TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 4, April 1998, Pages 1519–1553
S 0002-9947(98)01872-8

THE TRACE OF JET SPACE $J^k\Lambda^\omega$ TO AN ARBITRARY CLOSED SUBSET OF $\mathbb{R}^n$

YURI BRUDNYI AND PAVEL SHVARTSMAN

Abstract. The classical Whitney extension theorem describes the trace $J^k|_X$ of the space of k-jets generated by functions from $C^k(\mathbb{R}^n)$ to an arbitrary closed subset $X \subset \mathbb{R}^n$. It establishes existence of a bounded linear extension operator as well. In this paper we investigate a similar problem for the space $C^k\Lambda^\omega(\mathbb{R}^n)$ of functions whose higher derivatives satisfy the Zygmund condition with majorant ω. The main result states that the vector function $\vec{f} = (f_\alpha\colon X \to \mathbb{R})_{|\alpha|\le k}$ belongs to the corresponding trace space if the trace $\vec{f}|_Y$ to every subset $Y \subset X$ of cardinality $3 \cdot 2^\ell$, where $\ell = \binom{n+k-1}{k+1}$, can be extended to a function $f_Y \in C^k\Lambda^\omega(\mathbb{R}^n)$ and $\sup_Y |f_Y|_{C^k\Lambda^\omega} < \infty$. The number $3 \cdot 2^l$ generally speaking cannot be reduced. The Whitney theorem can be reformulated in this way as well, but with a two-pointed subset $Y \subset X$. The approach is based on the theory of local polynomial approximations and a result on Lipschitz selections of multivalued mappings.

1. Introduction

We let $J^k\Lambda^\omega(\mathbb{R}^n)$ denote the space of k-jets generated by functions of the space $C^k\Lambda^\omega(\mathbb{R}^n)$. Let us recall that this space consists of k-times continuously differentiable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ satisfying

$$|f|_{C^k\Lambda^\omega} := \sum_{|\alpha|=k} \sup_{x,y\in\mathbb{R}^n} \frac{|D^\alpha f(x) - 2D^\alpha f\left(\frac{x+y}{2}\right) + D^\alpha f(y)|}{\omega(|x-y|)} < \infty.$$

Here $\omega\colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous non-decreasing function such that $\omega(t)/t^2$ is non-increasing and $\omega(0) = 0$. Throughout the paper the symbol ":=" indicates that the statement is a definition.

Thus the space $J^k\Lambda^\omega(\mathbb{R}^n)$ consists of k-jets $\vec{f} = (f_\alpha\colon |\alpha| \le k)$ such that

$$f_\alpha = D^\alpha f, \quad |\alpha| \le k$$

for some function f from $C^k\Lambda^\omega(\mathbb{R}^n)$. We equip this space with the seminorm

$$|\vec{f}|_{J^k\Lambda^\omega} := |f|_{C^k\Lambda^\omega}.$$

Now let X be an arbitrary closed subset of $\mathbb{R}^n$.

Received by the editors February 28, 1995 and, in revised form, July 25, 1996.
1991 *Mathematics Subject Classification.* Primary 46E35.
Key words and phrases. Trace spaces of smooth functions, Whitney's extension theorem, finiteness property, Lipschitz selections of multivalued mappings.
The first-named author was supported by the Fund for Promotion of Research at the Technion and the J. & S. Frankel Research Fund. The second-named author was supported by the Center for Absorption in Science, Israel Ministry of Immigrant Absorption.

Problem 1. Given a positive integer k and an arbitrary vector function $\vec{f} = (f_\alpha)_{|\alpha|\le k}$ defined on X, does there exist a k-jet $\vec{F} \in J^k\Lambda^\omega(\mathbb{R}^n)$ such that $\vec{F}|_X = \vec{f}$?

In other words, in this case we have to find a function $F \in C^k\Lambda^\omega(\mathbb{R}^n)$ such that

$$D^\alpha F|_X = f_\alpha, \quad \text{for all } \alpha, |\alpha| \le k.$$

Problem 2. Does there exist a linear bounded extension operator $E_X\colon J^k\Lambda^\omega|_X \to J^k\Lambda^\omega(\mathbb{R}^n)$?

Recall that for a (semi)-normed space A of (possibly vector-valued) functions defined on $\mathbb{R}^n$, $A|_X$ denotes the trace space; i.e., $A|_X := \{F|_X\colon F \in A\}$ with the (semi-)norm

$$\|f\|_{A|_X} := \inf\{\|F\|_A\colon F \in A, F|_X = f\}.$$

Since the space $\Lambda^\omega (:= C^0\Lambda^\omega)$ for $\omega(t) = t^\theta$ $(0 < \theta < 1)$ coincides (up to factorization by polynomials of degree 1) with the Lipschitz space Lip_θ, in this case the solutions to the above problems have been given by the classical Whitney extension theorem [W]; see also G. Glaeser [G]. For our purpose it would be appropriate to formulate this result in the following equivalent way.

For the sake of brevity we will denote by W the space $J^k\,\mathrm{Lip}_\theta(\mathbb{R}^n)$. Now let Y be an arbitrary subset of X. We denote by W_Y the space of vector functions $\vec{f} = (f_\alpha)_{|\alpha|\le k}$ *defined on* X, which is equipped with the seminorm

$$|\vec{f}|_Y := \inf\{|\vec{F}|_W\colon \vec{F}|_Y = \vec{f}|_Y, \vec{F} \in W\}. \tag{1.1}$$

Let us recall that the norm of intersection $\bigcap_{i\in I} B_i$ of a family of Banach spaces $\{B_i\}_{i\in I}$ is defined as $\sup_{i\in I}\|\cdot\|_{B_i}$.

Theorem A (H. Whitney)**.** (a) *The following isomorphism is valid with constants of equivalence depending on n and k only:*

$$W|_X = \bigcap_{Y\subset X} W_Y, \tag{1.2}$$

where Y runs over all two-point subsets of X.

(b) *There exists a linear bounded extension operator $E_X\colon W|_X \to W$. The operator norm of E_X is bounded by a constant depending on n and k only.*

Thus the statement (a) of this theorem means that a vector function $\vec{f}$ defined on X belongs to $W|_X$ if and only if for some constant $\lambda = \lambda(\vec{f}) > 0$ the restriction $\vec{f}|_{\{x,y\}}$ to an *arbitrary two-point* subset $\{x, y\}$ of X can be extended to a k-jet $\vec{F}_{\{x,y\}} \in W$ with the seminorm $|\vec{F}_{\{x,y\}}|_W \le \lambda$. It is not hard to see that the latter condition is equivalent to the inequalities

$$|f_\alpha(x) - \sum_{|\alpha+\beta|\le k} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta| \le \gamma(n,k)\lambda(\vec{f})|x-y|^{k-|\alpha|+\theta} \quad (|\alpha| \le k) \tag{1.3}$$

which precisely coincide with the statement of Whitney's extension theorem.

The above reformulation gives a direction for seeking a solution to the problems formulated. As will be shown below, both statements of Theorem A are valid also in the case of the jet-space $J^k\Lambda^\omega(\mathbb{R}^n)$. The only distinction is in cardinality of the set Y in (1.2). Now this number, rather surprisingly, depends on n and k and grows extra-exponentially with the dimension. The difference in order of magnitude of this number for these two cases is somehow a measure of the comparative difficulty of dealing with the spaces $J^k\Lambda^\omega(\mathbb{R}^n)$.

To formulate our main result we denote by $\mathcal{J}$ the space $J^k\Lambda^\omega(\mathbb{R}^n)$ and set $N(n,k) = \binom{n+k-1}{k+1}$.

Theorem 1.1. (a) (*finiteness*). *The following isomorphism is valid with constants of equivalence depending on n and k only, and*

$$\mathcal{J}|_X = \bigcap_{Y\subset X} \mathcal{J}_Y,$$

where Y runs over all subsets of X, consisting of $3\cdot 2^{N(n,k)}$ points.

(b) (*linearity*). *There exists a linear bounded extension operator $E_X : \mathcal{J}|_X \to \mathcal{J}$. The operator norm of E_X is bounded by a constant depending on n and k only.*

Remark 1.2. It is readily seen that the number of points in part (a) is generally speaking greater than the dimension $\binom{n+k+1}{n}$ of the space $\mathcal{P}_{k+1}(\mathbb{R}^n)$ of polynomials of degree $\le k+1$. On the other hand, in case $k = 0$, the number $3\cdot 2^{N(n,0)} = 3\cdot 2^{n-1}$ cannot be lowered (see [Sh1]). In the forthcoming publication we will prove that this number is sharp for $k > 0$ as well.

It is also worth mentioning that the large number of points in the finiteness property (see assertion (a)) leads to an essential complication of the construction of the linear extension operator. For the same reason a "constructive" characterization of the elements of the trace space $\mathcal{J}|_X$ cannot be obtained in the same simple way as in Whitney's criteria (1.3). We intend to devote a forthcoming paper to the problem of a constructive description of the trace space $\mathcal{J}|_X$ (according to assertion (a) it suffices to solve the problem for a set with $3\cdot 2^{N(n,k)}$ points).

The main result of this paper was announced in [BS]. In the special case $k = 0$ the finiteness property of Theorem 1.1(a) was conjectured by the first-named author and was proved in the exact form by the second (see [Sh], [Sh1]). The assertion (b) of Theorem 1.1 (linearity) for $k = 0$ was announced in [BS1] and proved in [BS2]. The relatively simple case $n = 1$, k is arbitrary was then treated in the paper [H], where a one-dimensional analog of Theorem 1.1 was obtained in a different but equivalent formulation.

The paper is organized as follows. Section 2 contains the proof of Theorem 1.1 except for the proof of two (rather technical) assertions concerning a description of the space $J^k\Lambda^\omega$ in terms of local polynomial approximations. The first of these, Proposition 2.4, we prove in Section 3. Finally, Section 4 is devoted to the proof of the second, Proposition 2.5.

2. Proof of Theorem 1.1

2.1. **Proof of the finiteness property (Assertion (a) of Theorem 1.1).** The proof consists of the following two main stages:

(1) We describe the space $J^k\Lambda^\omega(\mathbb{R}^n)|_X$ in terms of the theory of local polynomial approximations (Propositions 2.3–2.5).

(2) Using the above description of the trace space, we reduce the proof of the finiteness property to the problem of the existence of a Lipschitz selection for some

multivalued mapping generated by a vector function $\vec{f} = (f_\alpha\colon X \to \mathbb{R})_{|\alpha|\le k}$ (Proposition 2.8). Finally we make use of the criteria of the existence of a Lipschitz selection (Theorem 2.10), and in this way the proof will be complete.

In the first stage the following notion plays the key role.

Let $\mathcal{P}_k \subset \mathbb{R}[x_1, \dots, x_n]$ be the space of polynomials of degree $\le k$. Let f be a locally bounded measurable function on $\mathbb{R}^n$.

Definition 2.1. *The local polynomial approximation of f* of order $k \in \mathbb{N} \cup \{0\}$ is the mapping $Q \to E_k(f; Q)$ defined on the set $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$ of n-cubes Q by

$$E_k(f; Q) := \inf_{p \in \mathcal{P}_{k-1}} \sup_{x \in Q} |f(x) - p(x)|.$$

Remark 2.2. (a) Throughout the paper $Q = Q_r(c)$ denotes a cube of radius r centered in c; i.e.,

$$Q_r(c) := \{x \in \mathbb{R}^n; |x - c| \le r\},$$

where $|x| := \max_{1\le i\le n} |x_i|$ is the ℓ_∞-norm on $\mathbb{R}^n$.

(b) $\mathcal{P}_{k-1} := \{0\}$ for $k \le 0$.

Based on this notion, as a consequence of the main result of [B] we get the following

Proposition 2.3.

$$|f|_{\Lambda^\omega(\mathbb{R}^n)} \approx \sup_{Q_r \in \mathcal{K}} \frac{E_2(f; Q_r)}{\omega(r)}$$

with[1] *the constants independent of f and ω.*

This assertion leads us to the next description of the trace space $J^k\Lambda^\omega(\mathbb{R}^n)|_X$ in terms of the local polynomial approximations. For its formulation we let $\mathcal{K}_X$ denote the family of cubes centered in X. As before, for the sake of brevity we also put

$$\mathcal{J} = J^k\Lambda^\omega(\mathbb{R}^n).$$

Proposition 2.4. *A vector function $\vec{f} = (f_\alpha\colon X \to \mathbb{R})_{|\alpha|\le k}$ belongs to $\mathcal{J}|_X$ if and only if there exist a positive constant λ and a family of polynomials $\{P_Q\colon Q \in \mathcal{K}_X\} \subset \mathcal{P}_{k+1}$ such that*

(i) *$(D^\alpha P_Q)(c_Q) = f_\alpha(c_Q)$ for all α, $|\alpha| \le k$ and $Q \in \mathcal{K}_X$. Here and below, c_Q denotes the center of Q and r_Q its radius;*

(ii) *for every pair of cubes $Q, Q' \in \mathcal{K}_X$ satisfying $Q' \subset Q$ the following inequality*

$$\sup_{Q'} |P_{Q'} - P_Q| \le \lambda\{|c_Q - c_{Q'}| + r_{Q'}\}^k \omega(r_Q)$$

holds. Moreover, the trace norm of $\vec{f}$ satisfies

$$|\vec{f}|_{\mathcal{J}|_X} \approx \inf \lambda.$$

[1] Here and below "$f \approx g$" means that $c^{-1}f \le g \le cf$ with some constant $c > 0$ depending on n and k only.

See Section 3 for the proof.

For the purposes of the second stage we now have to choose from the above family of polynomials $\{P_Q\}$ a subfamily which also gives a description of $\mathcal{J}|_X$ but has some additional "interpolation" properties. To this end we consider the set

$$U(X) := \{(x, y) \in X \times X : x \neq y\}.$$

We shall regard $U(X)$ as a graph whose vertices $u = (x, y)$ and $u' = (x', y')$ are considered to be joined by an edge if $\{x, y\} \cap \{x', y'\} \neq \varnothing$ (written $u \leftrightarrow u'$). Let ψ be a function defined on pairs $u = (x, y)$ and $u' = (x', y')$ of vertices joined by an edge by the formula

$$\psi(u, u') := \int_{\min(|x-y|,|x'-y'|)}^{|x-y|+|x'-y'|} \frac{\omega(t)}{t^2} dt \quad \text{for } u \neq v \text{ and } \psi(u, u) := 0. \tag{2.1}$$

Proposition 2.5. *A vector function $\vec{f} = (f_\alpha : X \to \mathbb{R})_{|\alpha| \le k}$ belongs to the trace space $\mathcal{J}|_X$ if and only if there exist a positive constant λ and a mapping $u \to P_u$ from $U(X)$ into $\mathcal{P}_{k+1}$ such that:*

(i) *For every $u = (x, y) \in U(X)$*

$$(D^\alpha P_u)(x) = f_\alpha(x)$$

for all α with $|\alpha| \le k$ and

$$(D^\alpha P_u)(y) = f_\alpha(y)$$

for α, $|\alpha| = k$.

(ii)

$$|f_\alpha(y) - (D^\alpha P_u)(y)| \le \lambda |x - y|^{k-|\alpha|} \omega(|x - y|)$$

for all α with $|\alpha| \le k$ and $u = (x, y) \in U(X)$.

