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THE POSSIBLE ORDERS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract. We find specific information about the possible orders of transcendental solutions of equations of the form $f^{(n)}+p_{n-1}(z)f^{(n-1)}+\cdots+p_0(z)f=0$, where $p_0(z),p_1(z),\ldots,p_{n-1}(z)$ are polynomials with $p_0(z)\not\equiv 0$. Several examples are given.

1. Introduction

For $n\geq 2$, consider a linear differential equation of the form

$$f^{(n)}+p_{n-1}(z)f^{(n-1)}+\cdots+p_0(z)f=0, \tag{1.1}$$

where $p_0(z)$, ..., $p_{n-1}(z)$ are polynomials with $p_0(z)\not\equiv 0$. It is well known that every solution f of equation (1.1) is an entire function of finite rational order; see [7], [8], [3, pp. 199–209], [6, pp. 106–108], [9, pp. 65–67].

For equation (1.1), set

$$\lambda = 1+\max_{0\leq k\leq n-1}\frac{\deg p_k}{n-k}. \tag{1.2}$$

Let $\rho(f)$ denote the order of an entire function f. It is known [4, p. 127] that for any solution f of (1.1),

$$\rho(f)\leq\lambda. \tag{1.3}$$

Wittich obtained the following result.

Theorem A [8], [9, pp. 65–67]. *For a given equation of the form (1.1), there exists a set of positive rational numbers χ_1, χ_2, ..., χ_k, where $k\leq n$, such that if f is any transcendental solution of equation (1.1), then*

$$\rho(f)=\chi_j \tag{1.4}$$

for some j, $1\leq j\leq k$.

In his proof of Theorem A, Wittich used the method of Frobenius, the Wiman-Valiron theory, the theory of algebraic functions, and the Newton-Puiseux diagram, where the rational numbers χ_1, χ_2, ..., χ_k are determined from the Newton-Puiseux diagram. Helmrath and Nikolaus [2] and Jank and Volkmann [3, pp. 199–209] also gave a proof of Theorem A, each on a more general equation than (1.1), where their proofs use the Wiman-Valiron theory, the theory of algebraic functions, and the Newton-Puiseux diagram.

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Theorem 1 in §2 gives a list of positive rational numbers that includes all the possible orders of transcendental solutions of equation (1.1). This list of rational numbers is obtained from simple arithmetic with the degrees of the polynomial coefficients in (1.1). We do not appeal to the Newton-Puiseux diagram to obtain this list of rational numbers or to prove Theorem 1.

Some natural questions can be asked. For example:

(i) What is the maximum number of possible distinct orders of transcendental solutions of a given equation of the form (1.1)?

(ii) Consider (1.3). Is the upper bound λ always reached? In other words, for any given equation of the form (1.1), does there always exist a solution f of (1.1) that satisfies $\rho(f) = \lambda$, where λ is the constant in (1.2)?

(iii) What is the smallest possible sum of the orders of a fundamental set of solutions of a given equation of the form (1.1)?

(iv) What is the maximum number of linearly independent polynomial solutions that an equation of the form (1.1) can possess?

In this paper we answer these four questions, and we also give related results. Several examples are given to illustrate our results.

2. Statement of results

Consider equation (1.1). For convenience, set $d_j = \deg p_j$ if $p_j \not\equiv 0$ and $d_j = -\infty$ if $p_j \equiv 0,\ 0 \leq j \leq n-1$.

We define a strictly decreasing finite sequence of non-negative integers

$$s_1 > s_2 > \cdots > s_p \geq 0 \tag{2.1}$$

in the following manner. We choose s_1 to be the unique integer satisfying

$$\frac{d_{s_1}}{n-s_1} = \max_{0\leq k\leq n-1} \frac{d_k}{n-k} \quad \text{and} \quad \frac{d_{s_1}}{n-s_1} > \frac{d_k}{n-k} \quad \text{for all } 0 \leq k < s_1. \tag{2.2}$$

Then given s_j, $j \geq 1$, we define s_{j+1} to be the unique integer satisfying

$$\begin{aligned} &\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} = \max_{0\leq k< s_j} \frac{d_k - d_{s_j}}{s_j - k} > -1 \quad \text{and} \\ &\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} > \frac{d_k - d_{s_j}}{s_j - k} \quad \text{for all} \quad 0 \leq k < s_{j+1}. \end{aligned} \tag{2.3}$$

For a certain p, the integer s_p will exist, but the integer s_{p+1} will not exist, and then the sequence $s_1, s_2, \ldots, s_p$ terminates with s_p. Obviously, $p \leq n$, and we also see that (2.1) holds.

Correspondingly, define for $j = 1, 2, ..., p$,

$$\alpha_j = 1 + \frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j}, \tag{2.4}$$

where we set

$$s_0 = n \quad \text{and} \quad d_{s_0} = d_n = 0. \tag{2.5}$$

From (2.3) and (2.4), we observe that $\alpha_j > 0$ for each j, $1 \leq j \leq p$.

We mention that the integers s_1, s_2, ..., s_p in (2.1) can also be expressed in the following manner:

$$s_1 = \min\left\{ j : \frac{d_j}{n-j} = \max_{0\leq k\leq n-1} \frac{d_k}{n-k} \right\};$$

and given s_j, $j \geq 1$, we have

$$s_{j+1} = \min\left\{ i : \frac{d_i - d_{s_j}}{s_j - i} = \max_{0 \leq k < s_j} \frac{d_k - d_{s_j}}{s_j - k} > -1 \right\}.$$

We prove the following result.

Theorem 1. *For equation (1.1), the following conclusions hold:*

(i) *If f is a transcendental solution of (1.1), then $\rho(f) = \alpha_j$ for some j, $1 \leq j \leq p$.*

(ii) *If $s_1 \geq 1$ and $p \geq 2$, then the following inequalities hold:*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p \geq \frac{1}{s_{p-1} - s_p} \geq \frac{1}{s_1 - s_p} \geq \frac{1}{s_1}.$$

(iii) *If $s_1 = 0$, then any nontrivial solution f of (1.1) satisfies $\rho(f) = 1 + d_0/n$.*

Theorem 1(iii) is known [9, Chap. V]. We will, however, give a new proof of this result by using sharp estimates of logarithmic derivatives; see the remark at the end of §6.

Regarding Theorem 1(ii), in the case when $s_1 \geq 1$ and $p = 1$, we obtain from Theorem 1(i), (2.2), (2.4), and (2.5) that any transcendental solution f of (1.1) satisfies $\rho(f) = \lambda$, where λ is the constant in (1.2).

We mention that Pöschl [5] gave a detailed analysis of the possible orders of transcendental solutions of (1.1) in the case when $n = 3$, and Theorem 1 gives an improvement of this result of Pöschl.

Since $p \leq n$ in (2.4), a corollary of Theorem 1 is the known result that there can exist at most n distinct possible orders of transcendental solutions of equation (1.1); see Theorem A. However, Theorem 1 yields more than this result. We observe from (2.2) and (2.3) that the integer p satisfies $p \leq s_1 + 1$. Then from Theorem 1 and the construction of α_j in (2.4), we deduce the following result.

Corollary 1. *There can exist at most $s_1 + 1$ distinct orders of transcendental solutions of (1.1). Furthermore, if an equation of the form (1.1) possesses $s_1 + 1$ transcendental solutions that have $s_1 + 1$ distinct orders, then these $s_1 + 1$ orders must be the following numbers:*

$$1 + \frac{d_{s_1}}{n - s_1},\ 1 + d_{s_1 - 1} - d_{s_1},\ 1 + d_{s_1 - 2} - d_{s_1 - 1},\ \ldots,\ 1 + d_0 - d_1.$$

Consequently, in this case, $d_{s_1} \leq d_{s_1 - 1} \leq \cdots \leq d_1 \leq d_0$.

Examples 2 through 5 in §8 illustrate the sharpness of Corollary 1. In the special case when $s_1 = 1$, it follows from Corollary 1 and Theorem 1 that there can exist at most two distinct orders of transcendental solutions of (1.1), and these two possible orders are $1 + d_1/(n-1)$ and $1 + d_0 - d_1$.

Next, from (2.2) we see that

$$s_1 \leq n - 1. \tag{2.6}$$

Combining (2.6) with Theorem 1 yields the following result.

Corollary 2. *Every transcendental solution f of (1.1) satisfies*

$$\rho(f) \geq \frac{1}{n-1}. \tag{2.7}$$

Wittich [8], [9, pp. 65–68] proved Corollary 2, and he also gave an example to indicate that Corollary 2 is sharp for all $n \geq 2$ (see the remark at the end of Example 1 in §8). In his proof of Corollary 2, Wittich used the method of Frobenius and the Newton-Puiseux diagram, which we do not appeal to here. In Example 1 in §8, we exhibit an equation of the form (1.1) where $n = 3$, which possesses a special contour integral solution $f = G(z)$ satisfying $\rho(G) = 1/2$, which gives equality in (2.7). Of course all nontrivial solutions of equations of the form (1.1) with constant coefficients have order one, which gives equality in (2.7) when $n = 2$.

We prove the following result.

Theorem 2. *For any $j = 1, 2, ..., p$, there can exist* at most *s_j linearly independent solutions f of (1.1) satisfying $\rho(f) < \alpha_j$.*

Examples 2 through 6 in §8 illustrate the sharpness of Theorem 2.

