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A CLASSIFICATION THEOREM FOR ALBERT ALGEBRAS

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ABSTRACT. Let k be a field whose characteristic is different from 2 and 3 and let L/k be a quadratic extension. In this paper we prove that for a fixed, degree 3 central simple algebra B over L with an involution σ of the second kind over k, the Jordan algebra $J(B,\sigma,u,\mu)$, obtained through Tits' second construction is determined up to isomorphism by the class of (u,μ) in $H^1(k,SU(B,\sigma))$, thus settling a question raised by Petersson and Racine. As a consequence, we derive a "Skolem Noether" type theorem for Albert algebras. We also show that the cohomological invariants determine the isomorphism class of $J(B,\sigma,u,\mu)$, if (B,σ) is fixed.

INTRODUCTION

Let k be a field with characteristic different from 2 and 3. Exceptional simple Jordan algebras over k are called *Albert algebras*. There are rational constructions due to Tits of all Albert algebras over k, referred to as the *first* and the *second* constructions. We begin by recalling the second construction briefly. Let L/k be a quadratic extension, bar denoting its nontrivial k-automorphism. Let B be a degree 3 central simple (associative) algebra over L with an involution σ of the second kind over k. Let u be a unit of B with $\sigma(u)=u$ and $N(u)=\mu\overline{\mu},\ \mu\in L$, where N denotes the reduced norm on B. Let $\mathcal{H}(B,\sigma)=\{x\in B|\sigma(x)=x\}$. Let $J(B,\sigma,u,\mu)=\mathcal{H}(B,\sigma)\oplus B$ be the Jordan algebra with the multiplication given by

$$(b_0,b)(b_0',b')=(b_0.b_0'+\widetilde{bu\sigma(b')}+\widetilde{b'u\sigma(b)},\ \widetilde{b_0}b'+\widetilde{b_0'}b+\overline{\mu}(\sigma(b)\times\sigma(b'))u^{-1}),$$

where $x.y=\frac{1}{2}(xy+yx),\ \widetilde{x}=\frac{1}{2}(t(x)-x)$ and

$$x\times y=x.y-\frac{1}{2}t(x)y-\frac{1}{2}t(y)x+\frac{1}{2}(t(x)t(y)-t(x\cdot y)),$$

t denoting the reduced trace of B. The cubic norm of an element (a_0,a) of $J(B,\sigma,u,\mu)$ is given by

$$(*_1)\qquad n(a_0,a)=N(a_0)+\mu N(a)+\overline{\mu}N(\sigma(a))-t(a_0au\sigma(a)).$$

For any unit $w\in B$, we have an isomorphism

$$J(B,\sigma,u,\mu)\longrightarrow J(B,\sigma,wu\sigma(w),N(w)\mu)$$

given by $(a_0,a)\mapsto(a_0,aw)$. The following converse problem is posed in [P-R].

Question. If $J(B,\sigma,u,\mu)\simeq J(B,\sigma,u',\mu')$, then does there exist a unit $w\in B$ such that $u'=wu\sigma(w),\ \mu'=N(w)\mu$?

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In this paper we answer this question in the affirmative (§2, Theorem 2.7). As a consequence, we prove a "Skolem Noether" type theorem on extensions of isomorphisms on certain types of simple Jordan subalgebras of an Albert algebra (§3, Theorem 3.1). There are cohomological invariants $g_3 \in H^3(k, \mathbb{Z}/3)$ and $f_3,\ f_5$ in $H^3(k, \mathbb{Z}/2)$ and $H^5(k, \mathbb{Z}/2)$ respectively, attached to an Albert algebra ([S]). Serre raised the question whether these invariants determine the isomorphism class of the Albert algebra. We indeed show that if $J(B, \sigma, u, \mu)$ and $J(B, \sigma, u', \mu')$ have the same invariants, then they are isomorphic (§2, Theorem 2.8).

1. Coordinatization of a certain Tits second construction

Let L/k be a quadratic extension. Let $*$ denote the involution on $M_3(L)$ given by $X^* = \Gamma^{-1}\overline{X}^t\Gamma$, where $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle,\ \gamma_i \in k$ with $\gamma_1\gamma_2\gamma_3 = 1$ and bar denoting the entrywise action of the automorphism bar of L. Let $V \in GL_3(L)$ with $V^* = V$. Suppose further that $det\ V = \mu\overline{\mu}$ for some $\mu \in L^*$. Then one has the second Tits' construction $J(M_3(L), *, V, \mu)$ with the underlying space $\mathcal{H}(M_3(L), *) \oplus M_3(L)$.

