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BEREZIN'S QUANTIZATION ON FLAG MANIFOLDS AND SPHERICAL MODULES

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Abstract. We show that the theory of spherical Harish-Chandra modules naturally gives rise to Berezin's symbol quantization on generalized flag manifolds. It provides constructions of symbol algebras and of their representations for covariant and contravariant symbols, and also for symbols which so far have no explicit definition. For all these symbol algebras we give a general proof of the correspondence principle.

Introduction

Berezin's quantization on a symplectic manifold Ω (see [1]) is given by a family of associative algebras $\{\mathcal{A}_h\}$, whose elements are smooth functions on Ω, together with their representations in Hilbert spaces H_h, where the parameter h plays the role of the Planck constant. The multiplication $*_h$ in $\mathcal{A}_h$ must satisfy the so-called correspondence principle, as $h \to 0$. (A formal definition is given in Section 1).

Representation of the algebra $\mathcal{A}_h$ in H_h maps a function $f \in \mathcal{A}_h$ to the operator $\hat{f}$ in H_h. In that case the function f is called a symbol of the operator $\hat{f}$. The concrete examples of quantizations are usually based on explicit constructions of symbols (see [1], [2]).

In this paper we show how the theory of spherical Harish-Chandra modules naturally provides the algebras of covariant and contravariant symbols on generalized flag manifolds. Moreover, it also gives rise to the algebras of symbols which so far have no general construction or definition, as in the case of covariant and contravariant symbols. For all these symbol algebras we give a general proof of the correspondence principle. We also prove the conjecture from [3] on rational dependence of the product $f *_h g$ on h on an arbitrary generalized flag manifold.

1. The definition of quantization

We will give now a formal definition of quantization on a symplectic manifold Ω. Let F be a set of positive numbers with a limit point 0. For each $h \in F$ let $\mathcal{A}_h$ be an algebra, which is a linear subspace of $C^\infty(\Omega)$, with a multiplication $*_h$ and a given representation in a Hilbert space H_h. Define $\mathcal{A} = \bigcup_h \mathcal{A}_h$ and let $\mathcal{A}$ be dense in $C^\infty(\Omega)$. These data determine a quantization on Ω, if for $f, g \in \mathcal{A}$ and $h \to 0$ the correspondence principle holds,

$$f *_h g \to fg, \qquad h^{-1}(f *_h g - g *_h f) \to i\{f, g\}_\Omega. \tag{1}$$

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Here $\{\cdot,\cdot\}_\Omega$ is the Poisson bracket on Ω. The limits in (1) make sense if for each $f \in \mathcal{A}$ one has $f \in \mathcal{A}_h$ for sufficiently small values of h (a weak nesting property). This condition is valid if $\mathcal{A}_h \supset \mathcal{A}_{h'}$ holds for $h < h'$ (a strong nesting property).

2. Covariant and contravariant symbols

Suppose a compact Lie group K has an irreducible unitary representation in a q-dimensional vector space E with a hermitian scalar product $\langle\cdot,\cdot\rangle$, $v \in E$ is a unit vector and dk is the Haar measure on K such that the total measure of K is 1. For $k \in K$ denote $v_k = kv$. Then the orthogonality relations for matrix coefficients immediately imply that for all $\xi, \eta \in E$ the Parseval identity holds:

$$\langle \xi, \eta \rangle = q \int \langle \xi, v_k \rangle \langle v_k, \eta \rangle dk.$$

Thus the K-orbit of v, $\{v_k\}$, forms a supercomplete system of vectors in the sense of [2], and one can associate with it the constructions of covariant and contravariant symbols on K. Let P_k be the orthogonal projection operator in E to the vector v_k.

Definition. A function $f(k)$ on K is called the covariant symbol of an operator $A \in End\ E$, associated with the supercomplete system $\{v_k\}$, if $f(k) = tr\ AP_k = \langle Av_k, v_k \rangle$.

Such symbols for a compact semisimple Lie group K were studied in [4].

Definition. A measurable function $g(k)$ on K is called a contravariant symbol of an operator $B \in End\ E$, associated with the supercomplete system $\{v_k\}$, if $B = q \int g(k) P_k dk$.

The proofs of the following two lemmas are trivial.

Lemma 1. *A measurable function $g(k)$ is a contravariant symbol of an operator $B \in End\ E$ iff for any operator $A \in End\ E$ with the covariant symbol $f(k)$ the equality $q \int f(k) g(k) dk = tr\ AB$ holds.*

Let $k_0 \in K$. Denote $v' = k_0 v$ and consider the supercomplete system $\{v'_k = kv'\}$.

Lemma 2. *Let $f(k)$ be the covariant (contravariant) symbol of an operator $A \in End\ E$, associated with the supercomplete system $\{v_k\}$. Then $f(kk_0)$ is the covariant (contravariant) symbol of A, associated with the supercomplete system $\{v'_k\}$.*

3. A spherical principal series of Harish-Chandra modules

We give a construction of a spherical principal series of Harish-Chandra modules, following [5].

Let g_c be a complex semisimple Lie algebra, h_c its Cartan subalgebra, W the Weyl group, $\triangle$, $\triangle^+$, Σ the sets of all nonzero, positive and simple roots respectively, ρ the half-sum of positive roots. For each $\alpha \in \triangle$ choose weight elements $X_\alpha \in g_c$ such that $[H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}$ for $H_\alpha = [X_\alpha, X_{-\alpha}]$. An element $\lambda \in h_c^*$ is called dominant if $\lambda(H_\alpha) \geq 0$ for all $\alpha \in \Sigma$, and is a weight if $\lambda(H_\alpha) \in \mathbf{Z}$ for all $\alpha \in \Sigma$. Denote n_c, n_c^- the complex subalgebras of g_c spanned by $\{X_\alpha\}$ for α running over $\triangle^+$, $-\triangle^+$ respectively. Then $g_c = n_c^- \oplus h_c \oplus n_c$ is the Gauss decomposition.

Let g_r, h_r, n_r, n_r^- denote the Lie algebras g_c, h_c, n_c and n_c^- respectively, considered as real Lie algebras, and let θ be an automorphism of g_r, antilinear with respect to the complex structure on g_c and such that $\theta X_\alpha = -X_{-\alpha}$, $\theta H_\alpha = -H_\alpha$.

The subspace of θ-invariant elements of g_r, $k_r = g_r^\theta$, is a compact form of g_c. Denote by a_r the subalgebra of g_r generated by $\{H_\alpha\}$. Then $g_r = k_r \oplus a_r \oplus n_r$ is the Iwasawa decomposition.

Now let G be the complex semisimple connected simply connected Lie group with the Lie algebra g_r, considered as a real Lie group, and $G = KAN$ be the corresponding Iwasawa decomposition. Then K is a compact semisimple connected simply connected Lie group. Let $m_r = k_r \cap h_r$, M the maximal torus in K with the Lie algebra m_r, M' its normalizer in K, so that $W = M'/M$.