Note that $\alpha_1 = \lambda$, where α_1 is defined in (2.4) and λ is the constant in (1.2). From (1.3), the order of any solution of (1.1) cannot be greater than λ. From (2.6) and Theorem 2 (with $j = 1$), we obtain the following result, which says that this maximum possible order λ is always reached.

Corollary 3. *For any given equation of the form (1.1), there must exist a solution of (1.1) that satisfies $\rho(f) = \lambda$, where λ is the constant in (1.2).*

Thus from (1.3) and Theorem 1, Corollary 3 shows that there always exists a solution of equation (1.1) that has the maximum possible order $\lambda = \alpha_1$. This shows, among other things, that it is not possible for an equation of the form (1.1) to have only polynomials for solutions. Moreover, we observe from Theorem 1 that any solution $f \not\equiv 0$ of (1.1) satisfying $\rho(f) < \alpha_p$ must be a polynomial. Combining this with Theorem 2 yields the following result.

Corollary 4. *There can exist at most s_p linearly independent polynomial solutions of (1.1).*

Corollary 4 is sharp; see Example 6 in §8.

Wittich proved the following result.

Theorem B [8]. *Suppose that every nontrivial solution of (1.1) is transcendental. If $\{f_1, f_2, ..., f_n\}$ is any fundamental set of solutions of (1.1), then*

$$\sum_{k=1}^{n} \rho(f_k) \geq n. \tag{2.8}$$

We can improve Theorem B by appealing to Theorem 2. Specifically, suppose that $\{f_1, f_2, ..., f_n\}$ is any fundamental set of solutions of equation (1.1), where we allow the possibility that an f_k might be a polynomial. From Theorem 2 and Corollary 4, it can be deduced that

$$\sum_{k=1}^{n} \rho(f_k) \geq (n - s_1)\alpha_1 + (s_1 - s_2)\alpha_2 + \cdots + (s_{p-1} - s_p)\alpha_p + s_p \cdot 0. \tag{2.9}$$

From (2.4), we obtain that the right side of (2.9) equals $n + d_{s_p} - s_p$, and so

$$\sum_{k=1}^{n} \rho(f_k) \geq n + d_{s_p} - s_p. \tag{2.10}$$

Now $d_{s_p} - s_p \geq d_0$, because if $s_p = 0$, then $d_{s_p} - s_p = d_0$, while if $s_p \geq 1$, then from (2.3) and the fact that s_p is the last element in the sequence $s_1, s_2, ..., s_p$, we obtain $(d_0 - d_{s_p})/(s_p - 0) \leq -1$. Combining this with (2.10) gives the following result.

Corollary 5. *If $\{f_1, f_2, ..., f_n\}$ is any fundamental set of solutions of (1.1), then*

$$\sum_{k=1}^{n} \rho(f_k) \geq n + d_0. \tag{2.11}$$

Corollary 5 is sharp. This is illustrated by Examples 2 through 6 in §8 and by the situation in Theorem 1(iii). Corollary 5 is an improvement of Theorem B because (2.11) improves (2.8), and also because nontrivial polynomial solutions of (1.1) are allowed.

In §§3–7 we give the proofs of Theorems 1 and 2. In §8 we give several examples to illustrate the sharpness of our results, and also to exhibit some possibilities that can occur.

We mention that we prove Theorem 1(ii) first (in §3), because we use Theorem 1(ii) in the proof of Lemma 4.2 in §4, and Lemma 4.2 is used in the proof of Theorem 1(i) in §5.

3. Proof of Theorem 1(ii)

We first prove $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. From (2.1), (2.2), (2.3), and (2.5), we obtain, for any $j = 1, 2, ..., p-1$,

$$s_j > s_{j+1} \qquad \text{and} \qquad \frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} > \frac{d_{s_{j+1}} - d_{s_{j-1}}}{s_{j-1} - s_{j+1}},$$

which yields

$$-d_{s_j} s_{j+1} - d_{s_{j-1}}(s_j - s_{j+1}) > d_{s_{j+1}}(s_{j-1} - s_j) - d_{s_j} s_{j-1}. \tag{3.1}$$

Adding $d_{s_j} s_j$ to both sides of (3.1) gives

$$(d_{s_j} - d_{s_{j-1}})(s_j - s_{j+1}) > (d_{s_{j+1}} - d_{s_j})(s_{j-1} - s_j),$$

$$\frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} > \frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}}. \tag{3.2}$$

From the definition of α_j in (2.4), we obtain immediately from (3.2) that $\alpha_j > \alpha_{j+1}$. This proves that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p. \tag{3.3}$$

From (2.1) we have

$$\frac{1}{s_{p-1} - s_p} \geq \frac{1}{s_1 - s_p} \geq \frac{1}{s_1}, \tag{3.4}$$

and so to complete the proof of Theorem 1(ii), we need only to prove that

$$\alpha_p \geq \frac{1}{s_{p-1} - s_p}. \tag{3.5}$$

From (2.3) we obtain

$$\frac{d_{s_p} - d_{s_{p-1}}}{s_{p-1} - s_p} > -1,$$

$$d_{s_p} - d_{s_{p-1}} + s_{p-1} - s_p > 0. \tag{3.6}$$

Since the left side of (3.6) is an integer, we have $d_{s_p} - d_{s_{p-1}} + s_{p-1} - s_p \geq 1$. Hence from (2.4), we see that (3.5) holds. By combining (3.5), (3.4), and (3.3), we obtain Theorem 1(ii).

4. Lemmas for the proof of Theorem 1(i)

We use the three lemmas in this section in the proof of Theorem 1(i).

Lemma 4.1. *For any fixed $j = 0, 1, ..., p-1$, let α be any real number satisfying $\alpha > \alpha_{j+1}$, and let k be any integer satisfying $0 \leq k < s_j$. Then*

$$n - k + d_k + k\alpha < n - s_j + d_{s_j} + s_j\alpha. \tag{4.1}$$

Proof. Since $n - k + d_k + k\alpha = (n - s_j + d_{s_j} + s_j\alpha) + \alpha(k - s_j) + d_k - d_{s_j} + s_j - k$, we obtain

$$n - k + d_k + k\alpha < (n - s_j + d_{s_j} + s_j\alpha) + \alpha_{j+1}(k - s_j) + d_k - d_{s_j} + s_j - k. \tag{4.2}$$

Now from the definition of α_{j+1} in (2.4), we obtain

$$\alpha_{j+1}(k - s_j) + d_k - d_{s_j} + s_j - k = (k - s_j)\left(\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} - \frac{d_k - d_{s_j}}{s_j - k}\right). \tag{4.3}$$

Since $0 \leq k < s_j$, it follows from the definition of s_{j+1} in (2.2) and (2.3) that

$$\frac{d_{s_{j+1}} - d_{s_j}}{s_j - s_{j+1}} \geq \frac{d_k - d_{s_j}}{s_j - k}. \tag{4.4}$$

From (4.4) and (4.3) we obtain

$$\alpha_{j+1}(k - s_j) + d_k - d_{s_j} + s_j - k \leq 0. \tag{4.5}$$

Then (4.1) follows from (4.2) and (4.5). ∎

Lemma 4.2. *For any fixed $j = 1, 2, ..., p$, let α be any real number satisfying $\alpha < \alpha_j$, and let k be any integer satisfying $s_j < k \leq n$. Then*

$$n - k + d_k + k\alpha < n - s_j + d_{s_j} + s_j\alpha. \tag{4.6}$$

Proof. We consider two separate cases.

Case (i). Suppose that $s_j < k \leq s_{j-1}$.

This case uses an argument similar to the proof of Lemma 4.1. As in the proof of Lemma 4.1, we have

$$n - k + d_k + k\alpha < (n - s_j + d_{s_j} + s_j\alpha) + \alpha_j(k - s_j) + d_k - d_{s_j} + s_j - k. \tag{4.7}$$

Then from the definition of α_j in (2.4), we obtain

$$\alpha_j(k - s_j) + d_k - d_{s_j} + s_j - k = (k - s_j)\frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} + d_k - d_{s_j}. \tag{4.8}$$

If $k = s_{j-1}$, then the right side of (4.8) equals zero. Then (4.6) follows from (4.7).

On the other hand, if $s_j < k < s_{j-1}$, then from the definition of s_j in (2.2) and (2.3), we obtain

$$\begin{aligned}(k - s_j)\frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} + d_k - d_{s_j} &= (k - s_{j-1})\left(\frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} - \frac{d_k - d_{s_{j-1}}}{s_{j-1} - k}\right) \\ &\leq 0.\end{aligned} \tag{4.9}$$

Combining (4.9), (4.8), and (4.7) gives (4.6). This proves Lemma 4.2 for Case (i).

Case (ii). Suppose that $s_{j-1} < k \le n$.

