the supersolution $u_{i,+} \geq u_-$ on Ω_i. It follows from the method of supersolution-subsolution that equation $(*)$ has a bounded positive solution on Ω_i which is bounded from below by u_-. Lemma 1.6 implies that the boundary value problem

$$\begin{cases} \Delta u_i + k u_i - K u_i^p = 0, \\ u_i|_{\partial\Omega_i} = \infty \end{cases}$$

has a positive C^2 solution u_i on Ω_i such that $u_i \geq u_-$.

By Lemma 1.3, for each $i_0 \geq 1$, $\{u_i\}_{i\geq i_0}$ is a sequence of decreasing solutions of equation $(*)$ on Ω_{i_0}. Since all u_i are bounded from below by $u_- > 0$, the limiting function $u = \lim_{i\to\infty} u_i \geq u_- > 0$ exists, and standard elliptic regularity theory shows that it is a positive C^2 solution of equation $(*)$ on Ω. u is obviously a maximal solution on Ω, because if v is any positive C^2 solution of equation $(*)$ on Ω, then for any $x_0 \in \Omega$ there exists $i_0 \geq 1$ such that $x_0 \in \Omega_i$ for all $i \geq i_0$. Now Lemma 1.3 implies that $v \leq u_i$ on Ω_i for all $i \geq i_0$. In particular, $v(x_0) \leq \lim_{i\to\infty} u_i(x_0) = u(x_0)$. This completes the proof of the theorem. □

Corollary 1.7. *Let $K_1 \geq K_2 \geq 0$ be permissible functions of equation $(*)$ on a connected Riemannian manifold Ω. If equation $(*)$ has a positive C^2 solution u_1 for $K = K_1$, then equation $(*)$ has a positive C^2 solution u_2 for $K = K_2$ such that $u_2 \geq u_1$.*

Proof. Since $K_1 \geq K_2$, $\Delta u_1 + k u_1 - K_2 u_1^p = (K_1 - K_2)u_1^p \geq 0$. Thus u_1 is a positive C^2 subsolution of equation $(*)$ for $K = K_2$. Hence the corollary follows from Theorem 1.2. □

By Corollary 1.7, the following definition is meaningful.

Definition 1.3. Let M be a complete noncompact n-dimensional Riemannian manifold. The *upper* (*lower*) *critical exponential growth rate* of the function K for equation $(*)$ on M is defined by the extended real number

$$\sup\{\alpha\} \qquad (\inf\{\beta\}),$$

where the supremum is taken for all α such that the equation $\Delta u + ku - e^{\alpha r(x)}u^p = 0$ has a positive C^2 solution on M and the infimum is taken for all β such that the equation $\Delta u + ku - e^{\beta r(x)}u^p = 0$ has no positive C^2 solution on M.

We will see that the upper (lower) critical exponential growth rate of $(*)$ on $\mathbf{R}^n$ and $\mathbf{H}^n$ are closely related to the first eigenvalue of the Laplacian and the first eigenvalue of the operator $\mathbf{L} = -\Delta - k$.

2. First eigenvalues and the existence of supersolutions

Let k and $K \geq 0$ be Hölder continuous functions on a complete noncompact n-dimensional Riemannian manifold (M, g). Let $\mathfrak{K}_0 = \{x \in M | K(x) = 0\}$ be the zero set of the function K. In this section, we will study the relationship between the first Dirichlet eigenvalue $\lambda_{\mathfrak{K}_0}$ of the linear operator $\mathbf{L}u = -\Delta u - ku$ on $\mathfrak{K}_0$ and the existence of a positive supersolution of

$$(*) \qquad \Delta u + ku - Ku^p = 0.$$

The basic result is that the existence or non-existence of a supersolution of equation $(*)$ on a compact domain X is completely determined by the sign of the first Dirichlet eigenvalue $\lambda_{\mathfrak{K}_0 \cap X}$ of the linear operator $\mathbf{L}u = -\Delta u - ku$ with $\lambda_{\mathfrak{K}_0 \cap X}$ defined suitably.

For any nonempty smooth open set $\Omega \subseteq M$, the first eigenvalue λ_Ω of the operator $\mathbf{L}u = -\Delta u - ku$ on Ω is given by the variational principal

$$\lambda_\Omega = \inf \frac{\int_\Omega (|\nabla f|^2 - kf^2)\, dvol}{\int_\Omega f^2\, dvol},$$

where the infimum is taken over all smooth functions f with nontrivial compact support in Ω. In fact (see [6]), there exists a unique eigenfunction v defined on the closure of Ω so that

$$\Delta v + kv + \lambda_\Omega v = 0,$$

$$v > 0 \qquad \text{on } \Omega,$$

and

$$v|_{\partial\Omega} = 0.$$

For an arbitrary set in M, we will define the first eigenvalue by an approximation procedure.

Definition 2.1. Let $S \subset M$ be an arbitrary bounded subset in M. The first eigenvalue λ_S of the operator $\mathbf{L} = -\Delta - k$ on S is defined by

$$\lambda_S = \sup \lambda_\Omega,$$

where the sup is taken over all smooth open sets $\Omega \subset M$ such that $S \subset \Omega$. In particular, if $S = \emptyset$ then $\lambda_S = \infty$.

Definition 2.2. Let $S \subset M$ be an arbitrary unbounded subset of M. The first eigenvalue λ_S of the operator $\mathbf{L} = -\Delta - k$ on S is defined by

$$\lambda_S = \lim_{r \to \infty} \lambda_{S_r},$$

where $S_r = S \cap \overline{B}(o, r)$ for all $r > 0$ and $o \in M$ is a fixed point.

Remark 2.1. It follows from the monotonicity property of the first eigenvalue [6] that λ_S is well defined and is independent of the base point o. It is easy to show that Definitions 2.1 and 2.2 are equivalent to the usual definition for open sets.

Theorem 2.1. *Let $K \geq 0$ be a Hölder continuous nonnegative function on M. Let $\mathfrak{K}_0 = \{x \in M | K(x) = 0\}$. If the equation $(*)$ has a positive supersolution on a bounded open domain X, then $\lambda_{\mathfrak{K}_0 \cap X} \geq 0$. Conversely, if $\lambda_{\mathfrak{K}_0 \cap X} > 0$, then $(*)$ has a positive supersolution on X.*

Proof. Let X be a bounded open domain. Suppose $(*)$ has a positive supersolution u on X and $\lambda_{\mathfrak{K}_0 \cap X} < -2a < 0$ for some positive constant $a > 0$. By the definition of $\lambda_{\mathfrak{K}_0 \cap X}$ and the fact that the first eigenvalue is continuous with respect to C^0 deformations of the domain, there is a bounded open domain $\Omega \subset X$ such that $\mathfrak{K}_0 \cap X \subset \Omega$ and $\lambda_{\mathfrak{K}_0 \cap \Omega} < -2a$. Hence we can find a sequence of bounded open domains $\Omega_i \subset X$ such that $\bigcap_i \Omega_i = \mathfrak{K}_0 \cap X$ and $\lambda_i = \lambda_{\Omega_i} \leq -a$. In particular, there are positive eigenfunctions v_i on Ω_i satisfying

$$\Delta v_i + k v_i + \lambda_i v_i = 0$$

and $v_i = 0$ on $\partial \Omega_i$. By assumption, $u > 0$ satisfies

$$\Delta u + k u \leq K u^p$$

on X. Therefore, we have

$$\int_{\Omega_i} (u \Delta v_i - v_i \Delta u) + \int_{\Omega_i} (\lambda_i u v_i + K u^p v_i) \geq 0.$$

Using the facts that $u > 0$, $v_i = 0$ on $\partial\Omega$, and $v_i > 0$ in Ω_i, so that $\frac{\partial v_i}{\partial \nu} \leq 0$ on $\partial \Omega_i$, we have

$$\int_{\Omega_i} \left(\lambda_i + K u^{p-1}\right) u v_i \geq 0. \tag{2.1}$$

Note that $\lambda_i \leq -a$, and $K = 0$ on $\bigcap_i \Omega_i$. We have

$$-a + K u^{p-1} < 0$$

in Ω_i, provided that i is large enough. This contradicts (2.1).

Note that in the above argument, if $\mathfrak{K}_0 \cap X$ is an open domain, then we can simply argue by using the eigenfunction v of $\mathfrak{K}_0 \cap X$. In this case, if $\lambda_{\mathfrak{K}_0 \cap X} = 0$, then we can conclude that the boundary integral

$$\int_{\partial(\mathfrak{K}_0 \cap X)} u \frac{\partial v}{\partial \nu} = 0.$$

This implies that $u|_{\partial(\mathfrak{K}_0 \cap X)} = 0$, which violates the assumption that u is positive on X. However, we do not know if it is possible to lose the positivity of λ once we take the limit of λ_i when $\mathfrak{K}_0 \cap X$ is not an open domain.

Conversely, suppose $\lambda_{\mathfrak{K}_0 \cap X} > 0$. Then there are bounded open domains Ω, Ω' such that

$$\mathfrak{K}_0 \cap X \subset\subset \Omega' \subset\subset \Omega$$

and $\lambda = \lambda_\Omega > 0$. Let u_1 be a positive function such that $\Delta u_1 + k u_1 + \lambda u_1 = 0$ in Ω. Let $0 \leq \phi \leq 1$ be a nonnegative smooth function such that $\phi = 1$ in Ω' and $\phi = 0$ outside Ω. By the definition of $\mathfrak{K}_0$,

$$\inf_{X \setminus \Omega'} K > 0. \tag{2.2}$$

From this it is easy to see that if N is a sufficiently large positive constant, then $u_2 = N$ is a positive supersolution of $(*)$ in $X \setminus \Omega'$. Let us define $u = a\,(\phi u_1 + (1-\phi)u_2)$, where a is a positive constant to be determined later. One checks readily that for any $a \geq 1$, u is a positive supersolution of $(*)$ on Ω' and on $X \setminus \Omega$. On $(\Omega \setminus \Omega') \cap X$, there is a positive constant C such that

$$(\Delta + k)\,(\phi u_1 + (1-\phi)u_2) \leq C.$$

By (2.2), and the definitions of u_1, u_2 and ϕ, we may choose C sufficient large, so that $K\,(\phi u_1 + (1-\phi)u_2)^p \geq C^{-1}$ in $(\Omega \setminus \Omega') \cap X$. Setting $a = C^{2/(p-1)}$, we see that on $(\Omega \setminus \Omega') \cap X$,

$$\Delta u + ku - Ku^p \leq a(C - C^{-1}a^{p-1}) \leq 0.$$

This completes the proof of the theorem. □

Corollary 2.2. *Let K be a Hölder continuous, essentially positive function on a complete noncompact manifold M. Let $\mathfrak{K}_0 = \{x \in M |\ K(x) = 0\}$ be the zero set of K. Suppose the first eigenvalue $\lambda_{\mathfrak{K}_0}$ of $\mathbf{L}u = -\Delta u - ku$ is positive. Then K is permissible for $(*)$. In particular, if $K > 0$ everywhere, then K is permissible for $(*)$.*

Corollary 2.3. *Using the same notation as in Corollary 2.2, suppose $(*)$ has a positive supersolution on M. Then $\lambda_{\mathfrak{K}_0} \geq 0$.*

As an application of the above results, let us consider the following situation. For $n \geq 3$, let $p = \frac{n+2}{n-2}$ and $k = k_s \equiv -\frac{(n-2)s}{4(n-1)}$, where s is the scalar curvature function of the Riemannian manifold (M, g). Equation $(*)$ becomes the prescribing scalar curvature equation on (M, g). In [17], it was claimed (Theorems 3 and 4) that for a simply connected complete Riemannian manifold (M, g) with sectional curvature bounded both from above and from below by negative constants, the equation

$$\Delta u + k_s u - K u^{\frac{n+2}{n-2}} = 0$$

has a positive solution on (M, g) for all nonnegative functions K which satisfies certain decay conditions at ∞. That is, $\mathfrak{K}_0 = \{x \in M | K(x) = 0\}$ may be any large compact set in M. We will show by examples that Theorems 3 and 4 in [17] are false. Notice that by Corollary 2.3, these theorems imply the following statement:

Let (M, g) be a simply connected complete Riemannian manifold with sectional curvature bounded both from above and below by negative constants. Then the first eigenvalue of the eigenvalue problem

$$\Delta u + k_s u + \lambda u = 0 \tag{2.3}$$

on (M, g) is nonnegative.