(iii) *For every $u, v \in U(X)$ joined by an edge and all α with $|\alpha| = k + 1$, the set of numbers $\{D^\alpha P_u\}_{|\alpha|=k+1}$ satisfies*

$$|D^\alpha P_u - D^\alpha P_v| \le \lambda \psi(u, v).$$

Moreover, the trace norm

$$|\vec{f}|_{\mathcal{J}|_X} \approx \inf \lambda.$$

See Section 4 for the proof.

Let us now reformulate Proposition 2.5 to a form which will be applied in the next step of the proof. To this end we will introduce some notation. Let $\mathcal{H}_k = \mathcal{H}_k(\mathbb{R}^n)$ be the space of homogeneous polynomials of degree k. We define the norm in this space by

$$\|H\|_{\mathcal{H}_k} := \max_{|\alpha|=k} |D^\alpha H|, \quad H \in \mathcal{H}_k.$$

Proposition 2.6. *A vector function $\vec{f} = (f_\alpha : X \to \mathbb{R})_{|\alpha| \le k}$ belongs to $\mathcal{J}|_X$ if and only if there exist a positive constant λ and a mapping $H : U(X) \to \mathcal{H}_{k+1}$ such that:*

(i) *For every two-point set $Y \subset X$*

$$|\vec{f}|_Y \le \lambda \tag{2.2}$$

(see (1.1) for definition of $|\cdot|_Y$).

(ii) *For every* $u=(x,y)\in U(X)$ *and every* α, $|\alpha|=k$

$$(D^{\alpha}H(u))(x-y)=f_{\alpha}(x)-f_{\alpha}(y).$$

(iii) *For every pair* $u,v\in U(X)$ *joined by an edge* (*i.e.*, $u\leftrightarrow v$)

$$\|H(u)-H(v)\|_{\mathcal{H}_{k+1}}\le\lambda\psi(u,v).$$

Besides the trace norm

$$|\vec{f}|_{\mathcal{J}|_X}\approx\inf\lambda.$$

Proof (*necessity*). Let $\vec{f}\in\mathcal{J}|_X$. Since $|\vec{f}|_Y\le|\vec{f}|_{\mathcal{J}|_X}$, $Y\subset X$, the assertion (i) of the proposition under consideration holds. Let us prove the assertions (ii) and (iii).

By Proposition 2.5 there exist a constant $\lambda>0$ and a mapping $U(X)\ni u\to P_u\in\mathcal{P}_{k+1}$ such that the assertions (i)–(iii) of the proposition hold.

We introduce the required mapping $H\colon U(X)\to\mathcal{H}_{k+1}$ by

$$(H(u))(z):=\sum_{|\alpha|=k+1}\frac{(D^{\alpha}P_u)(z)}{\alpha!}z^{\alpha}\quad(z\in\mathbb{R}^n).$$

Then for arbitrary α, $|\alpha|=k$, we have

$$\begin{aligned}(D^{\alpha}P_u)(z)&=\sum_{i=1}^{n}(D^{\alpha+\ell_i}P_u)(z_i-x_i)+(D^{\alpha}P_u)(x)\\&=(D^{\alpha}H(u))(z-x)+(D^{\alpha}P_u)(x).\end{aligned}$$

Setting $z=y$ and applying (i) of Proposition 2.5, we obtain the desired statement (ii). The remaining statement (iii) is a simple consequence of the inequality (iii) of Proposition 2.5. From this inequality and the definition of the norm in $\mathcal{H}_k$, it follows that

$$\begin{aligned}\|H(u)-H(v)\|_{\mathcal{H}_{k+1}}&=\max_{|\alpha|=k+1}|D^{\alpha}(H(u))-D^{\alpha}(H(v))|\\&=\max_{|\alpha|=k+1}|D^{\alpha}P_u-D^{\alpha}P_v|\le\lambda\psi(u,v).\end{aligned}$$

Proof (*sufficiency*). Suppose that all the assumptions of the proposition have held and define a polynomial $P_u\in\mathcal{P}_{k+1}$, $u=(x,y)\in U(X)$ by

$$P_u(z)=(H(u))(z-x)+\sum_{|\alpha|\le k}\frac{f_{\alpha}(x)}{\alpha!}(z-x)^{\alpha}.\tag{2.3}$$

Let us show that the family $\{P_u\colon u\in U(X)\}$ satisfies the conditions (i)–(iii) of Proposition 2.5. By definition

$$(D^{\beta}P_u)(x)=f_{\beta}(x),\quad|\beta|\le k,\tag{2.4}$$

and for every α, $|\alpha|=k$,

$$(D^{\alpha}P_u)(y)=(D^{\alpha}H(u))(y-x)+f_{\alpha}(x)=f_{\alpha}(y)-f_{\alpha}(x)+f_{\alpha}(x)=f_{\alpha}(y).\tag{2.5}$$

The statement (i) of Proposition 2.5 is fulfilled. To prove inequality (ii) of this proposition we set $Y=\{x,y\}$ and apply the assertion (i) of the proposition under consideration. By this assertion, there exists a vector function

$$\vec{f}_Y=(\tilde{f}_{\alpha}\colon\mathbb{R}^n\to\mathbb{R})_{|\alpha|\le k}\in\mathcal{J}$$

such that $|\vec{f}_Y|_{\mathcal{J}} \le \lambda$. Therefore by Proposition 2.5 (necessity part) there exists a polynomial $\widetilde{P}_u \in \mathcal{P}_{k+1}$, $u = (x, y) \in U(X)$, such that $(D^\beta \widetilde{P}_u)(x) = f_\beta(x)$ for all β with $|\beta| \le k$, $D^\alpha \widetilde{P}_u(y) = f_\alpha(y)$ for α, $|\alpha| = k$, and

$$|f_\alpha(y) - (D^\alpha \widetilde{P}_u(y))| \le \gamma(n,k)\lambda|x-y|^{k-|\alpha|}\omega(|x-y|), \quad |\alpha| \le k.$$

So, it suffices to prove that

$$D^\alpha \widetilde{P}_u(y) = D^\alpha P_u(y) \tag{2.6}$$

for all α with $|\alpha| \le k$ and the desired inequality (ii) of Proposition 2.5 follows. To do this we remark that the polynomial $S = P(\cdot + x) - \widetilde{P}(\cdot + x)$ belongs to $\mathcal{H}_{k+1}$ and $D^\alpha S(a) = 0$, $|\alpha| = k$, where $a = y - x$. Then $D^\alpha S(a) = 0$ for all α with $|\alpha| \le k$, which[2] implies the property (2.6).

The remaining inequality (iii) of Proposition 2.5 is a simple consequence of the identity

$$D^\alpha P_u = D^\alpha(H(u)), \quad |\alpha| = k+1,$$

and the statement (iii) of the proposition under consideration. □

Remark 2.7. Suppose that H maps $U(X) \times \mathcal{J}|_X$ into $\mathcal{H}_{k+1}$ and H is *linear* in the second argument; i.e.,

$$H(u, s\vec{f} + t\vec{g}) = sH(u, \vec{f}) + tH(u, \vec{g}), \quad s, t \in \mathbb{R}.$$

Let H satisfy the conditions (ii) and (iii) of Proposition 2.6 with the constant $\lambda = \lambda(\vec{f})$ satisfying

$$\lambda(\vec{f}) \le \gamma|\vec{f}|_{\mathcal{J}|_X}, \quad \vec{f} \in \mathcal{J}|_X.$$

Then there exists a linear extension operator $E_X \colon \mathcal{J}|_X \to \mathcal{J}(\mathbb{R}^n) := J^k\Lambda^\omega(\mathbb{R}^n)$ with the norm[3]

$$\|E_X\| \le O(1)\gamma.$$

To prove this variant of the sufficiency part of Proposition 2.6 we first remark that in this situation the polynomials P_u in (2.3) are also linear functions of $\vec{f}$. Therefore the extension operator constructed in Section 3, formula (3.5) is also linear in $\vec{f}$; see the sufficiency part of Section 4, the formulas (4.11)–(4.14). The inequality for the norm of this extension operator directly follows from the proof of the sufficiency part of Propositions 2.4 and 2.5; see Sections 3 and 4. □

We shall now consider the second stage of the proof of Theorem 1.1.

Let $F \colon \mathcal{M} \to S(\mathcal{N})$ be a "multivalued" mapping of a metric space $(\mathcal{M}, \rho)$ into the family of subsets of a normed space $\mathcal{N}$. Recall that a mapping $f \colon \mathcal{M} \to \mathcal{N}$ is called a *Lipschitz selection* of F if

$$f(x) \in F(x) \quad (x \in \mathcal{M})$$

and besides the seminorm

$$|f|_{\mathrm{Lip}(\mathcal{M},\mathcal{N})} := \sup_{x \ne y} \frac{\|f(x) - f(y)\|_{\mathcal{N}}}{\rho(x,y)}$$

is finite.

---

[2]By invariance of this assertion with respect to the orthogonal transformations, the proof is reduced to the case $a = (1, 0, \ldots, 0)$, which can be verified by straightforward calculation.

[3]Here and below $O(1)$ denotes a positive constant depending only on n and k (and in Section 3 also on m).

Let us now describe the particular spaces $\mathcal{M}$ and $\mathcal{N}$ and the set-valued map F which we shall need in this step. Namely, let $\mathcal{M}$ be the graph $U(X)$ with the "geometric" metric ρ_X defined by

$$\rho_X(u,v) := \inf\left\{\sum_{i=0}^{m-1}\psi(u_i,u_{i+1})\right\} \tag{2.7}$$

where the infimum is taken over all finite paths $\{u_0,\dots,u_m\}$ joining u and v (i.e., $u_0=u$, $u_m=v$ and u_i is joined to u_{i+1} by an edge). Let $\mathcal{N}$ be the normed space $\mathcal{H}_{k+1}$ as well.

Finally, for a fixed vector function $\vec{f}=(f_\alpha\colon X\to\mathbb{R})_{|\alpha|\le k}$ we define a multivalued map $L_{\vec{f}}$ from $(U(X),\rho_X)$ into the set $\operatorname{Aff}(\mathcal{H}_{k+1})$ of all affine manifolds of $\mathcal{H}_{k+1}$ by

$$L_{\vec{f}}(u) := \{h\in\mathcal{H}_{k+1}\colon (D^\alpha h)(x-y)=f_\alpha(x)-f_\alpha(y) \text{ for all } \alpha, |\alpha|=k\} \tag{2.8}$$

where $u=(x,y)\in U(X)$. Let us note here that

$$\dim L_{\vec{f}}(u) = N(n,k) = \binom{n+k-1}{k+1} \tag{2.9}$$

for all $u\in U(X)$ (see Remark 2.17).

Within this notion Proposition 2.6 can be reformulated as follows.

Proposition 2.8. *$\vec{f}\in\mathcal{J}|_X$ if and only if there exists a positive constant λ such that the inequality* (2.2) *holds and $L_{\vec{f}}$ has a Lipschitz selection $h\colon U(X)\to\mathcal{H}_{k+1}$ with the seminorm*

$$|h|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})}\le\lambda.$$

Moreover, the trace norm

$$|\vec{f}|_{\mathcal{J}|_X}\approx\inf\lambda. \quad \square$$

Thus the proof of Theorem 1.1 is reduced to the existence problem of a Lipschitz selection of $L_{\vec{f}}$. The solution of the problem is contained in Theorem 2.10 stated below (see [Sh2] and [Sh3] for a more general situation). For its formulation we shall need the following

Definition 2.9. A subset U' of $U(X)$ is said to be *admissible* if, being regarded as a subgraph of the graph $U(X)$, it has no isolated vertices.

Theorem 2.10. *If the trace $L_{\vec{f}}|_{U'}$ to every admissible subset $U'\subset U$ consisting of at most $2^{N(n,k)+1}$ points has a Lipschitz selection $h_{U'}$ with*

$$|h_{U'}|_{\mathrm{Lip}(U',\mathcal{H}_{k+1})}\le\lambda,$$

then $L_{\vec{f}}$ has a Lipschitz selection h such that

$$|h|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})}\le O(1)\lambda.$$

Recall that $N(n,k)$ is defined by (2.9) and $O(1)$ denotes a constant depending on n and k.

We are now in a position to prove Theorem 1.1. Since for an arbitrary subset Y of X the embedding

$$\mathcal{J}|_X\hookrightarrow\mathcal{J}_Y$$

is obvious, it remains to prove the embedding

$$\bigcap_{Y\subset X}\mathcal{J}_Y \hookrightarrow \mathcal{J}|_X \tag{2.10}$$

where Y runs over all subsets of Y of cardinality $\operatorname{card} Y \le 3\cdot 2^{N(n,k)}$.

To prove (2.10) we need an auxiliary result. For its formulation we set

$$\mathcal{J}^m(X) := \bigcap_{Y\subset X}\mathcal{J}_Y \tag{2.11}$$

where $Y\subset X$ runs over all m-pointed subsets and $\mathcal{J}_Y$ is defined by (1.1).

Lemma 2.11. *If $\vec{f}\in\mathcal{J}^m(X)$, then $L_{\vec{f}}|_V$ has a Lipschitz selection h_V with the seminorm*

$$|h_V|_{\mathrm{Lip}(V,\mathcal{H}_{k+1})} \le O(1)|\vec{f}|_{\mathcal{J}^m(X)}$$

for any admissible $V\subset U(X)$ of cardinality $\le \frac{2}{3}m$.

Proof. We make use of the following simple statement of the graph theory. Let Γ be a graph with v vertices and r edges. Suppose that Γ has no isolated edges. Then

$$v\le \frac{3}{2}r. \tag{2.12}$$

The proof is left to the reader.

Consider now the set

$$X_V := \{x\in X;\ \text{there exists } v\in V \text{ such that } v=(x,y) \text{ or } v=(y,x)\}$$

and prove that

$$\operatorname{card} X_V \le m. \tag{2.13}$$

Let us equip X_V with the oriented graph structure induced by V; i.e., (x,y) is an oriented edge of the graph if $(x,y)\in V$. Because of admissibility of V this graph has no isolated edges. Hence, applying (2.12) we deduce that the inequality (2.13) holds.

From here and definition (2.11) it follows that the trace $\vec{g} := \vec{f}|_{X_V}$ of the vector function $\vec{f}$ belongs to $\mathcal{J}|_{X_V}$ and besides

$$|\vec{g}|_{\mathcal{J}|_{X_V}} \le |\vec{f}|_{\mathcal{J}^m(X)}.$$

Applying now the necessity part of Proposition 2.8 to the set X_V and the function $\vec{g}$, we conclude that the multivalued mapping $L_{\vec{g}}\colon U(X_V)\to \mathrm{Aff}(\mathcal{H}_{k+1})$ has a Lipschitz selection h with

$$|h|_{\mathrm{Lip}(U(X_V),\mathcal{H}_{k+1})} \le O(1)|\vec{f}|_{\mathcal{J}^m(X)}. \tag{2.14}$$

But $\vec{g}=\vec{f}|_{X_V}$ and therefore

$$L_{\vec{g}} = (L_{\vec{f}})|_{U(X_V)}.$$

On the other hand, by definition, $V\subset U(X_V)$ and consequently

$$(L_{\vec{f}})|_V = (L_{\vec{g}})|_V.$$

So, if we now set $h_V := h|_V$, we get a selection of $(L_{\vec{f}})|_V$ and, moreover,

$$|h_V|_{\mathrm{Lip}(V,\mathcal{H}_{k+1})} \le |h|_{\mathrm{Lip}(U(X_V),\mathcal{H}_{k+1})}.$$

This inequality and (2.14) imply the desired inequality of Lemma 2.11. $\square$

We are now in a position to complete the proof of the finiteness property of Theorem 1.1. We have to show that if

$$\vec{f} \in \mathcal{J}^m(X), \quad m = 3 \cdot 2^{N(n,k)},$$

where $N(n,k) := \binom{n+k-1}{k+1}$, then $\vec{f}$ belongs to the trace space $\mathcal{J}|_X$. To accomplish this it suffices to verify that $\vec{f}$ satisfies the conditions of Proposition 2.8. Since $\vec{f} \in \mathcal{J}^m(X)$ and $m \geq 2$, the inequality (2.2) holds with $\lambda \leq |\vec{f}|_{\mathcal{J}^m(X)}$.