The matrix $U = V\Gamma^{-1}$ is hermitian, i.e, $\overline{U}^t = U$. Further, $det\ U = det\ V = \mu\overline{\mu}$. Let h denote the hermitian form on L^3 given by $h(x, y) = x\overline{U}\overline{y}^t$. Then the discriminant of h denoted by $disc\ h$ is trivial. Let $\psi : (\bigwedge^3 L^3, \bigwedge^3 h) \simeq (L, \langle 1 \rangle)$ be the trivialization of $disc\ h$ given by $e_1 \wedge e_2 \wedge e_3 \mapsto \overline{\mu},\ e_i$ being the standard basis vectors of L^3. We then have the Cayley algebra (cf. [T]), $C = C(L^3, h, \psi) = L \oplus L^3$, with multiplication defined by

$$(a, v)(a', v') = (aa' - h(v, v'),\ av' + \overline{a'}v + \theta(v, v')),$$

where θ is defined by the identity

$$h(v'', \theta(v, v')) = \psi(v'' \wedge v \wedge v'),$$

for all $v, v', v'' \in L^3$. Also, the norm n_C is given by $n_C(a, v) = n_{L/k}(a) + h(v)$, where $h(v) = h(v, v)$. Then one has the reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$ which consists of all 3×3 matrices

$$X = \begin{pmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\overline{b} \\ \gamma_2^{-1}\gamma_1\overline{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\overline{a} & \alpha_3 \end{pmatrix},$$

where $\alpha_i \in k,\ a, b, c \in C$, with the multiplication $(X, Y) \mapsto \frac{1}{2}(XY + YX)$. Here the bar denotes the involution on $C = L \oplus L^3$ given by $\overline{(\alpha, v)} = (\overline{\alpha}, -v)$. Note that $\mathcal{H}_3(C, \Gamma)$ contains $\mathcal{H}_3(L, \Gamma) = \mathcal{H}(M_3(L), *)$ as a Jordan subalgebra. The cubic norm of any element X in $\mathcal{H}_3(C, \Gamma)$ as above, is given by

$(*_2)$
$$n(X) = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n_C(a) - \gamma_1^{-1}\gamma_3\alpha_2 n_C(b) - \gamma_2^{-1}\gamma_1\alpha_3 n_C(c) + t_C((ca)b),$$

where, for $(\alpha, v) \in C, t_C(\alpha, v) = t_{L/k}(\alpha)$. With this notation we have the following.

Theorem 1.1. *The map $\Phi : J(M_3(L), *, V, \mu) \to \mathcal{H}_3(C, \Gamma)$, induced by the natural inclusion $\mathcal{H}(M_3(L), *) \hookrightarrow \mathcal{H}_3(C, \Gamma)$ and the map $M_3(L) \to \mathcal{H}_3(C, \Gamma)$ given by*

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\gamma_1^{-1}\overline{\mathbf{a}_3} & \gamma_1^{-1}\overline{\mathbf{a}_2} \\ \gamma_2^{-1}\overline{\mathbf{a}_3} & 0 & -\gamma_2^{-1}\overline{\mathbf{a}_1} \\ -\gamma_3^{-1}\overline{\mathbf{a}_2} & \gamma_3^{-1}\overline{\mathbf{a}_1} & 0 \end{pmatrix},$$

is an isomorphism of Jordan algebras, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denoting the rows of a matrix in $M_3(L)$.

Proof. Clearly Φ is an isomorphism of vector spaces. Hence by ([MC-1], p. 507), to check that Φ is an isomorphism of Jordan algebras, it suffices to check that it preserves the cubic norms and maps identity to identity. It is clear from the definition that $\Phi(1) = 1$. Let

$$(A_0, A) = \left(\begin{pmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\overline{b} \\ \gamma_2^{-1}\gamma_1\overline{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\overline{a} & \alpha_3 \end{pmatrix} , \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \right)$$

be any element of $J(M_3(L), *, V, \mu) = \mathcal{H}_3(L, \Gamma) \oplus M_3(L)$. Equating the norms of (A_0, A) and $\Phi(A_0, A)$ (cf. $(*_1)$, $(*_2)$), we need to verify that