Since $s_j < s_{j-1} < \cdots < s_1 < s_0 = n$ and $s_{j-1} < k \le n$, it follows that $j \ge 2$ and there exists an integer m, $1 \le m \le j-1$, such that $s_{j-m} < k \le s_{j-m-1}$. Also, from Theorem 1(ii), which we proved in §3, we have

$$\alpha_j < \alpha_{j-1} < \cdots < \alpha_{j-m}. \tag{4.10}$$

Since $\alpha < \alpha_j$, we have $\alpha < \alpha_{j-m}$. Hence we can apply Case (i) to obtain that

$$n - k + d_k + k\alpha < n - s_{j-m} + d_{s_{j-m}} + s_{j-m}\alpha. \tag{4.11}$$

Now from successive applications of Case (i), we obtain the following inequalities:

$$\begin{gathered} n - s_{j-1} + d_{s_{j-1}} + s_{j-1}\alpha < n - s_j + d_{s_j} + s_j\alpha \qquad \text{for} \quad \alpha < \alpha_j, \\ n - s_{j-2} + d_{s_{j-2}} + s_{j-2}\alpha < n - s_{j-1} + d_{s_{j-1}} + s_{j-1}\alpha \qquad \text{for} \quad \alpha < \alpha_{j-1}, \\ \cdots \quad \cdots \quad \cdots \\ n - s_{j-m} + d_{s_{j-m}} + s_{j-m}\alpha < n - s_{j-m+1} + d_{s_{j-m+1}} + s_{j-m+1}\alpha \\ \text{for} \quad \alpha < \alpha_{j-m+1}. \end{gathered}$$

It follows from (4.10) that all of the above inequalities hold for $\alpha < \alpha_j$. Therefore, by combining these inequalities with (4.11), we obtain (4.6). This proves Case (ii), and completes the proof of Lemma 4.2. ∎

Lemma 4.3. *Let $\alpha > 0$. Then for any integer k satisfying $0 \le k < s_p$, we have*

$$n - k + d_k + k\alpha < n - s_p + d_{s_p} + s_p\alpha. \tag{4.12}$$

Proof. Since s_p is the last element in the sequence s_1, s_2, ..., s_p, it follows from the construction of s_p in (2.2) and (2.3) that for any $k < s_p$,

$$\frac{d_k - d_{s_p}}{s_p - k} \le -1.$$

This gives $d_k - k \le d_{s_p} - s_p$. Since $k\alpha < s_p\alpha$, we obtain (4.12). ∎

5. Proof of Theorem 1(i)

Let f be a transcendental (entire) solution of (1.1) with order $\rho(f)$. The statements in (5.1), (5.2), and (5.3) below are well known; see [3, pp. 199–209], [6, pp. 105–108], and [9, pp. 65–67]. We have

$$0 < \rho(f) < \infty. \tag{5.1}$$

Furthermore, if $V(r)$ denotes the central index of f, then

$$V(r) = (1 + o(1))\, Cr^{\alpha} \tag{5.2}$$

as $r \to \infty$, where $\alpha = \rho(f)$ and C is a positive constant. In addition, from the Wiman-Valiron theory it follows that there exists a set $E_0 \subset (0, \infty)$ that has finite logarithmic measure, such that for all $q = 1, 2, ..., n$ we have

$$\frac{f^{(q)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V(r)}{z_r} \right)^q \tag{5.3}$$

as $r \to \infty$, $r \notin E_0$, where z_r is a point on the circle $|z| = r$ that satisfies $|f(z_r)| = M(r, f)$. Here $M(r, f)$ denotes the usual maximum modulus function: $M(r, f) = \max_{|z|=r} |f(z)|$, $0 < r < \infty$.

Now in equation (1.1), for each k, $0 \leq k \leq n-1$, let b_k denote the leading coefficient of the polynomial $p_k(z)$, and set $a_k = C^k|b_k|$, where $C > 0$ is the constant in (5.2). Also set $a_n = C^n$. We now divide equation (1.1) by f, and then substitute (5.3) and (5.2) into (1.1). This yields an equation whose right side is zero and whose left side consists of a sum of $n+1$ terms whose absolute values are asymptotic (as $r \to \infty$, $r \notin E_0$) to the following $n+1$ terms:

$$a_n r^{n\alpha},\ a_{n-1} r^{1+d_{n-1}+(n-1)\alpha},\ \cdots,\ a_k r^{n-k+d_k+k\alpha},\ \cdots,\ a_0 r^{n+d_0}. \tag{5.4}$$

Now from (1.2) and (1.3), the order of any solution of (1.1) is at most λ, and $\lambda = \alpha_1$ from (2.4) and (2.2). Thus $\alpha \leq \alpha_1$.

Now suppose that $\alpha_{j+1} < \alpha < \alpha_j$ for some $j = 1, 2, ..., p-1$. Then from Lemma 4.1 and Lemma 4.2, we obtain that

$$n-k+d_k+k\alpha < n-s_j+d_{s_j}+s_j\alpha \qquad \text{for any} \quad k \neq s_j. \tag{5.5}$$

But from inspection of (5.5) and (5.4), we see that there will exist exactly one dominant term (as $r \to \infty$, $r \notin E_0$) in (5.4). Specifically, there exists exactly one term in (5.4) with exponent $n-s_j+d_{s_j}+s_j\alpha$, where $a_{s_j} \neq 0$, such that the exponent $n-s_j+d_{s_j}+s_j\alpha$ is greater than all the other exponents of the terms in (5.4). This is impossible.

On the other hand, suppose that $\alpha < \alpha_p$. Then from Lemma 4.2 and Lemma 4.3, we obtain

$$n-k+d_k+k\alpha < n-s_p+d_{s_p}+s_p\alpha \qquad \text{for any} \quad k \neq s_p. \tag{5.6}$$

Again, by the same reasoning, (5.6) is impossible, because otherwise (5.4) would have exactly one dominant term as $r \to \infty$, $r \notin E_0$.

Therefore, the only admissible values for α, the order of f, are α_1, α_2, ..., α_p. This proves Theorem 1(i).

6. Lemmas for the proof of Theorem 2

For the rest of the paper we make the following two conventions: (i) A meromorphic function will always be meromorphic in the whole complex plane. (ii) We will let $E = E_0 \cup [0, 1]$, where E_0 is a set in $0 < r < \infty$ that has finite logarithmic measure, and the set E may not necessarily be the same set each time it appears.

We also mention that the definition of the order of a meromorphic function is the standard definition from Nevanlinna theory (see [4, p. 24]), and this definition generalizes the definition of the order of an entire function. As with entire functions, we again use $\rho(f)$ to denote the order of a meromorphic function f.

Our proof of Theorem 2 will consist of combining the standard method of reduction of order for linear differential equations with the following result.

Lemma 6.1 [1]. *Let $f \not\equiv 0$ be a meromorphic function of finite order β, and let $k \geq 1$ be an integer. Then for any given $\varepsilon > 0$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{k(\beta-1)+\varepsilon}, \qquad |z| \notin E. \tag{6.1}$$

We remark that the estimate (6.1) is sharp in the sense that we cannot replace (6.1) with the statement '$|f^{(k)}(z)/f(z)| \leq C\,|z|^{k(\beta-1)}$, where $C > 0$ is some constant' (see [1, §9]). We need this sharpness in our proof of Theorem 2, and also in our new proof of Theorem 1(iii).

Lemmas 6.2 to 6.5 below are concerned with the method of reduction of order.

Lemma 6.2. *Let f_1, f_2, ..., f_N $(N \geq 2)$ be N linearly independent meromorphic functions. Set $h_j = (f_{j+1}/f_1)'$ for $j = 1, 2, ..., N-1$. Then h_1, h_2, ..., h_{N-1} are $N-1$ linearly independent meromorphic functions.*

Proof. Suppose that h_1, h_2, ..., h_{N-1} are linearly dependent. Then there exist $N-1$ constants $c_1, c_2, ..., c_{N-1}$, which are not all zero, such that

$$c_1 h_1 + c_2 h_2 + \cdots + c_{N-1} h_{N-1} \equiv 0,$$

i.e.,

$$\frac{d}{dz}\left(\frac{c_1 f_2 + c_2 f_3 + \cdots + c_{N-1} f_N}{f_1}\right) \equiv 0.$$

It follows that there exists some constant c_0 such that

$$c_0 f_1 + c_1 f_2 + c_2 f_3 + \cdots + c_{N-1} f_N \equiv 0.$$

Hence, f_1, f_2, ..., f_N are linearly dependent, which contradicts our assumption. ■

Lemma 6.3. *Let f and g be two linearly independent meromorphic solutions of an equation of the form*

$$y^{(n)} + A_{n-1}(z) y^{(n-1)} + \cdots + A_1(z) y' + A_0(z) y = 0, \tag{6.2}$$

where $A_0(z)$, $A_1(z)$, ..., $A_{n-1}(z)$ are meromorphic functions. Set $u = (f/g)'$. Then $y = u(z)$ satisfies the equation

$$y^{(n-1)} + B_{n-2}(z) y^{(n-2)} + \cdots + B_1(z) y' + B_0(z) y = 0,$$

where

$$B_j(z) = \sum_{k=j+1}^{n} \binom{k}{j+1} A_k(z) \frac{g^{(k-j-1)}(z)}{g(z)}, \qquad j = 0, 1, ..., n-2.$$

Here $\binom{k}{j+1}$ denotes the binomial coefficient, and $A_n(z) \equiv 1$.