So it remains to establish the existence of a Lipschitz selection of $L_{\vec{f}}$. But according to Lemma 2.11 the restriction $L_{\vec{f}}|_V$ of $L_{\vec{f}}$ to an arbitrary admissible subset $V \subset U(X)$ of cardinality

$$\operatorname{card} V \leq 2/3m := 2^{N(n,k)+1} \tag{2.15}$$

has a Lipschitz selection h_V with the seminorm

$$|h_V|_{\mathrm{Lip}(V,\mathcal{H}_{k+1})} \leq O(1)|\vec{f}|_{\mathcal{J}^m(X)}. \tag{2.16}$$

Furthermore, the set $L_{\vec{f}}(u)$, $u \in U(X)$, is an affine submanifold of $\mathcal{H}_{k+1}$ of dimension $N(n,k)$ (see (2.9)). Therefore (2.15), (2.16) and Theorem 2.10 imply the existence of a Lipschitz selection $h\colon U(X) \to \mathcal{H}_{k+1}$ of $L_{\vec{f}}$ such that

$$|h|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})} \leq O(1)|\vec{f}|_{\mathcal{J}^m(X)}.$$

Thus, all the assumptions of the sufficiency part of Proposition 2.8 are fulfilled. According to this proposition $\vec{f} \in \mathcal{J}|_X$ and its trace norm satisfies

$$|\vec{f}|_{\mathcal{J}|_X} \leq O(1)|\vec{f}|_{\mathcal{J}^m(X)}.$$

The proof of statement (a) of Theorem 1.1 is complete.

2.2. **Linearity (Part (b) of the theorem).** The main goal of this proof is linearization of the extension operator constructed (implicitly) in Section 2.1. To this end we make use of the next "linearized" version of Proposition 2.8. It would be appropriate to divide this result into two parts.

Proposition 2.12. *If $\vec{f}$ belongs to the trace space $\mathcal{J}|_X$, then the set-valued mapping $L_{\vec{f}}$ has a Lipschitz selection $h\colon U(X) \to \mathcal{H}_{k+1}$ such that*

$$|h|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})} \leq O(1)|\vec{f}|_{\mathcal{J}|_X}.$$

This is obviously the necessity part of Proposition 2.8.

The next result is a direct reformulation of the statement of Remark 2.7.

Proposition 2.13. *Let L be the set-valued map defined on the set $U(X) \times \mathcal{J}|_X$ by*

$$L(u;\vec{f}) := L_{\vec{f}}(u)$$

(see (2.8)). Suppose that L has a Lipschitz selection $H\colon U(X) \times \mathcal{J}|_X \to \mathcal{H}_{k+1}$ which is linear in $\vec{f}$ and satisfies

$$|H|_{\mathrm{Lip}} \leq \gamma |f|_{\mathcal{J}|_X}, \tag{2.17}$$

where $\gamma > 0$ is a constant. Then there exists a linear bounded extension operator $E_X\colon \mathcal{J}|_X \to \mathcal{J}(\mathbb{R}^n)$ such that

$$\|E_X\| \leq O(1)\gamma.$$

According to Proposition 2.12, $L_{\vec f}$ has a Lipschitz selection depending on $\vec f$, which is, generally speaking, *nonlinear*. If we could find such a selection of $L_{\vec f}$ which depends on $\vec f$ *linearly* and satisfies (2.17), then the statement (b) of Theorem 1.1 would follow from Proposition 2.13.

Thus the proof is reduced to the problem of linear choice of a Lipschitz selection of the multivalued mapping L. For the solution of the problem we will need some notation and auxiliary results. Let $u = (x, y) \in U(X)$ and

$$F(u) := \{h \in \mathcal{H}_{k+1} \colon (D^\alpha h)(x-y) = 0, |\alpha| = k\}. \tag{2.18}$$

Definition 2.14. $\Sigma(X)$ is the space of mappings $G\colon U(X) \to \mathcal{H}_{k+1}$ such that the set-valued function $\mathcal{N}_G := G + F$ has a Lipschitz selection h. We equip this space with the seminorm

$$|G|_{\Sigma(X)} := \inf\{|h|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})} \colon h \text{ is a selection of } \mathcal{N}_G\}.$$

Proposition 2.15. *There exits a linear bounded operator*

$$T_X \colon \Sigma(X) \to \mathrm{Lip}(U(X), \mathcal{H}_{k+1})$$

such that $T_X(G)$ is a selection of $\mathcal{N}_G$ for any $G \in \Sigma(X)$. Moreover, $\|T_X\| \le O(1)$.

Proof. Theorem 4.7 of the paper [BS2] states that the following property of the metric space $(U(X), \rho_X)$ is sufficient for the existence of T_X. ☐

Definition 2.16 ($\mathcal{L}$-property)**.** For every $V \subset U(X)$ there exists a linear bounded extension operator $E_V\colon\ \mathrm{Lip}(V, \rho_X) \to \mathrm{Lip}(U(X), \rho_X)$ with the norm $\|E_V\| \le O(1)$.

As can be seen below, $(U(X), \rho_X)$ possesses the $\mathcal{L}$-property (see Proposition 2.21). But first of all we will explain how the existence of the desired extension operator E_X follows from Proposition 2.15.

We will make use of the following

Lemma 2.17. *For every $a \in \mathbb{R}^n\backslash\{0\}$ there exists a linear operator $S_a\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ such that*

$$D^\alpha(S_a h)(a) = D^\alpha h, \quad \textit{for all } \alpha, \qquad |\alpha| = k \tag{2.19}$$

and, moreover,

$$\|S_a\| \le O(1)\frac{1}{\|a\|}. \tag{2.20}$$

Proof. We let S_a denote the operator

$$(S_a h)(x) := \int_0^{\langle a,x\rangle} h(x - ta^*)\, dt$$

where $\langle a, x\rangle := \sum_{i=1}^n a_i x_i$ and $a^* := a/\langle a, a\rangle$. It is readily seen that $S_a\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ and that the inequality (2.20) holds.

So, it remains to check the property (2.19). Without loss of generality we can assume that $\|a\| = 1$, i.e., $a^* = a$. We can then write the difference $S_a h - h$ ($h \in \mathcal{H}_k$) as follows:

$$\int_1^{\langle a,x\rangle} h(x - ta)dt + \int_0^1 (h(x - ta) - h(x))dt =: H + P.$$

By Taylor's formula P is a polynomial of degree $k - 1$. On the other hand it is not hard to see that for arbitrary $t \in \mathbb{R}$

$$H(tx + a) = t^{k+1} H(x + a)$$

and, consequently, the polynomial $G := H(\cdot + a)$ belongs to $\mathcal{H}_{k+1}$. Finally,

$$D^\alpha(S_a h - h)(a) = (D^\alpha G)(0) = 0, \quad |\alpha| = k,$$

and the lemma follows. □

Remark 2.18. Using the lemma we can compute

$$\dim F(u) = \dim\{h \in \mathcal{H}_{k+1}; (D^\alpha h)(a) = 0, \text{ for all } \alpha, |\alpha| = k\}.$$

Here $u = (x, y) \in U(x)$ and $a := x - y \neq 0$ (see (2.18)). Really, $F(u)$ coincides with Ker T_a, where $T_a\colon \mathcal{H}_{k+1} \to \mathcal{H}_k$ is the linear operator defined by the equality

$$D^\alpha(T_a h) = (D^\alpha h)(a), \quad |\alpha| = k.$$

By the lemma T_a is a surjection. Hence

$$\begin{aligned}\dim F(u) &= \dim \mathcal{H}_{k+1} - \dim(\mathrm{Im}\ T_a) = \dim \mathcal{H}_{k+1} - \dim \mathcal{H}_k \\ &= \binom{n+k}{n-1} - \binom{n+k-1}{n-1} = \binom{n+k-1}{k+1}.\end{aligned}$$

Let us note also that the affine manifold

$$L_{\vec{f}}(u) := \{h \in \mathcal{H}_{k+1}\colon D^\alpha h = f_\alpha(x) - f_\alpha(y), \text{ for all } \alpha,\ |\alpha| - k\}$$

(see (2.8)) has the same dimension as $F(u)$.

Suppose now that Proposition 2.15 holds. We then have to prove the existence of a linear extension operator

$$E_X\colon \mathcal{J}|_X \to \mathcal{J}(\mathbb{R}^n)$$

with the norm controlled by a constant depending on n and k only. To accomplish this for a fixed $\vec{f} = (f_\alpha)_{|\alpha| \le k} \in \mathcal{J}|_X$ and $u = (x, y) \in U(X)$, we consider a polynomial $h_{\vec{f}}$ defined by

$$h_{\vec{f}}(z) := \sum_{|\alpha| = k} \frac{(f_\alpha(x) - f_\alpha(y))}{\alpha!} z^\alpha \quad (z \in \mathbb{R}^n).$$

Applying to $h_{\vec{f}}$ the linear operator S_a, $a = x - y$, of Lemma 2.17, we define the operator $G\colon U(X) \times \mathcal{J}|_X \to \mathcal{H}_{k+1}$ which is linear in the second argument and satisfies

$$D^\alpha(G(u, \vec{f})) = f_\alpha(x) - f_\alpha(y), \quad u = (x, y), \qquad |\alpha| - k.$$

By definition (see (2.8) or Remark 2.18)

$$G(u, \vec{f}) \in L_{\vec{f}}(u) =: L(u, \vec{f}), \quad u \in U(X).$$

From here and the linearity of G in $\vec{f}$ it follows that

$$L(u, \vec{f}) = F(u) + G(u, \vec{f}), \quad u \in U(X)$$

(see (2.18) or Remark 2.18 for the definition of F).

Thus the norm of G in the space $\Sigma(X)$ (see Definition 2.14) is less than or equal to $|h|_{\mathrm{Lip}(U(X), \mathcal{H}_{k+1})}$ where h is an arbitrary Lipschitz selection of $L_{\vec{f}} := L(\cdot, \vec{f})$.But according to Proposition 2.12, $L_{\vec{f}}$ has such a selection with the Lipschitz seminorm controlled by $O(1)|\vec{f}|_{\mathcal{J}|_X}$. Therefore,

$$|G|_{\Sigma(X)} \le O(1)|\vec{f}|_{\mathcal{J}|_X}.$$

Now let $T_X\colon \Sigma(X) \to \mathrm{Lip}(U(X), \mathcal{H}_{k+1})$ be the linear operator from Proposition 2.15. By the proposition, the composition $T \circ G(\cdot, \vec{f})$ is a selection of $L_{\vec{f}}$ and

$$|T \circ G(\cdot, \vec{f})|_{\mathrm{Lip}(U(X),\mathcal{H}_{k+1})} \le \|T\| \cdot |G(\cdot, \vec{f})|_{\Sigma(X)} \le O(1)|\vec{f}|_{\mathcal{J}|_X}.$$

Thus $T \circ G(\cdot, \vec{f})$ satisfies all the assumptions of Proposition 2.13, and consequently the desired extension operator E_X exists.

So, it remains to prove that the metric space $(U(X), \rho_X)$ has the $\mathcal{L}$-property (see Definition 2.16). To do this we first introduce the following graph structure on the set $\mathcal{K}_X$ of all cubes of $\mathbb{R}^n$ centered in X. Vertices (cubes) Q, Q' of $\mathcal{K}_X$ are said to be joined by an edge (written $Q \leftrightarrow Q'$) if one of them contains another. Let us further introduce a metric d_X on $\mathcal{K}_X$ using the function $\hat{\omega}$ on $\mathcal{K}_X \times \mathcal{K}_X$ defined by

$$\hat{\omega}(Q, Q') := \begin{cases} \frac{\omega(r_{Q\cup Q'})}{r_{Q\cap Q'}}, & \text{for } Q \leftrightarrow Q', \\ 0, & \text{otherwise.} \end{cases}$$

Recall that $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the function from the definition of $\mathcal{J}(\mathbb{R}^n) := J^k\Lambda^\omega(\mathbb{R}^n)$. Then

$$d_X(Q, Q') := \inf_{\{Q_i\}} \sum_{i=1}^{m-1} \hat{\omega}(Q_i, Q_{i+1})$$

where the infimum is taken over all paths $\{Q_i\}_{i=1}^m$ in $\mathcal{K}_X$ connecting Q' and Q (i.e., $Q_0 = Q$, $Q_m = Q'$ and $Q_i \leftrightarrow Q_{i+1}$, $i = 0, 1, \dots, m-1$).

In the paper [BS2], Section 5, the following has been proved.

Proposition 2.19. *Let $Q, Q' \in \mathcal{K}_X$ and*

$$m(Q, Q') := \min(r_Q, r_{Q'}), \quad M(Q, Q'); = r_{Q'} + r_Q + |c_Q - c_{Q'}|.$$

Then, for every pair $Q, Q' \in \mathcal{K}_X$,

$$d_X(Q, Q') \approx \int_{m(Q,Q')}^{M(Q,Q')} \frac{\omega(t)}{t^2} dt \tag{2.21}$$

with constants depending on n only.

Recall that r_Q and c_Q denote the center and the radius of a cube Q.

From the proposition, it immediately follows that the metric space $(\mathcal{K}_X, d_X)$ is quasi-isometrically embedded into the metric space $(\mathcal{K}, d) := (\mathcal{K}_{\mathbb{R}^n}, d_{\mathbb{R}^n})$ and distortion is bounded by a constant depending on n only. So, it further suffices to deal with the space $(\mathcal{K}, d)$.

Let us now compare the metric ρ_X on $U(X)$ (see (2.7)) with the metric d on $\mathcal{K} := \mathcal{K}_{\mathbb{R}^n}$. To this end, for $u = (x, y) \in U(X)$ we set

$$|u| := |x - y| \text{ and } Q_u := Q_{|u|}(x).$$

Proposition 2.20. *The following equivalence*

$$\rho_X(u, v) \approx d(Q_u, Q_v) + \frac{\omega(|u|)}{|u|} + \frac{\omega(|v|)}{|v|}, \quad u, v \in U(X), u \ne v,$$

holds with constants depending on n only.

Proof. If $u, v \in U(X)$ are joined by an edge (i.e., $\{x_u, y_u\} \cap \{x_v, y_v\} \neq \varnothing$, where $u = (x_u, y_u)$, $v = (x_v, y_v)$), then the function

$$\psi(u,v) := \int_{\min(|u|,|v|)}^{|u|+|v|} \frac{\omega(t)}{t^2}\, dt$$

is equivalent to $d(Q_u, Q_v)$.This readily follows from (2.21) and from the fact that the functions $\omega(t)$ and $t^2/\omega(t)$ are non-decreasing. In particular, for such u and v we get

$$\psi(u,v) \geq \gamma_1 \left\{ \frac{\omega(|u|)}{|u|} + \frac{\omega(|v|)}{|v|} \right\} \tag{2.22}$$

with some absolute constant $\gamma_1, \gamma_1 > 0$.