Example 2.1. For $n \geq 3$, let $M = \mathbf{H}^2 \times \mathbf{R}^{n-2}$ with the product metric, where $\mathbf{H}^2$ is the hyperbolic disc with constant sectional curvature -1 and $\mathbf{R}^{n-2}$ is the $(n-2)$-dimensional Euclidean space. It is easy to see that the scalar curvature $s = -2$. Thus, $k_s = \frac{(n-2)}{2(n-1)}$. Since the first eigenvalue of the equation $\Delta u + \lambda u = 0$ on $\mathbf{H}^2$ and $\mathbf{R}^{n-2}$ are $\frac{1}{4}$ and 0, respectively, the product formula implies that the first eigenvalue of (2.3) on M is $\lambda_n = \frac{3-n}{4(n-1)}$. In particular, $\lambda_n < 0$ when $n > 3$. It follows from Corollary 2.3 that there exists a positive constant $r > 0$ such that if $K \geq 0$ and $K(x) = 0$ on a geodesic ball of radius r, then the equation

$$\Delta u + k_s u - K u^p = 0$$

has no positive supersolution on M, for all $p > 1$.

Notice that in the above example, M is diffeomorphic to $\mathbf{R}^n$ and hence simply connected. The sectional curvature is nonpositive. To obtain a simply connected counterexample with sectional curvature bounded from both above and below by negative constants, we shall construct a family of complete negatively curved metrics g_α, $0 \le \alpha \le 1$, on $\mathbf{R}^n$ such that g_0 is the product metric on M and g_α converges to g_0 uniformly on any compact set in $\mathbf{R}^n$.

Example 2.2. For any $0 \le \alpha \le 1$, the metric g_α on $\mathbf{R}^n$ is constructed as the warped product of $\mathbf{H}^2$ with $\mathbf{R}^{n-2}$. For convenience, we shall use the upper half plane model for $\mathbf{H}^2$. Thus, let $\mathbf{R}^2_+ = \{(t,s) \in \mathbf{R}^2 | t > 0\}$, and let $x = (x_1, x_2, \cdots, x_{n-2}) \in \mathbf{R}^{n-2}$. For $0 \le \alpha \le 1$, let

$$g_\alpha = t^{-2}(dt^2 + ds^2) + t^{-2\alpha} dx^2,$$

where $dx^2 = dx_1^2 + dx_2^2 + \cdots + dx_{n-2}^2$ is the Euclidean metric on $\mathbf{R}^{n-2}$. It is easy to see that g_α is a complete Riemannian metric for each $\alpha \in [0,1]$. g_0 is the product metric on $\mathbf{H}^2 \times \mathbf{R}^{n-2}$ and g_1 is the hyperbolic metric on $\mathbf{H}^n$ with constant sectional curvature -1. Let s_α, Ric_α, and K_α be the scalar curvature, the Ricci curvature, and the sectional curvature of g_α, respectively. A direct computation shows that

$$s_\alpha = -\{1 + (1 + (n-2)\alpha)^2 + (n-2)\alpha^2\},$$

$$-(1 + n\alpha) \le Ric_\alpha \le -\alpha(1 + n\alpha),$$

and

$$-1 \le K_\alpha \le -\alpha^2.$$

In particular, $(\mathbf{R}^n, g_\alpha)$ is of constant negative scalar curvature s_α, and s_α is a continuous function of α. For each $\alpha \in (0,1]$, the sectional curvature K_α is bounded from above and below by negative constants. Let $k_\alpha \equiv k_{s_\alpha} = -\frac{(n-2)s_\alpha}{4(n-1)}$. For $n > 3$, it follows from Example 2.1 that there is a number $r > 0$ such that the first eigenvalue of

$$\Delta u + k_0 u + \lambda u = 0$$

on a geodesic ball $B(o,r)$ centered at o with radius r in $(\mathbf{R}^n, g_0)$ is negative. By the continuity of the first eigenvalue in the metric, there is a positive constant $0 < \epsilon < 1$ such that for all $\alpha \in (0,\epsilon]$, the first eigenvalue of

$$\Delta u + k_\alpha u + \lambda u = 0 \tag{2.4}$$

on a geodesic ball $B(o,r)$ centered at o with radius r in $(\mathbf{R}^n, g_\alpha)$ is negative. Thus, monotonicity of the first eigenvalue implies that the first eigenvalue of (2.4) on $(\mathbf{R}^n, g_\alpha)$ is negative for all $\alpha \in (0,\epsilon]$.

3. Nonexistence of solutions

In this section, we will use the maximum principle to prove a nonexistence result for positive subsolutions of the equation $(*)$ on a complete noncompact manifold M. Throughout this section, we will assume that k and K are continuous functions on M with $K \ge 0$, and $p > 1$ is a constant. Using a similar method, we will also obtain an upper estimate for any positive subsolution of $(*)$. In §4 and §5 we will demonstrate that the results in this section yield extremely sharp nonexistence results and upper estimates of positive solutions for $(*)$ on Euclidean space $\mathbf{R}^n$ and hyperbolic space $\mathbf{H}^n$.

First, let us prepare by performing the following computation. Let w be a positive C^2 function on M, and let u be a positive subsolution of $(*)$. Let us define $v = u/w$. Then we have

$$\Delta v + 2\langle \nabla \log w, \nabla v\rangle + v\left(w^{-1}\Delta w + k\right) - Kw^{p-1}v^p \geq 0. \tag{3.1}$$

Lemma 3.1. *Suppose w satisfies*

$$\Delta w + kw \leq 0 \tag{3.2}$$

on a bounded domain Ω. If K is not identically 0 in Ω, then v cannot attain any local maximum inside Ω.

Proof. Combining (3.1) and (3.2), we have

$$\Delta v + 2\langle \nabla \log w, \nabla v\rangle \geq Kw^{p-1}v^p. \tag{3.3}$$

The strong maximum principle asserts that v cannot attain a local maximum unless v is identically constant. On the other hand, if v is a constant, then (3.3) implies that

$$0 \geq Kw^{p-1}v^p$$

on Ω. The assumption that $K \geq 0$ and not identically 0, and the facts that $w > 0$ and $v > 0$, provide a contradiction . □

Let us define the function $\log^{(m)} t$ inductively. For $m = 0$, define $\log^{(0)} t = t$. For $m > 0$, let $\log^{(m)} t = \log\left(\log^{(m-1)} t\right)$. Clearly, $\log^{(m)} t$ is well-defined for t sufficiently large.

Theorem 3.2. *Let M be a complete noncompact manifold with Ricci curvature bounded from below by $-a^2$ for some constant $a \geq 0$. Let w be a positive C^2 function such that $|\nabla \log w|$ is bounded and*

$$\Delta w + kw \leq 0.$$

Let $r(x)$ be the distance from x to some fixed point $o \in M$. Suppose

$$K(x)w^{p-1}(x) \geq C\left(\prod_{i=0}^{m} \log^{(i)}(r(x))\right)^{-1}$$

near infinity for some constant $C > 0$ and for some positive integer m. Then equation $(*)$ does not admit any positive subsolution.

Proof. Suppose u is a positive subsolution of $(*)$. Let $v = u/w$. Then (3.3) is true for v. Let $\phi \geq 0$ be a cutoff function and let $F = \phi v$. Let $F(x_0) = \max F$. Suppose that $F(x_0) > 0$; then at x_0, we have

$$\nabla F = 0,$$

and so

$$\nabla v = -\frac{v\nabla\phi}{\phi}. \tag{3.4}$$

Moreover, at x_0

$$\begin{aligned} 0 &\geq \Delta F \\ &= v\Delta\phi + 2\langle\nabla\phi, \nabla v\rangle + \phi\Delta v \\ &\geq v\Delta\phi - \frac{2v|\nabla\phi|^2}{\phi} + 2v\langle\nabla \log w, \nabla\phi\rangle + K\phi w^{p-1}v^p, \end{aligned}$$

where we have used (3.3) and (3.4). Multiplying the above inequality by ϕ^{p-1}, we have

$$0 \geq \left(\phi^{p-2}\Delta\phi\right) F - 2\left(\phi^{p-3}|\nabla\phi|^2 - 2\phi^{p-2}\langle\nabla \log w, \nabla\phi\rangle\right) F + (Kw^{p-1})F^p.$$

Hence, at the maximum point x_0, we have

$$(Kw^{p-1})F^p \leq \phi^{p-2}\left(-\Delta\phi + \frac{2|\nabla\phi|^2}{\phi} - 2\langle\nabla \log w, \nabla\phi\rangle\right).$$

Now let $\phi = \psi^\sigma$, where $\sigma > 2$ is a constant, and ψ is a cutoff function. At x_0 we have

$$(Kw^{p-1})F^p \leq \sigma\psi^{\sigma(p-1)-2}\left(-\psi\Delta\psi + (\sigma+1)|\nabla\psi|^2 - 2\psi\langle\nabla \log w, \nabla\psi\rangle\right). \tag{3.5}$$

Since $p > 1$, we can choose $\sigma > 2$ large enough so that $\sigma(p-1) - 2 > 0$. Let $\eta(t)$ be a smooth nonnegative nonincreasing function on $[0, \infty)$ such that $\eta(t) = 0$ for t large enough. Let $\psi(x) = \eta((r(x))$. Then at the maximum point x_0 of F, if x_0 is within the cut locus of o, we have

(3.6)
$$\begin{aligned} (Kw^{p-1})F^p &\leq \sigma\psi^{\sigma(p-1)-2}\left(-\psi\eta'\Delta r - \psi\eta'' + (\sigma+1)\left(\eta'\right)^2 - 2\psi\eta'\langle\nabla \log w, \nabla r\rangle\right) \\ &= \sigma\psi^{\sigma(p-1)-2}\left(-\psi\eta'\left(\Delta r + 2\langle\nabla \log w, \nabla r\rangle\right) - \psi\eta'' + (\sigma+1)(\eta')^2\right) \end{aligned}$$