Denote by Λ the space of functions on K, K-finite with respect to the left shifts and invariant with respect to the right shifts by the elements of M (in [5] it is denoted by $L(0)$). Let $X \in a_r$, $a = \exp X \in A$. Then for $\lambda \in h_c^*$ we have $a^\lambda = \mathrm{e}^{\langle \lambda, X\rangle}$. Denote by $\tilde\Lambda(\lambda)$ the space of functions f on G such that for $g \in G, m \in M, a \in A, n \in N$ we have $f(gman) = f(g)a^{2(\lambda-\rho)}$, and the restriction of f to K belongs to Λ (in the notation of [5], $L(0, -2\lambda)$). Define a (g_r, K)-module structure on $\tilde\Lambda(\lambda)$, for $f \in \tilde\Lambda(\lambda)$, $X \in g_r$, $k \in K$, by $Xf(g) = \frac{d}{dt}|_{t=0} f(\exp(-tX)g)$, $kf(g) = f(k^{-1}g)$. The module $\tilde\Lambda(\lambda)$ is spherical, i.e., its subspace of K-invariant vectors is one dimensional.

By virtue of the Iwasawa decomposition, the restriction to K determines a bijection of $\tilde\Lambda(\lambda)$ onto Λ. Denote by $\Lambda(\lambda)$ the space Λ, endowed with a (g_r, K)-module structure, induced from $\tilde\Lambda(\lambda)$. The (g_r, K)-modules $\Lambda(\lambda)$ (or $\tilde\Lambda(\lambda)$) form the spherical principal series of Harish-Chandra (g_r, K)-modules.

Let $A(\lambda)$ and $B(\lambda)$ denote respectively the cyclic submodule of $\Lambda(\lambda)$ generated by $1 \in \Lambda(\lambda)$, and its maximal proper submodule. Then $\hat\Lambda(\lambda) = A(\lambda)/B(\lambda)$ is the canonical simple subquotient of the principal series. It is known, that for a given $\lambda \in h_c^*$ the modules $\hat\Lambda(w\lambda)$, $w \in W$, are isomorphic to each other (see [5]).

4. An algebra structure on $A(\lambda + \rho)$

Let $\mathcal{U}(g_c)$ be the universal enveloping algebra of g_c. Define an action of g_r on $\mathcal{U}(g_c)$: for $X \in g_r$, $u \in \mathcal{U}(g_c)$ let $X : u \mapsto \theta(X)u - uX$, and let K act on $\mathcal{U}(g_c)$ by the adjoint action (denoted by Ad). If $X \in k_r$, then $\theta X = X$, so $X : u \mapsto Xu - uX$, i.e., k_r acts on $\mathcal{U}(g_c)$ by the adjoint action. Thus the actions of g_r and K agree on k_r and $\mathcal{U}(g_c)$ is a (g_r, K)-module.

Consider a morphism of (g_r, K)-modules $s_\lambda : \mathcal{U}(g_c) \to \Lambda(\lambda+\rho)$ such that $s_\lambda 1 = 1$. This requirement defines s_λ uniquely, since $1 \in \mathcal{U}(g_c)$ is cyclic. Thus s_λ coincides with the morphism $\Phi_{-\lambda}$, introduced in [6], Sect. 9.6.5. Let $L(\lambda)$ denote the simple highest weight g_c-module of weight $\lambda - \rho$, and $Ann\ L(\lambda)$ its annihilator.

Lemma 3. (i) *$Im\ s_\lambda = A(\lambda + \rho)$;* (ii) *$Ker\ s_\lambda = Ann\ L(\lambda + \rho)$.*

The proof of (i) is trivial, and (ii) is proved in lemma 9.6.5 of [6].

Corollary. *$A(\lambda + \rho)$ carries the structure of an algebra naturally isomorphic to $\mathcal{U}(g_c)/Ann\ L(\lambda + \rho)$.*

We will give now an explicit construction of the mapping s_λ. For $\lambda \in h_c^*$ define a functional φ_λ on $\mathcal{U}(g_c)$ (in the notation of [6], Sect.9.6.4., it corresponds to $\varphi_{-\lambda}$). It follows from the Gauss decomposition, that

$$\mathcal{U}(g_c) = (n_c^- \mathcal{U}(g_c) + \mathcal{U}(g_c) n_c) \oplus \mathcal{U}(h_c). \tag{2}$$

Denote the projection of $u \in \mathcal{U}(g_c)$ on $\mathcal{U}(h_c)$ by u_0. Since $\mathcal{U}(h_c)$ is naturally isomorphic to the symmetric algebra $S(h_c)$ (or to the algebra of polynomials on h_c^*), one can evaluate u_0 at the point $\lambda \in h_c^*$. Define $\varphi_\lambda(u) = u_0(\lambda)$.

For $u \in \mathcal{U}(g_c)$ define a function $\tilde{s}_\lambda u$ on G by the formula

$$\tilde{s}_\lambda u(kan) = \varphi_\lambda(Ad(k^{-1})u)a^{2\lambda},$$

where $k \in K,\ a \in A,\ n \in N$.

Proposition 1. *The mapping $\tilde{s}_\lambda : u \mapsto \tilde{s}_\lambda u$ is a morphism of (g_r, K)-modules from $\mathcal{U}(g_c)$ to $\tilde{\Lambda}(\lambda + \rho)$.*

Proof. The K-equivariance of $\tilde{s}_\lambda$ can be checked immediately. Now the K-finiteness of $\tilde{s}_\lambda u$ follows from the finiteness of the adjoint action of K in $\mathcal{U}(g_c)$. The adjoint action of M on $\mathcal{U}(g_c)$ preserves the decomposition (2) and is trivial on $\mathcal{U}(h_c)$, which implies the invariance of $\tilde{s}_\lambda u$ with respect to the right shifts by the elements of M. Thus $\tilde{s}_\lambda u \in \tilde{\Lambda}(\lambda + \rho)$. It remains to show that $\tilde{s}_\lambda$ is g_r-equivariant, that is, for $X \in g_r,\ u \in \mathcal{U}(g_c)$ the functions $X\tilde{s}_\lambda u$ and $\tilde{s}_\lambda(\theta(X)u - uX)$ coincide. Since the restriction of $\tilde{\Lambda}(\lambda+\rho)$ to K is a bijection onto $\Lambda(\lambda+\rho)$ and $\tilde{s}_\lambda$ is K-equivariant, it is enough to compare these functions at $e \in K$, that is, to check the equality

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{s}_\lambda u(\exp(-tX)) = \varphi_\lambda(\theta(X)u - uX) \tag{3}$$

for $X \in k_r, a_r, n_r$. Since the actions of g_r and K agree on k_r, (3) holds for $X \in k_r$. Let $X \in a_r$; then $\tilde{s}_\lambda u(\exp(-tX)) = \mathrm{e}^{-2t\langle\lambda, X\rangle}\varphi_\lambda(u)$, so the left-hand side of (3) equals $-2\langle\lambda, X\rangle\varphi_\lambda(u)$. On the other hand, $\theta X = -X$ for $X \in a_r$, so $\theta(X)u - uX = -2uX$. Since the multiplication by $X \in a_r$ preserves the decomposition (2), one has $\varphi_\lambda(-2uX) = -2\langle\lambda, X\rangle\varphi_\lambda(u)$. Now let $X \in n_r$. Then $\tilde{s}_\lambda u(\exp(-tX)) = \varphi_\lambda(u)$ does not depend on t, and thus the left-hand side of (3) is equal to 0. Since $\theta X \in n_r^-$, therefore $\theta(X)u - uX$ belongs to the sum in the parenthesis in (2), so the right-hand side of (3) is also 0. This concludes the proof. □

Corollary. *The mapping s_λ is given by the formula $s_\lambda u(k) = \varphi_\lambda(Ad(k^{-1})u)$.*

Proof. Since the restriction of $\tilde{s}_\lambda 1$ to K equals the constant 1, the morphism $\mathcal{U}(g_c) \ni u \mapsto \tilde{s}_\lambda u|_K \in \Lambda(\lambda+\rho)$ coincides with s_λ. Now the proof follows from the definition of $\tilde{s}_\lambda$. □

5. Symbol algebras

Let $\lambda \in h_c^*$ be a dominant weight. Consider the holomorphic irreducible representation of the group G with the highest weight λ in the space E^λ with a K-invariant hermitian scalar product $\langle\cdot,\cdot\rangle$, and a highest weight vector v of length 1. Denote the corresponding representation of $\mathcal{U}(g_c)$ in E^λ by π_λ. Let $w_0 \in W$ be the Coxeter element (the element of the maximal reduced length, which maps $\triangle^+$ to $-\triangle^+$). Set $\lambda' = -w_0\lambda$. It is known, that the module $E^{\lambda'}$ is dual to E^λ.