Now if $\{u_i\}$ is a path in the graph $U(X)$ connecting u and v, then the above equivalence implies

$$\sum_{k=0}^{m-1} \psi(u_i, u_{k+1}) \geq \gamma_2 \sum_{i=0}^{m-1} d(Q_{u_i}, Q_{u_{i+1}}) \geq \gamma_2 d(Q_u, Q_v)$$

with $\gamma_2 > 0$ depending on n only. By the definition of the metric ρ_X we hence deduce that

$$\rho_X(u,v) := \inf_{\{u_i\}} \sum_{i=0}^{m-1} \psi(u_i, u_{i+1}) \geq \gamma_2 d(Q_u, Q_v).$$

Besides, by (2.22) for the above infimum we have

$$\begin{aligned} \inf_{\{u_i\}} \sum_{i=0}^{m-1} \psi(u_i, u_{i+1}) &\geq \gamma_1 \inf_{\{u_i\}} \sum_{i=0}^{m-1} \left(\frac{\omega(|u_i|)}{|u_i|} + \frac{\omega(|u_{i+1}|)}{|u_{i+1}|} \right) \\ &\geq \gamma_1 \left(\frac{\omega(|u|)}{|u|} + \frac{\omega(|v|)}{|v|} \right). \end{aligned}$$

Together with the previous inequality, this leads to the desired lower estimate of ρ_X.

To prove the upper estimate we consider two points $u = (x_u, y_u)$, $v = (x_v, y_v)$ of $U(X)$. Without loss of generality one can regard that

$$|x_u - y_u| := |u| \leq |v| := |x_v - y_v|.$$

If now

$$|x_u - x_v| \leq |x_u - y_v|,$$

then we set

$$w := (x_u, y_v).$$

Otherwise we put $w := (x_u, x_v)$. By definition $w \leftrightarrow u$, $w \leftrightarrow v$, and therefore

$$\rho_X(u,v) \leq \psi(u,w) + \psi(w,v) \leq 2 \int_a^b \frac{\omega(t)}{t^2} dt,$$

where $a := |u|/2$ and $b := |u| + 2|v| + |x_u - x_v|$. Here we made use of the definition of ψ and the inequalities

$$|u| \leq |v| \leq 2|w|, \quad |w| \leq |x_u - x_v| + |v|.$$

Let us represent the latter integral as the sum $I_1 + I_2 + I_3$ by dividing the domain of integration into three parts:

$$\Delta_1 := [|u|/2, |u|], \quad \Delta_2 := [|u|, |u| + |v| + |x_u - x_v|],$$
$$\Delta_3 := [|u| + |v| + |x_u - x_v|, |u| + 2|v| + |x_u - x_v|].$$

Then $I_1 \approx \frac{\omega(|u|)}{|u|}$ and $I_2 \approx d(Q_u, Q_v)$; see (2.21). The function $t \to \frac{\omega(t)}{t^2}$ is non-increasing as well, and therefore

$$I_3 \le \int_{|v|}^{2|v|} \frac{\omega(t)}{t^2} dt \approx \frac{\omega(|v|)}{|v|}.$$

Collecting these estimates we get the desired upper estimate

$$\rho_X(u, v) \le \gamma \left\{ \frac{\omega(|u|)}{|u|} + \frac{\omega(|v|)}{|v|} + d(Q_u, Q_v) \right\}.$$

□

Proposition 2.21. *The metric space $(U(X), \rho_X)$ possesses the $\mathcal{L}$-property.*

Proof. From Proposition 7.1 and Theorem 4.7 of [BS2] it follows that the metric space $(U(X), \tilde{\rho})$ with

$$\tilde{\rho}(u, v) = \rho(u, v) + \chi(u) + \chi(v) \quad (u, v \in U(X))$$

possesses the $\mathcal{L}$-property if $(U(X), \rho)$ does. Here $\chi \colon U(X) \to \mathbb{R}_+$ is an arbitrary function. This statement and Proposition 2.20 imply that $(U(X), \rho_X)$ possesses the required property together with the space $(\mathcal{K}, d)$. But the latter space has the $\mathcal{L}$-property, as has been proved in [BS2], Corollary 5.3 and Proposition 5.4. □

So, to complete the proof of Theorem 1.1, it remains to prove Propositions 2.4 and 2.5. We will do this in the next two sections.

3. Proof of Proposition 2.4

Let $\mathcal{J}_m := J^k\Lambda^{m,\omega}(\mathbb{R}^n)$ be the space of k-jets generated by functions from the space $C^k\Lambda^{m,\omega}(\mathbb{R}^n)$ defined by the seminorm

$$|f|_{C^k\Lambda^{m,\omega}} := \sum_{|\alpha|=k} \sup_{x,h \in \mathbb{R}^n} \frac{|\Delta_h^m (D^\alpha f)(x)|}{\omega(|h|)}. \tag{3.1}$$

Here $\omega \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function such that the function $t \to \frac{\omega(t)}{t^m}$ is non-increasing and $\omega(0) = 0$. As usual, Δ_h^m denotes m-difference of step h; i.e.,

$$\Delta_h^m(f; x) := \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x + ih) \quad (x, h \in \mathbb{R}^n).$$

In view of some other applications we shall prove the next generalization of Proposition 2.4.

Proposition 3.1. *A vector function $\vec{f} = (f_\alpha \colon X \to \mathbb{R})_{|\alpha| \le k}$ belongs to the trace space $\mathcal{J}_m|_X$ if and only if there exist a positive constant λ and a family $\{P_Q \colon Q \in \mathcal{K}_X\}$ of polynomials of degree $k + m - 1$ such that:*

(i) $(D^\alpha P_Q)(c_Q) = f_\alpha(c_Q)$

for every $\alpha, |\alpha| \le k$, and every cube $Q \in \mathcal{K}_X$ (i.e., recall that the center c_Q of Q belongs to X).

(ii) *For every Q, Q' from $\mathcal{K}_X$ satisfying*

$$Q' \subset Q$$

the inequality

$$\sup_{Q'} |P_{Q'} - P_Q| \le \lambda\{|c_Q - c_{Q'}| + r_{Q'}\}^k \omega(r_Q)$$

holds.

Moreover, the trace norm of $\vec{f}$ satisfies

$$|\vec{f}|_{\mathcal{J}_m|_X} \approx \inf \lambda$$

with constants depending on k, m and n only.

Before we pass on to the proof, we remark that the case $m = 2$, $\omega(t) = t^\beta$, $0 < \beta \le 1$ (in formulation close to the above) was established earlier by A. Jonsson and H. Wallin [JW]. The case where $k = 0$, m and ω are arbitrary was obtained in [BS3]. For another characterization of the trace space $\mathcal{J}_m|_X$ for $m = 2$ see the paper of L. Hanin [H].

Proof (*necessity*). Let $\vec{f} \in \mathcal{J}_m|_X$ and its trace norm equal 1. Let us prove that the statements of Proposition 3.1 hold with $\lambda \le O(1)$.

By the definition of $\mathcal{J}_m|_X$ there is a function $\tilde{f} \in C^k\Lambda^{m,\omega}(\mathbb{R}^n)$ such that

$$D^\alpha \tilde{f}|_X = f_\alpha, \qquad |\alpha| \le k$$

and besides

$$|\tilde{f}|_{C^k\Lambda^{m,\omega}} \le 2.$$

To define the required family $\{P_Q : Q \in \mathcal{K}_X\}$, we make use of the main result of the paper [B] which states that

$$|g|_{\Lambda^{m,\omega}(\mathbb{R}^n)} \approx \sup_{Q \in \mathcal{K}} \frac{E_m(g; Q)}{\omega(r_Q)} \tag{3.2}$$

where $E_m(f; Q)$ is the best approximation of f in $C(Q)$ by polynomials of degree $\le m - 1$. This result and the definition (3.1) lead to the following

Lemma 3.2. *Let $Q = Q_r(c)$ be a cube ($\in \mathcal{K}$). Then for every $\alpha, |\alpha| = k$, there exists a polynomial $P_Q^{(\alpha)}$ of degree $m - 1$ such that*

$$\sup_Q |D^\alpha \tilde{f} - P_Q^{(\alpha)}| \le O(1)\omega(r).$$

We now put

$$P_Q(x) := \sum_{|\alpha| \le k} \frac{f_\alpha(c)}{\alpha!}(x - c)^\alpha + R_k(x, c) \tag{3.3}$$

where

$$R_k(x, c) := k \sum_{|\alpha| = k} \frac{(x - c)^\alpha}{\alpha!} \int_0^1 (1 - t)^{k-1}[P_Q^{(\alpha)}(c + t(x - c)) - P_Q^{(\alpha)}(c)]dt$$

and show that $\{P_Q : Q \in \mathcal{K}_X\}$ is the required family of polynomials.

On account of the definition (3.3) the property (i) of Proposition 3.1 holds. To prove (ii) let us first consider the case of cubes $Q' = Q_{r'}(c)$ and $Q = Q_r(c)$ with the same center and $r' \le r$. Applying Taylor's formula to the function $\tilde f$ at the point c, we obtain

$$\sup_{Q'}|\tilde f - P_Q| \le k \sup_{x\in Q'} \sum_{|\alpha|=k} \frac{|x-c|^\alpha}{\alpha!}\int_0^1 (1-t)^{k-1}|\varphi_\alpha(c+t(x-c))-\varphi_\alpha(c)|dt$$

where $\varphi_\alpha := D^\alpha \tilde f - P_Q^{(\alpha)}$. Besides, by Lemma 3.2,

$$\sup_{Q'}|\varphi_\alpha| \le O(1)\omega(r)$$

and consequently

$$\sup_{Q'}|\tilde f - P_Q| \le O(1)(r')^k\omega(r). \tag{3.4}$$

The general case can be reduced to this situation. In fact, if $Q' = Q_{r'}(c') \subset Q = Q_r(c)$, then

$$Q' \subset Q'' := Q_\rho(c) \subset Q,$$

where $\rho := |c-c'| + r'$. Then, according to (3.4), this yields

$$\begin{aligned}\sup_{Q'}|P_Q - P_Q'| &\le \sup_{Q'}|\tilde f - P_{Q'}| + \sup_{Q''}|\tilde f - P_Q| \\ &\le O(1)[(r')^k\omega(r') + \rho^k\omega(r)] \le O(1)\rho^k\omega(r).\end{aligned}$$

The proof of the necessity part is complete.

Proof (*sufficiency*). We will need a few auxiliary results. To formulate the first of them we let γQ denote a dilation of the cube Q with respect to its center by a factor of γ. Let W_X be the family of cubes of Whitney's decomposition of $X^c := \mathbb{R}^n \backslash X$. For the sake of convenience we will collect the properties of W_X in the next lemma (see, e.g., [St], Ch. 6).

Lemma 3.3. (a) *W_X is a covering of X^c.*

(b) *The interiors of cubes of W_X are pairwise disjoint.*

(c) $\operatorname{diam} Q \le \operatorname{dist}(Q, X) \le 4 \operatorname{diam} Q, Q \in W_X$.

Here the diameter and the distance are measured with respect to the Euclidean metric in $\mathbb{R}^n$.

(d) *If $Q, Q' \in W_X$ have a non-empty intersection, then*

$$1/4 \operatorname{diam} Q \le \operatorname{diam} Q' \le 4 \operatorname{diam} Q.$$

(e) *Let Q^* denote the cube $\frac{9}{8}Q$. Then $K, Q \in W_X$ have a non-empty intersection if and only if*

$$K^* \cap Q^* \ne \varnothing.$$

Let us now define the map

$$W_X \ni Q \to \widehat{Q} \in \mathcal{K}_X,$$

where the cube $\widehat{Q}$ is defined by the conditions:

(1) volume $|\widehat{Q}|$ of $\widehat{Q}$ equals $|Q|$,

(2) the center of $\widehat{Q}$ is the point of X nearest to the center of Q.

As an immediate consequence of the above lemma and the above definition, we get the following

Lemma 3.4. *Let $Q \in \mathcal{K}_X$, $K \in W_X$ and $K^* \cap Q \neq \varnothing$. Then there exists a constant $\gamma = \gamma(n)$ such that*

$$\widehat{K} \subset \gamma Q.$$

Now let $\{\psi_Q\colon Q \in W_X\}$ be a C^∞-partition of the unity subordinated to the covering W_X. The next proposition describes the properties of the partition (see, e.g., [St], Ch. 6).

Lemma 3.5. *Let $Q \in W_X$. Then*
(a) $\psi_Q \in C_0^\infty$ *and* $0 \le \psi_Q \le 1$,
(b) $\psi_Q = 1$ *on* Q *and* $\psi_Q = 0$ *out of* Q^*,
(c) *for every multiindex* $\alpha \in \mathbb{Z}_+^n$

$$\sup_{\mathbb{R}^n} |D^\alpha \psi_Q| \le c(\alpha)(r_Q)^{-|\alpha|},$$

(d) $\sum_{Q \in W_X} \psi_Q = 1$.

We are now in a position to begin the proof. Let the vector function $\vec{f} = (f_\alpha)_{|\alpha| \le k}$ and the family $\{P_Q\colon Q \in \mathcal{K}_X\}$ satisfy the conditions (i) and (ii) of the proposition. We then set

$$\tilde{f}(x) := \begin{cases} f_0(x), & \text{for } x \in X, \\ \sum_{Q \in W_X} (\psi_Q P_{\widehat{Q}})(x), & \text{for } x \in X^c. \end{cases} \tag{3.5}$$

It suffices to check that $\tilde{f}$ possesses the next two properties:
(A) $\tilde{f} \in C^k\Lambda^{m,\omega}(\mathbb{R}^n)$ and $|\tilde{f}|_{C^k\Lambda^{m,\omega}} \le O(1)\lambda$ where λ is a constant from the condition (ii) of Proposition 3.1,
(B) for every $\alpha, |\alpha| \le k$,

$$(D^\alpha \tilde{f})|_X = f_\alpha.$$

To accomplish this we make use of the following lemmas.

Lemma 3.6. *Let $Q, Q' \in \mathcal{K}_X$ and $Q' \subset Q$. Then for every multiindex α*

$$\sup_{Q'} |D^\alpha(P_Q - P_{Q'})| \le O(1)\lambda(r_Q^{k-|\alpha|} + r_{Q'}^{k-|\alpha|})\omega(r_Q).$$

Proof. If $Q = Q_r(x)$, $Q' = Q_{r'}(x')$ and $\widetilde{Q}$ denotes the cube $Q_{2r}(x')$, then

$$Q' \subset Q \subset \widetilde{Q}.$$

Then, by the Markov inequality,

$$\sup_{Q'} |D^\alpha(P_Q - P_{Q'})| \le O(1)\{(r')^{-|\alpha|} \sup_{Q'} |P_{Q'} - P_{\widetilde{Q}}| + r^{-|\alpha|} \sup_{Q} |P_Q - P_{\widetilde{Q}}|\}.$$

Finally, applying the inequality of the condition (ii) to the right-hand side, we get the desired estimate. □

To formulate the next result we let $\rho(y)$ denote $\inf_{x \in X} |y - x|$; recall that $|\cdot|$ is the uniform norm in $\mathbb{R}^n$. Let us also set

$$\tilde{f}_\alpha(x) := \begin{cases} f_\alpha(x), & x \in X, \\ (D^\alpha \tilde{f})(x), & x \in X^c, \end{cases} \tag{3.6}$$

where $|\alpha| \le k$. We will extend this definition to all multiindexes α, putting $f_\alpha \equiv 0$ for α, $|\alpha| > k$.