Proof. Set $v = f/g$. By substituting $f = vg$ into equation (6.2), and noting that $u = v'$, we deduce the result. ■

Lemma 6.4. *Let $f_{0,1}$, $f_{0,2}$, ..., $f_{0,m}$ be $m \geq 2$ linearly independent meromorphic solutions of an equation of the form*

$$y^{(n)} + A_{0,n-1}(z) y^{(n-1)} + \cdots + A_{0,0}(z) y = 0, \qquad n \geq m, \tag{6.3}$$

where $A_{0,0}(z), ..., A_{0,n-1}(z)$ are meromorphic functions. For $1 \leq q \leq m-1$, set

$$f_{q,j} = \left(\frac{f_{q-1,j+1}}{f_{q-1,1}}\right)', \qquad j = 1, 2, ..., m-q.$$

Then $f_{q,1}$, $f_{q,2}$, ..., $f_{q,m-q}$ are $m-q$ linearly independent meromorphic solutions of the equation

$$y^{(n-q)} + A_{q,n-q-1}(z) y^{(n-q-1)} + \cdots + A_{q,0}(z) y = 0, \tag{6.4}$$

where

$$A_{q,j}(z) = \sum_{k=j+1}^{n-q+1} \binom{k}{j+1} A_{q-1,k}(z) \frac{(f_{q-1,1})^{(k-j-1)}(z)}{f_{q-1,1}(z)} \tag{6.5}$$

for $j = 0, 1, ..., n-q-1$. Here we set $A_{k,n-k}(z) \equiv 1$ for all $k = 0, 1, ..., q$.

Moreover, suppose that for each j, $j = 0, 1, ..., n-1$, there exists a real number $\tau_{0,j}$ such that

$$|A_{0,j}(z)| \leq |z|^{\tau_{0,j}}, \qquad |z| \notin E. \tag{6.6}$$

Suppose further that each $f_{0,j}$ is of finite order $\rho(f_{0,j})$. Set $\beta = \max_{1\leq j\leq m} \{\rho(f_{0,j})\}$. Then for any given $\varepsilon > 0$, we have

$$|A_{q,j}(z)| \leq |z|^{\tau_{q,j}}, \qquad |z| \notin E, \tag{6.7}$$

where

$$\tau_{q,j} = \max_{q+j\leq k\leq n} \{\tau_{0,k} + (k-q-j)(\beta-1) + \varepsilon\} \tag{6.8}$$

for $j = 0, 1, ..., n-q-1$.

Proof. By applying Lemma 6.2 and Lemma 6.3 q times, we obtain (6.4) and (6.5). Therefore, we need only to prove (6.7) and (6.8). For this proof, we use induction on q.

First suppose that $q = 1$. Then (6.5) is

$$A_{1,j}(z) = \sum_{k=j+1}^{n} \binom{k}{j+1} A_{0,k}(z) \frac{f_{0,1}^{(k-j-1)}(z)}{f_{0,1}(z)}. \tag{6.9}$$

Hence

$$|A_{1,j}(z)| \leq \sum_{k=j+1}^{n} \binom{k}{j+1} |A_{0,k}(z)| \left| \frac{f_{0,1}^{(k-j-1)}(z)}{f_{0,1}(z)} \right|. \tag{6.10}$$

Since $\rho(f_{0,1}) \leq \beta$, it follows from (6.1), (6.6), and (6.10) that (6.7) and (6.8) hold for $q = 1$.

For the induction step, we now make the assumption that for any given $\varepsilon > 0$, (6.7) and (6.8) hold for $q-1$, i.e.,

$$|A_{q-1,j}(z)| \leq |z|^{\tau_{q-1,j}}, \qquad |z| \notin E, \tag{6.11}$$

where

$$\tau_{q-1,j} = \max_{q-1+j\leq k\leq n} \{\tau_{0,k} + (k-q+1-j)(\beta-1) + \varepsilon\} \tag{6.12}$$

for $j = 0, 1, ..., n-q$. We now show that (6.7) and (6.8) hold (for q). From (6.11) and $\rho(f_{q-1,1}) \leq \beta$, we apply the same argument as above to (6.5), and obtain

$$|A_{q,j}(z)| \leq |z|^{\mu_{q,j}}, \qquad |z| \notin E, \tag{6.13}$$

where

$$\mu_{q,j} = \max_{j+1\leq k\leq n-q+1} \{\tau_{q-1,k} + (k-j-1)(\beta-1) + \varepsilon\}. \tag{6.14}$$

From (6.14) and (6.12), we have

$$\begin{aligned} \mu_{q,j} = \max_{j+1\leq k\leq n-q+1} \Big\{ &\max_{q-1+k\leq l\leq n} \{\tau_{0,l} + (l-q+1-k)(\beta-1) + \varepsilon\} \\ &+ (k-j-1)(\beta-1) + \varepsilon \Big\} \\ \leq \max_{q+j\leq l\leq n} &\{\tau_{0,l} + (l-q-j)(\beta-1) + 2\varepsilon\}. \end{aligned} \tag{6.15}$$

From (6.15), (6.14), and (6.13), we see that (6.7) and (6.8) hold (for q). This proves the induction step, and therefore completes the proof of Lemma 6.4. ■

Next we analyze the particular coefficient $A_{q,0}(z)$ in (6.4), for use in the proof of Lemma 6.5 below. From (6.5), we have

$$\begin{aligned}
A_{q,0} &= A_{q-1,1} + \sum_{k=2}^{n-q+1} \binom{k}{1} A_{q-1,k} \frac{f_{q-1,1}^{(k-1)}}{f_{q-1,1}} \\
&= A_{q-2,2} + \sum_{k=3}^{n-q+2} \binom{k}{2} A_{q-2,k} \frac{f_{q-2,1}^{(k-2)}}{f_{q-2,1}} + \sum_{k=2}^{n-q+1} \binom{k}{1} A_{q-1,k} \frac{f_{q-1,1}^{(k-1)}}{f_{q-1,1}} \\
&= \cdots\cdots \\
&= A_{0,q} + \sum_{k=q+1}^{n} \binom{k}{q} A_{0,k} \frac{f_{0,1}^{(k-q)}}{f_{0,1}} + \sum_{k=q}^{n-1} \binom{k}{q-1} A_{1,k} \frac{f_{1,1}^{(k-q+1)}}{f_{1,1}} \\
&\qquad + \cdots + \sum_{k=3}^{n-q+2} \binom{k}{2} A_{q-2,k} \frac{f_{q-2,1}^{(k-2)}}{f_{q-2,1}} + \sum_{k=2}^{n-q+1} \binom{k}{1} A_{q-1,k} \frac{f_{q-1,1}^{(k-1)}}{f_{q-1,1}} \\
&= A_{0,q} + \sum_{j=2}^{q+1} H_j,
\end{aligned} \tag{6.16}$$

where

$$H_j(z) = \sum_{k=j}^{n-q+j-1} \binom{k}{j-1} A_{q-j+1,k}(z) \frac{f_{q-j+1,1}^{(k-j+1)}(z)}{f_{q-j+1,1}(z)}. \tag{6.17}$$

Then we have the following result.

Lemma 6.5. *Under the hypotheses of Lemma 6.4, we have*

$$A_{q,0}(z) = A_{0,q}(z) + G_q(z), \tag{6.18}$$

where $G_q(z) = \sum_{j=2}^{q+1} H_j(z)$ *with* $H_j(z)$ *given in (6.17). Moreover,* $G_q(z)$ *satisfies*

$$|G_q(z)| \le |z|^{\tau_q}, \qquad |z| \notin E, \tag{6.19}$$

where

$$\tau_q = \max_{q+1 \le k \le n} \left\{ \tau_{0,k} + (k-q)(\beta-1) + \varepsilon \right\}. \tag{6.20}$$

Proof. First note that (6.18) is (6.16) with (6.17). Thus we need only to prove (6.19) and (6.20).

Let j be fixed, $2 \le j \le q+1$. From (6.17), we have

$$|H_j(z)| \le \sum_{k=j}^{n-q+j-1} \binom{k}{j-1} |A_{q-j+1,k}(z)| \left| \frac{f_{q-j+1,1}^{(k-j+1)}(z)}{f_{q-j+1,1}(z)} \right|. \tag{6.21}$$

Since $\rho(f_{q-j+1,1}) \le \beta$, we obtain from (6.1), (6.7), and (6.21) that

$$|H_j(z)| \le |z|^{\mu_j}, \qquad |z| \notin E, \tag{6.22}$$

where

$$\mu_j = \max_{j \le k \le n-q+j-1} \left\{ \tau_{q-j+1,k} + (k-j+1)(\beta-1) + \varepsilon \right\}. \tag{6.23}$$

However, from (6.23) and (6.8), we have

$$\begin{aligned}\mu_j = \max_{j\le k\le n-q+j-1} \Big\{ &\max_{q+k-j+1\le l\le n} \big\{\tau_{0,l} + (l-q-k+j-1)(\beta-1)+\varepsilon\big\} \\ &+ (k-j+1)(\beta-1)+\varepsilon\Big\} \\ \le \max_{q+1\le l\le n} &\big\{\tau_{0,l} + (l-q)(\beta-1)+2\varepsilon\big\}.\end{aligned} \tag{6.24}$$

Since $G_q(z) = \sum_{j=2}^{q+1} H_j(z)$, (6.19) and (6.20) follow immediately from (6.22), (6.23), and (6.24). ■

Lemmas 6.6 to 6.9 contain properties of the integers s_j and the rational numbers α_j in §2, which we also use in the proof of Theorem 2.