Now let $\widetilde{\eta}(t)$ be a smooth nonincreasing function defined on $(-\infty, \infty)$ such that $\widetilde{\eta} \geq 0$, $\widetilde{\eta}(t) = 1$ for $t \leq 2$, $\widetilde{\eta}(t) = 0$ for $t \geq 3$. For any R large enough so that $\log^{(m+1)} R > 0$, it is easy to see that the function

$$\eta(t) = \widetilde{\eta}\left(\frac{\log^{(m+1)}(t)}{\log^{(m+1)} R}\right)$$

is well-defined and smooth on $[0, \infty)$, such that $\eta(t) = 1$ for $0 \leq t \leq R$, and $\eta(t) = 0$ for those t such that $\log^{(m+1)} t \geq 3\log^{(m+1)} R$. In fact, $\eta(t)$ is also equal to 1 for those t such that $R \leq t \leq R_1$, where R_1 is such that $\log^{(m+1)} R_1 = 2\log^{(m+1)} R$. Let $\psi(x) = \eta\left(r(x)\right)$, where $r(x)$ is the distance from a point x to o. By Lemma 3.1, the maximum of F cannot be attained inside $B_o(R_1)$. Therefore, if $\max F > 0$, then the maximum point x_0 satisfies $R_1 \leq r \leq R_2$, where $\log^{(n+1)} R_2 = 3\log^{(n+1)} R$ and $r = r(x_0)$. Hence if x_0 is within the cut locus of o, then ψ is smooth near x_0, and, at x_0,

$$\Delta r + 2\langle\nabla \log w, \nabla r\rangle \leq C_1 \tag{3.7}$$

for some constant C_1 depending only on the lower bound of the Ricci curvature of M and the upper bound of $|\nabla \log w|$. Also, at $t = r(x_0)$,

$$\begin{aligned} 0 &\le -\eta' \\ &\le C_2 \left(\log^{(m+1)} R \prod_{i=0}^{m} \log^{(i)} r \right)^{-1} \\ &\le 3C_2 \left(\prod_{i=0}^{m+1} \log^{(i)} r \right)^{-1} \end{aligned} \tag{3.8}$$

for some constant C_2 depending only on $\widetilde{\eta}$, where $r = r(x_0)$ and we have used the fact that

$$1 \le \frac{\log^{(m+1)} r}{\log^{(m+1)} R} \le 3.$$

Finally, at $t = r(x_0)$

$$|\eta''| \le C_3 \left(r \prod_{i=0}^{m+1} \log^{(i)} r \right)^{-1} \tag{3.9}$$

if R is large enough. Combining (3.6)–(3.9), at x_0,

$$Kw^{p-1} F^p \le C_4 \left(\prod_{i=0}^{m+1} \log^{(i)} r \right)^{-1}$$

for some constant C_4 which is independent of R, if R is large. By the assumption on Kw^{p-1}, we conclude that

$$\sup_{B_o(R)} v^p \le \max F^p \le \frac{C_5}{\log^{(m+1)} r} \tag{3.10}$$

for some $r \ge R$. If x_0 is on the cut locus of o, then we may use a trick of Calabi, as in [11], to show that (3.10) is still true. Let $R \to \infty$; we have $v \equiv 0$, and so $u \equiv 0$. Hence $(*)$ has no positive subsolution. □

Remark 3.1. From the proof of the theorem, it is easy to see that if we have a better upper estimate for $\Delta r + 2\langle \nabla \log w, \nabla r \rangle$, then we may get a better result. For example, if the above quantity is less than or equal to Cr^{-1} for some constant, then we may relax the assumption on K and w so that $Kw^{p-1} \ge C \left(r \prod_{i=0}^{m} \log^{(i)} r \right)^{-1}$, and the conclusion of the theorem is still true.

Theorem 3.3. *Let M be a complete noncompact manifold with Ricci curvature bounded from below by $-a^2$ for some $a \ge 0$. Let $w > 0$ be a C^2 function such that $w^{-1}\Delta w$ and $|\nabla \log w|$ are uniformly bounded from above. Let u be a positive subsolution of $(*)$, where $K \ge 0$ is a continuous function satisfying $Kw^{p-1} \ge C$ near infinity for some constant $C > 0$. Then $u \le C'w$ for some constant C'.*

Proof. Let $v = u/w$, and let $o \in M$ be a fixed point. Suppose there is a fixed $R_0 > 0$ such that $\sup_{B_o(R)} v = \sup_{B_o(R_0)} v$, for all $R > R_0$. Then the theorem is obviously true. Otherwise, we can find a sequence $R_i \to \infty$ such that, for all

$x \in B_o(R_i)$, $v(x) < \sup_{B_o(R_i)} v$ for all i. As in the proof of the previous theorem, one can show that for each i,

$$\sup_{B_o(R_i)} v^p \leq \frac{C_4}{K(x_i)w^{p-1}(x_i)}\left(\frac{1}{r_i} + C_5\right),$$

for some point x_i such that $r_i = r(x_i) \geq R_i$, and for some constants C_4 and C_5. Here we have used (3.1) and the assumption that $w^{-1}\Delta w$ and $|\nabla \log w|$ are uniformly bounded from above. Use the assumption that Kw^{p-1} is bounded away from 0 near infinity. The theorem follows. □

4. The structure of the equation $\Delta u + ku - Ku^p = 0$ on $\mathbf{R}^n$

The aim of this section is to use the results in §§1–3 to analyze the existence and nonexistence of positive solutions of the equation

$$(*) \qquad \Delta u + ku - Ku^p = 0$$

on the Euclidean space $\mathbf{R}^n$, assuming that k is a constant and $K \geq 0$ is a permissible function on $\mathbf{R}^n$. Recall that K is said to be permissible for equation $(*)$ on $\mathbf{R}^n$ if equation $(*)$ has a positive supersolution on every compact domain X of $\mathbf{R}^n$, and K is essentially positive, which means that there is a compact exhaustion Ω_i of $\mathbf{R}^n$ such that $K > 0$ on $\partial\Omega_i$. Since the first eigenvalue of the Laplacian of $\mathbf{R}^n$ is 0, the first eigenvalue of any compact domain X is positive. By Theorem 2.1, if $k \leq 0$, and if K is essentially positive, then K is permissible. If $k > 0$, then K is permissible if $\lambda_{\mathfrak{K}_0} > 0$, and K is essentially positive by Corollary 2.2. Here $\mathfrak{K}_0$ is the zero set of K, and $\lambda_{\mathfrak{K}_0}$ is the first eigenvalue of $\mathbf{L} = -\Delta - k$ on $\mathfrak{K}_0$ (see Definitions 2.1 and 2.2). In particular, if $K > 0$ everywhere, then K is permissible.

This section is presented in such a way that many of the results can be readily generalized to complete manifolds with nonnegative Ricci curvature by comparison arguments. Note that most of the work for equation $(*)$ has been concentrated on the special case $k = 0$ on the Euclidean space $\mathbf{R}^n$. The reader is referred to [7], [12], [19], [23], [28], [29], etc. Even on $\mathbf{R}^n$, however, very little is known for equation $(*)$ if $k \neq 0$.

By Theorem 1.2, if K is permissible, then $(*)$ has a maximal positive solution if $(*)$ has a positive subsolution. In order to obtain a sharp lower estimate for the maximal positive solution, we try to obtain a sharp subsolution. This can be done by analyzing the solution of the equation $\Delta u + ku = 0$. See Lemma 4.1.

By Theorem 3.2, precise nonexistence results will be obtained by the construction of suitable positive functions w on $\mathbf{R}^n$ such that $\Delta w + kw \leq 0$. These estimates will give us the asymptotic behavior of the maximal solutions of $(*)$ in terms of the asymptotic behavior of K, which is assumed to be permissible.

In the next section, we will carry out a similar analysis on the simply connected hyperbolic space $\mathbf{H}^n$. In these two sections, the analysis is rather lengthy and the results are stated in a more complicated manner. However, one of the main consequences of those results is the following:

Theorem A. *Let M be either the n-dimensional Euclidean space $\mathbf{R}^n$ $(n \geq 1)$ or the n-dimensional simply connected hyperbolic space $\mathbf{H}^n$ $(n \geq 2)$ of constant negative sectional curvature -1. Let k be a constant. Let ν be the first eigenvalue of the Laplacian on M, and let λ be the first eigenvalue of the associated linear operator $\mathbf{L} = -\Delta - k$ on M. If $\lambda < 0$, then the upper critical exponential growth rate of the* *function K for equation $(*)$ on M is ∞. If $\lambda \geq 0$ then the upper and lower critical exponential growth rate are the same and are both equal to $(p-1)(\sqrt{\nu} - \sqrt{\lambda})$.*

Recall from Definition 1.3 that the upper (lower) critical exponential growth rate of the function K for equation $(*)$ on M is defined by the extended real number

$$\sup\{\alpha\} \qquad (\inf\{\beta\}),$$

where the sup is taken for all α such that the equation $\Delta u + ku - e^{\alpha r(x)}u^p = 0$ has a positive C^2 solution on M and the inf is taken for all β such that the equation $\Delta u + ku - e^{\beta r(x)}u^p = 0$ has no positive C^2 solution on M.

Let us begin with a lemma that will give us a condition on K so that $(*)$ has positive subsolution.

Lemma 4.1. *Let u be a positive function on M satisfying $\Delta u + ku \geq \phi u$ for some positive function ϕ. Suppose K is a function on M such that $Ku^{p-1} \leq C\phi$ for some constant C. Then ϵu is a subsolution of $(*)$, if $\epsilon > 0$ is small enough.*

Proof. Let $v = \epsilon u$ with $\epsilon > 0$. Then

$$\begin{aligned}\Delta v + kv - Kv^p &\geq \phi v - Kv^p \\ &= v(\phi - \epsilon^{p-1}Ku^{p-1}) \\ &\geq \phi u(1 - C\epsilon^{p-1}) \\ &\geq 0\end{aligned}$$

if $\epsilon > 0$ is small enough. Here we have used the fact that $p > 1$. □

By Lemma 4.1 and Theorem 3.2, in order to obtain existence or nonexistence of solutions of $(*)$, it is sufficient to study the subsolutions or supersolutions of $\Delta u + ku = 0$.

The following lemma is proved by a straightforward computation.

Lemma 4.2. *Let $m \geq 1$ be an integer and let $w = e^{Ar}r^B\left(\log^{(m)} r\right)^C$, where A, B, C are constants, and r is the distance function to a fixed point on $\mathbf{R}^n$ or $\mathbf{H}^n$. Then*

$$\begin{aligned}\frac{\Delta w}{w} = &\left(A + Br^{-1} + C\tilde{b}_{m+1}^{-1}(r)\right)\Delta r - C\tilde{b}_{m+1}^{-1}(r)\left(\sum_{i=1}^{m+1}\tilde{b}_i^{-1}(r)\right) \\ &+ \left(A + Br^{-1} + C\tilde{b}_{m+1}^{-1}(r)\right)^2 - Br^{-2},\end{aligned}$$

for $r > 0$, where $\tilde{b}_k(r) = \prod_{i=0}^k \log^{(k)} r$. Here Δ is the Laplacian on $\mathbf{R}^n$ or $\mathbf{H}^n$.