Lemma 3.7. *Let $Q \in \mathcal{K}_X$ and $y \in Q$. Then for every multiindex α*

$$|\tilde{f}_\alpha(y) - D^\alpha P_Q(y)| \le O(1)\lambda(\rho(y)^{k-|\alpha|} + r_Q^{k-|\alpha|})\omega(r_Q).$$

Proof. We will divide the proof into two parts.

(a) $y \in Q \cap X$. Since $\rho(y) = 0$, it suffices to consider the case $|\alpha| \le k$ only. Let $\widetilde{Q} := Q_{2r}(y)$ where $r := r_Q$. Then $Q \subset \widetilde{Q}$, and by (3.6) and the property (i) of the proposition

$$\tilde{f}_\alpha(y) = f_\alpha(y) = (D^\alpha P_{\widetilde{Q}})(y).$$

Hence we deduce that

$$|\tilde{f}_\alpha(y) - (D^\alpha P_Q)(y)| \le \sup_Q |D^\alpha(P_{\widetilde{Q}} - P_Q)|.$$

It remains to apply Lemma 3.6, and the case (a) follows.

(b) $y \in Q \cap X^c$. We let $\widehat{y} \in X$ denote the point of X nearest to y (measured in the uniform metric). Since $\rho(y) \le r_Q$

$$Q^{(1)}, Q \subset Q^{(2)}, \text{ where } Q^{(1)} := Q_{\rho(y)}(\hat{y}), Q^{(2)} := Q_{3r}(\hat{y}) \quad (r := r_Q).$$

Therefore

$$\begin{aligned}(\tilde{f}_\alpha - D^\alpha P_Q)(y) = (\tilde{f}_\alpha - D^\alpha P_{Q^{(1)}})(y) + D^\alpha(P_{Q^{(1)}} - P_{Q^{(2)}})(y)\\ + D^\alpha(P_{Q^{(2)}} - P_Q)(y).\end{aligned}$$

The two latter terms on the right-hand side can be estimated in the required way with the help of Lemma 3.6. So it remains to prove the inequality

$$|(\tilde{f}_\alpha - D^\alpha P_{Q^{(1)}})(y)| \le O(1)\lambda\rho(y)^{k-|\alpha|}\omega(r_Q). \tag{3.7}$$

To accomplish this we set

$$S_y := \{K \in W_X : y \in K^*\}.$$

On account of properties (b), (d) and (e) of Lemma 3.3 this set is finite, and moreover

$$\operatorname{card} S_y \le \gamma(n). \tag{3.8}$$

Let us now check that

$$r_K \approx \rho(y) \tag{3.9}$$

for every cube $K \in S_y$. In fact, by Lemma 3.4, $\widehat{K} \subset \gamma(n)Q^{(1)}$, and therefore

$$r_K =: r_{\widehat{K}} \le \gamma(n) r_{Q^{(1)}} = \gamma(n)\rho(y).$$

On the other hand, for every $K \in S_y$ and every cube $K' \in W_X$ containing y (and therefore, belonging to S_y) by the properties (d) and (e) of Lemma 3.3, we have

$$r_K \approx r_{K'}.$$

From here and the property (e) of this lemma, we deduce that

$$\begin{aligned}\rho(y) &\le \inf_{x \in X} \|x - y\| \le \operatorname{dist}(K', X) + \operatorname{diam}(K')\\ &\le 5 \operatorname{diam} K' \le \gamma_1(n) r_{K'} \le \gamma_2(n) r_K.\end{aligned}$$

Here the distances and the diameters are measured in the Euclidean norm $\|\cdot\|$.

So, (3.9) is established and we can continue the proof of (3.7). By the definitions (3.5) and (3.6) and the property (d) of Lemma 3.5, we can write the left-hand side of (3.7) as $D^\alpha \sum_{K\in W_X} \psi_K(y)(P_{\widehat{K}} - P_{Q^{(1)}})(y)$. Applying the Leibnitz formula and taking into account the properties (b) and (c) of Lemma 3.5, we obtain the estimate

$$|(\tilde{f}_\alpha - D^\alpha P_{Q^{(1)}})(y)| \le O(1) \sum_{K\in S_y} \sum_{\alpha=\alpha_1+\alpha_2} (r_K)^{-|\alpha_1|} |[D^{\alpha_2}(P_{\widehat{K}} - P_{Q^{(1)}})](y)|.$$

But $y \in Q^{(1)}$, and therefore

$$|D^{\alpha_2}(P_{\widehat{K}} - P_{Q^{(1)}})(y)| \le \sup_{Q^{(1)}} |D^{\alpha_2}(P_{\widehat{K}} - P_{Q^{(1)}})|.$$

From here, (3.8) and (3.9), we get

$$\begin{aligned} |(\tilde{f}_\alpha - D^\alpha P_{Q^{(1)}})(y)| \le O(1) \max_{K\in S_y} &\sum_{\alpha=\alpha_1+\alpha_2} \rho(y)^{-|\alpha_1|} \\ &\cdot \{\sup_{Q^{(1)}} |D^{\alpha_2}(P_{\widehat{K}} - P_{\gamma Q^{(1)}})| + \sup_{Q^{(1)}} |D^{\alpha_2}(P_{\gamma Q^{(1)}} - P_{Q^{(1)}})|\}, \end{aligned}$$

where the constant $\gamma = \gamma(n)$ is taken from Lemma 3.4. According to this lemma, $\widehat{K} \subset \gamma Q^{(1)}$ for every $K \in S_y$. We also point out that by (3.9)

$$r_{\widehat{K}} := r_K \approx \rho(y) =: r_{Q^{(1)}} \approx r_{\gamma Q^{(1)}}. \tag{3.10}$$

Therefore the first summand in the right-hand side of the above inequality can be estimated as follows:

$$\begin{aligned} \sup_{Q^{(1)}} |D^{\alpha_2}(P_{\widehat{K}} - P_{\gamma Q^{(1)}})| &\le \sup_{\gamma Q^{(1)}} |D^{\alpha_2}(P_{\widehat{K}} - P_{\gamma Q^{(1)}})| \\ &\le O(1) \sup_{\widehat{K}} |D^{\alpha_2}(P_{\widehat{K}} - P_{\gamma Q^{(1)}})|. \end{aligned}$$

Now by the property (ii) of Proposition 3.1 and the Markov inequality we finally get

$$\begin{aligned} &|(\tilde{f}_\alpha - D^\alpha P_{Q^{(1)}})(y)| \\ &\quad\le O(1)\lambda \max_{K\in S_y} \max_{\alpha_1+\alpha_2=\alpha} \{\rho(y)^{-|\alpha_1|}[r_{\widehat{K}}^{-|\alpha_2|}(r_{\widehat{K}} + r_{\gamma Q^{(1)}})^k + r_{Q^{(1)}}^{k-|\alpha_2|}]\omega(r_{\gamma Q^{(1)}})\}. \end{aligned}$$

This inequality, (3.10) and the inequality $\rho(y) \le r_Q$ (since $y \in Q$) imply the required estimate (3.7). $\square$

Corollary 3.8. *Under the condition of Lemma 3.7*

$$|(\tilde{f}_\alpha - D^\alpha P_Q)(y)| \le O(1)\lambda r_Q^{k-|\alpha|}\omega(r_Q)$$

for every α, $|\alpha| \le k$.

For the proof it suffices to note that

$$\rho(y) \le r_Q$$

for $y \in Q$ and $Q \in \mathcal{K}_X$. $\square$

We are now in a position to prove statement (B). We have to establish that the function $\tilde{f}$ defined by (3.5) has all derivatives up to the order k inclusive and that

$$D^\alpha \tilde{f} = \tilde{f}_\alpha \text{ for all } \alpha, |\alpha| \le k.$$

According to (3.6) it suffices to check only that

$$(D^\alpha \tilde{f})(x) = f_\alpha(x), \quad x \in X, \tag{3.11}$$

for every α, $|\alpha| \le k$.

Let α, $|\alpha| \le k-1$, be an arbitrary multiindex and $\alpha^{(i)} := \alpha + e^{(i)}$ where $\{e^{(1)}, \dots, e^{(n)}\}$ is the standard basis in $\mathbb{R}^n$. Then (3.11) will follow immediately from the estimate

$$\tilde{f}_\alpha(y) = f_\alpha(x) + \sum_{i=1}^n f_{\alpha^{(i)}}(x)(y_i - x_i) + o(|x-y|)$$

as $y \to x \in X$.

To prove this we let Q denote the cube $Q_r(x)$ with $r := |y-x|$. Then, by Corollary 3.8 we have

$$|\tilde{f}_\alpha(y) - D^\alpha P_Q(y)| \le O(1)\lambda r^{k-|\alpha|}\omega(r).$$

Since $k - |\alpha| \ge 1$ and $\omega(r) = o(1)$, $r \to 0$, the right-hand side of the above inequality is $o(r)$ as $r \to 0$. Besides, by the property (i) of the proposition,

$$D^{\alpha^{(i)}} P_Q(x) = f_{\alpha^{(i)}}(x), \quad 1 \le i \le n,$$

and therefore the difference

$$\tilde{f}_\alpha(y) - f_\alpha(x) - \sum_{i=1}^n f_{\alpha^{(i)}}(x)(y_i - x_i)$$

is majorized by

$$\begin{aligned}
&|(\tilde{f}_\alpha - D^\alpha P_Q)(y)| + |(D^\alpha P_Q)(y) - [(D^\alpha P_Q)(x) + \langle \operatorname{grad}(D^\alpha P_Q)(x), y - x\rangle]| \\
&\quad \le O(1)\lambda r^{k-|\alpha|}\omega(r) + \sum_{|\beta|\ge 2} \frac{1}{\beta!}|(D^{\alpha+\beta} P_Q)(x)| r^{|\beta|}.
\end{aligned}$$

Here, as usual, $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$.

Thus it remains to prove that for every β, $|\beta| > 1$,

$$r^{|\beta|}(D^{\alpha+\beta} P_Q)(x) = o(r) \text{ as } r \to 0. \tag{3.12}$$

By the property (i) of the proposition, the left-hand side is equal to $r^{|\beta|} f_{\alpha+\beta}(x)$ if $|\alpha + \beta| \le k$, so in this case (3.12) holds. Now let $|\alpha+\beta| > k$ and $Q_i := Q_{r_i}(x)$, where $r_i = 2^i r$, $i = 0, 1, \dots$.

We can regard below that $0 < r < 1/2$. Let us choose a positive integer ℓ such that $r_\ell \le 1 \le r_{\ell+1}$. Then

$$r^{|\beta|}|(D^{\alpha+\beta}P_Q)(x)| \le r^{|\beta|}\sum_{i=0}^{\ell-1} |D^{\alpha+\beta}(P_{Q_i} - P_{Q_{i+1}})(x)| + r^{|\beta|}|(D^{\alpha+\beta}P_{Q_\ell})(x)|.$$

The second summand in the right-hand side is $o(r)$ as $r \to 0$. By the Markov inequality and the property (ii) of the proposition, the first summand can be estimated as follows:

$$\begin{aligned}
r^{|\beta|}\sum_{i=0}^{\ell-1} \sup_{Q_i} |D^{\alpha+\beta}(P_{Q_i} - P_{Q_{i+1}})| &\le O(1) r^{|\beta|} \sum_{i=0}^{\ell-1} r_i^{-|\alpha|-|\beta|} \sup_{Q_i} |P_{Q_i} - P_{Q_{i+1}}| \\
&\le O(1)\lambda r^{|\beta|}\sum_{i=0}^{\ell-1} r_i^{k-|\alpha|-|\beta|}\omega(r_i).
\end{aligned}$$

Since $\omega(t)$ and $t^2/\omega(t)$ are non-decreasing functions, $k-|\alpha|\geq 1$ and $r_\ell \approx 1$,

$$r^{|\beta|}\sum_{i=0}^{\ell-1} r_i^{k-|\alpha|-|\beta|}\omega(r_i) \leq O(1)r^{|\beta|}\int_r^1 t^{k-|\alpha|}\frac{\omega(t)}{t^{|\beta|}}\frac{dt}{t} \leq O(1)r^{|\beta|}\int_r^1 \frac{\omega(t)}{t^{|\beta|}}dt.$$

It is not hard to see that under conditions $|\beta|>1$ and $\omega(r)\to 0$, $r\to 0$, the last integral is $o(r^{1-|\beta|})$ as $r\to 0$. Thus (3.12) holds and the property (B) follows.

To prove the statement (A) we have to check that $\tilde{f}\in C^k\Lambda^{m,\omega}(\mathbb{R}^n)$ and its seminorm in this space is majorized by $O(1)\lambda$. But according to (B), $\tilde{f}\in C^k(\mathbb{R}^n)$ and $D^\alpha\tilde{f}=\tilde{f}_\alpha$, $|\alpha|\leq k$ (see definition (3.6)). Therefore it remains to prove that the best approximation $E_m(\tilde{f}_\alpha, Q)$ of $\tilde{f}_\alpha$ in $C(Q)$ by polynomials of degree $m-1$ can be estimated as follows:

$$E_m(\tilde{f}_\alpha;Q)\leq O(1)\lambda\omega(r_Q) \text{ for all } \alpha, |\alpha|=k \text{ and } Q\in\mathcal{K}. \tag{3.13}$$

Thus by applying (3.2), we can conclude that $D^\alpha\tilde{f}=\tilde{f}_\alpha\in\Lambda^{m,\omega}(\mathbb{R}^n)$ if $|\alpha|=k$ and its seminorm in this space $\leq O(1)\lambda$.

We will divide the proof of (3.13) into a sequence of lemmas. To formulate the first of them, let us divide the family $\mathcal{K}$ of cubes of $\mathbb{R}^n$ into two classes $\mathcal{K}_I$ and $\mathcal{K}_{II}$ in the following way.

A cube Q is said to belong to $\mathcal{K}_I$ if

$$Q\subset X^c \text{ and } Q\subset K^* \tag{3.14}$$

for the Whitney cube $K\in W_X$ containing the center of Q. Then we set

$$\mathcal{K}_{II}:=\mathcal{K}\backslash\mathcal{K}_I.$$

Now from the definition of $\mathcal{K}_{II}$ and Lemma 3.3 it immediately follows that for cubes of $\mathcal{K}_{II}$ the following result holds.

Lemma 3.9. *Let $Q\in\mathcal{K}_{II}$. Then*

$$\operatorname{dist}(Q,X)\leq\gamma\operatorname{diam}Q.$$

Moreover, for every $Q'\in W_X$ intersecting Q, we have

$$\operatorname{diam}Q'\leq\gamma\operatorname{diam}Q.$$

Here γ is an absolute constant.