$$det\ A_0 + \mu det\ A + \overline{\mu} det\ A^* - t(A_0 AVA^*) = \alpha_1\alpha_2\alpha_3$$
$$-\gamma_3^{-1}\gamma_2\alpha_1 n_C(a, -\gamma_2^{-1}\overline{\mathbf{a}_1}) - \gamma_1^{-1}\gamma_3\alpha_2 n_C(b, -\gamma_3^{-1}\overline{\mathbf{a}_2}) - \gamma_2^{-1}\gamma_1\alpha_3 n_C(c, -\gamma_1^{-1}\overline{\mathbf{a}_3})$$
$$+t_C(((c, -\gamma_1^{-1}\overline{\mathbf{a}_3})(a, -\gamma_2^{-1}\overline{\mathbf{a}_1}))(b, -\gamma_3^{-1}\overline{\mathbf{a}_2}))$$

i.e.,

$$det\ A_0 + \mu det\ A + \overline{\mu} det\ A^* - t(A_0 AVA^*) = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2\alpha_1 n(a) - \gamma_1\alpha_1 h(\overline{\mathbf{a}_1})$$
$$-\gamma_1^{-1}\gamma_3\alpha_2 n(b) - \gamma_2\alpha_2 h(\overline{\mathbf{a}_2}) - \gamma_2^{-1}\gamma_1\alpha_3 n(c) - \gamma_3\alpha_3 h(\overline{\mathbf{a}_3}) + t_C((ca - \gamma_3 h(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}),$$
$$-\gamma_2^{-1}c\overline{\mathbf{a}_1} - \gamma_1^{-1}\bar{a}\bar{\mathbf{a}}_3 + \gamma_3\theta(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}))(b, -\gamma_3^{-1}\overline{\mathbf{a}_2})).$$

We have

$$det\ A_0 = \alpha_1\alpha_2\alpha_3 - \gamma_3^{-1}\gamma_2 n(a) - \gamma_1^{-1}\gamma_3\alpha_2 n(b) - \gamma_2^{-1}\gamma_1\alpha_3 n(c) + t_{L/k}(cab).$$

Further, $t_C(\alpha, v) = t_{L/k}(\alpha)$, so that we are reduced to verifying the following equality:

$(*_3)$

$$\mu det\ A + \overline{\mu} det\ A^* - t(A_0 AVA^*) = -\alpha_1\gamma_1 h(\overline{\mathbf{a}_1}) - \alpha_2\gamma_2 h(\overline{\mathbf{a}_2}) - \alpha_3\gamma_3 h(\overline{\mathbf{a}_3})$$
$$- \gamma_3 t_{L/k}(h(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1})b) - \gamma_1 t_{L/k}(h(c\overline{\mathbf{a}_1}, \overline{\mathbf{a}_2}))$$
$$- \gamma_2 t_{L/k}(h(\bar{a}\bar{\mathbf{a}}_3, \overline{\mathbf{a}_2})) + t_{L/k}(h(\theta(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}), \overline{\mathbf{a}_2})).$$

We first compute $t(A_0 AVA^*)$. We have

$$AVA^* = AV\Gamma^{-1}\overline{A}^t\Gamma = AU\overline{A}^t\Gamma = (a_{ij})\Gamma,$$

where $a_{ij} = h(\overline{\mathbf{a}_j}, \overline{\mathbf{a}_i})$. This gives

$$t(A_0 AVA^*) = \alpha_1\gamma_1 h(\overline{\mathbf{a}_1}) + \gamma_1 c h(\overline{\mathbf{a}_1}, \overline{\mathbf{a}_2}) + \gamma_3\overline{b} h(\overline{\mathbf{a}_1}, \overline{\mathbf{a}_3}) + \gamma_1\overline{c} h(\overline{\mathbf{a}_2}, \overline{\mathbf{a}_1})$$
$$+\alpha_2\gamma_2 h(\overline{\mathbf{a}_2}) + \gamma_2 a h(\overline{\mathbf{a}_2}, \overline{\mathbf{a}_3}) + \gamma_3 b h(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}) + \gamma_2\overline{a} h(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_2}) + \alpha_3\gamma_3 h(\overline{\mathbf{a}_3}).$$

Comparing the above with $(*_3)$, it only remains to verify that

$$\mu det\ A + \overline{\mu} det\ A^* = t_{L/k}(h(\theta(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}), \overline{\mathbf{a}_2})).$$

By definition of θ, we have

$$h(\theta(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}), \overline{\mathbf{a}_2}) = \overline{\psi(\overline{\mathbf{a}_2} \wedge \overline{\mathbf{a}_3} \wedge \overline{\mathbf{a}_1})} = \overline{\overline{\mu} det\ \overline{A}} = \mu det\ A.$$

Hence

$$t_{L/k}(h(\theta(\overline{\mathbf{a}_3}, \overline{\mathbf{a}_1}), \overline{\mathbf{a}_2})) = \mu det\ A + \overline{\mu} det\ \overline{A} = \mu det\ A + \overline{\mu} det\ A^*.$$

□

For a Jordan algebra $J \simeq \mathcal{H}_3(C, \Gamma)$, C is called the *coordinate algebra* of J. Its isomorphism class is uniquely determined by J (cf. [A-J]).