Lemma 6.6. *Suppose that $s_{m-j} \le k < n$, where $1 \le j \le m-1$. Then*

$$d_{s_{m-j-1}} + (s_{m-j-1}-k)(\alpha_{m-j}-1) \le d_{s_{m-j}} + (s_{m-j}-k)(\alpha_{m-j+1}-1). \tag{6.25}$$

Proof. If $k = s_{m-j}$, then (6.25) follows directly from the definition of α_{m-j} in (2.4). On the other hand, if $s_{m-j} < k < n$, then using (2.4) we obtain that (6.25) holds

$$\begin{aligned}&\iff (s_{m-j-1}-k)(\alpha_{m-j}-1) \le d_{s_{m-j}} - d_{s_{m-j-1}} + (s_{m-j}-k)(\alpha_{m-j+1}-1) \\ &\iff (s_{m-j-1}-k)(\alpha_{m-j}-1) \\ &\qquad \le (s_{m-j-1}-s_{m-j})(\alpha_{m-j}-1) + (s_{m-j}-k)(\alpha_{m-j+1}-1) \\ &\iff (s_{m-j}-k)(\alpha_{m-j}-1) \le (s_{m-j}-k)(\alpha_{m-j+1}-1) \\ &\iff \alpha_{m-j} \ge \alpha_{m-j+1},\end{aligned}$$

which is true from Theorem 1(ii). Lemma 6.6 follows. ■

Lemma 6.7. *Suppose that $s_{m-j} \le k < n$, where $1 \le j \le m-1$. Then*

$$d_{s_{m-j-1}} + (s_{m-j-1}-k)(\alpha_{m-j}-1) \le d_{s_{m-1}} + (s_{m-1}-k)(\alpha_m-1).$$

Proof. Since $s_{m-j} > s_{m-j+1} > \cdots > s_{m-1}$, applying Lemma 6.6 repeatedly yields

$$\begin{aligned}d_{s_{m-j-1}} &+ (s_{m-j-1}-k)(\alpha_{m-j}-1) \le d_{s_{m-j}} + (s_{m-j}-k)(\alpha_{m-j+1}-1) \\ &\le d_{s_{m-j+1}} + (s_{m-j+1}-k)(\alpha_{m-j+2}-1) \le \cdots\cdots \\ &\le d_{s_{m-1}} + (s_{m-1}-k)(\alpha_m-1). \quad ■\end{aligned}$$

Lemma 6.8. *Suppose that $s_m+1 \le k < n$ for two positive integers m and k. Then*

$$d_k \le d_{s_{m-1}} + (s_{m-1}-k)(\alpha_m-1). \tag{6.26}$$

Proof. If $s_m+1 \le k < s_{m-1}$, then by the definition of s_m in (2.2) and (2.3), we have

$$\frac{d_k - d_{s_{m-1}}}{s_{m-1}-k} \le \frac{d_{s_m} - d_{s_{m-1}}}{s_{m-1}-s_m} = \alpha_m - 1.$$

Thus (6.26) holds for such k.

On the other hand, if $s_{m-1} \le k < n$, then $m \ge 2$ and $s_{m-j} \le k < s_{m-j-1}$ for some $j = 1, 2, ..., m-1$. Then by the definition of s_{m-j}, we obtain

$$\frac{d_k - d_{s_{m-j-1}}}{s_{m-j-1}-k} \le \frac{d_{s_{m-j}} - d_{s_{m-j-1}}}{s_{m-j-1}-s_{m-j}} = \alpha_{m-j} - 1. \tag{6.27}$$

Applying Lemma 6.7 to (6.27) gives

$$d_k \leq d_{s_{m-j-1}} + (s_{m-j-1} - k)(\alpha_{m-j} - 1) \leq d_{s_{m-1}} + (s_{m-1} - k)(\alpha_m - 1),$$

which is (6.26). Lemma 6.8 is proved. ■

Lemma 6.9. *For all $m = 1, 2, ..., p$, we have*

$$(n - s_m)d_{s_{m-1}} \geq (n - s_{m-1})d_{s_m}. \tag{6.28}$$

Proof. We prove Lemma 6.9 by induction on m. Obviously, (6.28) holds for $m = 1$, since $s_0 = n$ and $d_{s_0} = d_n = 0$ from (2.5).

Suppose now that (6.28) holds for $m = j$, $1 \leq j \leq p - 1$, i.e., suppose that

$$d_{s_{j-1}} \geq \frac{n - s_{j-1}}{n - s_j} d_{s_j}. \tag{6.29}$$

We will show that (6.28) also holds for $m = j+1$. From the definition of s_j in (2.2) and (2.3), we have

$$\frac{d_{s_j} - d_{s_{j-1}}}{s_{j-1} - s_j} > \frac{d_{s_{j+1}} - d_{s_{j-1}}}{s_{j-1} - s_{j+1}}.$$

Hence

$$(s_{j-1} - s_{j+1})d_{s_j} > (s_{j-1} - s_j)d_{s_{j+1}} + (s_j - s_{j+1})d_{s_{j-1}}. \tag{6.30}$$

Substituting (6.29) into (6.30) and simplifying gives

$$(s_{j-1} - s_j)(n - s_{j+1})d_{s_j} > (s_{j-1} - s_j)(n - s_j)d_{s_{j+1}},$$

which means (6.28) holds for $m = j + 1$, since the common factor $s_{j-1} - s_j$ (> 0) on both sides can be deleted. This proves the induction step, and completes the proof of Lemma 6.9. ■

Remark. As mentioned in §2, we now give a new proof of Theorem 1(iii) by using Lemma 6.1. We show that if $s_1 = 0$, then every nontrivial solution of (1.1) has order $1 + d_0/n$.

Suppose to the contrary that there exists a nontrivial solution f of (1.1) which satisfies $\rho(f) < 1 + d_0/n$. Set $\beta = \rho(f)$. Then

$$\beta = 1 + \tfrac{d_0}{n} - \tau, \tag{6.31}$$

where τ is a positive constant. We will show that this results in a contradiction.

Since $s_1 = 0$, from (2.2) we have

$$d_k \leq \frac{n - k}{n} d_0, \qquad k = 1,\ 2,\ ...,\ n - 1. \tag{6.32}$$

Since f is a solution of (1.1), we obtain

$$-p_0(z) = \frac{f^{(n)}}{f} + p_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + p_1(z)\frac{f'}{f},$$

from which it follows that

$$|p_0(z)| \leq \sum_{k=1}^{n} \left| p_k(z) \frac{f^{(k)}(z)}{f(z)} \right|, \tag{6.33}$$

where we set $p_n(z) \equiv 1$.

Since $p_k(z)$ is a polynomial of degree d_k and f is of order β, from (6.1) and (6.33) we obtain that for any given $\varepsilon > 0$,

$$|p_0(z)| \le \sum_{k=1}^{n} |z|^{d_k+k(\beta-1)+2\varepsilon}, \qquad |z| \notin E. \tag{6.34}$$

From (6.34), (6.32), and (6.31), we have

$$|p_0(z)| \le \sum_{k=1}^{n} |z|^{d_0-k\tau+2\varepsilon} \le n\,|z|^{d_0-\tau+2\varepsilon}, \qquad |z| \notin E,$$

which is impossible if we choose $2\varepsilon < \tau$, since $d_0 = \deg p_0(z)$. This proves Theorem 1(iii).

7. Proof of Theorem 2

Assume the contrary, i.e., suppose that for some integer m satisfying $1 \le m \le p$, an equation of the form (1.1) admits $s_m + 1$ linearly independent solutions with order less than α_m. We show that this assumption results in a contradiction.

We consider two separate cases.

Case (i). Suppose that $s_m \ge 1$.

We denote these $s_m + 1$ linearly independent solutions of (1.1) by $f_{0,1}$, $f_{0,2}, \ldots,$ f_{0,s_m+1}, and we define β to be the maximum order of these s_m+1 solutions. Then our assumption is that

$$\beta = \max_{1\le k\le s_m+1} \{\rho(f_{0,k})\} < \alpha_m. \tag{7.1}$$

Denote $A_{0,k}(z) = p_k(z)$ for $k = 0, 1, \ldots, n-1$. We now perform the method of reduction of order on equation (1.1), by using these solutions $\{f_{0,j}\}$ of (1.1). As in Lemma 6.4, for $1 \le q \le s_m$, set

$$f_{q,j} = \left(\frac{f_{q-1,j+1}}{f_{q-1,1}}\right)', \qquad j = 1, 2, \ldots, s_m + 1 - q. \tag{7.2}$$

From Lemma 6.4, $f_{q,1}$, $f_{q,2}$, ..., f_{q,s_m+1-q} are linearly independent meromorphic solutions of equation (6.4). Taking $q = s_m$ and using (7.2) and Lemma 6.4, we obtain that the function $f = f_{s_m,1}(z)$ is a nontrivial solution of an equation of the form

$$f^{(n-s_m)} + A_{s_m,n-s_m-1}(z) f^{(n-s_m-1)} + \cdots + A_{s_m,0}(z) f = 0, \tag{7.3}$$

where the coefficients $A_{s_m,n-s_m-1}(z)$, ..., $A_{s_m,0}(z)$ are meromorphic functions which for any given $\varepsilon > 0$ satisfy

$$|A_{s_m,j}(z)| \le |z|^{\tau_{s_m,j}}, \qquad |z| \notin E, \tag{7.4}$$

where

$$\tau_{s_m,j} = \max_{s_m+j\le k\le n} \left\{d_k + (k - s_m - j)(\beta-1) + \varepsilon\right\} \tag{7.5}$$

for $j = 0, 1, \ldots, n - s_m - 1$, and $d_k = \deg p_k(z) = \deg A_{0,k}(z)$.