Lemma 4.3. *Consider $\mathbf{R}^n$. Let k be a constant.*

(1) *If $k > 0$, then for any constants $\alpha \neq 0$, β, and integer $m > 0$, there is a positive function $v(x)$ such that*

$$\Delta v + kv \geq \frac{1}{2}kv$$

and $v(x) \sim \left(e^{\alpha r}\left(\log^{(m)} r\right)^\beta\right)^{-1/(p-1)}$, *(here and below, $r = r(x)$).*

(2) *If $k < 0$, and if $\alpha > 1$ is a constant, then there is a positive function v such that $\Delta v + kv > 0$, $\Delta v(x) + kv(x) \geq C \left(\prod_{i=0}^{m+1} \log^{(i)}(r)\right)^{-1} \cdot v(x)$ near ∞ for some positive constant C, and*

$$v(x) \sim e^{\sqrt{-k}r} r^{-(n-1)/2} \left(\log^{(m)}(r)\right)^{\beta}$$

near ∞, where $\beta = \frac{\alpha-1}{p-1} > 0$.

(3) *If $k = 0$ and $n \geq 3$, then for any positive integer m and $\beta > 0$, there is a positive function v such that $\Delta v(x) \geq C \left(r \prod_{i=0}^{m+1} \log^{(i)} r\right)^{-1} \cdot v(x)$ near infinity for some positive constant $C > 0$, $\Delta v > 0$, and $v(x) \sim \left(\log^{(m)} r\right)^{\beta}$.*

Proof. To prove (1), let $w_1(x) = \left(e^{\alpha r}\left(\log^{(m)} r\right)^{\beta}\right)^{-1/(p-1)}$. By Lemma 4.2,

$$\begin{aligned}\frac{\Delta w_1}{w_1} &= \left(-\frac{\alpha}{p-1} + C\tilde{b}_{m+1}^{-1}(r)\right) \cdot \frac{n-1}{r} - C\tilde{b}_{m+1}^{-1}(r)\left(\sum_{i=1}^{m+1} \tilde{b}_i^{-1}(r)\right) \\ &\quad + \left(\frac{\alpha}{p-1} + C\tilde{b}_{m+1}^{-1}(r)\right)^2 \\ &\geq 0\end{aligned}$$

if r is large enough, because $\alpha \neq 0$, $p > 1$, and $\tilde{b}_i(r) \to \infty$ as $r \to \infty$ for all $i \geq 1$. We extend w_1 to be a positive smooth function on $\mathbf{R}^n$. On the other hand, if $\epsilon > 0$ and $w_2 = e^{-\epsilon r^2}$, then

$$\frac{\Delta w_2}{w_2} = \epsilon(-\Delta(r^2) + \epsilon^2|\nabla(r^2)|^2 \geq -2n\epsilon.$$

If we choose $2n\epsilon = \frac{k}{4}$, then $\Delta w_2 \geq -\frac{k}{4} w_2$. Hence if we choose $\gamma > 0$ large enough and $\delta > 0$ small enough, and let $v = \delta(w_1 + \gamma w_2)$, then v will be the required function.

To prove (2), let $\epsilon = \sqrt{-k} > 0$, $t = \theta \ln \cosh(\epsilon r(x))$, and define

$$v(x) = \delta \cosh(\epsilon r)(N + r^2)^{-(n-1)/4}(a + a_m(t))^{\beta},$$

where θ, $\delta > 0$ and $N > 0$ are constants to be chosen. Here $a_1(t) = \log(a + t)$ and $a_m(t) = \log(a + a_{m-1}(t))$, where a is a large positive constant, so that $a_m(t)$ is defined and smooth for all $t \geq 0$. If we set $b_i(t) = 1/a_i'(t)$, direct computation gives

$$\begin{aligned}v^{-1}(\Delta v + kv) &= (n-1)N\epsilon r^{-1}(N + r^2)^{-1}\frac{\sinh(\epsilon r)}{\cosh(\epsilon r)}(1 + \beta\theta b_{m+1}^{-1}(t)) \\ &\quad + \beta\theta\epsilon^2 \frac{\sinh^2(\epsilon r)}{\cosh^2(\epsilon r)} b_{m+1}^{-1}(t)\{1 - \theta(\sum_{i=1}^{m+1} b_i^{-1}(t) - \beta b_{m+1}^{-1}(t))\} \\ &\quad + \beta\theta\epsilon^2 b_{m+1}^{-1}(t) - \frac{n-1}{4}(N + r^2)^{-2}(2nN + (n-3)r^2) \\ &\geq \beta\theta\epsilon^2 \frac{\sinh^2(\epsilon r)}{\cosh^2(\epsilon r)} b_{m+1}^{-1}(t)\{1 - \theta(\sum_{i=1}^{m+1} b_i^{-1}(t) - \beta b_{m+1}^{-1}(t))\} \\ &\quad + \{\beta\theta\epsilon^2 b_{m+1}^{-1}(t) - \frac{n-1}{4}(N + r^2)^{-2}(2nN + (n-3)r^2)\}.\end{aligned}$$

We can first choose $\theta > 0$ small enough that the first term in the last inequality above is nonnegative. Next we can choose $N > 0$ large enough that the second term is positive and

$$v^{-1}(\Delta v + kv) \geq \frac{1}{2}\beta\theta\epsilon^2 b_{m+1}^{-1}(t).$$

This is possible since $\lim_{r\to\infty} r^2 b_{m+1}^{-1}(\theta \ln\cosh(\epsilon r)) = \infty$. This completes the proof of (2).

To prove (3), let $A = B = 0$ and $C = \beta$ in Lemma 4.1. Let $w_1 = \left(\log^{(m)} r\right)^{\beta}$. Then w_1 satisfies

$$(w_1)^{-1}\Delta w_1 = \beta\tilde{b}_{m+1}^{-1}\left(\Delta r - \sum_{i=1}^{m+1}\tilde{b}_i^{-1}(r)\right).$$

Since $n \geq 3$, by the definition of $\tilde{b}_i$ it is easy to see that

$$(w_1)^{-1}\Delta w_1 \geq C\left(r\tilde{b}_{m+1}(r)\right)^{-1}$$

if $r \geq R$ for some R. Extend w_1 to be positive inside $B(R)$, and let $f \geq 0$ be a smooth function with compact support, such that $f \geq -\Delta w_1 + 1$ in $B(R)$. Since $n \geq 3$, we can find a bounded positive function w_2 such that $\Delta w_2 = f$. Then $v = w_1 + w_2$ is the required function. □

The following lemma will be used to find a condition on K so that $(*)$ has no positive subsolution.

Lemma 4.4. *Consider* $\mathbf{R}^n$. *Let* $k < 0$ *be a constant. Then for any* $\delta > 0$, *there is a positive* w *such that* $\Delta w + kw \leq 0$, $|\nabla \log w|$ *is bounded and* $w \sim e^{\sqrt{-k}r} r^{-\frac{n-1}{2}}\left(\log^{(m+2)} r\right)^{-\delta}$, *where* $r = r(x)$.

Proof. Let $w_1 = e^{\sqrt{-k}r} r^{-\frac{n-1}{2}}\left(\log^{(m+2)}(r)\right)^{-\delta}$, where $\delta > 0$ is a constant. By Lemma 4.2, we have

$$\begin{aligned}\frac{\Delta w_1}{w_1} &= \Big(\sqrt{-k} - \frac{n-1}{2}r^{-1} + \delta\tilde{b}_{m+3}^{-1}(r)\Big)\cdot\frac{n-1}{r} + \frac{n-1}{2}r^{-2} \\ &\quad - \delta\tilde{b}_{m+3}^{-1}(r)\left(\sum_{i=1}^{m+3}\tilde{b}_i^{-1}(r)\right) + \Big(\sqrt{-k} - \frac{n-1}{2}r^{-1} - \delta\tilde{b}_{m+3}^{-1}\Big)^2 \\ &\quad - \frac{(n-1)(n-2)}{2}r^{-2} + \frac{(n-1)^2}{4}r^{-2} + \delta\tilde{b}_{m+3}^{-1}(r)\left(\sum_{i=1}^{m+3}\tilde{b}_i^{-1}(r)\right) \\ &\quad - k + \delta^2\tilde{b}_{m+3}^{-2} - 2\sqrt{-k}\,\delta\tilde{b}_{m+3}^{-1}\end{aligned}$$

where $\tilde{b}_i$ is as in Lemma 4.2. Since r^{-2} is small compared with $\tilde{b}_{m+3}^{-1}(r)$, and $\tilde{b}_i(r) \to \infty$, we see that

$$\frac{\Delta w_1}{w_1} + k \leq 0,$$

if r is large enough. Let us extend w_1 to be a positive smooth function on $\mathbf{R}^n$. Since $k < 0$, therefore if $\gamma > 0$ is a large enough constant, and if $w = w_1 + \gamma$, then $\Delta w + kw \leq 0$ in $\mathbf{R}^n$. It is easy to see that $|\nabla \log w|$ is bounded. □

Theorem 4.5. *Let $k > 0$ be a constant. Assume that $K \geq 0$ is a continuous function on $\mathbf{R}^n$. For any constants α, β, and a positive integer m, let*

$$K_{\alpha,\beta,m}(r) = e^{\alpha r}\left(\log^{(m)} r\right)^{\beta}.$$

1. *Suppose there are constants $C > 0$, α, β and a positive integer m such that*
$$K(x) \geq CK_{\alpha,\beta,m}(r(x))$$
near ∞. Then there is a positive constant $N > 1$ such that each positive C^2 solution u of equation $(*)$ on $\mathbf{R}^n$ is bounded from above by
$$u(x) \leq NK_{\alpha,\beta,m}^{-1/(p-1)}(r(x))$$
near ∞.
2. *If there are constants $C > 0$, α, β, and a positive integer m such that*
$$K(x) \leq CK_{\alpha,\beta,m}(r(x))$$
near ∞, then $(*)$ has a positive subsolution $v(x)$ such that
$$v(x) \sim C_1 K_{\alpha,\beta,m}^{-1/(p-1)}(r(x))$$
near ∞ for some positive constant C_1. If K is also a permissible function for equation $(*)$ on $\mathbf{R}^n$, then $(*)$ has a positive C^2 solution u on $\mathbf{R}^n$ such that $u(x) \geq v(x)$.
3. *Suppose that K is also a permissible function for equation $(*)$ on $\mathbf{R}^n$. If $K(x) \sim CK_{\alpha,\beta,m}(r(x))$ near ∞ for some positive constant C, then equation $(*)$ has a positive maximal C^2 solution u such that*
$$u(x) = C(x)K_{\alpha,\beta,m}^{-1/(p-1)}(r(x))$$
near ∞ for some positive function $C(x)$ which is bounded from above and below by positive constants.

Proof. (1) Let $t = \ln\cosh r(x)$. Let v be the function

$$v(x) = \left(\cosh^{\alpha} t(a + a_m(t))^{\beta}\right)^{-\frac{1}{p-1}}.$$

Here $a_m(t)$ is defined as in the proof of Lemma 4.3. That is, $a_1(t) = \log(c+t)$ and $a_{j+1}(t) = \log(c + a_j(t))$, with c a large positive constant, so that $a_m(t)$ is smooth for $t \geq 0$. It is easy to see that $v^{-1}\Delta v$ and $|\nabla \log v|$ are uniformly bounded from above. By the assumption on K, we have $Kv^{p-1} \geq C$ near infinity for some positive constant $C > 0$. Hence (1) follows from Theorem 3.3.

(2) follows from Lemma 4.1, 4.3.(1), and Theorem 1.2.