Corollary 1.2. *Let C be any coordinatization of $J(M_3(L), *, V, \mu)$. The norm form n_C of C is given by $n_C = tr_{L/k}(\langle 1\rangle \perp h)$, where h is the hermitian form given by the matrix $\overline{V\Gamma^{-1}}$.*

Proof. By the uniqueness of the coordinate algebra of a reduced Albert algebra (cf. [A-J]) and by Theorem 1.1 we get $C \simeq C(L^3, h, \psi)$. The corollary now follows (cf. [T], p. 5124).

2. Classification of Albert algebras

Let k be a field with characteristic different from 2 and 3. Let J be an Albert algebra over k. Then there exist a 3-fold Pfister form ϕ_3 and a 5-fold Pfister form ϕ_5 over k such that

$$Q_J \perp \phi_3 \simeq \langle 2,2,2\rangle \perp \phi_5,$$

Q_J denoting the trace quadratic form of J. This property characterizes ϕ_3 and ϕ_5 up to isomorphism (cf. [S]). Let G be the group of automorphisms of the split Albert algebra over k. We have the mod 2 cohomological invariants

$$f_3 : H^1(k,G) \to H^3(k,\mathbb{Z}/2)$$

and

$$f_5 : H^1(k,G) \to H^5(k,\mathbb{Z}/2),$$

defined as the Arason invariants of ϕ_3 and ϕ_5 respectively. If $J \simeq \mathcal{H}_3(C,\Gamma)$, then $f_3(J)$ is the Arason invariant of n_C and $f_5(J)$ is the Arason invariant of $n_C \otimes \langle 1, \gamma_1^{-1}\gamma_2\rangle \otimes \langle 1, \gamma_2^{-1}\gamma_3\rangle$. Following a suggestion of Serre, Rost ([R]) constructed a mod 3 invariant

$$g_3 : H^1(k,G) \to H^3(k,\mathbb{Z}/3).$$

If $J = J(B,\sigma,u,\mu)$ is a second Tits' construction Albert algebra corresponding to a quadratic extension L of k, then $g_3(J) \in H^3(k,\mathbb{Z}/3)$ maps to $[B]\cup[\mu] \in H^3(L,\mathbb{Z}/3)$; more precisely,

$$g_3(J) = -cor_{L/k}([B]\cup[\mu]).$$

We begin by reviewing a result for Albert algebras belonging to Tits' first construction and which motivated the question posed in the introduction.

Let k be a field of characteristic different from 2 and 3. Let A be a degree 3 central simple (associative) algebra over k and let $\mu \in k^*$. Let $A_i = A$ for $i = 0,1,2$. On the $k-$vector space $J(A,\mu) = A_0 \oplus A_1 \oplus A_2$, we define a multiplication by

$$(a_0,a_1,a_2)(a_0',a_1',a_2') = (a_0.a_0' + \widetilde{a_1a_2'} + \widetilde{a_1'a_2},\ \widetilde{a_0}a_1' + \widetilde{a_0'}a_1$$
$$+\mu^{-1}a_2 \times a_2',\ a_2\widetilde{a_0'} + a_2'\widetilde{a_0} + \mu a_1 \times a_1'),$$

where $x.y,\ \widetilde{x},\ x\times y$ are defined as in the introduction. With this multiplication, $J(A,\mu)$ is an Albert algebra. Given $\mu,\mu' \in k^*$ with $\mu' = N(w)\mu$ for some unit $w \in A$, there is an isomorphism of $J(A,\mu)$ with $J(A,\mu')$ given by $(a_0,a_1,a_2) \mapsto (wa_0w^{-1}, wa_1, a_2w^{-1})$. Using the Rost invariant for Albert algebras, Petersson and Racine ([P-R]) proved that the converse is true, i.e., if $J(A,\mu) \simeq J(A,\mu')$, then there exists a unit $w \in A$ such that $\mu' = N(w)\mu$.