Note that $\rho(f_{s_m,1}) \le \beta$. Hence it follows from (7.4), (7.5), and (6.1) that for $j = 0, 1, \ldots, n - s_m - 1$,

$$\left| A_{s_m,j}(z) \frac{f^{(j)}_{s_m,1}(z)}{f_{s_m,1}(z)} \right| \le |z|^{\sigma_j}, \qquad |z| \notin E, \tag{7.6}$$

where

$$\sigma_j = \max_{s_m+j\le k\le n} \bigl\{d_k + (k-s_m)(\beta-1) + 2\varepsilon\bigr\}. \tag{7.7}$$

However, from (7.3) we have

$$-A_{s_m,0}(z) = \frac{f_{s_m,1}^{(n-s_m)}}{f_{s_m,1}} + \sum_{j=1}^{n-s_m-1} A_{s_m,j}(z)\frac{f_{s_m,1}^{(j)}}{f_{s_m,1}}. \tag{7.8}$$

Therefore, from (7.6), (7.7), and (7.8), we obtain

$$|A_{s_m,0}(z)| \le |z|^{\eta}, \qquad |z| \notin E, \tag{7.9}$$

where

$$\eta = \max_{s_m+1\le k\le n} \bigl\{d_k + (k-s_m)(\beta-1) + 2\varepsilon\bigr\}. \tag{7.10}$$

However, from Lemma 6.5, we have

$$A_{0,s_m}(z) = A_{s_m,0}(z) - G_{s_m}(z), \tag{7.11}$$

where $G_{s_m}(z)$ satisfies (6.19) and (6.20) with q replaced by s_m. Hence from (7.9), (7.11), (6.19), and (6.20) (with $q = s_m$), we obtain

$$|A_{0,s_m}(z)| \le |z|^{\eta}, \qquad |z| \notin E, \tag{7.12}$$

where η is the number in (7.10). Here, note that $\tau_{0,k} = d_k + \varepsilon$ in (6.20).

Finally, we will show that (7.12) results in a contradiction. To this end, we will prove that, for any k satisfying $s_m + 1 \le k \le n$,

$$d_k + (k-s_m)(\beta-1) \le d_{s_m} - \alpha, \tag{7.13}$$

where $\alpha = \alpha_m - \beta > 0$ from (7.1). Once (7.13) has been established, then a contradiction will follow immediately, since from (7.13), (7.12), and (7.10) we obtain

$$|A_{0,s_m}(z)| \le |z|^{d_{s_m}-\alpha+2\varepsilon},$$

which is impossible when $2\varepsilon < \alpha$, because $A_{0,s_m}(z) = p_{s_m}(z)$ is a polynomial of degree d_{s_m}.

To prove (7.13), we will use Lemmas 6.8 and 6.9. We consider the cases $s_m+1 \le k < n$ and $k = n$ separately.

If $s_m + 1 \le k < n$, then from Lemma 6.8 and the definition of α_m in (2.4), we obtain

$$\begin{aligned} d_k + (k-s_m)(\beta-1) &\le d_{s_{m-1}} + (s_{m-1}-k)(\alpha_m-1) + (k-s_m)(\beta-1) \\ &= d_{s_m} - \alpha(k-s_m) \\ &\le d_{s_m} - \alpha. \end{aligned} \tag{7.14}$$

On the other hand, from Lemma 6.9, we obtain

$$(n-s_m)(d_{s_m} - d_{s_{m-1}}) \le (s_{m-1}-s_m)d_{s_m}.$$

Noting that $d_n = 0$, this gives

$$\begin{aligned} d_n + (n-s_m)(\beta-1) &= (n-s_m)(\alpha_m-1) - \alpha(n-s_m) \\ &\le (n-s_m)\frac{d_{s_m}-d_{s_{m-1}}}{s_{m-1}-s_m} - \alpha \\ &\le d_{s_m} - \alpha. \end{aligned} \tag{7.15}$$

Therefore, from (7.15) and (7.14), we see that (7.13) holds for all $s_m+1 \le k \le n$. This proves (7.13), which completes the proof of Case (i).

Case (ii). Suppose that $s_m = 0$.

In this case, $m = p$ and $s_p = 0$. We thus need to show that equation (1.1) does not admit a nontrivial solution with order less than α_p.

To this end, we assume that there exists a solution $f_0 \not\equiv 0$ of (1.1) with $\rho(f_0) < \alpha_p$. Then, by Theorem 1(i), f_0 must be a polynomial.

From (1.1), we obtain

$$-p_0(z) = p_1(z)\frac{f_0'}{f_0} + \cdots + p_{n-1}(z)\frac{f_0^{(n-1)}}{f_0} + \frac{f_0^{(n)}}{f_0}. \tag{7.16}$$

Since f_0 is a polynomial, it follows from (7.16) that

$$d_0 \le \max_{1\le k\le n-1}\{d_k - k\}. \tag{7.17}$$

Applying Lemma 6.8 with $m = p$ gives

$$d_k \le d_{s_{p-1}} + (s_{p-1} - k)(\alpha_p - 1) \tag{7.18}$$

for all $1 \le k \le n-1$, since $s_p = 0$. Therefore, from (7.18), the definition of α_p in (2.4), and the fact that $s_p = 0$, we obtain for any $1 \le k \le n-1$,

$$\begin{aligned} d_k - k &\le d_{s_{p-1}} - k + (s_{p-1} - k)(\alpha_p - 1) \\ &= d_0 + \frac{k}{s_{p-1}}(d_{s_{p-1}} - d_0 - s_{p-1}). \end{aligned} \tag{7.19}$$

Since $\alpha_p > 0$ and $s_p = 0$, it follows from the definition of α_p in (2.4) that $d_{s_{p-1}} < s_{p-1} + d_0$. Hence from (7.19), we obtain $d_k - k < d_0$ for all $k = 1, 2, ..., n-1$. But this contradicts (7.17).

This proves Case (ii), and thus completes the proof of Theorem 2.

8. Examples

In this section we give several examples which illustrate the sharpness of our results and which also exhibit some possibilities that can occur.

Example 1. We now exhibit a special contour integral solution $f = G(z)$ of the third order differential equation

$$f''' - zf'' - f = 0 \tag{8.1}$$

which satisfies $\rho(G) = 1/2$. This gives equality in (2.7) when $n = 3$.

We define the function $G(z)$ as follows:

$$G(z) = \int_{K_R} \exp\left\{\frac{z}{w} - \frac{1}{2w^2} - w\right\} dw, \qquad \text{where} \quad K_R : |w| = R \quad (R > 0). \tag{8.2}$$

For any fixed z, the integrand in (8.2) is analytic for all $w \neq 0$. From Cauchy's theorem it follows that for any fixed z, the value of $G(z)$ is independent of $R > 0$.

From (8.2) we obtain

$$\begin{aligned} G'''(z) - zG''(z) - G(z) &= \int_{K_R} \left(\frac{1}{w^3} - \frac{z}{w^2} - 1 \right) \exp\left\{ \frac{z}{w} - \frac{1}{2w^2} - w \right\} dw \\ &= \int_{K_R} \frac{\partial}{\partial w} \exp\left\{ \frac{z}{w} - \frac{1}{2w^2} - w \right\} dw \\ &= \left[\exp\left\{ \frac{z}{w} - \frac{1}{2w^2} - w \right\} \right]_{K_R} = 0, \end{aligned}$$

since K_R is a closed curve. Thus $f = G(z)$ is a solution of equation (8.1).

Now note that equation (8.1) cannot possess a nontrivial polynomial solution. Hence it follows from Theorem 1(i) that we must have exactly one of the following:

$$\rho(G) = 2, \quad \rho(G) = 1/2, \quad \text{or} \quad G \equiv 0. \tag{8.3}$$

We will show that $\rho(G) = 1/2$.