Lemma 3.10. *The inequality* (3.13) *is valid for $Q\in\mathcal{K}_{II}$.*

Proof. Suppose first that a cube $Q\in\mathcal{K}_X\cap\mathcal{K}_{II}$; i.e., its center belongs to X. Applying Corollary 3.8 with $|\alpha|=k$ and taking into account that $D^\alpha P_Q\in\mathcal{P}_{m-1}$, we deduce that

$$E_m(\tilde{f}_\alpha;Q)\leq\sup_Q|\tilde{f}_\alpha-D^\alpha P_Q|\leq O(1)\lambda\omega(r_Q). \tag{3.15}$$

In the remaining case $c_Q\notin X$ we let $\hat{c}_Q$ denote the point of X nearest to c_Q (in the uniform norm). If $K=K_Q$ is the smallest cube with the center $\hat{c}_Q$ containing Q, then $K\in\mathcal{K}_{II}$ and by Lemma 3.9,

$$\operatorname{diam}K\leq O(1)\operatorname{diam}Q.$$

From this and (3.15) it follows that

$$E_m(\tilde{f}_\alpha;Q)\leq E_m(\tilde{f}_\alpha;K)\leq O(1)\lambda\omega(r_K)\leq O(1)\lambda\omega(r_Q). \quad \square$$

Now let $Q\in\mathcal{K}_I$ and $K:=K_Q$ be defined as in the proof of Lemma 3.10.

Lemma 3.11. *For every α, $|\alpha| = k$,*

$$E_m(\tilde{f}_\alpha; Q) \le O(1)\lambda r_Q^m \frac{\omega(r_K)}{r_K^m}. \tag{3.16}$$

Proof. Since $Q \subset X^c$ we deduce from (3.5) that $\tilde{f}_\alpha|_Q = D^\alpha \tilde{f}|_Q \in C^\infty(Q)$. Therefore, for every α, $|\alpha| = k$, we can apply (3.2) to $\tilde{f}_\alpha$ and obtain

$$\begin{aligned} E_m(\tilde{f}_\alpha; Q) &\le O(1) \sup_{x, x+mh \in Q} |\Delta_n^m \tilde{f}_\alpha| \\ &\le O(1) r_Q^m \sum_{|\beta|=m} \sup_Q |D^\beta \tilde{f}_\alpha| = O(1) r_Q^m \sum_{|\beta|=m} \sup_Q |\tilde{f}_{\alpha+\beta}|. \end{aligned}$$

Since $|\alpha + \beta| = k + m$, in this case $D^{\alpha+\beta} P_K = 0$. So, taking $y \in Q$ and applying Lemma 3.7, we can conclude that

$$|\tilde{f}_{\alpha+\beta}(y)| = |(\tilde{f}_{\alpha+\beta} - D^{\alpha+\beta} P_K)(y)| \le O(1)\lambda\{\rho(y)^{-m} + r_K^{-m}\}\omega(r_K)$$

where recall that $\rho(y) := \inf_{x\in X} |y - x|$. This, and the above estimate, will lead to the required inequality of the lemma if we prove that

$$r_K \le O(1)\rho(y) \ (y \in Q). \tag{3.17}$$

To prove this we recall that the center c_K of K is the point of X nearest to c_Q. Therefore

$$|z - c_K| \le r_Q + \rho(c_Q) \le 2r_Q + \rho(y)$$

for every $z \in Q$. On the other hand, by this inequality and the definition of the cube $K := K_Q$ we now conclude that

$$r_K \le 2r_Q + \rho(y).$$

Thus, it remains to prove that the first term on the right is majorized by the second one. To accomplish this we consider the cube R from W_X such that $Q \subset R^*$ and $c_Q \in R$ (see (3.14)). If the point y belongs to another Whitney cube, say S, then $R^* \cap S^* \ne \varnothing$ and therefore, by Lemma 3.3(e), $R \cap S \ne \varnothing$ as well. Applying the statement (d) of this lemma, we get

$$\operatorname{diam} R \approx \operatorname{diam} S.$$

Finally, according to the statement (d) of the same lemma, we conclude that

$$\begin{aligned} r_Q \le r_{R^*} := 9/8 r_R &\le O(1) \operatorname{diam} R \le O(1) \operatorname{diam} S \\ &\le O(1) \operatorname{dist}(S, X) \le O(1) \operatorname{dist}(y, X) \le O(1)\rho(y). \end{aligned}$$

Thus we have proved that (3.17) holds, and the lemma follows. □

Now let $Q \in \mathcal{K}_I$. Since $r_Q \le r_K$ and the function $t \to \omega(t)/t^m$ is non-increasing, the right-hand side of (3.16) does not exceed

$$O(1)\lambda r_Q^m \frac{\omega(r_Q)}{r_Q^m} = O(1)\lambda\omega(r_Q).$$

Thus the inequality (3.13) holds for all cubes, and the proof of the statement (A) is complete. □

4. Proof of Proposition 2.5

(*Necessity*). Let $\vec{f} = (f_\alpha \colon X \to \mathbb{R})_{|\alpha|\le k}$ belong to the trace space $\mathcal{J}|_X$ and its trace norm be equal to 1.

We have to construct the mapping $U(X) \ni u \to P_u \in \mathcal{P}_{k+1}$ satisfying the conditions (i)–(iii) of the proposition (with $\lambda \le O(1)$).

To accomplish this we define the mapping

$$U(X) \ni u \to Q^u \in \mathcal{K}_X$$

where Q^u is the cube with center x_u and radius $|u| := |x_u - y_u|$. Recall that $u = (x_u, y_u)$, where $x_u, y_u \in X$ and $x_u \ne y_u$. Let $\{P_Q \colon Q \in \mathcal{K}_X\}$ be the family of polynomials of degree $\le k+1$ satisfying the conditions of Proposition 2.4. We let h_u denote the polynomial

$$h_u(x) := \sum_{|\alpha|=k} (f_\alpha - D^\alpha P_{Q^u})(y_u)\frac{x^\alpha}{a!}.$$

where S_a is the linear operator introduced in Lemma 2.17. On account of the lemma

$$(D^\alpha G_u)(y_u - x_u) = (f_\alpha - D^\alpha P_{Q^u})(y_u) \tag{4.1}$$

for every α, $|\alpha| = k$, and besides

$$|D^\alpha G_u| \le O(1)|u|^{-1} \max_{|\beta|=k} |(f_\beta - D^\beta P_{Q^u})(y_u)| \tag{4.2}$$

for every α, $|\alpha| = k+1$.

Finally, we put

$$P_u(x) := P_{Q^u}(x) + G_u(x - x_u), \quad u \in U(X). \tag{4.3}$$

Let us show that this is the required family of polynomials. To check the statement (i) of the proposition, we make use of (4.1) and the statement (i) of Proposition 2.4. Then we get

$$(D^\alpha P_u)(x_u) = (D^\alpha P_{Q^u})(x_u) + (D^\alpha G_u)(0) = f_\alpha(x_u)$$

for all α, $|\alpha| \le k$. Moreover, for every α, $|\alpha| = k$, by (4.1) and (4.3)

$$\begin{aligned}(D^\alpha P_u)(y_u) &= (D^\alpha P_{Q^u})(y_u) + (D^\alpha G_u)(y_u - x_u)\\ &= (D^\alpha P_{Q^u})(y_u) + (f_\alpha - D^\alpha P_{Q^u})(y_u) = f_\alpha(y_u)\end{aligned}$$

which completes the proof of (i).

To prove the inequality of the statement (ii) of the proposition, we first note that for every α, $|\alpha| \le k$,

$$\begin{aligned}|(f_\alpha - D^\alpha P_u)(y_u)| &\le |(f_\alpha - D^\alpha P_{Q^u})(y_u)| + |D^\alpha(P_{Q^u} - P_u)(y_u)|\\ &= I_1 + I_2.\end{aligned}$$

Then, by (4.3)

$$I_2 = |(D^\alpha G_u)(y_u - x_u)|$$

where G_u is a homogeneous polynomial of degree $\le k+1$ whose coefficients are estimated by (4.2). Hence we have

$$I_2 \le O(1)|u|^{k-|\alpha|} \max_{|\beta|=k} |(f_\beta - D^\beta P_{Q^u})(y_u)|.$$

To estimate the right-hand side of the inequality (and consequently I_1), we put $\widetilde{Q} := 2Q_{|u|}(y_u)$. Then $Q^u := Q_{|u|}(x_u) \subset \widetilde{Q}$ and applying successively the statements (i) and (ii) of Proposition 2.4 with $|\beta| \le k$ and the Markov inequality, we obtain

$$\begin{aligned} |(f_\beta - D^\beta P_{Q^u})(y_u)| &= |D^\beta(P_{\widetilde{Q}} - P_{Q^u})(y_u)| \\ &\le \sup_{Q^u} |D^\beta(P_{\widetilde{Q}} - P_{Q^u})| \le O(1)|u|^{-\beta} \sup_{Q^u} |P_{\widetilde{Q}} - P_{Q^u}| \\ &\le O(1)|u|^{k-|\beta|}\omega(|u|)|\vec{f}|_{\mathcal{J}|_X} = O(1)|u|^{k-|\beta|}\omega(|u|). \end{aligned} \tag{4.4}$$

Hence it follows that I_1 and I_2 are majorized by $O(1)|u|^{k-|\alpha|}\omega(|u|)$. Thus

$$|(f_\alpha - D^\alpha P_u)(y_u)| \le O(1)|u|^{k-|\alpha|}\omega(|u|)$$

and the statement (ii) follows.

To prove the statement (iii) of the proposition, we first establish the inequality

$$|D^\alpha(P_{Q^{(1)}} - P_{Q^{(2)}})| \le O(1)\int_{r_1}^{r_1+r_2} \frac{\omega(t)}{t^2}\,dt \quad (|\alpha| = k+1) \tag{4.5}$$

where $Q^{(i)} := Q_{r_i}(x)$, $i = 1, 2, x \in X$ and $r_1 \le r_2$. Let ℓ be the positive integer satisfying

$$2^{\ell-1}r_1 \le r_2 \le 2^\ell r_1. \tag{4.6}$$

Let us define the increasing chain $\{K^{(i)}\}_{i=0}^\ell$ of cubes by

$$K^{(0)} := Q^{(1)}, \quad K^{(\ell)} := Q^{(2)}, \quad K^{(i)} := Q_{2^i r_1}(x).$$

Applying successively the Markov inequality and the inequality of the statement (ii) of Proposition 2.4, we conclude that the left-hand side of (4.5) is majorized by

$$\begin{aligned} \sum_{i=0}^{\ell-1} |D^\alpha(P_{K^{(i)}} - P_{K^{(i+1)}})| &\le O(1)\sum_{i=1}^{\ell-1}(2^i r_1)^{-k-1} \sup_{K^{(i)}} |P_{K^{(i)}} - P_{K^{(i+1)}}| \\ &\le O(1)\sum_{i=1}^{\ell-1} \frac{\omega(2^i r_1)}{2^i r_1}. \end{aligned}$$

Together with (4.6) it gives the estimate (4.5).

Now let $u, v \in U(X)$ be joined by an edge, i.e., $\{x_u, y_u\} \cap \{x_v, y_v\} \ne \varnothing$, and $|\alpha| = k+1$. The desired property (iii) of the proposition is equivalent to the inequality

$$|D^\alpha P_u - D^\alpha P_v| \le O(1)\psi(u, v) := O(1)\int_{|u|}^{|u|+|v|} \frac{\omega(t)}{t^2}dt. \tag{4.7}$$

To prove (4.7) let us suppose that

$$|x_u - y_u| =: |u| \le |v| := |x_v - y_v|,$$

and denote by z the common point of $\{x_u, y_u\}$ and $\{x_v, y_v\}$. We also set

$$K^u := Q_{2|u|}(z) \subset Q_{2|v|}(z) =: K^v.$$

According to (4.2)–(4.4) we have

$$|D^\alpha P_u - D^\alpha P_{Q^u}| = |D^\alpha G_u| \le O(1)\frac{\omega(|u|)}{|u|}, \qquad |\alpha| = k+1.$$

Since $Q^u \subset K^u$, the Markov inequality and the inequality of the statement (ii) of Proposition 2.4 now give

$$\begin{aligned}|D^\alpha(P_u - P_{K^u})| &\le |D^\alpha(P_u - P_{Q^u})| + |D^\alpha(P_{Q^u} - P_{K^u})| \\ &\le O(1)\left\{\frac{\omega(|u|)}{|u|} + |u|^{-k-1}\sup_{Q^u}|P_{Q^u} - P_{K^u}|\right\} \le O(1)\frac{\omega(|u|)}{|u|}.\end{aligned}$$

The same holds for P_v and therefore

$$|D^\alpha(P_u - P_v)| \le O(1)\left\{\frac{\omega(|u|)}{|u|} + \frac{\omega(|v|)}{|v|} + |D^\alpha(P_{K^u} - P_{K^v})|\right\}.$$

In order to obtain the desired inequality (4.7) it remains to estimate the third summand on the right-hand side by (4.5).

This completes the proof of necessity.

(*Sufficiency*). Let $\{P_u : u \in U(X)\} \subset \mathcal{P}_{k+1}$ be the family of polynomials satisfying the conditions (i)–(iii) of Proposition 2.5 with respect to a fixed vector function $\vec{f} = (f_\alpha : X \to \mathbb{R})_{|\alpha| \le k}$. To prove that $\vec{f}$ belongs to the trace space $\mathcal{J}|_X$ with the norm majorized by $O(1)\lambda$, it suffices to construct the family $\{P_Q : Q \in \mathcal{K}_X\} \subset \mathcal{P}_{k+1}$ of polynomials satisfying the conditions (i) and (ii) of Proposition 2.4.

To this end we will introduce the family S of cubes of $\mathcal{K}_X$ by

$$S := \{Q \in \mathcal{K}_X : (\tfrac{1}{4}Q) \cap X = 4Q \cap X\} \tag{4.8}$$

Let us define the equivalence relation "$\sim$" in this family setting:

$$Q_1 \sim Q_2 \Leftrightarrow Q_1 \cap X = Q_2 \cap X \quad (Q_1, Q_2 \in S). \tag{4.9}$$

We let $[Q]$ denote the equivalence class of $Q \in S$ and choose points $x_{[Q]}$, $y_{[Q]}$ from $Q \cap X$ such that

$$|x_{[Q]} - y_{[Q]}| = d(Q \cap X) =: \sup_{x,y \in Q\cap X} |x - y|.$$

By definition it is clear that $Q \cap X$ depends on the class $[Q]$ only.

Further, let $K_{[Q]}$ denote the cube with center $x_{[Q]}$ and radius $d(X \cap Q)$. We remark that $K_{[Q]}$ may coincide with the point $\{x_{[Q]}\}$ (if $X \cap Q$ consists of one point).

Let $\widetilde{K}_{[Q]}$ be the smallest cube with center $x_{[Q]}$ satisfying

$$K_{[Q]} \subset \widetilde{K}_{[Q]} \text{ and } \widetilde{K}_{[Q]} \cap X \ne K_{[Q]} \cap X.$$

It should be noted that if X is bounded, then the cube $\widetilde{K}_{\{Q\}}$ is undefined for some $[Q]$ (e.g., satisfying $\frac{1}{4}Q \supset X$). In this case we put $\widetilde{K}_{[Q]} := \mathbb{R}^n$.

It is clear that if $\operatorname{card} X > 1$ (we omit the trivial case $\operatorname{card} X = 1$), then at least one of the conditions

$$|K_{[Q]}| > 0 \text{ and\textbackslash or } |\widetilde{K}_{[Q]}| < \infty$$

holds.