Let L/k be a quadratic extension and $J(B,\sigma,u_i,\mu_i)$ be the Tits' second construction with respect to units $u_i \in B, \sigma(u_i) = u_i$ and $\mu_i \in L^*$ with $N(u_i) = \mu_i\overline{\mu_i}$. The map

$$(\mathcal{H}(B,\sigma)\oplus B)\otimes_k L \to B_0 \oplus B_1 \oplus B_2,$$

given by

$$(a_0, a) \otimes \lambda \mapsto (\lambda a_0, \lambda a,\ \overline{\lambda} u_i \sigma(a)),$$

where B_1, B_2, B_3 are copies of B, gives an isomorphism $J(B, \sigma, u_i, \mu_i) \otimes_k L \simeq J(B, \mu_i)$, $i = 1, 2$. Thus, in view of the above discussion for algebras of the first kind, we have

Proposition 2.1 (cf. [P-R], p. 205)**.** *If $g_3(J(B, \sigma, u_1, \mu_1)) = g_3(J(B, \sigma, u_2, \mu_2))$, then $\mu_1^{-1}\mu_2 \in Nrd(B^*)$.*

Proof. Since $g_3(J(B, \sigma, u_1, \mu_1)) = g_3(J(B, \sigma, u_2, \mu_2))$, we have $[B] \cup [\mu_1] = [B] \cup [\mu_2]$, which gives $[B] \cup [\mu_1 \mu_2^{-1}] = 0$. Hence the algebra $J(B, \mu_1 \mu_2^{-1})$ is reduced ([R]). This implies $\mu_1 \mu_2^{-1} \in Nrd(B^*)$. □

Proposition 2.2. *If $f_3(J(B, \sigma, u_1, \mu_1)) = f_3(J(B, \sigma, u_2, \mu_2))$, then the rank 1 hermitian forms $\langle u_1 \rangle$ and $\langle u_2 \rangle$ over (B, σ) are equivalent.*

Proof. In view of ([BF-L]), it suffices to prove this after an odd degree base change of k. Let M be an odd degree extension of k such that $B_M = B \otimes_k M$ is split over $L_M = L \otimes_k M$. We may therefore assume that $(B, \sigma) = (M_3(L), *)$, where $*$ is given by $X^* = \Gamma^{-1} \overline{X}^t \Gamma,\ \Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, with $\gamma_i \in k^*$. We may further assume that $\gamma_1 \gamma_2 \gamma_3 = 1$. Let u_1 correspond to the matrix V_1 and u_2 to the matrix V_2 in $M_3(L)$. Then $det(V_i) = \mu_i \overline{\mu_i},\ i = 1, 2$. By Theorem 1.1,

$$J(M_3(L), *, V_1, \mu_1) \simeq \mathcal{H}_3(C_1(L^3, \overline{V_1 \Gamma^{-1}}, \psi_1), \Gamma)$$

and

$$J(M_3(L), *, V_2, \mu_2) \simeq \mathcal{H}_3(C_2(L^3, \overline{V_2 \Gamma^{-1}}, \psi_2), \Gamma),$$

where ψ_i is the trivialization of $disc(\overline{V_i \Gamma^{-1}})$ given by $\overline{\mu_i},\ i = 1, 2$. We therefore have

$$f_3(\mathcal{H}_3(C_1(L^3, \overline{V_1 \Gamma^{-1}}, \psi_1), \Gamma)) = f_3(\mathcal{H}_3(C_2(L^3, \overline{V_2 \Gamma^{-1}}, \psi_2), \Gamma)),$$

which implies that $n_{C_1} \simeq n_{C_2}$. So, by Corollary 1.2,

$$tr_{L/k}(\langle 1 \rangle \perp h_1) \simeq tr_{L/k}(\langle 1 \rangle \perp h_2),$$

where h_i is the hermitian form given by the matrix $\overline{V_i \Gamma^{-1}}$. Thus, $tr_{L/k} \circ h_1 \simeq tr_{L/k} \circ h_2$. By a theorem of Jacobson (cf. Appendix 2 of [M-H]), $h_1 \simeq h_2$. So there exists $W \in GL_3(L)$ such that $W \overline{V_1 \Gamma^{-1}} \overline{W}^t = \overline{V_2 \Gamma^{-1}}$, i.e., $\overline{W} V_1 (\Gamma^{-1} W^t \Gamma) = V_2$. Hence V_1 and V_2 are hermitian equivalent in $(M_3(L), *)$, which implies that the hermitian forms $\langle u_1 \rangle$ and $\langle u_2 \rangle$ over (B, σ) are equivalent. □