Endow the space $End\ E^\lambda$ with a (g_r, K)-module structure. For $A \in End\ E^\lambda$, $X \in g_r$ and $k \in K$, let $X : A \mapsto \pi_\lambda(\theta X)A - A\pi_\lambda(X)$ and $k : A \mapsto kAk^{-1}$.

The complexification of the algebra g_r is isomorphic to $g_c \times g_c$. Consider an imbedding of g_r in $g_c \times g_c$, $X \mapsto (\theta X, X)$. Then $End\ E^\lambda$ is isomorphic to $E^\lambda \otimes E^{\lambda'}$ as a $g_c \times g_c$-module, and therefore is simple.

It is easy to check that $\pi_\lambda : \mathcal{U}(g_c) \to End\ E^\lambda$ is a surjective morphism of (g_r, K)-modules.

Proposition 2. *Let $\lambda \in h_c^*$ be a dominant weight, $u \in \mathcal{U}(g_c)$. The function $s_\lambda u$ is the covariant symbol of the operator $\pi_\lambda(u) \in End\ E^\lambda$, associated with the supercomplete system $\{v_k = kv\}$.*

Proof. Suppose $Ad(k^{-1})u$ is decomposed according to (2), $Ad(k^{-1})u = u_0 + Xu_1 + u_2Y$, where $u_0 \in \mathcal{U}(h_c)$, $X \in n_c^-$, $Y \in n_c$. Let us calculate the covariant symbol of the operator $\pi_\lambda(u)$, associated with $\{v_k\}$:

$$\begin{aligned} \langle \pi_\lambda(u)v_k, v_k\rangle &= \langle \pi_\lambda(Ad(k^{-1})u)v, v\rangle \\ &= \langle \pi_\lambda(u_0)v, v\rangle + \langle \pi_\lambda(Xu_1)v, v\rangle + \langle \pi_\lambda(u_2Y)v, v\rangle. \end{aligned} \tag{4}$$

Since v is the highest weight vector of the weight λ, the first summand in the right-hand side of (4) is equal to $s_\lambda(k)$, and the third one is 0. Finally, since $\theta X \in n_c$, we have $\langle \pi_\lambda(Xu_1)v, v\rangle = -\langle \pi_\lambda(u_1)v, \pi_\lambda(\theta X)v\rangle = 0$.

The proposition is proved. □

Denote by ζ the mapping which maps an operator $A \in End\ E^\lambda$ to its covariant symbol, $\zeta A(k) = \langle Akv, kv\rangle$.

Corollary. *ζ is an isomorphism of the (g_r, K)-modules and associative algebras $End\ E^\lambda$ and $A(\lambda + \rho)$.*

Proof. Proposition 2 asserts that $s_\lambda = \zeta\pi_\lambda$. It follows from the surjectivity of π_λ and from the fact that the image of s_λ coincides with $A(\lambda + \rho)$, that the image of ζ coincides with $A(\lambda + \rho)$ as well. The mapping ζ is injective since $End\ E^\lambda$ is a simple module. Finally, since s_λ and π_λ are morphisms of (g_r, K)-modules and homomorphisms of associative algebras, the same is true for ζ. □

The shifted action of the Weyl group W on h_c^* is defined as follows. For $\lambda \in h_c^*$, $w \in W$ we put $w\cdot\lambda = w(\lambda+\rho)-\rho$. For a dominant weight $\lambda \in h_c^*$, the (g_r, K)-module $\hat\Lambda(\lambda + \rho) = A(\lambda + \rho)/B(\lambda + \rho)$, together with all $\hat\Lambda(w(\lambda + \rho))$, $w \in W$, are isomorphic to $End\ E^\lambda$. This implies that, since the modules $A(w(\lambda + \rho))$, $w \in W$, and $End\ E^\lambda$ are spherical, one can choose a surjective morphism $\tau_{w\cdot\lambda} : A(w(\lambda + \rho)) \to End\ E^\lambda$ such that $\tau_{w\cdot\lambda}1 = 1$. It is clear, that $\tau_{w\cdot\lambda}s_{w\cdot\lambda} = \pi_\lambda$, and the (g_r, K)-module morphism $\tau_{w\cdot\lambda}$ is also an algebra homomorphism, i.e. a representation of $A(w(\lambda + \rho))$ in E^λ.

Now we can introduce the following

Definition. A function $f \in A(w(\lambda + \rho))$ is called a mixed symbol of an operator $A \in End\ E^\lambda$, corresponding to the element $w \in W$, if $A = \tau_{w\cdot\lambda}f$.

Let $\mu \in h_c^*$, $f \in \Lambda(\mu)$, $g \in \Lambda(-\mu)$, and q be a nonzero constant. It is known that $(f, g)_q = q\int_K f(k)g(k)dk$ is a pairing of (g_r, K)-modules $\Lambda(\mu)$ and $\Lambda(-\mu)$ (see [5]). Since the module $B(\mu)$ does not contain $1 \in A(\mu)$, it is orthogonal to $1 \in A(-\mu)$, and therefore to the whole module $A(-\mu)$, with respect to $(\cdot,\cdot)_q$. Besides, $(1, 1)_q = q$. This implies that $(\cdot,\cdot)_q$ induces a nondegenerate pairing of the simple (g_r, K)-modules $\hat\Lambda(\mu)$ and $\hat\Lambda(-\mu)$, and each pairing of these modules can be obtained by choosing the appropriate value of q.

Now let $\lambda \in h_c^*$ be a dominant weight. The module $\hat\Lambda(\lambda + \rho)$ is isomorphic to $End\ E^\lambda$ and dual to $\hat\Lambda(-\lambda - \rho)$. Since $-\lambda - \rho = w_0(\lambda' + \rho)$, the module $\hat\Lambda(-\lambda - \rho) = \hat\Lambda(w_0(\lambda' + \rho))$ is isomorphic to $End\ E^{\lambda'}$. On the other hand, since the g_c-modules E^λ and $E^{\lambda'}$ are dual, an operator $A \in End\ E^\lambda$ has the transpose $A^t \in End\ E^{\lambda'}$. It is easy to check, that for $A \in End\ E^\lambda$ and $B \in End\ E^{\lambda'}$, $tr\ AB^t$

also defines a nondegenerate pairing of the (g_r, K)-modules $End\ E^{\lambda}$ and $End\ E^{\lambda'}$. This implies

Lemma 4. *Let $f \in A(\lambda + \rho)$, $g \in A(w_0(\lambda' + \rho))$, and q be the dimension of E^{λ}. Then*

$$q \int_K f(k)g(k)dk = tr(\tau_\lambda f)(\tau_{w_0 \cdot \lambda'} g)^t.$$

To prove the assertion of the lemma it is enough to check it for $f = g = 1$.