We first prove that $G \not\equiv 0$. To prove this, we show that $G'(0) \neq 0$. Since the value of $G(z)$ is independent of $R > 0$ in (8.2), we use $R = 1$ and obtain

$$G'(0) = \int_{K_1} \frac{1}{w} \exp\left\{ -\frac{1}{2w^2} - w \right\} dw = \int_{-\pi}^{\pi} i \exp\left\{ -\frac{1}{2e^{i2\theta}} - e^{i\theta} \right\} d\theta. \tag{8.4}$$

We break this expression into real and imaginary parts. The real part of the integrand is an odd function, while the imaginary part of the integrand is an even function. This gives

$$G'(0) = 2i \int_0^{\pi} \exp\left\{ -\frac{\cos 2\theta}{2} - \cos\theta \right\} \cos L(\theta)\, d\theta, \tag{8.5}$$

where

$$L(\theta) = \frac{\sin 2\theta}{2} - \sin\theta. \tag{8.6}$$

Consider now the function $L(\theta)$ in (8.6). Since

$$L'(\theta) = \cos 2\theta - \cos\theta = 2\cos^2\theta - 1 - \cos\theta = (2\cos\theta + 1)(\cos\theta - 1),$$

we obtain $L'(\theta) < 0$ for $0 < \theta < \frac{2\pi}{3}$ and $L'(\theta) > 0$ for $\frac{2\pi}{3} < \theta < \pi$. Also, $L(0) = L(\pi) = 0$ and $L(\frac{2\pi}{3}) < 0$. Hence, $L(\theta) < 0$ for $0 < \theta < \pi$. More specifically, we have

$$-\tfrac{\pi}{2} < -\tfrac{3\sqrt{3}}{4} = L(\tfrac{2\pi}{3}) \le L(\theta) < 0 \quad \text{for} \quad 0 < \theta < \pi,$$

which implies that $\cos L(\theta) > 0$ for $0 < \theta < \pi$. Then from (8.5) we obtain that $G'(0) \neq 0$. Hence

$$G \not\equiv 0. \tag{8.7}$$

We next show that $\rho(G) \le 1/2$. To see this, let $z \neq 0$ be fixed, and choose $R = \sqrt{|z|} > 0$ in (8.2), i.e., $K_R : |w| = \sqrt{|z|}$. Then

$$|G(z)| \le \int_{K_R} \exp\left\{ \frac{|z|}{|w|} + \frac{1}{2|w|^2} + |w| \right\} |dw| = 2\pi\sqrt{|z|} \exp\left\{ 2\sqrt{|z|} + \frac{1}{2|z|} \right\}.$$

By letting $z \to \infty$, we see that $\rho(G) \le 1/2$. By combining this fact with (8.7) and (8.3), we obtain that $\rho(G) = 1/2$. Since $f = G(z)$ satisfies equation (8.1), this is an example where the inequality (2.7) becomes an equality in the case when $n = 3$.

We also mention the following observation. By differentiating equation (8.1) and then adding the resulting equation to (8.1), we obtain that $f = G(z)$ is a solution of the fourth order equation

$$f^{(4)} + (1 - z)f''' - (1 + z)f'' - f' - f = 0.$$

Continuing in this manner, we see that for any $n \geq 3$ we can obtain a particular equation of the form (1.1) that possesses a solution of order $1/2$, namely, $f = G(z)$.

Remark. For $n \geq 2$, consider an equation of the form

$$f^{(n)} + p(z)f^{(n-1)} + c_{n-2}f^{(n-2)} + \cdots + c_1 f' + c_0 f = 0, \tag{8.8}$$

where c_0, c_1, ..., c_{n-2} are all constants ($c_0 \neq 0$) and $p(z)$ is a polynomial of degree $n - 2$. From Theorem 1(i), there are two possible orders $\alpha_1 = n - 1$ and $\alpha_2 = 1/(n - 1)$ of transcendental solutions of equation (8.8). Wittich [8], [9, pp. 65–68] indicates that there always exists a solution $f = \psi(z)$ of equation (8.8) that satisfies $\rho(\psi) = 1/(n - 1)$, which gives an equality in (2.7) for all $n \geq 2$.

Examples 2 through 6 below illustrate the sharpness of Corollary 1, Theorem 2, and Corollary 5, and are also examples of Theorem 1.

Example 2. Let q_1, $q_2, ..., q_n$ be any n distinct integers that satisfy

$$0 < q_1 < q_2 < ... < q_n.$$

We construct an equation of the form (1.1) which possesses n distinct solutions f_1, $f_2, ..., f_n$ satisfying $\rho(f_k) = q_k$ for $1 \leq k \leq n$. This illustrates the sharpness of Corollary 1 and Theorem 2. Furthermore, the equation of the form (1.1) that we construct satisfies $d_0 = \deg p_0 = q_1 + q_2 + \cdots + q_n - n$, which means that this is a sharp example for Corollary 5, because (2.11) becomes an equality ($= q_1 + q_2 + \cdots + q_n$).

We construct the equation as follows. Set

$$A_{1,0}(z) = -q_1 z^{q_1 - 1}.$$

Then $f_1(z) = \exp(z^{q_1})$ is a solution of the equation

$$L_1(f) = f' + A_{1,0}(z)f = 0, \tag{8.9}$$

and $\rho(f_1) = q_1$.

Since f_1 is a solution of (8.9), f_1 is also a solution of the two equations

$$f'' + A_{1,0}(z)f' + A_{1,0}'(z)f = 0 \tag{8.10}$$

and

$$z^{q_2 - 1}f' + z^{q_2 - 1}A_{1,0}(z)f = 0. \tag{8.11}$$

By adding (8.10) and (8.11), we obtain that f_1 is a solution of the equation

$$L_2(f) = f'' + A_{2,1}(z)f' + A_{2,0}(z)f = 0, \tag{8.12}$$

where $A_{2,1}(z) = z^{q_2 - 1} + A_{1,0}(z)$ and $A_{2,0}(z) = z^{q_2 - 1}A_{1,0}(z) + A_{1,0}'(z)$. Note that

$$L_2(f) = z^{q_2 - 1}L_1(f) + \frac{d}{dz}(L_1(f)).$$

Since $0 < q_1 < q_2$, $A_{2,1}(z)$ is a polynomial of degree $q_2 - 1$. We also note that $\deg A_{2,0} = q_1 + q_2 - 2$. Now let f_2 be any solution of (8.12) such that f_1 and f_2

are linearly independent. From (8.12) and Abel's identity [4, p.16], the Wronskian $W(f_1,\, f_2)$ satisfies

$$W(f_1,\, f_2) = C\,\exp\left(\int -A_{2,1}(z)\,dz\right)$$

for some constant $C \neq 0$. Since $\rho(f_1) = q_1$ and $\rho(W(f_1,\, f_2)) = 1 + \deg A_{2,1} = q_2 > q_1$, we conclude that $\rho(f_2) \geq q_2$. However, from (1.2) and (1.3), we have

$$\rho(f_2) \leq 1 + \max\left\{\deg A_{2,1},\ \frac{\deg A_{2,0}}{2}\right\} = 1 + \max\left\{q_2 - 1,\ \frac{q_1 + q_2 - 2}{2}\right\} = q_2.$$

Hence $\rho(f_2) = q_2$. (Note: Alternatively, it also follows from Corollary 3 that we must have $\rho(f_2) = q_2$.) Therefore, f_1 and f_2 are solutions of equation (8.12) such that $\rho(f_k) = q_k$ for $k = 1, 2$.

Continuing in this manner, we can contruct the desired general example. Namely, we define

$$L_j(f) = z^{q_j - 1} L_{j-1}(f) + \frac{d}{dz}\left(L_{j-1}(f)\right)$$

for $3 \leq j \leq n$. Then by the same argument as above, it can be shown that the j-th order linear differential equation $L_j(f) = 0$ admits j solutions $f_1, f_2, ..., f_j$ such that $\rho(f_k) = q_k$ for $k = 1, 2, ..., j$. Furthermore, the coefficient of f in the differential equation $L_j(f) = 0$ is a polynomial of degree $q_1 + q_2 + \cdots + q_j - j$.

Example 3. The equation

$$f''' - f'' + zf' - zf = 0 \tag{8.13}$$

possesses the solution $f_1 = e^z$, and, by Corollary 3, a solution f_2 satisfying $\rho(f_2) = 3/2$. For this equation we have $p = 2$, $s_1 = 1$, $s_2 = 0$, $\alpha_1 = 3/2$, and $\alpha_2 = 1$. This gives a sharp example for Corollary 1 and Theorem 2.

It can also be deduced that this gives a sharp example for Corollary 5. Specifically, since $\alpha_1 = 3/2$ and $\alpha_2 = 1$, it follows from Theorem 2 or Corollary 5 that there exists a third solution f_3 of (8.13) that satisfies $\rho(f_3) = 3/2$, such that $\{f_1,\, f_2,\, f_3\}$ is a fundamental set of solutions of (8.13). Hence for the fundamental solution set $\{f_1,\, f_2,\, f_3\}$, we obtain the equality $4 = 4$ in (2.11).

Example 4. The equation

$$f^{(4)} - f''' + zf' - zf = 0$$

possesses the solution $f_1 = e^z$, and, by Corollary 3, a solution f_2 satisfying $\rho(f_2) = 4/3$. For this equation we have $p = 2$, $s_1 = 1$, $s_2 = 0$, $\alpha_1 = 4/3$, and $\alpha_2 = 1$. This gives a sharp example for Corollary 1 and Theorem 2. By reasoning similar to that in Example 3, we can deduce that this also gives a sharp example for Corollary 5, because we can obtain the equality $5 = 5$ in (2.11).

Example 5. The equation

$$\begin{aligned} f^{(4)} &+ (8z^3 - 13)f'' - (16z^4 + 16z^3 + 12z^2 + 4z + 2)f' \\ &+ (16z^4 + 8z^3 + 12z^2 + 4z + 14)f = 0 \end{aligned}$$

possesses the solutions $f_1 = e^z$ and $f_2 = e^{z^2}$, and, by Corollary 3, a solution f_3 satisfying $\rho(f_3) = 5/2$. For this equation we have $p = 3$, $s_1 = 2$, $s_2 = 1$, $s_3 = 0$, $\alpha_1 = 5/2$, $\alpha_2 = 2$, and $\alpha_3 = 1$. This gives a sharp example for Corollary 1, Theorem 2, and (by reasoning similar to that in Example 3) Corollary 5.