Proposition 2.3. *Let u_1, u_2 be units in (B, σ) with $\sigma(u_i) = u_i,\ n(u_i) = \mu_i \overline{\mu_i},\ \mu_i \in L^*$. Let $\langle u_1 \rangle$ and $\langle u_2 \rangle$ be hermitian equivalent over (B, σ) and let $\mu_1^{-1}\mu_2$ be a reduced norm from B. Then there exists $w \in B$ such that*

$$u_2 = w u_1 \sigma(w),\ \ \mu_2 = n(w) \mu_1.$$

Proof. We introduce an equivalence $\sim$ among the pairs (u, μ), where $u \in B$ with $\sigma(u) = u$ and $n(u) = \mu \overline{\mu},\ \mu \in L^*$, as follows: $(u, \mu) \sim (u', \mu') \Leftrightarrow$ there exists $w \in B$ with $u' = w u \sigma(w)$ and $\mu' = n(w)\mu$. We show that $(u_1, \mu_1) \sim (u_2, \mu_2)$.

Let $\mu_2 = n(w_1)\mu_1$. Then $(u_1, \mu_1) \sim (u', \mu_2)$, where $u' = w_1 u_1 \sigma(w_1)$. Since $\langle u_1 \rangle$ and $\langle u_2 \rangle$ are hermitian equivalent, we get that $\langle u' \rangle$ is hermitian equivalent to $\langle u_2 \rangle$. Let $w' \in B$ be such that $u_2 = w' u' \sigma(w')$. Now

$$n(u') = n(w_1) \overline{n(w_1)} \mu_1 \overline{\mu_1} = \mu_2 \overline{\mu_2} = n(u_2).$$

Thus, $n(w')\overline{n(w')} = 1$. Let $\lambda = n(w')$. By the following lemma due to Rost, there exist $w'' \in B$ such that $\lambda = n(w'')$ and $w''u_2\sigma(w'') = u_2$. Thus

$$(u', \mu_2) \sim (u_2, \lambda^{-1}\mu_2) \sim (w''u_2\sigma(w''), n(w'')\lambda^{-1}\mu_2) = (u_2, \mu_2).$$

Therefore, $(u_1, \mu_1) \sim (u_2, \mu_2)$. □

Lemma 2.4 (Rost)**.** *Let L/k be a quadratic extension. Let A be a degree 3 central simple (associative) algebra over L with an involution σ of the second kind. Let $x \in L^*$ be such that $x\sigma(x) = 1$. Assume that $x = N(a)$ for some $a \in A$. Then there exists $a' \in A$ with $x = N(a')$ and $a'\sigma(a') = 1$.*

In fact, we have the following more general lemma due to Suresh, the proof of which uses the following result of Merkurjev on norm principle for unitary groups.

Lemma 2.5 (Merkurjev)**.** *Let $x \in L^*$. Then $x = N(v\sigma(v)^{-1})$ for some $v \in A^*$ if and only if there exists $u \in A^*$ with $u\sigma(u) = 1$ and $x = N(u)$.*

Proof. (cf. Theorem 5.1.3 of [BF-P]).

Lemma 2.6 (Suresh)**.** *Let L/k be a quadratic extension and A a central simple (associative) algebra over L of odd degree with an involution of the second kind. Let $x \in L$ be such that $x\overline{x} = 1$. If x is the reduced norm of some element of A, then there exists an element $u \in A^*$ such that $u\sigma(u) = 1$ and $N(u) = x$.*

Proof. Let $x \in L^*$ be such that $x\overline{x} = 1$ and $x = N(v)$ for some $v \in A^*$. Then there exists $y \in L^*$ such that $x = y(\overline{y})^{-1}$. Since σ is the identity on k, for $\lambda \in k^*$ we have $x = \lambda y(\overline{\lambda y})^{-1}$. Therefore, by Lemma 2.5, it is enough to show that λy is a reduced norm of some element of A^* for some $\lambda \in k^*$. Let $[A : L] = (2r+1)^2$. We have

$$\begin{aligned} N(v^r\overline{y}) &= N(v)^r(\overline{y})^{2r+1} \qquad (\text{since } \sigma(y) \in L^*) \\ &= x^r(\overline{y})^{2r+1} \\ &= (y^r(\overline{y})^{-r})(\overline{y})^{2r+1} \\ &= (y\overline{y})^r\overline{y}. \end{aligned}$$

Let $\lambda = (y\overline{y})^r \in k$. Then it follows that $\lambda\overline{y}$ is a reduced norm from A^*, and hence λy is a reduced norm from A^*. This completes the proof of lemma. □