It follows from Proposition 2, that a function f is the covariant symbol of the operator $\tau_\lambda f$, associated with the supercomplete system $\{v_k = kv\}$. Now Lemma 1 and Lemma 4 imply that a function g is a contravariant symbol of the operator $(\tau_{w_0 \cdot \lambda'} g)^t$, also associated with the supercomplete system $\{v_k\}$. Taking into account that for a dominant weight λ we have $\tau_{w \cdot \lambda} s_{w \cdot \lambda} = \pi_\lambda$, we get the following:

Lemma 5. *For $u \in \mathcal{U}(g_c)$ the function $s_{w_0 \cdot \lambda'} u$ is a contravariant symbol of the operator $(\pi_{\lambda'}(u))^t \in End\ E^{\lambda}$, associated with the supercomplete system $\{v_k\}$.*

Let $u \mapsto \check{u}$ be an involutive anti-automorphism of $\mathcal{U}(g_c)$, such that for $X \in g_c$ we have $\check{X} = -X$. Since the g_c-modules E^{λ} and $E^{\lambda'}$ are dual, the transpose operator to $\pi_{\lambda'}(X) \in End\ E^{\lambda'}$ is equal to $-\pi_\lambda(X) \in End\ E^{\lambda}$, which implies that $(\pi_{\lambda'}(u))^t = \pi_\lambda(\check{u})$. Therefore it follows from Lemma 5, that the function $s_{w_0 \cdot \lambda'} u$ is a contravariant symbol of the operator $\pi_\lambda(\check{u})$, associated with $\{v_k\}$.

Lemma 6. *Let $\tilde{w}_0 \in M'$ be a representative of the Coxeter element $w_0 \in W$, $u \in \mathcal{U}(g_c)$. Then $s_{w_0 \cdot \lambda'} u(k) = s_{w_0 \cdot \lambda} \check{u}(k \tilde{w}_0^{-1})$.*

Proof. Since the mappings $u \mapsto \check{u}$ and $u \mapsto s_\lambda u$ are K-equivariant, it is enough to check the equality of the lemma at the point $k = e$. Let u be decomposed in accordance with (2), $u = u_0 + X u_1 + u_2 Y$, where $u_0 \in \mathcal{U}(h_c)$, $X \in n_c^-$, $Y \in n_c$. Then, taking into account that $w_0 \cdot \lambda' = -\lambda - 2\rho$, we get $s_{w_0 \cdot \lambda'} u(e) = u_0(-\lambda - 2\rho)$. Since the anti-automorphism $u \mapsto \check{u}$ for $u \in \mathcal{U}(h_c)$ reduces to the change of a sign in the argument when $\mathcal{U}(h_c)$ is identified with $S(h_c)$, then $u_0(-\lambda - 2\rho) = \check{u}_0(\lambda + 2\rho)$. On the other hand, $\check{u} = \check{u}_0 - \check{u}_1 X - Y \check{u}_2$, so

$$Ad(\tilde{w}_0)\check{u} = Ad(\tilde{w}_0)\check{u}_0 - Ad(\tilde{w}_0)(\check{u}_1 X + Y \check{u}_2). \tag{5}$$

Since $Ad(\tilde{w}_0)X \in n_c$, and $Ad(\tilde{w}_0)Y \in n_c^-$, the subtrahend in the right-hand side of (5) belongs to the sum in the parenthesis in (2), therefore

$$s_{w_0 \cdot \lambda} \check{u}(\tilde{w}_0^{-1}) = \varphi_{w_0 \cdot \lambda}(Ad(\tilde{w}_0)\check{u}) = (Ad(\tilde{w}_0)\check{u}_0)(w_0 \cdot \lambda) = \check{u}_0(w_0(w_0 \cdot \lambda)).$$

To conclude the proof it is enough to notice that $w_0(w_0 \cdot \lambda) = \lambda + 2\rho$. □

Consider a vector $v' = \tilde{w}_0 v \in E^{\lambda}$. Since $v \in E^{\lambda}$ is the highest weight vector, then v' is the lowest weight one.

Theorem 1. (i) *A mixed symbol of an operator $A \in End\ E^{\lambda}$, corresponding to the unit element $e \in W$, is the covariant symbol of A, associated with the supercomplete system $\{v_k = kv\}$.*

(ii) *A mixed symbol of an operator A, corresponding to the element $w_0 \in W$, is a contravariant symbol of A, associated with the supercomplete system $\{v'_k = kv'\}$.*

Proof. The assertion (i) is proved in Proposition 2. It follows from Lemmas 5 and 6 that for $u \in \mathcal{U}(g_c)$ the function $s_{w_0 \cdot \lambda} u(k \tilde{w}_0^{-1})$ is a contravariant symbol of the operator $\pi_\lambda(u)$, associated with the supercomplete system $\{v_k\}$. Now it follows from Lemma 2, that the function $s_{w_0\cdot\lambda}u(k)$ is a contravariant symbol of the operator $\pi_\lambda(u)$, associated with the supercomplete system $\{v'_k\}$, which concludes the proof. $\square$

6. Symbol algebras on flag manifolds

Denote by a_r^* the real subspace of h_c^*, formed by those functionals which are real on a_r. In particular, the roots and weights of g_c belong to a_r^*. We will give a condition on $\lambda \in a_r^*$, under which one can define a pushforward of the algebra $A(\lambda+\rho)$ to an orbit of the adjoint representation of K in k_r. These orbits are generalized flag manifolds.

The Killing form $(\cdot,\cdot)$ on g_c is positive definite on a_r and negative definite on m_r. Let $\lambda \in a_r^*$. Define a mapping $\Psi_\lambda : K \mapsto g_c$ such that for each $k \in K$ and $X \in g_c$ we have $is_\lambda X(k) = (X, \Psi_\lambda(k))$. The following lemma is straightforward.

Lemma 7. *For $k, l \in K$, $\Psi_\lambda(k^{-1}l) = Ad(k)\Psi_\lambda(l)$.*

The lemma asserts that Ψ_λ is a K-equivariant mapping. For each $\lambda \in a_r^*$ choose $H^\lambda \in a_r$ such that $\lambda(H) = (H, H^\lambda)$ for $H \in h_c$. Notice, that $H^\alpha = (2/(H_\alpha, H_\alpha))H_\alpha$ for $\alpha \in \triangle$. Since $ia_r = m_r$, for $\lambda \in a_r^*$ we have $iH^\lambda \in m_r$.