(3) follows immediately from (1) and (2). □

Theorem 4.6. *Let $k < 0$ be a constant. Assume that $K \geq 0$ is a continuous function on $\mathbf{R}^n$. Let $\phi_k(r) = r^{-(n-1)/2}e^{\sqrt{-k}r}$ and let*

$$K_m(r) = \phi_k^{1-p}(r)\left(\prod_{i=0}^{m}\log^{(i)} r\right)^{-1}.$$

1. *If there exist a positive constant $C > 0$ and a positive integer m such that*
$$K(x) \geq C\left(\log^{(m+1)} r(x)\right)^{-1} K_m(r(x))$$
near ∞, then equation $(*)$ has no positive C^2 solution (subsolution) on $\mathbf{R}^n$.

2. *If there exist positive constants $\alpha > 1$, $C > 0$, and a positive integer m such that*

$$K(x) \le C \left(\log^{(m+1)} r(x)\right)^{-\alpha} K_m(r(x))$$

near ∞, then $(*)$ has a positive subsolution v such that

$$v(x) \sim C_1 \left(\log^{(m+1)} r(x)\right)^{\frac{\alpha-1}{p-1}} \phi_k(r(x))$$

near ∞, where C_1 is a positive constant. Moreover, if K is also permissible, then $(*)$ has a positive maximal solution u on $\mathbf{R}^n$ such that $u(x) \ge v(x)$.

3. *If*

$$K(x) \ge C \left(\log^{(m+1)} r(x)\right)^{-\alpha} K_m(r(x))$$

near ∞ for some positive constants $\alpha > 1$, $C > 0$, and a positive integer m, then for each positive C^2 solution u of equation $(*)$ on $\mathbf{R}^n$ there is a positive constant $N > 1$ such that

$$u(x) \le N \left(\log^{(m+1)} r(x)\right)^{\frac{\alpha-1}{p-1}} \phi_k(r(x))$$

near ∞.

4. *If K is permissible and*

$$K(x) \sim C \left(\log^{(m+1)} r(x)\right)^{-\alpha} K_m(r(x))$$

near ∞ for some positive constants $\alpha > 1$, $C > 0$, and a positive integer m, then $(*)$ has a maximal solution u such that

$$u(x) = C(x) \left(\log^{(m+1)} r(x)\right)^{\frac{\alpha-1}{p-1}} \phi_k(r(x))$$

near ∞, where $C(x)$ is a positive function bounded both from above and from below by positive constants.

Proof. (1) follows from Theorem 3.2 and Lemma 4.4. (2) follows from Lemma 4.1 and Lemma 4.3. The proof of (3) is similar to the proof in Theorem 4.5(1). (4) follows from (2) and (3). □

Now we assume that $k = 0$. Then equation $(*)$ reduces to

$$\Delta u - Ku^p = 0. \tag{4.1}$$

This equation has been studied extensively by many authors (cf. [7], [12], [19], [23], [28], [29], etc). The first main existence result is due to W. M. Ni [29], who proved that if $K(x)$ is a Hölder continuous nonnegative function on $\mathbf{R}^n$, $x = (x_1, x_2) \in \mathbf{R}^{n-m} \times \mathbf{R}^m = \mathbf{R}^n$ for some $m \ge 3$, and

$$K(x_1, x_2) \le C|x_2|^{-l}$$

for $|x_2|$ large, $C > 0$, and $l > 2$, then equation (4.1) has infinitely many bounded solutions in $\mathbf{R}^n$ with positive lower bounds. This result was improved later by M. Naito [28]. Complementary nonexistence results are proved by W. M. Ni [29] and F. H. Lin [23]. Some more general nonexistence results were obtained subsequently by K. S. Cheng and J. T. Lin [7]. Our approach in this paper is quite different from theirs. Besides being able to treat the general equation $(*)$, both our existence and our nonexistence results for equation (4.1) are in many cases stronger than their results. Our main purpose here for $k = 0$ is to construct sharp subsolutions so that the results can later be generalized to manifolds with nonnegative Ricci curvature by a comparison argument.

Theorem 4.7. *Let K be a nonnegative continuous function on $\mathbf{R}^n$. Let m be a positive integer and $k = 0$. Let $r = r(x)$.*

1. *Assume that there exists a positive constant $C > 0$ such that*
$$K(x) \geq C \left(r \prod_{i=0}^{m+1} \log^{(i)} r \right)^{-1}$$
near ∞. Then equation (4.1) has no positive C^2 solution (subsolution) on $\mathbf{R}^n$.
2. *Assume that $n \geq 3$ and there exist positive constants $\alpha > 1$ and $C > 0$ such that*
$$K(x) \leq C \left(\log^{(m+1)} r \right)^{-\alpha} \left(r \prod_{i=0}^{m} \log^{(i)} r \right)^{-1}.$$
Then (4.1) has a positive subsolution v such that $v(x) \sim C_1 \left(\log^{(m+1)} r \right)^{\frac{\alpha-1}{p-1}}$ for some positive constant C_1. Moreover, if K is also permissible, then (4.1) has a positive maximal C^2 solution which is bounded from below by the subsolution v.

Proof. (1) follows from Theorem 3.2 and the fact that the function $w = 1$ satisfies $\Delta w = 0$. (2) follows from Lemma 4.1 and Lemma 4.3. □

One should notice that the existence result in Theorem 4.7 is not true for both $n = 1$ and $n = 2$, and the nonexistence result in Theorem 4.7 is not sharp for both $n = 1$ and $n = 2$.

To obtain sharp existence and nonexistence results for the two exceptional cases $n = 1$ and $n = 2$, we first analyze the case $n = 1$. Then equation (4.1) is reduced to

$$u'' - Ku^p = 0 \tag{4.2}$$

on the real line $\mathbf{R}$.

Theorem 4.8. *Let K be a nonnegative continuous function on the real line $\mathbf{R}$.*

1. *If*
$$K(x) \geq C(|x|)^{-p} \left(\prod_{i=0}^{m+1} \log^{(i)} |x| \right)^{-1}$$
near ∞ for some positive integer m and positive constant C, then equation (4.2) has no positive C^2 solution (subsolution) on the real line $\mathbf{R}$.
2. *Let K satisfy one of the following two conditions:*
   (a) *There are positive constants $\alpha > 1$ and C such that*
$$K(x) \leq C|x|^{-p} \left(\log^{(m+1)} |x| \right)^{-\alpha} \left(\prod_{i=0}^{m} \log^{(i)} |x| \right)^{-1}$$
near ∞.

(b) There are constants $\alpha \neq 0$, β, and $C > 0$ such that

$$K(x) \leq Ce^{\alpha x}|x|^{\beta}$$

near ∞.

Then equation (4.2) has a positive C^2 subsolution on the real line $\mathbf{R}$. Moreover, if K is also a permissible function on $\mathbf{R}$, then equation (4.2) has a positive maximal C^2 solution on $\mathbf{R}$.

Proof. To prove (1), suppose that there is a positive C^2 subsolution u on $\mathbf{R}$. For $r \geq 0$, let

$$w(r) = 1/2\,(u(r) + u(-r)).$$

By the assumption on K and u, we have

$$w'' \geq 0,$$

$$w'(0) = 0,$$

and, for r sufficiently large,

$$w''(r) \geq Cr^{-p}\left(\prod_{i=0}^{m+1} \log^{(i)} r\right)^{-1} w^p(r). \tag{4.3}$$

Therefore, there exist positive constants r_1 and δ such that, for $r \geq r_1$, we have $w'(r) \geq \delta$, $w(r) \geq \delta r$. Hence,

$$\begin{aligned} w'(r) &\geq \int_{r_1}^{r} w'' dt \\ &\geq C\delta^p \int_{r_1}^{r} \left(\prod_{i=0}^{m+1} \log^{(i)} t\right)^{-1} dt \\ &= C\delta^p \left(\log^{(m+2)} r - \log^{(m+2)} r_1\right). \end{aligned}$$

It follows that $\lim_{r\to\infty} w'(r) = \infty$ and $\lim_{r\to\infty} r^{-1}w(r) = \infty$. Define a positive C^2 function v on $[r_1, \infty)$ by $v(r) = r^{-1}w(r)$. Since $\lim_{r\to\infty} v(r) = \infty$, there exists a constant $r_0 \geq r_1$ such that $v'(r_0) > 0$. Since $w(r) = rv(r)$ and $w''(r) = rv''(r) + 2v'(r)$, it follows from (4.3) that

$$\begin{cases} v'' + 2r^{-1}v' \geq Cr^{-1}\left(\prod_{i=0}^{m+1} \log^{(i)} r\right)^{-1} v^p, \\ v'(r_0) > 0. \end{cases}$$

The maximum principle implies that v attains no local maximum on $[r_0, \infty)$. Since $v'(r_0) > 0$, v must be strictly increasing on $[r_0, \infty)$ and v attains a strict maximum at ∞. As in the proof of Theorem 3.2, we can show that v is bounded, and this is a contradiction. To prove (2), let $\beta = \frac{\alpha-1}{p-1}$ and $t = \ln\cosh x$. For $\delta > 0$ sufficiently small and $a \geq e$ sufficiently large, it is easy to verify that, if K satisfies the condition in (a), then

$$u(x) = \delta(1 + x^2)^{1/2}(a + a_m(t))^{\beta}$$

is a positive subsolution.

For part (b), let $\alpha_1 = \frac{\alpha}{p-1}$, $\beta_1 = \frac{\beta}{p-1}$, $t = \ln\cosh(\epsilon r(x))$, and let $\delta > 0$ be a small constant. Then it is easy to verify that, if K satisfies the condition in (b), then

$$u(x) = \delta e^{-\alpha_1 x}(a+t)^{-\beta_1}$$

is a positive subsolution for $\epsilon > 0$ sufficiently small. The last statement follows from Theorem 1.3. □

The following result concerns the two dimensional case.

Theorem 4.9. *Let K be a nonnegative continuous function on $\mathbf{R}^2$. Let $m \geq 2$ be a positive integer and C a positive constant. Let $r = r(x)$ be the distance function to the origin.*

1. *Assume that*

$$K(x) \geq C(\log r)^{-p}\left(r\prod_{i=0}^{m+1}\log^{(i)} r\right)^{-1}$$

near ∞. Here and below, $r = r(x)$. Then equation (4.1) has no positive C^2 solution (subsolution) on $\mathbf{R}^2$.

2. *Assume that there is a constant $\alpha > 1$ such that*

$$K(x) \leq C(\log r)^{-p}\left(\log^{(m+1)} r\right)^{-\alpha}\left(r\prod_{i=0}^{m}\log^{(i)} r\right)^{-1}.$$

Then equation (4.1) has a positive C^2 subsolution on $\mathbf{R}^2$. If K is also essentially positive on $\mathbf{R}^2$, then equation $(*)$ has a positive C^2 maximal solution on $\mathbf{R}^2$.