Theorem 2.7. *Let k be a field with characteristic different from 2 and 3. Let L/k be a quadratic extension, B a degree 3 central simple algebra over L with an involution σ of the second kind. Let u_i, $i = 1, 2$, be units in B with $\sigma(u_i) = u_i$ and $n(u_i) = \mu_i\overline{\mu_i}$, $\mu_i \in L^*$. Then $J(B, \sigma, u_1, \mu_1) \simeq J(B, \sigma, u_2, \mu_2)$ if and only if there is $w \in B$ with $u_2 = wu_1\sigma(w)$ and $\mu_2 = n(w)\mu_1$.*

Proof. If $u_2 = wu_1\sigma(w)$ and $\mu_2 = n(w)\mu_1$ then, as was remarked in the introduction, the map $(a_0, a) \mapsto (a_o, aw)$ gives an isomorphism of $J(B, \sigma, u_1, \mu_1)$ with $J(B, \sigma, u_2, \mu_2)$.

Conversely, suppose $J_1 = J(B, \sigma, u_1, \mu_1) \simeq J_2 = J(B, \sigma, u_2, \mu_2)$. Then $f_3(J_1) = f_3(J_2)$ and $g_3(J_1) = g_3(J_2)$. The theorem now follows from Propositions 2.1, 2.2 and 2.3. □

Theorem 2.8. *Let $J = J(B, \sigma, u, \mu)$, $J' = J(B, \sigma, u', \mu')$ be Albert algebras such that $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Then $J \simeq J'$.*

Proof. The proof follows from Propositions 2.1, 2.2, 2.3 and the 'if' part of Theorem 2.7. □

Remark. Theorem 2.7 asserts that the map $H^1(k, SU(B,\sigma)) \to H^1(k, F_4)$, on the Galois cohomology, corresponding to the second Tits' construction, is injective.

3. A Skolem Noether type theorem

In this section we prove the following "Skolem Noether" type theorem.

Theorem 3.1. *Let L/k be a quadratic field extension. Let $(B,\sigma), (B',\sigma')$ be degree 3 central simple algebras over L with involutions of the second kind over k. Let $\mathcal{H}(B,\sigma), \mathcal{H}(B',\sigma')$ denote the 9-dimensional Jordan algebras over k associated to the symmetric elements in $(B,\sigma), (B',\sigma')$ respectively. Suppose that $\mathcal{H}(B,\sigma)$ and $\mathcal{H}(B',\sigma')$ are subalgebras of an Albert algebra J over k and $\alpha : \mathcal{H}(B,\sigma) \simeq \mathcal{H}(B',\sigma')$ is an isomorphism of Jordan algebras. Then α extends to an automorphism of J.*

For the proof we need some lemmas (which are also consequences of a more general result proved by Jacobson [J], p. 210, Theorem 11. However, we give direct proofs for the sake of completeness.)

Lemma 3.2. *Let L/k be a quadratic field extension. Let B be a degree 3 central simple algebra with an involution σ of the second kind. Let $\alpha : \mathcal{H}(B,\sigma) \to \mathcal{H}(B,\sigma)$ be an automorphism of Jordan algebras. Then α is the restriction of an isomorphism $\widetilde{\alpha} : (B,\sigma) \simeq (B,\sigma)$ or $\widetilde{\alpha} : (B,\sigma) \simeq (B^{op},\sigma)$ of associative algebras with involutions.*

Proof. Consider the map $\alpha \otimes 1 : \mathcal{H}(B,\sigma) \otimes_k L \simeq \mathcal{H}(B,\sigma) \otimes_k L$. The map $\psi : \mathcal{H}(B,\sigma) \otimes_k L \to B$, given by $(x,\lambda) \mapsto x\lambda$, is an isomorphism of Jordan algebras over L. Let $\widetilde{\alpha} = \psi \circ (\alpha \otimes 1) \circ \psi^{-1} : B \simeq B$. Then $\widetilde{\alpha}$ restricts on $\mathcal{H}(B,\sigma)$ to α. By a theorem of Ancochea ([A]), $\widetilde{\alpha}$ is an automorphism or an anti-automorphism of the algebra B. We show that $\widetilde{\alpha}$ commutes with σ. For $x \in \mathcal{H}(B,\sigma), \sigma(x) = x$, so that

$$\widetilde{\alpha}\sigma(x) = \widetilde{\alpha}(x) = \alpha(x) = \sigma\widetilde{\alpha}(x).$$

Let $L = k(j)$ with $j^2 \in k$. Then, $\widetilde{\alpha}\sigma(j) = \widetilde{\alpha}(-j) = -j = \sigma\widetilde{\alpha}(j)$. Since $\mathcal{H}(B,\sigma)$ generates B as an L-algebra, it follows that $\widetilde{\alpha}$ commutes with σ on the whole of B. Hence $\widetilde{\alpha}$ has the required properties. □