Lemma 8. *Let $\lambda \in a_r^*$. Then $\Psi_\lambda(e) = iH^\lambda$, so the image of Ψ_λ is the orbit of the point $iH^\lambda \in m_r$ with respect to the adjoint action of K on g_c, and it lies completely in k_r.*

Proof. For any $X \in g_c$ we have $(X, \Psi_\lambda(e)) = is_\lambda X(e) = i\varphi_\lambda(X)$. If $X \in n_c \oplus n_c^-$, then $i\varphi_\lambda(X) = 0$ and $(X, H^\lambda) = 0$. And if $X \in h_c$, then $i\varphi_\lambda(X) = i\lambda(X) = (X, iH^\lambda)$. Thus $(X, \Psi_\lambda(e)) = (X, iH^\lambda)$ holds for each $X \in g_c$, which implies the assertion of the lemma. $\square$

Denote by $\triangle(\lambda)$ the set of roots $\alpha \in \triangle$ such that $\lambda(H_\alpha) = 0$ for $\lambda \in a_r^*$, and let $\triangle^+(\lambda) = \triangle^+ \cap \triangle(\lambda)$. It is evident, that $\lambda(H_\alpha) = 0$ iff $\lambda(H^\alpha) = 0$.

The centralizer of H^λ in g_c is generated by the Cartan subalgebra h_c and the elements X_α, $\alpha \in \triangle(\lambda)$. Denote by $k_r^\lambda \subset k_r$ the centralizer of iH^λ in k_r. Then k_r^λ is generated by the subalgebra m_r and the elements $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha})$, $\alpha \in \triangle^+(\lambda)$. Since the stabilizer $K^\lambda \subset K$ of the element $iH^\lambda \in k_r$ with respect to the adjoint action of K is connected, it is the integral subgroup of the subalgebra $k_r^\lambda \subset k_r$. The orbit $\Omega_\lambda \subset k_r$ of the point iH^λ under the adjoint action of K is isomorphic to K/K^λ as a homogeneous space, and is a generalized flag manifold.

Let $\sigma \subset \Sigma$ be a set of simple roots. Denote by $\langle\sigma\rangle$ the set of roots which are linear combinations of elements of σ.

Definition. An element $\lambda \in a_r^*$ is called regular relative to $\sigma \subset \Sigma$, if $\triangle(\lambda) = \langle\sigma\rangle$.

Definition. An element $\lambda \in a_r^*$ is called relatively regular, if it is regular relative to some $\sigma \subset \Sigma$.

Lemma 9. *An element $\lambda \in a_r^*$ is relatively regular iff for all $\alpha, \beta \in \triangle^+$ such that $\alpha + \beta \in \triangle^+(\lambda)$ we have $\alpha, \beta \in \triangle^+(\lambda)$.*

Proof. The necessity is evident. Let us prove the sufficiency. Set $\sigma = \Sigma \cap \triangle(\lambda)$. We will show that $\triangle(\lambda) = \langle\sigma\rangle$. Since $\triangle(\lambda) = -\triangle(\lambda)$, it is enough to consider elements $\gamma \in \triangle^+(\lambda)$. If $\gamma \in \triangle^+$, then either $\gamma \in \Sigma$, or one can find $\alpha \in \triangle^+$, $\beta \in \Sigma$ such that $\gamma = \alpha + \beta$. It then follows from $\gamma \in \triangle^+(\lambda)$ that either $\gamma \in \sigma$, or $\alpha \in \triangle^+(\lambda)$, and $\beta \in \sigma$. It remains to use the induction over the number of summands in the decomposition of γ into a sum of simple roots. □

Lemma 10. *Any dominant element $\lambda \in a_r^*$ is relatively regular.*

Since the Killing form is positive definite on a_r, $\lambda \in a_r^*$ is dominant iff for all $\alpha \in \triangle^+$ we have $\lambda(H^\alpha) \geq 0$. Let $\alpha, \beta \in \triangle^+$ be such that $\gamma = \alpha + \beta \in \triangle^+(\lambda)$. Then $\lambda(H^\alpha) + \lambda(H^\beta) = \lambda(H^\gamma) = 0$, and since $\lambda(H^\alpha) \geq 0$, $\lambda(H^\beta) \geq 0$, then $\lambda(H^\alpha) = \lambda(H^\beta) = 0$, which means that $\alpha, \beta \in \triangle^+(\lambda)$. The lemma follows now from Lemma 9. □

Lemma 11. *All the elements of $A(\lambda + \rho)$, $\lambda \in a_r^*$, are invariant under the right shifts by the elements of K^λ only if λ is relatively regular.*

Proof. Assume $\alpha, \beta \in \triangle^+$ are such that $\gamma = \alpha + \beta \in \triangle^+(\lambda)$, and, say, $\alpha \notin \triangle^+(\lambda)$. We will show that the function $s_\lambda X_\alpha X_\beta$ is not constant on $K^\lambda \subset K$. In order to do that, let us calculate the infinitesimal action of the element $X_\gamma - X_{-\gamma} \in k_r^\lambda$ on $s_\lambda X_\alpha X_\beta$ at the point $e \in K$:

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} (s_\lambda X_\alpha X_\beta)(\exp(-t(X_\gamma - X_{-\gamma}))) \\ &= \frac{d}{dt}\bigg|_{t=0} \varphi_\lambda(Ad(\exp t(X_\gamma - X_{-\gamma}))X_\alpha X_\beta) = \varphi_\lambda([X_\gamma - X_{-\gamma}, X_\alpha X_\beta]). \end{aligned} \tag{6}$$

Since $X_{-\gamma}X_\alpha X_\beta$, $X_\gamma X_\alpha X_\beta$ and $X_\alpha X_\beta X_\gamma$ belong to the kernel of φ_λ, the expression in (6) is equal to $\varphi_\lambda(X_\alpha X_\beta X_{-\gamma}) = \varphi_\lambda(X_\alpha X_{-\gamma} X_\beta) + \varphi_\lambda(X_\alpha[X_\beta, X_{-\gamma}])$. Since $\beta - \gamma = -\alpha \in \triangle$, it follows that $[X_\beta, X_{-\gamma}] = c \cdot X_{-\alpha}$, where c is a nonzero constant. Finally, the expression in (6) is equal to $\varphi_\lambda(X_\alpha[X_\beta, X_{-\gamma}]) = c\varphi_\lambda(X_\alpha X_{-\alpha}) = c\varphi_\lambda([X_\alpha, X_{-\alpha}]) = c\varphi_\lambda(H_\alpha) = c\lambda(H_\alpha) \neq 0$, which implies the assertion of the lemma. □

Let us now show that the inverse to Lemma 11 is also true.

Let $\lambda \in a_r^*$ be a dominant weight, π_λ the representation of $\mathcal{U}(g_c)$ in the module E^λ with the highest weight λ and the highest weight vector v of length 1.