Proof. To prove (1), suppose that there is a positive C^2 subsolution u on $\mathbf{R}^2$ for equation (4.1). Let (r,θ) be the polar coordinates for $\mathbf{R}^2$. For $r \geq 0$, define

$$w(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta.$$

By Stokes' theorem,

$$\begin{aligned} rw'(r) &= \frac{1}{2\pi}\int_{B(r)}(\Delta u)\,rdrd\theta \\ &\geq \frac{1}{2\pi}\int_{B(r)} Ku^p\,rdrd\theta \\ &\geq 0, \end{aligned} \tag{4.4}$$

where $B(r)$ is the ball of radius r centered at the origin. Make the change of variable $r = e^s$ and let $v(s) = w(e^s)$. Then $v'(s) = rw'(r) \geq 0$ and

$$\begin{aligned} v''(s) &= r^2 w'' + rw' \\ &= r^2\Delta w \\ &\geq Cr\log^{-p} r\left(\prod_{i=0}^{m+1}\log^{(i)} r\right)^{-1} v^p \end{aligned} \tag{4.5}$$

near ∞ for some positive constant C. In view of (4.4) and (4.5), there exists a positive constant s_0 such that

$$\begin{cases} v''(s) & \geq Cs^{-p}\left(\prod_{i=0}^{m}\log^{(i)}(s)\right)^{-1} v^p(s) \\ v'(s_0) & > 0. \end{cases}$$

As in the proof of Theorem 4.8, the above equation has no positive C^2 solution. Therefore, equation (4.1) has no positive C^2 solution (subsolution) on $\mathbf{R}^2$.

To prove (2), let $\beta = \frac{\alpha-1}{p-1} > 0$ and let δ be a small positive constant. We can choose δ small enough and a large enough so that the positive C^2 function u defined by

$$u(x) = \delta(a + a_1(t))(a + a_{m+1}(t))^\beta$$

satisfies

$$\Delta u \geq \frac{1}{2}\beta u b_2^{-1}(t) b_{m+2}^{-1}(t),$$

where $t = \ln\cosh r(x)$, and $a_1(t) = \log(a+t)$, $a_j(t) = \log(a + a_{j-1}(t))$ for $j > 1$, and $b_j = 1/a_j'$. It follows that $\Delta u - Ku^p \geq 0$ for $\delta > 0$ sufficiently small, and u is a subsolution on $\mathbf{R}^2$ for equation (4.1). The second statement follows from the fact that K is permissible. □

Remark 4.1. The previous existence results can be extended to a product manifold $M = \mathbf{R} \times Y$ for some complete manifold Y. For example, for $k > 0$, if $K(x,y)$ is a nonnegative function on M such that $K(x,y) \leq Ce^{\beta r(x)}$ for some β, then $(*)$ has a positive subsolution on M. We leave the details to the readers.

Using a trick of W. M. Ni [29], the nonexistence results in this section and the next section can be slightly improved. Indeed, let u be a positive C^2 subsolution of equation $(*)$ on $\mathbf{R}^n$. Let the average of u and K on the boundary sphere of $B(r)$ be $\overline{u}(r)$ and $\overline{K}(r)$, respectively. Namely,

$$\overline{u}(r) = \frac{1}{\omega_n}\int_{S^{n-1}} u(r,\omega)\,d\omega,$$

$$\overline{K}(r) = \{\frac{1}{\omega_n}\int_{S^{n-1}} \frac{d\omega}{K(r,\omega)^{\sigma/p}}\}^{-p/\sigma},$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$, ω_n and $d\omega$ the volume and volume element of S^{n-1}, $(r,\omega) \in \mathbf{R}^+ \times S^{n-1}$ are the polar coordinates, and $1/p + 1/\sigma = 1$. Since k is a constant and $\Delta u + ku \geq Ku^p$, integrating the inequality on S^{n-1} and use the Hölder inequality yield

$$\begin{aligned} \Delta\overline{u} + k\overline{u} &= \frac{1}{\omega_n}\int_{S^{n-1}} (\Delta u + ku)\,d\omega \\ &\geq \frac{1}{\omega_n}\int_{S^{n-1}} Ku^p\,d\omega \\ &\geq \overline{K}\overline{u}^p. \end{aligned}$$

Thus, we may obtain corresponding nonexistence results if we replace K by $\overline{K}$ in the previous results.

As a second remark on the nonexistence results, recall that if $n \geq 3$, $p = \frac{n+2}{n-2}$, and $k = 0$, then $(*)$ is the equation for prescribing scalar curvature on $\mathbf{R}^n$. For $n = 2$, the equation for prescribing scalar curvature on $\mathbf{R}^2$ is of the form

$$\Delta u - Ke^u = 0.$$

It is shown in [21, 22] that there are both a global nonexistence result and a nonexistence result which is purely the property of the Euclidean end in $\mathbf{R}^2$ for the above equation. Thus, it is interesting to know whether the nonexistence results proved in this section are of a global nature. The following example shows that the nonexistence result in Theorem 4.3 is not a property of the Euclidean end in $\mathbf{R}^n$ for equation $(*)$ and therefore it should be viewed as a global nonexistence result. Similar examples can be constructed for the other nonexistence results in the next section.

Example 4.1. For any positive number β, let $K > 0$ be any smooth function on $\mathbf{R}^n$ such that $K(x) \leq Ce^{\beta r(x)}$. By Theorem 1.3.(b), K is a permissible function on $\mathbf{R}^n \backslash B(1)$, where $B(1)$ is the unit ball of radius 1 centered at the origin. Let

$$u(x) = \delta \sinh^{-\alpha} r(x).$$

It is easy to compute that

$$\begin{aligned} \Delta u &= \delta\alpha u\{(\alpha + 2 - n)\frac{\cosh^2 r(x)}{\sinh^2 r(x)} - 1\} \\ &\geq \delta\alpha u(\alpha + 1 - n). \end{aligned}$$

Hence, if $\delta > 0$ is a sufficiently small constant and $\alpha \geq n + |k| + \frac{\beta}{p-1}$, then

$$\begin{aligned} \Delta u + ku - Ku^p &\geq \delta u\{k + \alpha(\alpha + 1 - n) - K\delta^{p-1} \sinh^{-\alpha(p-1)} r\} \\ &\geq \delta u\{k + \alpha(\alpha + 1 - n) - C\delta^{p-1}\} \\ &\geq 0 \end{aligned}$$

and u is a positive smooth subsolution of equation $(*)$ on $\mathbf{R}^n \backslash B(1)$. It follows from Theorem 1.3 that equation $(*)$ has a smooth positive solution on $\mathbf{R}^n \backslash B(1)$. Since β is an arbitrary positive constant, we see that Theorem 4.6 (1) is not a property of the Euclidean end in $\mathbf{R}^n$ for equation $(*)$.

5. The structure of the equation $\Delta u + ku - Ku^p = 0$ on $\mathbf{H}^n$

In this section, we analyze the equation

$$(*) \qquad \Delta u + ku - Ku^p = 0$$

on the hyperbolic space $\mathbf{H}^n$, assuming that $n \geq 2$, k is a constant, and K is a nonnegative Hölder continuous function on $\mathbf{H}^n$. It is well known that the first eigenvalue of the Laplacian on $\mathbf{H}^n$ is $(n-1)^2/4$. Therefore, an essentially positive function K is permissible for equation $(*)$ on $\mathbf{H}^n$ if $k \leq (n-1)^2/4$. An everywhere positive Hölder continuous function K is always permissible. We also notice that

$$(5.1) \qquad \Delta r(x) = (n-1)\frac{\cosh r(x)}{\sinh r(x)}$$

on $\mathbf{H}^n$, where $r(x)$ is the distance function to some fixed point $o \in \mathbf{H}^n$. Below, $r(x)$ will often be abbreviated to r.

Almost all of the results obtained in this section are parallel to those in the previous section. All of the existence and nonexistence results in this section are extremely sharp. The only major difference from section 4 seems to be the fact that we need to use the full power of Theorem 3.2 (see Remark 3.1), in the proof of the nonexistence result for $k = (n-1)^2/4$.

Lemma 5.1. *Consider $\mathbf{H}^n$, and let k be a constant.*

1. *If $k > (n-1)^2/4$, then, for any constants α, β, and positive integer m, there exist a positive function v and a constant $\delta > 0$ such that $\Delta v + kv > \delta v$ and*
$$v(x) \sim e^{-\alpha r/(p-1)} \left(\log^{(m)} r\right)^{-\beta/(p-1)}$$
near ∞, where $r = r(x)$ is the distance from x to a fixed point in $\mathbf{H}^n$.
2. *If $k < (n-1)^2/4$, let $\sigma_k = \sqrt{(n-1)^2/4 - k} - (n-1)/2$. Then there is a positive function v such that $\Delta v + kv > 0$*
$$\Delta v(x) + kv(x) \geq C \left(\prod_{i=0}^{m+1} \log^{(i)} r\right)^{-1} \cdot v(x)$$
near ∞. If $k \leq n(n-2)/4$, then for any $\beta > 1$ we can choose v such that $v \sim e^{\sigma_k r} \left(\log^{(m)} r\right)^{(\beta-1)/(p-1)}$. If $(n-1)^2/4 > k > n(n-2)/4$, then we may choose v such that $v \sim e^{\sigma_k r} \left(\log^{(m+1)} r\right)^{\tau}$ for some large constant $\tau > 0$.
3. *If $k = (n-1)^2/4$, there is a positive function v such that $\Delta v + kv > 0$, $\Delta v + kv \geq C \left(\log r \prod_{i=0}^{m+1} \log^{(i)} r\right)^{-1}$ near ∞, and*
$$v \sim re^{-\frac{(n-1)}{2} r} \left(\log^{(m+1)} r\right)^{\beta}$$
near ∞, for some large constant β.

Proof. To prove (1), let $w_1 = e^{-\alpha r/(p-1)} \left(\log^{(m)} r\right)^{-\beta/(p-1)}$; then by Lemma 4.2 and (5.1), given any $\delta > 0$,

$$\begin{aligned}(w_1)^{-1}\Delta w_1 &\geq \left(\frac{\alpha}{p-1}\right)^2 - (n-1) \cdot \frac{\alpha}{p-1} - 2\delta \\ &\geq -\frac{(n-1)^2}{4} - 2\delta \\ &\geq -k + \delta\end{aligned}$$

for r sufficiently large and $\delta > 0$ small enough. Let $w_2 = e^{-\epsilon r^2}$ as in the above proof of Lemma 4.3; then

$$\begin{aligned}(w_2)^{-1}\Delta w_2 &= 2\epsilon \left(-\frac{(n-1)r \cosh r}{\sinh r} - 1 + 2\epsilon r^2\right) \\ &\geq 2\epsilon \left(-(n-1)(1+r) - 1 + 2\epsilon r^2\right) \\ &\geq -\frac{(n-1)^2}{4} - 2n\epsilon \\ &\geq -k + 2\delta,\end{aligned}$$

provided $\delta > 0$ and $\epsilon > 0$ are small enough, where we have used the fact that $\frac{r\cosh r}{\sinh r} \le 1 + r$. Let $\lambda > 0$ be a constant, and let $v = w_1 + \lambda w_2$; we see that, as before, if λ is large enough, then $\Delta u + ku \ge \delta u$.