It is convenient to extend the above definitions to cubes $Q \in \mathcal{K}_X \backslash S$. Namely, for such cube we set

$$[Q] := \{Q\}, K_{[Q]} = \widetilde{K}_{[Q]} := Q \qquad (Q \notin S).$$

We are now in a position to introduce the required family of polynomials $\{P_Q\colon Q\in\mathcal{K}_X\}$. We begin with the case of cube $Q\notin S$. By the definition of S there exists a point $y_Q\in X$ such that

$$\tfrac{1}{4}r_Q\le|c_Q-y_Q|\le 4r_Q. \tag{4.10}$$

Then $u=u(Q):=(c_Q,y_Q)$ belongs to $U(X)$ and we put

$$P_Q:=P_{u(Q)}. \tag{4.11}$$

Now let $Q\in S$. In this case the cubes $K_{[Q]}$ and $\widetilde{K}_{[Q]}$ do not belong to S (see Lemma 4.1 below), and therefore the polynomials $P_{K_{[Q]}}$ and $P_{\widetilde{K}_{[Q]}}$ have been defined by (4.11). Let

$$H_Q:=(1-\tau_Q)P_{\widetilde{K}_{[Q]}}+\tau_Q P_{K_{[Q]}}, \tag{4.12}$$

where we put

$$\tau_Q:=\left(\int_{r_Q}^{\tilde r_{[Q]}}\frac{\omega(t)}{t^2}dt\right)\Big/\left(\int_{r_{[Q]}}^{\tilde r_{[Q]}}\frac{\omega(t)}{t^2}\,dt\right). \tag{4.13}$$

Here $r_{[Q]}$ and $\tilde r_{[Q]}$ denote radii of $K_{[Q]}$ and $\widetilde{K}_{[Q]}$ respectively. In the cases $r_{[Q]}=0$ or $\tilde r_{[Q]}=\infty$ we supplement the definition of τ_Q by

$$\tau_Q:=\begin{cases}0, & \text{for } r_{[Q]}=0,\\ 1, & \text{for } \tilde r_{[Q]}=\infty.\end{cases}$$

Let us now define the required polynomial P_Q by the conditions

$$\begin{aligned}(D^\alpha P_Q)(c_Q)&=f_\alpha(c_Q), &&|\alpha|\le k,\\ D^\alpha P_Q&=D^\alpha H_Q, &&|\alpha|=k+1.\end{aligned} \tag{4.14}$$

By the formulas (4.11)–(4.14) the family $\{P_Q\colon Q\in\mathcal{K}_X\}$ of polynomials is completely defined. To check the required properties of the family, we need the following

Lemma 4.1. (a) *For any $Q\in\mathcal{K}_X$*

$$K_{[Q]},\widetilde{K}_{[Q]}\notin S;$$

(b) *either $Q\notin S$ or*

$$K_{[Q]}\subset Q'\subset\tfrac{1}{2}\widetilde{K}_{[Q]}$$

for every $Q'\in[Q]$;

(c) *let $Q',Q''\in\mathcal{K}_X$, $[Q']\ne[Q'']$ and $Q'\subset Q''$. Then either*

$$\widetilde{K}_{[Q']}\subset 6K_{[Q'']} \tag{4.15}$$

or

$$Q''=K_{[Q'']}\subset 5K_{Q'}. \tag{4.16}$$

Proof. (a). If $Q\notin S$, then by definition $K_{[Q]}=\widetilde{K}_{[Q]}$ $(=Q)\notin S$ as well. Now let $Q\in S$. Then according to the definition of $K_{[Q]}$ the boundary of this cube contains a point of X. Consequently

$$\tfrac{1}{4}K_{[Q]}\cap X\ne 4K_{[Q]}\cap X,$$

and the same is valid for the cube $\widetilde{K}_{[Q]}$. Therefore by (4.8) $K_{[Q]}$ and $\widetilde{K}_{[Q]}$ do not belong to S.

(b) Let us prove first that

$$K_{[Q]} \cap X = Q' \cap X \tag{4.17}$$

for any cube $Q' \in [Q]$. This property is obvious in the case $Q \notin S$ (since here $[Q] := \{Q\} = \{K_{[Q]}\}$). Now if $Q \in S$, then by (4.8) and the definition of $K_{[Q]}$ for any $Q' \in [Q]$

$$r_{K_{[Q]}} := d(Q \cap X) = d(Q' \cap X) = d(\tfrac{1}{4}Q' \cap X) \leq \frac{1}{2} r_{Q'}.$$

Hence

$$|c_{K_{[Q]}} - c_{Q'}| \leq d(Q \cap X) = d(Q' \cap X) \leq \tfrac{1}{2} r_{Q'}. \tag{4.18}$$

Consequently,

$$K_{[Q]} \subset Q' \tag{4.19}$$

and therefore $K_{[Q]} \cap X \subset Q' \cap X$.

On the other hand, since $c_{K_{[Q]}} \in Q \cap X$ and $r_{K_{[Q]}} = d(Q \cap X)$,

$$K_{[Q]} \supset Q \cap X = Q' \cap X,$$

and (4.17) is proved.

Now let $Q \in S$. By (4.19) it remains to prove that $Q' \subset \frac{1}{2}\widetilde{K}_{[Q]}$ for $Q' \in [Q]$. Recall that by the definition of $\widetilde{K}_{[Q]}$ its center $c_{\widetilde{K}_{[Q]}}$ coincides with $c_{K_{[Q]}}$ and

$$r_{\widetilde{K}_{[Q]}} := \rho(c_{K_{[Q]}}, X \backslash Q) := \inf_{x \in X \backslash Q} |c_{K_{[Q]}} - x|.$$

Since $Q' \in [Q]$ by (4.8) and (4.9) we have

$$X \backslash Q = X \backslash Q' = X \backslash (4Q').$$

Together with (4.18) it gives

$$\begin{aligned} r_{\widetilde{K}_{[Q]}} &= \rho(c_{K_{[Q]}}, X \backslash (4Q')) \geq \rho(c_{Q'}, X \backslash (4Q')) - |c_{Q'} - c_{K_{[Q]}}| \\ &\geq 4 r_{Q'} - \tfrac{1}{2} r_{Q'} = \tfrac{7}{2} r_{Q'}. \end{aligned}$$

From here and (4.18) we also get

$$|c_{\widetilde{K}_{[Q]}} - c_{Q'}| = |c_{K_{[Q]}} - c_{Q'}| \leq \tfrac{1}{2} r_{Q'} \leq \tfrac{1}{7} r_{\widetilde{K}_{[Q]}}.$$

Finally,

$$Q' \subset (\tfrac{2}{7} + \tfrac{1}{7}) \widetilde{K}_{[Q]} \subset \tfrac{1}{2} \widetilde{K}_{[Q]},$$

and (b) is proved.

(c) First let $Q', Q'' \in S$. By (4.17) and the relations $[Q'] \neq [Q'']$, $Q' \subset Q''$,

$$Q' \cap X \subsetneqq Q'' \cap X = K'' \cap X \tag{4.20}$$

where for the sake of brevity we put

$$K' := \widetilde{K}_{[Q']}, \quad K'' := K_{[Q'']}.$$

Besides, by the property (b) proved above,

$$Q' \subset \tfrac{1}{2} K', \quad K'' \subset Q''.$$

We point out also that by (4.20)

$$c_{K'} \subset Q' \cap X \subset K''$$

and therefore

$$K'' \subset \widetilde{Q} := 2Q_{r_{K''}}(c_{K'}) \subset 3K''.$$

On the other hand,

$$Q' \cap X \subsetneqq K'' \cap X \subset \widetilde{Q} \cap X$$

and consequently, by the definition of the cube $K' := \widetilde{K}_{[Q']}$,

$$K' \subset \widetilde{Q} \subset 3K''.$$

Thus, in the case under consideration, (4.15) holds.

Let us now show that (4.15) is also valid in the case where

$$Q' \notin S, \quad Q'' \in S.$$

Now, since $Q' \subset Q''$,

$$4Q' \subset 4Q''$$

as well. On the other hand, $Q'' \in S$, and therefore, by (4.8).

$$4Q' \cap X \subset 4Q'' \cap X = \tfrac{1}{4}Q'' \cap X. \tag{4.21}$$

Since $Q' \notin S$ there exists a point $y_{Q'} \in X$ such that

$$y_{Q'} \in (4Q' \setminus \tfrac{1}{4}Q') \cap X.$$

By (4.21) it follows that $y_{Q'} \in Q'' \cap X$. Besides, by (b),

$$K_{[Q'']} \supset Q'' \cap X \supset Q' \cap X \ni c_{Q'}. \tag{4.22}$$

Since $c_{Q'}$ belongs to $Q' \cap X$ and $y_{Q'} \notin \frac{1}{4}Q'$ we get

$$r_{Q'} \le 4|c_{Q'} - y_{Q'}| \le 4d(Q' \cap X) =: 4r_{K_{[Q'']}}.$$

From here and (4.22) we finally get

$$\widetilde{K}_{[Q']} := Q' \subset 5K_{[Q'']},$$

and (4.15) follows.

So it remains to consider the case

$$Q' \in S, \quad Q'' \notin S.$$

By (4.8) there exists a point $y_{Q''}$ satisfying

$$y_{Q''} \in (4Q'' \setminus \tfrac{1}{4}Q'') \cap X.$$

Suppose now that $c_{Q''}$ or $y_{Q''}$ does not belong to $Q' \cap X$, and show that in this case (4.15) holds. In fact, by the assumptions and the definition of the cube $K' := \widetilde{K}_{[Q']}$

$$r_{K'} \le \max\{|c_{K'} - y_{Q''}|, |c_{K'} - c_{Q''}|\}.$$

But $c_{K'} \in Q' \cap X \subset Q''$ and $y_{Q''} \in 4Q''$. Therefore,

$$r_{K'} \le 5r_{Q''},$$

and consequently

$$K' := \widetilde{K}_{[Q']} \subset 6Q'' =: 6K_{[Q']}.$$

Thus, it remains to consider the last case,

$$\{c_{Q''}, y_{Q''}\} \subset Q' \cap X.$$

Here,

$$r_{K_{[Q']}} := d(Q' \cap X) \geq |c_{Q''} - y_{Q''}| \geq \tfrac{1}{4} r_{Q''}.$$

On the other hand,

$$K_{[Q']} \supset Q' \cap X \ni c_{Q''},$$

which, together with the above inequality, gives

$$K_{[Q'']} := Q'' \subset 5K_{[Q']}.$$

Thus, in this case, (4.16) holds and the proof is complete. □

Lemma 4.2. *Let $Q', Q \in \mathcal{K}_X$, $Q', Q \notin S$ and $Q' \subset 6Q$. Then, for any multiindex α, $|\alpha| = k+1$,*

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \leq O(1)\lambda \int_{r_{Q'}}^{12 r_Q} \frac{\omega(t)}{t^2} dt. \tag{4.23}$$

Proof. Recall that in the case $Q \notin S$ we define the polynomial P_Q as $P_Q := P_{u(Q)}$, where $u(Q) := (c_Q, y_Q) \in U(X)$ and y_Q is a point of X satisfying (4.10). Analogously $P_{Q'} := Q_{u(Q')}$, where $u(Q') = (c_{Q'}, y_{Q'}) \in U(X)$ and

$$\tfrac{1}{4} r_{Q'} \leq |c_{Q'} - y_{Q'}| \leq 4 r_{Q'}. \tag{4.24}$$

Now let $a \in \{c_{Q'}, y_{Q'}\}$ satisfy

$$|c_Q - a| = \max\{|c_Q - c_{Q'}|, |c_Q - y_{Q'}|\}. \tag{4.25}$$

We put

$$\tilde{r} := r_{Q'} + |c_Q - c_{Q'}|$$

and

$$\tilde{u} := (c_Q, a) \in U(X).$$

Then, by (4.24) and (4.25) we have

$$\tilde{r} \leq 4|c_{Q'} - y_{Q'}| + |c_Q - c_{Q'}| \leq 5|c_{Q'} - c_Q| + |c_Q - y_{Q'}| \leq 6|c_Q - a|$$

and

$$|c_Q - a| \leq |c_{Q'} - c_Q| + |y_{Q'} - c_Q| \leq 2|c_{Q'} - c_Q| + |y_{Q'} - c_{Q'}| \leq 4\tilde{r}.$$

Thus,

$$\tilde{r}/6 \leq |\tilde{u}| := |c_Q - a| \leq 4\tilde{r}. \tag{4.26}$$

We mention also that $u(Q) \leftrightarrow \tilde{u}$ (i.e., $\{c_Q, y_Q\} \cap \{c_Q, a\} \neq \varnothing$) and $\tilde{u} \leftrightarrow u(Q')$. Therefore, by the property (iii) of the proposition,

$$\begin{aligned} |D^\alpha P_{Q'} - D^\alpha P_Q| &:= |D^\alpha P_{u(Q')} - D^\alpha P_{u(Q)}| \\ &\leq |D^\alpha P_{u(Q')} - D^\alpha P_{\tilde{u}}| + |D^\alpha P_{\tilde{u}} - D^\alpha P_{u(Q)}| \\ &\leq \lambda\{\psi(|u(Q')|, |\tilde{u}|) + \psi(|\tilde{u}|, |u(Q)|)\}, \end{aligned} \tag{4.27}$$

where, recall, $|u| := |x_u - y_u|$ for $u = (x_u, y_u) \in U(X)$ and the function ψ is defined by (2.1).

Let us now estimate the summands in the right-hand side of (4.27). By (4.24) and (4.26)

$$\min\{|u(Q')|,|\tilde{u}|\} := \min\{|c_{Q'}-y_{Q'}|,|c_Q-a|\} \geq \min\left\{\tfrac{1}{4}r_{Q'},\tfrac{1}{6}\tilde{r}\right\} \geq \tfrac{1}{6}r_{Q'}.$$

Besides, by the condition of the lemma, $Q' \subset 6Q$ and therefore

$$\tilde{r} := r_{Q'} + |c_Q - c_{Q'}| \leq 6r_Q.$$

Using again (4.24) and (4.26), we get

$$|u(Q')| + |\tilde{u}| \leq 4r_{Q'} + 4\tilde{r} := 8r_{Q'} + 4|c_Q - c_{Q'}| \leq 48r_Q.$$

Now, according to the definition (2.1)

$$\psi(|u(Q')|,|u(Q)|) := \int_{\min(|u(Q')|,|\tilde{u}|)}^{|u(Q')|+|\tilde{u}|} \frac{\omega(t)}{t^2}\,dt \leq \int_{\frac{1}{6}r_{Q'}}^{48r_Q} \frac{\omega(t)}{t^2}dt. \tag{4.28}$$

To estimate $\psi(|\tilde{u}|,|u(Q)|)$ we make use of (4.10), (4.25) and the inequalities $r_{Q'} \leq 6r_Q$ and $\tilde{r} \leq 6r_Q$. Then we obtain

$$\min(|\tilde{u}|,|u(Q)|) \geq \min(\tfrac{1}{6}\tilde{r},\tfrac{1}{4}r_Q) \geq \tfrac{1}{24}r_{Q'},$$

and

$$|\tilde{u}| + |u(Q)| \leq 4\tilde{r} + 4r_Q \leq 28r_Q.$$

Therefore, by (2.1)

$$\psi(|\tilde{u}|,|u(Q)|) \leq \int_{\frac{1}{24}r_{Q'}}^{28r_Q} \frac{\omega(t)}{t^2}dt.$$

From here, (4.27) and (4.28) we get

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \leq 2\lambda \int_{\frac{1}{24}r_{Q'}}^{48r_Q} \frac{\omega(t)}{t^2}dt.$$

It remains to make use of the non-decreasing properties of the functions ω and $t \to t^2/\omega(t)$, and the desired inequality (4.23) follows. □