Lemma 12. *Let $\alpha \in \triangle^+(\lambda)$. Then $\pi_\lambda(X_{-\alpha})v = 0$.*

Proof. Consider E^λ as a module over the sl_2-subalgebra of g_c generated by the elements $X_{-\alpha}, H_\alpha, X_\alpha$. Assume $\pi_\lambda(X_{-\alpha})v \neq 0$. Since

$$\pi_\lambda(X_\alpha X_{-\alpha})v = \pi_\lambda(X_{-\alpha}X_\alpha)v + \pi_\lambda(H_\alpha)v = \lambda(H_\alpha)v = 0,$$

and

$$\pi_\lambda(H_\alpha X_{-\alpha})v = \pi_\lambda(X_{-\alpha}H_\alpha)v + \pi_\lambda(-2X_{-\alpha})v = -2\pi_\lambda(X_{-\alpha})v,$$

then $\pi_\lambda(X_{-\alpha})v$ generates an infinite dimensional sl_2-submodule of E^λ, which contradicts the fact that E^λ is finite dimensional. □

Lemma 13. *The highest weight vector $v \in E^\lambda$ is an eigenvector for the subgroup $K^\lambda \subset K$.*

Since K^λ is the integral subgroup of the subalgebra $k_r^\lambda \subset k_r$, it is enough to check the assertion of the lemma for the subalgebra k_r^λ. It is generated by the subalgebra m_r, for which v is an eigenvector, and by the elements $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha})$, $\alpha \in \triangle^+(\lambda)$, which annihilate v according to Lemma 12. □

Lemma 14. *Let $f(k)$ be the covariant symbol of an operator $A \in End\ E^{\lambda}$. Then the function $f(k)$ is invariant under the right shifts by the elements of K^{λ}.*

Proof. It follows from Lemma 13 and the unitarity of the representation of K in E^{λ} that $lv = cv$ for $l \in K^{\lambda}$, where c is a constant of modulus 1. Therefore, $f(kl) = \langle Aklv, klv\rangle = \langle Akv, kv\rangle = f(k)$. The lemma is proved. □

Theorem 2. *All the functions from $A(\lambda+\rho)$, $\lambda \in a_r^*$, are invariant under the right shifts by the elements of K^{λ} iff λ is relatively regular.*

Proof. The necessity is proved in Lemma 11. Let us prove the sufficiency. Assume that λ is regular relative to $\sigma \in \Sigma$. Denote by a_{σ}^* the set of $\mu \in a_r^*$ such that $\mu(H_{\alpha}) = 0$ for $\alpha \in \sigma$. In particular, $\lambda \in a_{\sigma}^*$. Denote by ω^{α} the fundamental weight, corresponding to $\alpha \in \Sigma$ (i.e. $\omega^{\alpha}(H_{\beta}) = \delta_{\beta}^{\alpha}$ for $\alpha, \beta \in \Sigma$, where δ_{β}^{α} is the Kronecker symbol). The fundamental weights ω^{α}, $\alpha \in \Sigma\backslash\sigma$, form a basis in a_{σ}^*, and their linear combinations with coefficients in $\mathbf{N}$ are the dominant weights, which are regular relative to σ. Consider $s_{\mu}(k)$ and $s_{\mu}(kl)$ for fixed $k \in K, l \in K^{\lambda}$ and $u \in \mathcal{U}(g_c)$ as polynomials in μ on a_{σ}^*. It follows from Proposition 2 and Lemma 14, that these polynomials coincide on the dominant weights μ, which are regular relative to σ. It is easy to conclude from the above, that these polynomials coincide identically, which proves the theorem. □

Corollary. *Let $\Omega_{\lambda} \subset k_r$ be the adjoint orbit of the point iH_{λ}. The algebra $A(\lambda+\rho)$ can be pushed forward to Ω_{λ} by the mapping Ψ_{λ} iff λ is relatively regular.*

7. Quantization on flag manifolds

Let d be a nonnegative integer. Denote by $\mathcal{U}_d(g_c)$ the subspace of $\mathcal{U}(g_c)$ generated by all monomials of the form $X_1 \ldots X_j$, where $X_1, \ldots, X_j \in g_c$ and $j \le d$. The subspaces $\{\mathcal{U}_d(g_c)\}$ determine the canonical filtration on $\mathcal{U}(g_c)$ (see [6]).

The symmetric algebra $S(g_c)$ can be identified with the space of polynomials on k_r, so that an element $X \in g_c$ corresponds to the linear functional on k_r given by $\tilde{X}(Y) = (X, Y)$, $Y \in k_r$. Let $S^d(g_c)$ be the space of homogeneous polynomials on k_r of degree d. The graded algebra associated with the canonical filtration is canonically isomorphic to $S(g_c)$, so that $\mathcal{U}_d(g_c)/\mathcal{U}_{d-1}(g_c)$ corresponds to $S^d(g_c)$. For $u \in \mathcal{U}_d(g_c)$ let $\underline{u}^{(d)}$ denote the corresponding element of $S^d(g_c)$. Usually we will omit the superscript d in $\underline{u}^{(d)}$. If $u = X_1 \ldots X_d \in \mathcal{U}_d(g_c)$, then $\underline{u} = \tilde{X}_1 \ldots \tilde{X}_d$. We say that a monomial $u = X_1 \ldots X_d \in \mathcal{U}(g_c)$ has canonical form, if for some integers k, l such that $0 \le k \le l \le d$, an element X_j belongs to n_c^-, h_c, n_c if $0 < j \le k$, $k < j \le l$, $l < j \le d$, respectively. Recall that the projection of $u \in \mathcal{U}(g_c)$ onto $\mathcal{U}(h_c)$ in (2) is denoted by u_0.

Proposition 3. *Let $t \in \mathbf{R}$, $\lambda \in a_r^*$, $u \in \mathcal{U}_d(g_c)$, $k \in K$. Then*

$$\lim_{t\to\infty} t^{-d} s_{t\lambda} u(k) = i^{-d} \underline{u}^{(d)}(\Psi_{\lambda}(k)).$$

Proof. Since s_{λ}, Ψ_{λ} and the mapping $u \mapsto \underline{u}$ are K-equivariant, it is enough to check the equality in the proposition only at the point $e \in K$, i.e. one has to show that

$$\lim_{t\to\infty} t^{-d} u_0(t\lambda) = i^{-d} \underline{u}(iH^{\lambda}). \tag{7}$$

Assume that $u \in \mathcal{U}_{d-1}(g_c)$. Then $u_0(t\lambda)$ is a polynomial in t of degree less than d; hence the left-hand side in (7) is 0. On the other hand $\underline{u} = 0$, since $\mathcal{U}_{d-1}(g_c)$ is the kernel of the mapping $u \mapsto \underline{u}^{(d)}$. Now it remains to check (7) for a monomial $u = X_1 \dots X_d \in \mathcal{U}_d(g_c)$ which has canonical form. Since

$$i^{-d}\underline{u}(iH^\lambda) = i^{-d}\tilde{X}_1(iH^\lambda)\dots\tilde{X}_d(iH^\lambda) = (X_1, H^\lambda)\dots(X_d, H^\lambda), \tag{8}$$

the expression in (8) for $u \in \mathcal{U}(h_c)$ is equal to $u_0(\lambda)$. In this case $u_0(\lambda)$ is homogeneous of degree d, so the expression under the limit on the left-hand side of (7) does not depend on t and the left-hand side of (7) is also equal to $u_0(\lambda)$. Assume now that $u \notin \mathcal{U}(h_c)$. Then the factor (X_j, H^λ) on the right-hand side of (8), for which $X_j \notin h_c$, is equal to 0. On the other hand, in this case $u_0 = 0$, which concludes the proof. □

Let V be a K-module, $\nu \in a_r^*$ a dominant weight. Denote by V^ν the isotropic component of V, corresponding to the representation with the highest weight ν. Let $\Omega \subset k_r$ be an adjoint orbit of the group K. Denote by $Q(\Omega)$ the space of K-finite functions on Ω, and by $R(\Omega)$ the space of regular functions on Ω, i.e., the space of restrictions of the functions from $S(g_c)$ on Ω.