Theorem 3.3. *Let B be a central simple L-algebra of degree 3 with involutions σ, σ' of the second kind. Let $Q_\sigma, Q_{\sigma'}$ denote the restrictions to $\mathcal{H}(B,\sigma)$ and $\mathcal{H}(B,\sigma')$, respectively, of the trace form of B. Then the involutions σ and σ' are isomorphic if and only if Q_σ and $Q_{\sigma'}$ are isometric.*

Proof. See [H-K-R-T], Proposition 4. □

Lemma 3.4. *Let L/k be a quadratic extension of k. Let $(B,\sigma), (B',\sigma')$ be degree 3 central simple (associative) algebras over L with involutions of the second kind. If $\alpha : \mathcal{H}(B,\sigma) \simeq \mathcal{H}(B',\sigma')$ is an isomorphism of Jordan algebras, then there exists an isomorphism $\widetilde{\alpha} : (B,\sigma) \to (B',\sigma')$ or $\widetilde{\alpha} : (B,\sigma) \to (B'^{op},\sigma')$ of associative algebras with involutions, which restricts to α.*

Proof. The isomorphism α extends to a Jordan algebra isomorphism

$$\mathcal{H}(B,\sigma) \otimes_k L \simeq \mathcal{H}(B',\sigma') \otimes_k L,$$

which in turn gives an isomorphism of the Jordan algebra B with B'. By a theorem of Ancochea, B is isomorphic or anti-isomorphic to B'. Replacing B' by B'^{op}, if necessary, we may assume that B is isomorphic to B'. In view of the fact that α is an isomorphism of Jordan algebras, we get that $Q_\sigma \simeq Q_{\sigma'}$. By Theorem 3.3 and the fact that B is isomorphic to B' we conclude that there exists an isomorphism $\widetilde{\beta}:(B,\sigma)\simeq(B',\sigma')$ or $(B,\sigma)\simeq(B'^{op},\sigma')$ of associative algebras with involutions. Thus $\widetilde{\beta}$ restricts to $\beta:\mathcal{H}(B,\sigma)\simeq\mathcal{H}(B',\sigma')$. Consider $\beta^{-1}\circ\alpha:\mathcal{H}(B,\sigma)\to\mathcal{H}(B,\sigma)$. By Lemma 3.2, there exists $\gamma:(B,\sigma)\simeq(B,\sigma)$ or $(B,\sigma)\simeq(B^{op},\sigma)$, which restricts to $\beta^{-1}\circ\alpha$. Let $\widetilde{\alpha}=\widetilde{\beta}\circ\gamma$. Then $\widetilde{\alpha}$ satisfies the requirements in the lemma. □

Proof of Theorem 3.1. By Lemma 3.4 there is an isomorphism $\widetilde{\alpha}:(B,\sigma)\to(B',\sigma')$ or $(B,\sigma)\to(B'^{op},\sigma')$ which restricts to α. By ([MC-2]), there are isomorphisms

$$\phi_1:J(B,\sigma,u,\mu)\simeq J,\ \phi_2:J(B',\sigma',u',\mu')\simeq J$$

for suitable u,μ,u',μ', which restrict to the inclusions of $\mathcal{H}(B,\sigma)$ and $\mathcal{H}(B',\sigma')$ in J. We have an isomorphism $J(\widetilde{\alpha}):J(B,\sigma,u,\mu)\simeq J(B',\sigma',\alpha(u),\mu)$ given by $(a_0,a)\mapsto(\alpha(a_0),\alpha(a))$. But $J(B',\sigma',\alpha(u),\mu)\simeq J(B',\sigma',u',\mu')$, since both are isomorphic to J. By Theorem 2.2 of §2, there exists $w'\in B'$ such that $u'=w'\alpha(u)\sigma'(w'),\ \mu'=N(w')\mu$. Let $\psi:J(B',\sigma',\alpha(u),\mu)\simeq J(B',\sigma',u',\mu')$ be given by $(a_0',a')\mapsto(a_0',a'w')$. Then ψ restricts to the identity map on $\mathcal{H}(B',\sigma')$. Let $\phi=\phi_2\circ\psi\circ J(\widetilde{\alpha})\circ\phi_1^{-1}$. Then for $x\in\mathcal{H}(B,\sigma)$, we have $\phi(x)=\phi_2\psi(\alpha(x))=\alpha(x)$. Thus ϕ is an automorphism of J extending α. □

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