To prove (2), first assume that $k \le n(n-2)/4$. For any $\beta > 1$, let $\beta_1 = (\beta - 1)/(p-1) > 0$, and let

$$v(x) = \delta \cosh^{\alpha}(\epsilon r(x))(a + a_m(t))^{\beta_1},$$

where $t = \theta \ln \cosh(\epsilon r(x))$, $\alpha = \epsilon^{-1}\sigma_k$, and with $\delta > 0$, $\theta > 0$ and $\epsilon > 0$ to be chosen. Here, as before, $a_1(t) = \log(a+t)$, $a_j(t) = \log(a + a_j(t))$, $b_j = 1/a_j'$, and a is a large positive constant, so that a_m is smooth for $t \ge 0$. It is easy to compute that

$$\begin{aligned} v^{-1}\Delta v + k \ge{}& \sigma_k\{(n-1)(\frac{\cosh r \sinh(\epsilon r)}{\sinh r \cosh(\epsilon r)} - 1) - \sigma_k \cosh^{-2}(\epsilon r)\} \\ &+ \beta_1 \theta \epsilon b_{m+1}^{-1}(t)\{\epsilon \cosh^{-2}(\epsilon r) + 2\alpha\epsilon \frac{\sinh^2(\epsilon r)}{\cosh^2(\epsilon r)} \\ &- (m+1)\theta\epsilon \frac{\sinh^2(\epsilon r)}{\cosh^2(\epsilon r)} b_1^{-1}(t) + (n-1)\frac{\cosh r \sinh(\epsilon r)}{\sinh r \cosh(\epsilon r)}\}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &(n-1)(\frac{\cosh r \sinh(\epsilon r)}{\sinh r \cosh(\epsilon r)} - 1) - \sigma_k \cosh^{-2}(\epsilon r) \\ &= \frac{1}{2}\cosh^{-2}(\epsilon r)\sinh^{-1} r\{(n-1)\sinh((2\epsilon - 1)r) - (2\sigma_k + n - 1)\sinh r\}. \end{aligned} \tag{5.3}$$

If $k < 0$, then $\sigma_k > 0$ and $\alpha = \epsilon^{-1}\sigma_k > 0$. Now choose $\epsilon \ge \frac{\sigma_k}{n-1} + 1$ such that $\alpha \le 1$, and let $\theta = \frac{\alpha}{m+1}$. It follows from (5.2) and (5.3) that

$$\Delta v + kv \ge \frac{\beta_1}{m+1}\sigma_k^2 v b_{m+1}^{-1}(t) \ge 0,$$

where $b_j(t) = 1/a_i'(t)$, as before.

If $k = 0$, then $\sigma_k = 0$. Let $\alpha = 0$, $\epsilon = 1$, and $\theta = \frac{n-1}{2(m+1)}$. It follows from (5.2) and (5.3) that

$$\Delta v + kv \ge \frac{\beta_1 (n-1)^2}{4(m+1)} v b_{m+1}^{-1}(t) \ge 0.$$

If $n > 2$ and $0 < k \le n(n-2)/4$, then $-(n-2)/2 \le \sigma_k < 0$ and $\alpha = \epsilon^{-1}\sigma_k < 0$. Choose $\epsilon = 1/2$ and $\theta = \frac{1}{m+1}\sqrt{(n-1)^2/4 - k}$. It follows from (5.2) and (5.3) that

$$\Delta v + kv \ge \frac{\beta_1}{32(m+1)} v b_{m+1}^{-1}(t) \ge 0.$$

Now assume that $n(n-2)/4 < k < (n-1)^2/4$; then $-(n-1)/2 < \sigma_k < (2-n)/2 \le 0$ and $\alpha = \epsilon^{-1}\sigma_k < 0$. Let

$$v_1(x) = \delta \cosh^{\sigma_k} r(x)(a + a_{m+1}(t))^{\tau}.$$

Then

(5.4)

$$
\begin{aligned}
(v_1)^{-1}\Delta v_1 + k = {} & \sigma_k(1-\sigma_k)\cosh^{-2} r + n\tau\cosh^{-2} r b_{m+2}^{-1}(t) \\
& + \tau\frac{\sinh^2 r}{\cosh^2 r} b_{m+2}^{-1}(t)\{\tau b_{m+2}^{-1}(t) + 2\sqrt{(n-1)^2/4-k} - \sum_{i=1}^{m+2} b_i^{-1}(t)\} \\
& \geq \tau b_{m+2}^{-1}(t)\{2\sqrt{(n-1)^2/4-k} - \sum_{i=1}^{m+2} b_i^{-1}(t)\} + \sigma_k(1-\sigma_k)\cosh^{-2} r.
\end{aligned}
$$

Therefore, we can first choose $a \geq e$ sufficiently large that

$$\sqrt{(n-1)^2/4-k} - 2\sum_{i=1}^{m+2} b_i^{-1}(t) \geq 0,$$

and next choose $\tau > 0$ sufficiently large that

$$\tau\sqrt{(n-1)^2/4-k}\, b_{m+2}^{-1}(t) + 2\sigma_k(1-\sigma_k)\cosh^{-2} r \geq 0.$$

It follows from (5.4) that

$$\Delta v_1 + kv_1 \geq \tau\sqrt{(n-1)^2/4-k}\, v_1 b_{m+2}^{-1}(t) \geq 0.$$

To prove (3), let

$$v(x) = \delta\cosh^{-(n-1)/2}(r(x))(a+t)(a+a_{m+1}(t))^\beta,$$

where $\delta > 0$ and β are constants to be chosen, and $t = \ln\cosh r(x)$. Here a_{m+1} is as before, and a is a large constant. Direct computation yields

$$
\begin{aligned}
v^{-1}\Delta v + k \geq {} & \frac{1}{4}\beta b_1^{-1}(t) b_{m+2}^{-1}(t) + \frac{1}{4}\{\beta b_1^{-1}(t) b_{m+2}^{-1}(t) - (n^2-1)\cosh^{-2} r\} \\
& + \frac{1}{2}\beta\frac{\sinh^2 r}{\cosh^2 r} b_{m+2}^{-1}(t)\{b_1^{-1}(t) - 2\sum_{i=2}^{m+2} b_i^{-1}(t)\}.
\end{aligned} \tag{5.5}
$$

Now we can first choose $a \geq e$ so large that the last term in (5.5) is nonnegative and then choose β so large that the term before the last in (5.5) is nonnegative. The lemma is proved. □

Lemma 5.2. *Let k be a constant.*

1. *If $k < (n-1)^2/4$, let $\sigma_k = \sqrt{(n-1)^2/4-k} - (n-1)/2$. There is a positive function w $\Delta w + kw \leq 0$, such that $|\nabla \log w|$ is bounded and $w \sim e^{\sigma_k r}$ near infinity.*
2. *If $k = (n-1)^2/4$, then there exists a positive function w such that $\Delta w + kw \leq 0$, $\Delta r + \langle \nabla r, \nabla \log w\rangle \leq Cr^{-1}$ near infinity, and $w(x) \sim re^{-\frac{n-1}{2}r}$. Here $r = r(x)$.*

Proof. To prove (1), consider the function w defined by

$$w(x) = \cosh^\alpha(\epsilon r(x))$$

where $\alpha = \sigma_k/\epsilon$ and $0 < \epsilon \leq 1$ is to be determined. Since $k = -\sigma_k(\sigma_k + n - 1)$, we have

$$w^{-1}\Delta w + k = \sigma_k\{(\epsilon - \sigma_k)\cosh^{-2}(\epsilon r) + (n-1)(\frac{\cosh r\sinh(\epsilon r)}{\sinh r\cosh(\epsilon r)} - 1)\}. \tag{5.6}$$

If $k < 0$, then $\sigma_k = \sqrt{(n-1)^2/4 - k} - \frac{n-1}{2} > 0$. If we also have $\sigma_k \geq 1$, then choose $\epsilon = 1$. (5.6) becomes

$$w^{-1}\Delta w + k = \sigma_k(1-\sigma_k)\cosh^{-2} r \leq 0. \tag{5.7}$$

Otherwise, $0 < \sigma_k < 1$; choose $\epsilon = \sigma_k < 1$. Then

$$w^{-1}\Delta w + k = \sigma_k(n-1)\left(\frac{\cosh r \sinh(\epsilon r)}{\sinh r \cosh(\epsilon r)} - 1\right) \leq 0$$

since $\frac{\sinh(\epsilon r)}{\cosh(\epsilon r)}$ is an increasing function in ϵ.

If $0 \leq k < (n-1)^2/4$, then $\sigma_k \leq 0$. Choose $\epsilon = 1$; then the inequality (5.7) remains true. Hence, in any case, we can choose the constant $0 < \epsilon \leq 1$ such that (5.6) is nonpositive. It is easy to see that $|\nabla \ln w| \leq |\alpha\epsilon| = |\sigma_k|$.

To prove (2), let w be the function defined by

$$w(x) = \cosh^{-(n-1)/2}(r(x))(a+t),$$

where $t = \ln\cosh r(x)$ and a is a positive constant. It is easy to see that, for $a \geq \frac{4n}{n^2-1}$,

$$w^{-1}\Delta w + k = -\frac{1}{4}\cosh^{-2} r\{n^2 - 1 - 4n(a+t)^{-1}\} \leq 0.$$

Since

$$\nabla \ln w(x) = \frac{\sinh(r(x))}{\cosh(r(x))}\{-(n-1)/2 + (a+t)^{-1}\}\nabla r(x),$$

it follows from (5.1) that

$$\Delta r(x) + 2\langle \nabla r(x) \cdot \nabla \log w(x)\rangle = 2\frac{\sinh(r(x))}{\cosh(r(x))}(a+t)^{-1} \leq 3r^{-1}(x)$$

for $r(x)$ sufficiently large. The result follows. □

By Lemmas 5.1, 5.2, 4.1, and Theorems 3.1 and 3.2, we have the following existence and nonexistence results as in §4, except in the proof of Theorem 5.5.1 we have to use Remark 3.1 instead of Theorem 3.2. We omit the details of the proofs.

Theorem 5.3. *Let $k > (n-1)^2/4$ be a constant and let $K \geq 0$ be a continuous function on $\mathbf{H}^n$. For any constants α, β, and any positive integer m, let*

$$\phi_{\alpha,\beta,m}(r) = e^{\alpha r}\left(\log^{(m)} r\right)^{\beta}.$$

1. *If there are constants $C > 0$, α, β, and a positive integer m such that*

$$K(x) \leq C\phi(r(x))$$

near ∞, then $(*)$ has a positive subsolution u and a positive constant C_1 such that

$$u(x) \sim C_1\phi^{-1/(p-1)}(r(x)).$$

2. *Assume that K is a permissible function for equation $(*)$ on $\mathbf{H}^n$ and*

$$K(x) \geq C\phi(r(x))$$

near ∞ for some constants α, β, $C > 0$ and a positive integer m. Then every positive C^2 solution u of equation $(*)$ on $\mathbf{H}^n$ is bounded from above by $u(x) \leq N\phi^{-1/(p-1)}(r(x))$ near ∞ for some constant $N > 0$.

3. *If K is permissible, and $K(x) \sim C\phi(r(x))$ near ∞, then equation $(*)$ has a positive C^2 maximal solution u such that $u(x) \sim C_1\phi^{-1/(p-1)}(r(x))$ near ∞ for some positive constant C_1.*

Theorem 5.4. *Let $k < (n-1)^2/4$ be a constant and let $\sigma_k = \sqrt{(n-1)^2/4 - k} - \frac{n-1}{2}$. Let $K \geq 0$ be a continuous function on $\mathbf{H}^n$. For each positive integer m, let*

$$H_m(r) = \left(e^{(p-1)\sigma_k r} \prod_{i=0}^{m} \log^{(i)} r \right)^{-1}.$$

1. *Suppose*

$$K(x) \geq C \left(\log^{(m+1)} r(x) \right)^{-1} H_m(r(x))$$

near ∞ for some positive constant C and some positive integer m. Then equation $(*)$ has no positive C^2 subsolutions (solutions) on $\mathbf{H}^n$.