Lemma 4.3. *Let Q' and Q be cubes from $\mathcal{K}_X$ such that $Q' \in [Q] \cup \{K_{[Q]}, \widetilde{K}_{[Q]}\}$ and $Q' \subset Q$. Then for any α, $|\alpha| = k+1$,*

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \leq O(1)\lambda \int_{r_{Q'}}^{r_Q} \frac{\omega(t)}{t^2}dt.$$

Proof. By the definitions (4.12)–(4.14)

$$\begin{aligned} |D^\alpha P_{Q'} - D^\alpha P_Q| &:= |D^\alpha H_{Q'} - D^\alpha H_Q| = (\tau_Q - \tau_{Q'})|D^\alpha P_{K_{[Q]}} - D^\alpha P_{\widetilde{K}_{[Q]}}| \\ &= \left\{\left(\int_{r_{Q'}}^{r_Q} \frac{\omega(t)}{t^2}dt\right) \Big/ \left(\int_r^{\tilde{r}} \frac{\omega(t)}{t^2}dt\right)\right\} |D^\alpha P_{K_{[Q]}} - D^\alpha P_{\widetilde{K}_{[Q]}}| \end{aligned} \tag{4.29}$$

where we set $r := r_{K_{[Q]}}, \tilde{r} := r_{\widetilde{K}_{[Q]}}$. Since $K_{[Q]}$ and $\widetilde{K}_{[Q]} \notin S$ (see Lemma 4.1(a)), by Lemma 4.2 we have

$$|D^\alpha P_{K_{[Q]}} - D^\alpha P_{\widetilde{K}_{[Q]}}| \leq O(1)\lambda \int_r^{12\tilde{r}} \frac{\omega(t)}{t^2}dt.$$

We mention also that without loss of generality one can regard that $Q \in S$ (otherwise $[Q] := \{K_{[Q]}\} := \{\widetilde{K}_{[Q]}\}$, and the statement of the lemma is obvious). In this case, by Lemma 4.1(b), $r \le \frac{1}{2}\tilde{r}$. Recall also that neither the function ω nor the function $t \to t^2/\omega(t)$ decreases. Hence it follows that the latter integral is majorized by $O(1)\int_r^{\tilde{r}}$. Together with (4.29) it gives the required inequality of the lemma. $\square$

Lemma 4.4. *Let Q', Q be two arbitrary cubes from $\mathcal{K}_X$ such that $Q' \subset Q$. Then*

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \le O(1)\lambda \frac{\omega(r_Q)}{r_{Q'}}. \tag{4.30}$$

Proof. Let us consider first the case $[Q'] = [Q]$. Then, by Lemma 4.3, we have

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \le O(1)\lambda \int_{r_{Q'}}^{r_Q} \frac{\omega(t)}{t^2}dt \le O(1)\lambda \frac{\omega(r_Q)}{r_{Q'}},$$

and (4.30) holds. Moreover, by the same lemma,

$$|D^\alpha P_{Q'} - D^\alpha P_K| \le O(1)\lambda \int_{r_K}^{r_Q} \frac{\omega(t)}{t^2}dt \le O(1)\lambda \frac{\omega(r_Q)}{r_K}, \tag{4.31}$$

and analogously

$$|D^\alpha_{Q'} - D^\alpha P_{K'}| \le O(1)\lambda \int_{r_{Q'}}^{r_{K'}} \frac{\omega(t)}{t^2}dt \le O(1)\lambda \frac{\omega(r_{K'})}{r_{Q'}}. \tag{4.32}$$

Here, for the sake of brevity, we set

$$K := K_{[Q]}, \quad K' := \widetilde{K}_{[Q']}.$$

Now let

$$[Q'] \ne [Q].$$

According to the property (c) of Lemma 4.1, the latter relation is possible in the following two cases only:

Case 1.

$$K' := \widetilde{K}_{[Q']} \subset 6K := 6K_{[Q]}. \tag{4.33}$$

Then

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \le |D^\alpha P_{Q'} - D^\alpha P_{K'}| + |D^\alpha P_{K'} - D^\alpha P_K| + |D^\alpha P_K - D^\alpha P_Q|.$$

By Lemma 4.1(a) the cubes K' and $K \notin S$; therefore, by Lemma 4.2 and (4.33), we have

$$|D^\alpha P_{K'} - D^\alpha P_K| \le O(1)\lambda \int_{r_{K'}}^{12 r_K} \frac{\omega(t)}{t^2}dt \le O(1)\lambda \frac{\omega(r_K)}{r_{K'}}.$$

Together with (4.31) and (4.32) it gives

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \le O(1)\lambda \left\{ \frac{\omega(r_{K'})}{r_{Q'}} + \frac{\omega(r_K)}{r_{K'}} + \frac{\omega(r_Q)}{r_K} \right\}. \tag{4.34}$$

It remains to note that by (4.33) and the properties (b) of Lemma 4.1

$$r_{Q'} \le 6\min\{r_{K'}, r_K\}, \quad r_Q \ge \tfrac{1}{6}\max\{r_{K'}, r_K\},$$

and (4.30) in the case under consideration follows.

Case 2.

$$Q = K_{[Q]} \subset 5K_{[Q']}.$$

Let us note first that by Lemma 4.1(b) $K_{[Q']} \subset Q'$ and consequently

$$r_{[Q']} \le r_{Q'} \le r_Q \le 5r_{[Q']} \tag{4.35}$$

(recall that $r_{[Q]}$ is the radius of the cube $K_{[Q]}$). Then

$$|D^\alpha P_{Q'} - D^\alpha P_Q| \le |D^\alpha P_{Q'} - D^\alpha P_{K_{[Q']}}| + |D^\alpha P_{K_{[Q']}} - D^\alpha P_Q|.$$

But the cubes $Q = K_{[Q]}, K_{[Q']} \notin S$ (see Lemma 4.1(a)); therefore by (4.31), (4.35) and Lemma 4.2, we finally get

$$\begin{aligned} |D^\alpha P_{Q'} - D^\alpha P_Q| &\le O(1)\lambda \left\{ \frac{\omega(r_{Q'})}{r_{[Q']}} + \int_{r_Q}^{12r_{[Q']}} \frac{\omega(t)}{t^2}\, dt \right\} \\ &\le O(1)\lambda \left\{ \frac{\omega(r_{Q'})}{r_{[Q']}} + \frac{\omega(r_{[Q']})}{r_Q} \right\} \le O(1)\lambda \frac{\omega(r_Q)}{r_{Q'}}. \quad \square \end{aligned}$$

We are now in a position to complete the sufficiency part of Proposition 2.5. So let $Q', Q \in \mathcal{K}_X$ and $Q' \subset Q$. We have to show that

$$\sup_{Q'} |P_{Q'} - P_Q| \le O(1)\lambda\{|c_{Q'} - c_Q| + r_{Q'}\}^k \omega(r_Q).$$

By the Taylor formula

$$\begin{aligned} \sup_{Q'} |P_{Q'} - P_Q| &\le \sup_{x \in Q'} \left\{ \left| \sum_{|\beta| \le k} (D^\beta(P_{Q'} - P_Q))(c_{Q'}) \frac{(x - c_{Q'})^\beta}{\beta!} \right| \right. \\ &\qquad \left. + \sum_{|\alpha|=k+1} |D^\alpha P_{Q'} - D^\alpha P_Q| \frac{|x - c_{Q'}|^{k+1}}{\alpha!} \right\} \\ &\le O(1) \left\{ \sum_{|\beta| \le k} |D^\beta(P_{Q'} - P_Q)(c_{Q'})| r_{Q'}^{|\beta|} + r_{Q'}^{k+1} \sum_{|\alpha|=k+1} |D^\alpha P_{Q'} - D^\alpha P_Q| \right\}. \end{aligned} \tag{4.36}$$

Let us estimate every summand in the right-hand side of (4.36). Applying Lemma 4.4 to the last summand, we have

$$r_{Q'}^{k+1} \sum_{|\alpha|=k+1} |D^\alpha P_{Q'} - D^\alpha P_Q| \le O(1)\lambda r_{Q'}^{k+1} \frac{\omega(r_Q)}{r_{Q'}} = O(1)\lambda r_{Q'}^k \omega(r_Q). \tag{4.37}$$

To estimate the first summand, we remark that by definition

$$D^\beta P_{Q'}(c_{Q'}) = f_\beta(c_{Q'}) \quad \text{for all } \beta, |\beta| \le k,$$

and therefore

$$\begin{aligned} |(D^\beta P_{Q'} - D^\beta P_Q)(c_{Q'})| &= |f_\beta(c_{Q'}) - D^\beta P_Q(c_{Q'})| \\ &\le |f_\beta(c_{Q'}) - D^\beta P_u(c_{Q'})| + |D^\beta(P_u - P_Q)(c_{Q'})|. \end{aligned} \tag{4.38}$$

Here and below $u := (c_Q, c_{Q'})$. Recall also that $|u| := |c_Q - c_{Q'}|$ and $Q' \subset Q$. Hence, in particular, we have

$$|u| \le r_Q. \tag{4.39}$$

Now, by the property (ii) of the proposition, we get

$$|f_\beta(c_{Q'}) - D^\beta P_u(c_{Q'})| \le O(1)\lambda |u|^{k-|\beta|} \omega(|u|). \tag{4.40}$$

On the other hand,

$$D^\beta P_u(c_Q) = D^\beta P_Q(c_Q)(= f_\beta(c_Q)), \text{ for any } \beta, |\beta| \le k.$$

Therefore,

$$(P_u - P_Q)(x) = \sum_{|\alpha|=k+1} \frac{D^\alpha(P_u - P_Q)}{\alpha!}(x - c_Q)^\alpha$$

and consequently

$$|D^\beta(P_u - P_Q)(c_{Q'})| \le O(1)|u|^{k+1-|\beta|} \sum_{|\alpha|=k+1} |D^\alpha(P_u - P_Q)|.$$

So it remains to estimate the value of $|D^\alpha(P_u - P_Q)|$, $|\alpha| = k+1$. For this purpose we set

$$Q^u := Q_{|u|}(c_Q) = \{y\colon |c_Q - y| \le |c_Q - c_{Q'}|\}.$$

Then, by the definition (4.8) the cube $Q^u \notin S$. Besides, according to (4.10) and (4.11) the polynomial P_{Q_u} is defined as

$$P_{Q_u} := P_{\tilde u}$$

where $\tilde u := (c_Q, \tilde y) \in U(X)$ and $\tilde y$ is a point of X satisfying

$$\tfrac{1}{4} r_{Q_u} := \tfrac{1}{4}|u| \le |\tilde u| := |c_Q - \tilde y| \le 4 r_{Q^u} := 4|u|.$$

From here and the property (iii) of the proposition we have

$$\begin{aligned} |D^\alpha P_u - D^\alpha P_{Q^u}| &= |D^\alpha P_u - D^\alpha P_{\tilde u}| \le \lambda \psi(|u|, |\tilde u|) \\ &:= \lambda \int_{\min(|u|,|\tilde u|)}^{|u|+|\tilde u|} \frac{\omega(t)}{t^2} dt \le \lambda \int_{\frac{1}{4}|u|}^{5|u|} \frac{\omega(t)}{t^2} dt \le O(1)\lambda \frac{\omega(|u|)}{|u|} \end{aligned}$$

for all α, $|\alpha| = k+1$. This inequality, Lemma 4.4 and the property $Q^u \subset Q$ give

$$\begin{aligned} |D^\alpha P_Q - D^\alpha P_u| &\le |D^\alpha P_Q - D^\alpha P_{Q^u}| + |D^\alpha P_{Q^u} - D^\alpha P_u| \\ &\le O(1)\lambda \left\{ \frac{\omega(r_Q)}{r_{Q^u}} + \frac{\omega(|u|)}{|u|} \right\} = O(1)\lambda |u|^{-1}(\omega(r_Q) + \omega(|u|)). \end{aligned}$$

Now combining this inequality and the inequalities (4.36)–(4.40), we finally get

$$\begin{aligned} \sup_{Q'} |P_{Q'} - P_Q| \le & O(1)\lambda \Bigg\{ \Bigg[\sum_{|\beta| \le k} (|u|^{k-|\beta|} \omega(|u|) \\ & \qquad + |u|^{k-|\beta|}(\omega(r_Q) + \omega(|u|)) r_{Q'}^{|\beta|} \Bigg] + r_{Q'}^k \omega(r_Q) \Bigg\} \\ & \le O(1)\lambda\omega(r_Q) \left(\sum_{|\beta| \le k} |u|^{k-|\beta|} r_{Q'}^{|\beta|} \right) \le O(1)\lambda(r_{Q'} + |u|)^k \omega(r_Q) \\ & := O(1)\lambda(r_{Q'} + |c_Q - c_{Q'}|)^k \omega(r_Q). \end{aligned}$$

Proposition 2.5 is completely proved.

References

[B] Yu. Brudnyi, *A multidimensional analog of a theorem of Whitney*, Mat. Sbornik, **82** (124) (1970), N2, 175–191; English transl. in Math. USSR Sbornik, **11** (1970), N2, 157–170.

[BS] Yu. Brudnyi and P. Shvartsman, *Generalizations of Whitney's Extension Theorem*, Internat. Math. Res. Notices, N3 (1994), 129–139. MR **95c:**58018

[BS1] ———, *A linear extension operator for a space of smooth functions defined on a closed subset of* $\mathbb{R}^n$, Dokl. Akad. Nauk SSSR **280:2** (1985), 268–272; English transl. in Soviet Math. Dokl. **31** (1985), 48–51. MR **86f:**46031

[BS2] ———, *The Whitney Problem of Existence of a Linear Extension Operator*, The Journal of Geometric Analysis (to appear).

[BS3] ———, "A description of the trace of a function from the generalized Lipschitz space to an arbitrary compact" in *Studies in the Theory of Functions of Several Real Variables*, Yaroslav State Univ., Yaroslavl, 1982, 16–24 (Russian).

[G] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Analyse Math. **6** (1958), 1–124. MR **21:**107

[H] L. G. Hanin, "A trace description of functions with high order derivatives from generalized Zygmund space to arbitrary closed subsets" in *Studies in the Theory of Functions of Several Real Variables*, Yaroslav. State Univ., Yaroslavl, 1987, 128–144 (Russian).

[JW] A Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Math. Rep. **2** (1984), Part 1. MR **87f:**46056

[Sh] P. Shvartsman, *Lipschitz sections of set-valued mappings and traces of functions from the Zygmund class on an arbitrary compactum*, Dokl. Akad. Nauk SSSR **276** (1984), 559–562; English transl. in Soviet Math. Dokl., **29** (1984), 565–568. MR **85j:**46057

[Sh1] ———, *Traces of functions of the Zygmund class*, Sibirsk. Mat. Zh. **28:5** (1987), 203–215; English transl. in Siberian Math. J. **28** (1987), 853–863. MR **89a:**46081

[Sh2] ———, "Lipschitz sections of multivalued mappings" in *Studies in the Theory of Functions of Several Real Variables*, Yaroslav. State Univ., Yaroslavl, 1986, 121–132 (Russian). MR **88e:**46032

[Sh3] ———, "K-functionals of weighted Lipschitz spaces and Lipschitz selections of multivalued mappings" in *Interpolation Spaces and Related Topics*, Israel Math. Conf. Proc. **5**, Weizmann, Jerusalem, 1992, 245–268. MR **94c:**46069

[St] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970. MR **44:**7280

[W] H. Whitney, *Analytic extension of differentiable function defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `ybrudnyi@techunix.technion.ac.il`

E-mail address: `pshv@techunix.technion.ac.il`