Example 6. Let q and n be integers satisfying $2 \leq q \leq n$, and consider the equation

$$f^{(n)} - \frac{1}{(q-1)!} z^{q-1} f^{(q-1)} + \frac{1}{(q-2)!} z^{q-2} f^{(q-2)} - \cdots + (-1)^{q-1} z f' + (-1)^q f = 0. \tag{8.14}$$

Observe that equation (8.14) has the $q-1$ linearly independent polynomial solutions $f_k(z) = z^k$, $k = 1, 2, ..., q-1$. For equation (8.14), we have $p = 1$, $s_1 = q-1$, and $\alpha_1 = n/(n-q+1)$. Therefore, we see that equation (8.14) gives a sharp example for Theorem 2 and Corollary 4.

Moreover, from Theorem 2 or Corollary 5, we deduce that there exist $n - q + 1$ linearly independent solutions g_1, g_2, ..., g_{n-q+1} of (8.14) that satisfy $\rho(g_j) = \alpha_1 = n/(n-q+1)$ for each $1 \leq j \leq n-q+1$, and that $\{g_1, g_2, ..., g_{n-q+1}, z, z^2, ..., z^{q-1}\}$ forms a fundamental set of solutions of (8.14). Hence equation (8.14) also gives a sharp example for Corollary 5.

In the special case when $q = n$, equation (8.14) is an equation of order n that possesses $n-1$ linearly independent polynomial solutions and a transcendental solution of order n.

Example 1 is an example of an equation of the form (1.1) that possesses a special contour integral solution of order 1/2. The next example is another equation of the form (1.1) that possesses a special transcendental contour integral solution with order less than one.

Example 7. We now show that there exists a special contour integral solution $f = H(z)$ of the fourth order equation

$$f^{(4)} - z f''' - f = 0 \tag{8.15}$$

which satisfies $\rho(H) = 2/3$. Then by the technique given at the end of Example 1, we see that for any $n \geq 4$, there will exist an equation of the form (1.1) that possesses a solution of order 2/3.

We define the function $H(z)$ as follows:

$$H(z) = \int_C \exp\left\{ \frac{z}{\sqrt{2w}} - \frac{1}{4w} - w \right\} dw, \tag{8.16}$$

where C is the contour defined by $C = C_1 + C_2 + C_3$ with

$$\begin{aligned} C_1: &\quad w = r\, e^{i\pi/4},\ r \text{ goes from } +\infty \text{ to } 1, \\ C_2: &\quad w = e^{i\theta},\ \theta \text{ goes from } \pi/4 \text{ to } 7\pi/4, \\ C_3: &\quad w = r\, e^{i7\pi/4},\ r \text{ goes from } 1 \text{ to } +\infty, \end{aligned}$$

and where $\sqrt{2w}$ is defined by the branch $\sqrt{\zeta} = \exp\{\frac{1}{2} \log |\zeta| + i \frac{1}{2} \arg \zeta\}$, $0 < \arg \zeta < 2\pi$.

From (8.16),

$$\begin{aligned} H^{(4)}(z) - zH'''(z) - H(z) &= \int_C \left(\frac{1}{4w^2} - \frac{z}{(2w)^{3/2}} - 1 \right) \exp\left\{ \frac{z}{\sqrt{2w}} - \frac{1}{4w} - w \right\} dw \\ &= \left[\exp\left\{ \frac{z}{\sqrt{2w}} - \frac{1}{4w} - w \right\} \right]_C = 0. \end{aligned}$$

Thus $f = H(z)$ is a solution of equation (8.15).

Since equation (8.15) cannot possess a nontrivial polynomial solution, it follows from Theorem 1(i) that we must have exactly one of the following:

$$\rho(H) = 2, \quad \rho(H) = 2/3, \quad \text{or} \quad H \equiv 0. \tag{8.17}$$

We will show that $\rho(H) = 2/3$.

We will first show that $H \not\equiv 0$ by calculating that $H(0) \neq 0$. From (8.16),

$$H(0) = \int_C \exp\left\{-\frac{1}{4w} - w\right\} dw. \tag{8.18}$$

By appealing to Cauchy's theorem, it can be deduced that the contour C in (8.18) can be replaced by the unit circle $|w| = 1$. Then, from the residue theorem, we obtain

$$H(0) = \int_{|w|=1} \exp\left\{-\frac{1}{4w} - w\right\} dw = 2\pi i \operatorname{Res}(g(w),\, 0), \tag{8.19}$$

where

$$g(w) = \exp\left\{-\frac{1}{4w} - w\right\} = \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4w} - w\right)^k}{k\,!}. \tag{8.20}$$

From inspection of (8.20), we see that the even terms in the series contribute only even powers of w and thus contribute 0 to $\operatorname{Res}(g(w),\, 0)$, while the odd terms in the series contribute only odd powers of w with negative constant factors. Therefore, $\operatorname{Res}(g(w),\, 0) < 0$. Hence from (8.19), $H(0) \neq 0$. Thus

$$H \not\equiv 0. \tag{8.21}$$

We next show that $\rho(H) \leq 2/3$. By appealing to Cauchy's theorem, it can be seen that we can replace the contour C in (8.16) with the following curve γ_R, $R > 1$: $\gamma_R = K_1 + K_2 + K_3$, where

$K_1: \quad w = x + i, x$ goes from $+\infty$ to $\sqrt{R^2 - 1}$,

$K_2: \quad w = R\,e^{i\theta}, \theta$ goes from $\arctan\left(\frac{1}{\sqrt{R^2-1}}\right)$ to $2\pi - \arctan\left(\frac{1}{\sqrt{R^2-1}}\right)$,

$K_3: \quad w = x - i, x$ goes from $\sqrt{R^2 - 1}$ to $+\infty$.

Then

$$H(z) = \int_{\gamma_R} \exp\left\{\frac{z}{\sqrt{2w}} - \frac{1}{4w} - w\right\} dw,$$

where for a fixed value of z, the value of $H(z)$ is independent of $R > 1$.

Now for any fixed z satisfying $|z| > 1$, we choose $R = |z|^{2/3}$. Then

$$|H(z)| \leq \int_{\gamma_R} \exp\left\{\frac{|z|}{\sqrt{2|w|}} + \frac{1}{4|w|}\right\} \left|e^{-w}\right| |dw| \leq \exp\left\{\frac{|z|^{2/3}}{\sqrt{2}} + \frac{1}{4}\right\} \int_{\gamma_R} \left|e^{-w}\right| |dw|$$

and

$$\int_{\gamma_R} \left|e^{-w}\right| |dw| < e^{|z|^{2/3}} \cdot 2\pi |z|^{2/3} + 2\int_1^{\infty} e^{-x} dx.$$

It follows that $\rho(H) \leq 2/3$. Therefore, from (8.21) and (8.17), we obtain that $\rho(H) = 2/3$. Since $f = H(z)$ satisfies (8.15) and $\rho(H) = 2/3$, this proves the assertion.

Example 8. Theorem 1 gives all the possible orders $\alpha_1, \alpha_2, ..., \alpha_p$ of transcendental solutions of equation (1.1). This example shows that *not* every value α_j, $j = 1, 2, ..., p$, is necessarily attained as the order of a solution of (1.1). Consider the equation

$$f'' + z^3 f' - (z^4 + z^2) f = 0. \tag{8.22}$$

Here $p = 2$, $\alpha_1 = 4$, and $\alpha_2 = 2$. So 4 and 2 are the only possible orders for transcendental solutions of (8.22). Clearly, (8.22) does not possess a nontrivial polynomial solution. Moreover, if f is any transcendental solution of (8.22), and if we set

$$v = f\, e^{-\frac{1}{2}z^2}, \tag{8.23}$$

then v is a solution of the equation

$$v'' + (z^3 + 2z) v' + v = 0. \tag{8.24}$$

Obviously, (8.24) does not admit a nontrivial polynomial solution. From Theorem 1(i), all transcendental solutions of (8.24) have order 4. Hence from (8.24) and (8.23), we conclude that $\rho(f) = 4$. Therefore, all solutions $f \not\equiv 0$ of (8.22) satisfy $\rho(f) = \alpha_1 = 4$, and (8.22) does not admit any solution with order $\alpha_2 = 2$.

Example 9. From Theorem 1(iii), if

$$\frac{\deg p_k}{n-k} \leq \frac{\deg p_0}{n} \tag{8.25}$$

holds for all $k = 1, 2, ..., n-1$, then all solutions $f \not\equiv 0$ of (1.1) satisfy $\rho(f) = \lambda = 1 + \deg p_0/n$. Example 8 shows that (8.25) is *not* a necessary condition to ensure that all solutions of (1.1) have the maximum possible order λ; see (1.2) and (1.3). Here is another example.

Consider the equation

$$f''' + z^5 f'' + z^2 f' + z^2 f = 0. \tag{8.26}$$

Here we have $p = 1$ and $\alpha_1 = \lambda = 6$. Hence, by Theorem 1(i), all transcendental solutions of (8.26) have order λ. Clearly, (8.26) does not have a nontrivial polynomial solution. Therefore, all nontrivial solutions of (8.26) have order $\lambda = 6$, and equation (8.26) does not satisfy the condition (8.25).

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