Lemma 15. (i) *The spaces $Q(\Omega)$ and $R(\Omega)$ coincide.*

(ii) *The space $R(\Omega)^\nu$ for each dominant weight ν is finite dimensional.*

(iii) *There exist homogeneous polynomials $\psi_1, \dots, \psi_n \in S(g_c)^\nu$ such that their restrictions on Ω form a basis in $R(\Omega)^\nu$.*

Proof. Since each element in $S(g_c)$ is K-finite, $R(\Omega) \subset Q(\Omega)$. It follows from the Frobenius reciprocity theorem, that the irreducible representation of K with the highest weight ν has a finite multiplicity in $Q(\Omega)$. Hence $Q(\Omega)^\nu$ and, therefore, $R(\Omega)^\nu$ are finite dimensional. Introduce a Hilbert space $L^2(\Omega, dx)$, where dx is a K-invariant measure on Ω. It contains $Q(\Omega)$. Assume $Q(\Omega)^\nu \backslash R(\Omega)^\nu$ is nonempty for some ν. Then a function $f \in Q(\Omega)^\nu$ orthogonal to $R(\Omega)^\nu$ is orthogonal to the whole space $R(\Omega)$. It follows from the Stone-Weierstrass theorem, that $R(\Omega)$ is dense in $C(\Omega)$ and, therefore, in $L^2(\Omega, dx)$; thus $f = 0$, which implies (i).

Finally, since $S(g_c)^\nu = \bigoplus (S^d(g_c))^\nu$, the polynomials from (iii) can be constructed inductively. □

Lemma 16. *Let $\lambda \in a_r^*$ be a relatively regular element, $\nu \in a_r^*$ a dominant weight, $u \in \mathcal{U}_d(g_c)^\nu$, $t \in \mathbf{R}$, $\{f_1, \dots, f_n\}$ a basis in $R(\Omega_\lambda)^\nu$. Then $s_{t\lambda}u$ can be realized as a linear combination of functions $f_j \circ \Psi_\lambda$, $j = 1, \dots, n$, with coefficients which are polynomial in t of a degree not greater than d.*

Proof. It is easy to notice that for a given λ the stabilizer $K^{t\lambda}$ coincides with K^λ for all real $t \neq 0$, and with K for $t = 0$. Since $t\lambda$ is relatively regular for all t, it follows from Theorem 2 that $s_{t\lambda}u$ is the pullback of some function ψ_t on Ω_λ, that is, $s_{t\lambda}u = \psi_t \circ \Psi_\lambda$. The K-equivariance of s_λ and Lemma 15 imply that $\psi_t \in R(\Omega_\lambda)^\nu$; thus there exist functions $a_j(t)$, $j = 1, \dots, n$, such that $\psi_t = \sum a_j(t) f_j$. Since $R(\Omega_\lambda)^\nu$ is n-dimensional, there exist points $x_1, \dots, x_n \in \Omega_\lambda$ such that the matrix $(f_i(x_j))$ is invertible. Let (b_{ij}) be its inverse; then $a_i(t) = \sum b_{ij}\psi_t(x_j)$. Now the assertion of the lemma follows from the fact that for each $k \in K$ the function $\psi_t(\Psi_\lambda(k)) = s_{t\lambda}u(k)$ is polynomial in t of degree not greater than d. □

Let $\beta : S(g_c) \to \mathcal{U}(g_c)$ be the symmetrization mapping (see [6]). It is K-equivariant, and for a homogeneous polynomial $\psi \in S^d(g_c)$ we have $\beta\psi \in \mathcal{U}_d(g_c)$ and $\underline{\beta\psi} = \psi$.

Lemma 17. *Let $\lambda \in a_r^*$ be relatively regular, $f \in R(\Omega_\lambda)$. There exist $u_j \in \mathcal{U}_{d(j)}(g_c)$, $j = 1, \ldots, m$, and rational functions $\{a_j(t)\}$ with no poles at infinity, such that for all $t \neq 0$ different from the poles of the functions $\{a_j(t)\}$, we have*

$$f \circ \Psi_\lambda = \sum a_j(t) t^{-d(j)} s_{t\lambda} u_j.$$

Proof. It is enough to prove the lemma for functions $f_1, \ldots, f_n$ which form a basis of $R(\Omega_\lambda)^\nu$ for an arbitrary ν. According to Lemma 15, one can assume that f_j is a restriction of a homogeneous polynomial ψ_j of the degree $d(j)$. It follows from Lemma 16 that there exist polynomials $b_{jl}(t)$ of degree not greater than $d(j)$, such that

$$(9) \qquad (\frac{i}{t})^{d(j)} s_{t\lambda} \beta \psi_j = \sum_l t^{-d(j)} b_{jl}(t) f_l \circ \Psi_\lambda.$$

Using Proposition 3 and the properties of β, pass to the limit on both sides of (9) as $t \to \infty$. Since the left-hand side of (9) tends to $f_j \circ \Psi_\lambda$, then the matrix of rational functions $(t^{-d(j)} b_{jl}(t))$ tends to the identity matrix as $t \to \infty$. Therefore its determinant is not identically zero, and its inverse matrix of rational functions, $(a_{lj}(t))$ exists, which also tends to the identity matrix as $t \to \infty$. The assertion of the lemma is obtained by applying the matrix $(a_{lj}(t))$ to both sides of (9) and setting $u_j = i^{d(j)} \beta \psi_j$.

It is known that there exists a natural Poisson structure on the symmetric algebra $S(g_c)$, which is considered as the algebra of polynomials on k_r . It is defined by the (real) Poisson bracket $\{\cdot, \cdot\}$ on k_r such that for $X, Y \in g_c$ and $Z = [X, Y]$, we have $\{\tilde{X}, \tilde{Y}\} = \tilde{Z}$. The symplectic leaves of that Poisson structure are the (co)adjoint orbits of K in k_r (the Lie algebra k_r is identified with its dual space k_r^* via the Killing form). The symplectic structure on a (co)adjoint orbit $\Omega \subset k_r$ is defined by the Kirillov symplectic form, and the corresponding Poisson bracket $\{\cdot, \cdot\}_\Omega$ is the restriction of $\{\cdot, \cdot\}$ on Ω, i.e. for $f, g \in S(g_c)$ we have $\{f, g\}|_\Omega = \{f|_\Omega, g|_\Omega\}_\Omega$ (see [7]).

The multiplication and the Poisson bracket on $S(g_c)$ are connected with the canonical filtration on $\mathcal{U}(g_c)$ in the following way (see [6]). Let $u_\varepsilon \in \mathcal{U}_{d_\varepsilon}(g_c)$, $\varepsilon = 1, 2$, $d = d_1 + d_2$. Then $u_1 u_2 \in \mathcal{U}_d(g_c)$, $u_1 u_2 - u_2 u_1 \in \mathcal{U}_{d-1}(g_c)$, and

$$(10) \qquad \underline{(u_1 u_2)}^{(d)} = \underline{u_1}^{(d_1)} \cdot \underline{u_2}^{(d_2)}, \quad \underline{(u_1 u_2 - u_2 u_1)}^{(d-1)} = \{\underline{u_1}^{(d_1)}, \underline{u_2}^{(d_2)}\}.$$