2. *Suppose*

$$K(x) \leq C \left(\log^{(m+1)} r(x) \right)^{-\alpha} H_m(r(x))$$

near ∞ for some positive constants C, $\alpha > 1$ and some positive integer m. Then $(*)$ has a positive subsolution u such that if $k \leq n(n-2)/4$, then $u(x) \sim C_1 e^{\sigma_k r(x)} \left(\log^{(m+1)} r(x) \right)^{(\alpha-1)/(p-1)}$, and if $(n-1)^2/4 > k > n(n-2)/4$, then $u(x) \sim C_1 e^{\sigma_k r(x)} \left(\log^{(m+2)} r(x) \right)^{\tau}$ for some constants $C_1 > 0$ and $\tau > 0$.

3. *Suppose*

$$K(x) \geq C \left(\log^{(m+1)} r(x) \right)^{-\alpha} H_m(r(x))$$

near ∞ for some positive constants C, $\alpha > 1$ and some positive integer m. Then there is a positive constant $N > 1$ such that every positive C^2 solution u of equation $(*)$ is bounded from above by

$$u(x) \leq N e^{\sigma_k r(x)} \left(\log^{(m+1)} r(x) \right)^{(\alpha-1)/(p-1)}.$$

.

4. *If K is also permissible, and*

$$K(x) \sim C \left(\log^{(m+1)} r(x) \right)^{-\alpha} H_m(r(x)),$$

then $(*)$ has a C^2 maximal solution u and a positive constant C_1 such that

$$u(x) \sim C_1 e^{\sigma_k r(x)} \left(\log^{(m+1)} r(x) \right)^{(\alpha-1)/(p-1)}$$

if $k \leq n(n-2)/4$; and

$$N^{-1} e^{\sigma_k r(x)} \left(\log^{(m+2)} r(x) \right)^{\tau} \leq u(x) \leq N e^{\sigma_k r(x)} \left(\log^{(m+1)} r \right)^{(\alpha-1)/(p-1)}$$

near ∞, for some large constants $N \geq 1$ and $\tau > 0$, if $(n-1)^2/4 > k > n(n-2)/4$.

Theorem 5.5. *Let $k = (n-1)^2/4$. Assume that K is a nonnegative continuous function on $\mathbf{H}^n$. Let $\psi_n(r) = re^{-\frac{(n-1)}{2}r}$. For each positive integer m, let*

$$H_{n,m}(r) = \psi_n^{1-p}(r)\left(r\prod_{i=0}^{m}\log^{(i)} r\right)^{-1}.$$

1. *Suppose*

$$K(x) \geq C\left(\log^{(m+1)} r(x)\right)^{-1} H_{n,m}(r(x))$$

near ∞ for some positive constant C and some positive integer m. Then equation $(*)$ has no positive C^2 subsolutions (solutions) on $\mathbf{H}^n$.

2. *Suppose*

$$K(x) \leq C\left(\log^{(m+1)} r(x)\right)^{-\alpha} H_{n,m}(r(x))$$

near ∞ for some positive constants C, $\alpha > 1$, and some positive integer m. Then $(*)$ has a positive subsolution u such that

$$u(x) \sim C_1\psi(r(x))\left(\log^{(m+2)} r(x)\right)^{\beta}$$

for some positive constants C_1 and β.

3. *Suppose*

$$K(x) \geq C\left(\log^{(m+1)} r(x)\right)^{-\alpha} H_{n,m}(r(x))$$

near ∞ for some positive constants C, $\alpha > 1$, and some positive integer m. Then each positive C^2 solution u of equation $(*)$ is bounded by

$$u(x) \leq N\psi(r(x))\left(\log^{(m+1)} r(x)\right)^{(\alpha-1)/(p-1)}$$

near ∞ for some positive constant N.

4. *Suppose K is a permissible function on $\mathbf{H}^n$ such that*

$$K(x) \sim C\left(\log^{(m+1)} r(x)\right)^{-\alpha} H_{n,m}(r(x))$$

near ∞ for some positive constant C. Then equation $(*)$ has a positive C^2 maximal solution u on $\mathbf{H}^n$ such that

$$N^{-1}\psi(r(x))\left(\log^{(m+2)} r(x)\right)^{\beta} \leq u(x) \leq N\psi(r(x))\left(\log^{(m+1)} r(x)\right)^{(\alpha-1)/(p-1)}$$

near ∞ for some $N > 1$ and $\beta > 0$.

Proof of Theorem A in §4. Theorem A follows from Theorems 4.5–4.9 and 5.3–5.5. □

References

[1] P. Aviles and R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Diff. Geom. **21** (1985), 269–281. MR **87e:**53058

[2] P. Aviles and R. McOwen, *Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds,*, J. Diff. Geom. **27** (1988), 225-239. MR **89b:**58225

[3] T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), 269-296. MR **55:**4288

[4] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, New York, Springer, 1982. MR **85j:**58002

[5] J. Bland and M. Kalka, *Complete metrics conformal to the hyperbolic disc*, Proc. Amer. Math. Soc. **97** (1986), 128-132. MR **87f:**53013

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volumes I, II*, Interscience, New York, 1953,1962. MR **16:**426a; MR **25:**4216

[7] K.S. Cheng and J.T. Lin, *On the elliptic equations* $\Delta u = K(x)u^\sigma$ *and* $\Delta u = K(x)e^{2u}$, Trans. Amer. Math. Soc. **304** (1987), 639-668. MR **88j:**35054

[8] K.S. Cheng and W.M. Ni, *On the structure of the conformal Gaussian curvature equation on* $\mathbf{R}^2$, Duke Math. J. **62** (1991), 721-737. MR **92f:**35061

[9] K.S. Cheng and W.M. Ni, *On the structure of the conformal scalar curvature equation on* $\mathbf{R}^n$, Indiana Univ. Math. J. **41** (1992), 261-278. MR **93g:**35040

[10] S.Y. Cheng, *Eigenfunctions and eigenvalues of Laplacian*, AMS Proc. of Symposia in Pure Math. **27** Part 2 (1973), 185-193. MR **51:**14172

[11] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure and Appl. Math. **28** (1975), 333-354. MR **52:**6608

[12] W.Y. Ding and W.M. Ni, *On the elliptic equation* $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ *and related topics*, Duke Math. J. **52** (1985), 485-506. MR **86k:**35040

[13] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Berlin, Springer, 1983. MR **86c:**35035

[14] C.F. Gui and X.F. Wang, *The critical asymptotics of scalar curvatures of the conformal complete metrics with negative curvature*, preprint.

[15] D. Hulin and M. Troyanov, *Prescribing curvature on open surfaces*, Math. Ann. **293** (1992), 277-315. MR **93d:**53047

[16] Z.R. Jin, *A counterexample to the Yamabe problem for complete noncompact manifolds*, Lecture Notes in Math. 1306 (1988), 93-101. MR **91a:**53065

[17] Z.R. Jin, *Prescribing scalar curvatures on the conformal classes of complete metrics with negative curvature*, Trans. Amer. Math. Soc. **340** (1993), 785-810. MR **94b:**53074

[18] J. Kazdan, *Prescribing the curvature of a Riemannian manifold*, NSF-CBMS Regional Conference Lecture Notes **57** (1985). MR **86h:**53001

[19] N. Kawano, T. Kusano and M. Naito, *On the elliptic equation* $\Delta u = \phi(x)u^\gamma$ *in* $\mathbf{R}^2$, Proc. Amer. Math. Soc. 93 (1985), 73-78. MR **86e:**35053

[20] J. Kazdan and F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Diff. Geom. **10** (1975), 113-134. MR **57:**1661

[21] M. Kalka and D.G. Yang, *On conformal deformation of nonpositive curvature on noncompact surfaces*, Duke Mathematical Journal, 72 (1993), 405-430. MR **94i:**53040

[22] M. Kalka and D.G. Yang, *On nonpositive curvature functions on noncompact surfaces of finite topological type*, Indiana Univ. Math. J. 43 (1994), 775-804. MR **95j:**53060

[23] F.H. Lin, *On the elliptic equation* $D_i[A_{ij}(x)D_jU] - k(x)U + K(x)U^p = 0$, Proc. Amer. Math. Soc. **95** (1985), 219-226. MR **86k:**35041

[24] Li Ma, *Conformal deformations on a noncompact Riemannian manifold*, Math. Ann. **295** (1993), 75-80. MR **93k:**53040

[25] J. Lee and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37-91. MR **88f:**53001

[26] P. Li, L.F. Tam, and D.G. Yang, *On the elliptic equation* $\Delta u + ku - Ku^p = 0$ *on complete Riemannian manifolds and their geometric applications: II*, in preparation.

[27] R. McOwen, *On the equation* $\Delta u + K(x)e^{2u} = f$ *and prescribed negative curvature on* $\mathbf{R}^2$, J. Math. Anal. Appl. **103** (1984), 365-370. MR **86c:**58155

[28] M. Naito, *A note on bounded positive entire solutions of semilinear elliptic equations*, Hiroshima Math. J. **14** (1984), 211-214. MR **86d:**35047

[29] W.M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$*, its generalizations, and applications in geometry*, Indiana Univ. Math. J. **31** (1982), 495-529. MR **84e:**35049

[30] W.M. Ni, *On the elliptic equation* $\Delta u + Ke^{2u} = 0$ *and conformal metrics with prescribed Gaussian curvatures*, Invent. Math. **66** (1982), 343-352. MR **84g:**58107

[31] E.S. Noussair, *On the existence of solutions of nonlinear elliptic boundary value problems*, J. Diff. Equations **34** (1979), 482-495. MR **81c:**35053

[32] R. Osserman, *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. **7** (1957), 1641-1647. MR **20:**4701

[33] D.H. Sattinger, *Conformal metrics in $\mathbf{R}^2$ with prescribed Gaussian curvature*, Indiana Univ. Math. J. **22** (1972), 1-4. MR **46:**4437

[34] R. Schoen, *Conformal deformations of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. **20** (1984), 479-495. MR **86i:**58137

[35] R. Schoen, *A report on some recent progress on nonlinear problems in geometry*, Surveys in Differential Geometry (Cambridge, MA, 1990), Suppl. No. 1 to J. Differential Geom., Lehigh Univ., Bethlehem, PA (distributed by Amer. Math. Soc.), 1991, pp. 201–241. MR **92m:**53069

[36] R. Schoen and S.T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47-71. MR **89c:**58139

[37] M. Troyanov, *The Schwarz lemma for nonpositively curved Riemannian surfaces*, Man. Math. **72** (1991), 251-256. MR **92f:**53043

[38] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265-274. MR **39:**2093

[39] H. Yamabe, *On deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21-37. MR **23:**A2847

[40] D.G. Yang, *A note on complete conformal deformation on surfaces of infinite topological type*, in preparation.

[41] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201-228. MR **55:**4042

Department of Mathematics, University of California, Irvine, California 92697-3875
E-mail address: `pli@math.uci.edu`

Department of Mathematics, Chinese University of Hong Kong, Shatin, NT, Hong Kong
E-mail address: `lftam@math.cuhk.edu.hk`

Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail address: `dgy@math.tulane.edu`