Let $\lambda \in a_r^*$ be relatively regular, $t \in \mathbf{R} \backslash \{0\}$. It follows from the fact that $K^{t\lambda} = K^\lambda$, and Theorem 2, that the algebra $A(t\lambda + \rho)$ can be pushed forward to the orbit Ω_λ via Ψ_λ. Denote by $\mathcal{A}_{1/t}^{(\lambda)}$ and $*_{1/t}^{(\lambda)}$ the pushforward of $A(t\lambda + \rho)$ on Ω_λ, and the multiplication in it, respectively. Usually the superscript λ in these notations will be omitted. Since each element in $A(t\lambda + \rho)$ is K-finite, Lemma 15 implies that $\mathcal{A}_{1/t}^{(\lambda)} \subset R(\Omega_\lambda)$. □

Theorem 3. *Let $\lambda \in a_r^*$ be relatively regular, $t \in \mathbf{R} \backslash \{0\}$.*

(i) *Each function $f \in R(\Omega_\lambda)$ belongs to $\mathcal{A}_{1/t}^{(\lambda)}$ for all but a finite number of values of t.*

(ii) *For $f_1, f_2 \in R(\Omega_\lambda)$, $x \in \Omega_\lambda$, the product $(f_1 *_{1/t} f_2)(x)$ is a rational function of t, with no pole at infinity.*

(iii) *For $f_1, f_2 \in R(\Omega_\lambda)$ the following limits hold:*

$$\lim_{t \to \infty} f_1 *_{1/t} f_2 = f_1 f_2; \quad \lim_{t \to \infty} t(f_1 *_{1/t} f_2 - f_2 *_{1/t} f_1) = i\{f_1, f_2\}_{\Omega_\lambda}.$$

Proof. Assertion (i) follows directly from Lemma 17. In accordance with Lemma 17, there exist the elements $u_{\varepsilon j} \in \mathcal{U}_{d_\varepsilon(j)}(g_c)$, $\varepsilon = 1,2$, $j = 1,\dots,n(\varepsilon)$, and rational functions $a_{\varepsilon j}(t)$, with no poles at infinity, such that for all $t \neq 0$, different from the poles of $\{a_{\varepsilon j}(t)\}$,

$$f_\varepsilon \circ \Psi_\lambda = \sum_{j=1}^{n(\varepsilon)} a_{\varepsilon j}(t) t^{-d_\varepsilon(j)} s_{t\lambda} u_{\varepsilon j}. \tag{11}$$

Passing to the limit in (11) as $t \to \infty$, and denoting $\lim_{t\to\infty} a_{\varepsilon j}(t) = a_{\varepsilon j}$, one gets from Proposition 3, that

$$f_\varepsilon \circ \Psi_\lambda = \sum_j a_{\varepsilon j} i^{-d_\varepsilon(j)} \underline{u}_{\varepsilon j} \circ \Psi_\lambda. \tag{12}$$

Let us calculate the product $f_1 *_{1/t} f_2$, pulling it back to K via Ψ_λ:

$$(f_1 *_{1/t} f_2) \circ \Psi_\lambda = \sum_{j,l} a_{1j}(t) a_{2l}(t) t^{-d(j,l)} s_{t\lambda}(u_{1j} u_{2l}), \tag{13}$$

where $d(j,l) = d_1(j) + d_2(l)$. Since $u_{1j}u_{2l} \in \mathcal{U}_{d(j,l)}(g_c)$, for each $k \in K$ the function $s_{t\lambda}(u_{1j}u_{2l})(k)$ is a polynomial in t of a degree not greater than $d(j,l)$. Thus each summand in (13) is a rational function of t, with no pole at infinity, which proves (ii).

Pass to the limit in (13) as $t \to \infty$, using Proposition 3 and taking into account (10), (11) and (12):

$$\lim_{t\to\infty} (f_1 *_{1/t} f_2) \circ \Psi_\lambda = \sum_{j,l} a_{1j} a_{2l} i^{-d(j,l)} (\underline{u_{1j} u_{2l}}) \circ \Psi_\lambda$$

$$= (\sum_{j,l} a_{1j} a_{2l} i^{-d_1(j)} \underline{u}_{1j} i^{-d_2(l)} \underline{u}_{2l}) \circ \Psi_\lambda = (f_1 f_2) \circ \Psi_\lambda.$$

Thus we obtain the first of the two limits in (iii). The second limit is calculated similarly:

$$\begin{aligned} &\lim_{t\to\infty} t(f_1 *_{1/t} f_2 - f_2 *_{1/t} f_1) \circ \Psi_\lambda \\ &\quad = \lim_{t\to\infty} \sum_{j,l} a_{1j}(t) a_{2l}(t) t^{-d(j,l)+1} s_{t\lambda}(u_{1j}u_{2l} - u_{2l}u_{1j}) \\ &\quad = \sum_{j,l} a_{1j} a_{2l} i^{-d(j,l)+1} (\underline{u_{1j}u_{2l} - u_{2l}u_{1j}}) \circ \Psi_\lambda \\ &\quad = i(\sum_{j,l} a_{1j} a_{2l} \{i^{-d_1(j)} \underline{u}_{1j}, i^{-d_2(l)} \underline{u}_{2l}\}) \circ \Psi_\lambda = i\{f_1, f_2\}_{\Omega_\lambda} \circ \Psi_\lambda. \end{aligned}$$

□

Remark 1. The assertion (i) of Theorem 3 means, that the algebras $\mathcal{A}_{1/t}^{(\lambda)}$ have the weak nesting property. It is proved in [8] that if λ is a dominant weight and $n \in \mathbf{N}$, then the $\mathcal{A}_{1/n}^{(\lambda)} \subset \mathcal{A}_{1/(n+1)}^{(\lambda)}$, i.e., the algebras of covariant symbols, $\{\mathcal{A}_{1/n}^{(\lambda)}\}$, have the strong nesting property.

Remark 2. If λ is a dominant weight, the assertion (ii) of Theorem 3 is proved when Ω_λ is a compact hermitian symmetric space, and conjectured for an arbitrary generalized flag manifold in [3].

Remark 3. The assertion (iii) of Theorem 3 is proved in [3] for the case when λ is a dominant weight.

Remark 4. The connection between the K-orbit of the highest weight vector $v \in E^\lambda$ and the adjoint orbit Ω_λ with their application to covariant symbols is studied in [4].

Now we can define Berezin's quantization on a generalized flag manifold, using the algebras of mixed symbols. Assume $\lambda \in a^*$ is a relatively regular weight, and there exists an element $w \in W$ such that the weights $w\lambda$ and $w \cdot \lambda$ are dominant. Then for all natural n the weights $w \cdot (n\lambda) = (n-1)w\lambda + w \cdot \lambda$ are dominant. The algebra $A(n\lambda + \rho)$ has the representation $\tau_{n\lambda}$ in the g_c-module $E^{w\cdot(n\lambda)}$. Pushing forward $A(n\lambda+\rho)$ together with its representation $\tau_{n\lambda}$ to Ω_λ, we obtain the algebra $\mathcal{A}^{(\lambda)}_{1/n}$ with a representation in $E^{w\cdot(n\lambda)}$. Denote $\Omega = \Omega_\lambda,\ F = \{1, 1/2, 1/3, \dots\},\ h = 1/n \in F,\ \mathcal{A}_h = \mathcal{A}^{(\lambda)}_{1/n},\ H_h = E^{w\cdot(n\lambda)}$. Then Theorem 3 implies

Theorem 4. *The algebras $\{\mathcal{A}_h\},\ h \in F$, together with their representations in H_h define Berezin's quantization on the (co)adjoint orbit Ω of the group K, which is a generalized flag manifold.*

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