EQUIVARIANT NOVIKOV CONJECTURE FOR GROUPS ACTING ON EUCLIDEAN BUILDINGS

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ABSTRACT. We prove the equivariant Novikov conjecture for groups acting on Euclidean buildings by using an equivariant Hilsum-Skandalis method. We also obtain an equivariant version of the Connes-Gromov-Moscovici theorem for almost flat C^*-algebra bundles.

1. INTRODUCTION

The well-known Novikov conjecture states that the higher signature of a closed oriented Riemannian manifold is a homotopy invariant. There are many versions of this conjecture, and a lot of work has been devoted to them (cf. [BCH] [BHM] [Cap] [CaP] [CoJ] [CoM] [CGM] [FaJ] [FeW] [Gong5] [Kas] [KaS] [Lus] [Mis1], [Mis2] [Roe] [Ros] [RoW2] [Wein]). In particular, we mention an equivariant Novikov conjecture proposed by Rosenberg and Weinberger [RoW2] which says that for a commutative diagram

(1)
$$
\begin{array}{ccc}
M & & \\
\Big\uparrow \scriptstyle{h} & \searrow \scriptstyle{\varphi} & \\
 & Y & \xrightarrow{f_Y} B\pi(Y), \\
\Big\downarrow & \nearrow \scriptstyle{\psi} & \\
N & &
\end{array}
$$

of closed oriented Riemannian G-manifolds M and N, a G-space Y with G-maps φ and ψ, and a G-pseudo-equivalence h, the equivariant signature elements $[D_M]$ and $[D_N]$ of M and N satisfy $(f_Y)_*\varphi_*([D_M]) = (f_Y)_*\psi_*([D_N])$ in $K_*^G(B\pi(Y))$, provided the equivariant K-homology $K_*^G(B\pi(Y))$ of $B\pi(Y)$ is finitely generated over the representation ring $R(G)$ of the compact Lie group G, where the actions of G on M and N are given by isometries and $B\pi(Y)$ is the classifying space of the fundamental groupoid $\pi(Y)$ of Y. We refer to Section 5 for a precise definition. Rosenberg and Weinberger proved this conjecture for Y a complete Riemannian G-manifold of nonpositive curvature with the finite generation condition on $K_*^G(B\pi(Y))$ [RoW2]. One goal of this paper is to verify this equivariant Novikov conjecture for groups acting on Euclidean buildings without the condition of finite

generation of $K_*^G(B\pi(Y))$. This extends the Rosenberg-Weinberger theorem by removing the finite generation of $K_*^G(B\pi(Y))$ and by allowing Y to be a more general G-space.

Our method of proving the above mentioned result is different from that of [RoW2]. We planned originally to use equivariant cyclic cohomology to consider the equivariant Novikov conjecture. For this reason, we have rather systematically developed equivariant cyclic cohomology and its application to equivariant index theory [Gong1]-[Gong4]. However, due to the lack of an equivariant version of Connes-Moscovici's Γ-index theorem [CoM] and of equivariant Mishchenko symmetric signature for a general compact Lie group (these are still open problems), we were forced to seek another approach. We then realized that the Hilsum-Skandalis method [HiS] can be adapted to the equivariant situation to overcome the above problems without assuming the finite generation of $K_*^G(B\pi(Y))$. Essentially, the equivariant Hilsum-Skandalis method is to compare directly two equivariant signature elements in an equivariant K-theory group. This enables us to avoid the problem of the equivariant Mishchenko symmetric signature. Furthermore, we are able to deal with much more general equivariant C^*-algebra bundles, to which the applicability of equivariant cyclic cohomology is still not clear. As a by-product, we also obtain an equivariant Connes- Gromov-Moscovici theorem which is interesting on its own, and which is the second goal of the present paper.

The main theorems of the paper are stated as follows.

Theorem 1. *Let M and N be two closed oriented Riemannian G-manifolds of even dimensions. Suppose that $h : N \to M$ is an orientation preserving homotopy equivalence which is equivariant. Then for every unitary flat G-$\mathcal{A}$-bundle E over M, the equivariant analytic signatures of M and N with coefficients in E and $h^*(E)$, resp., are equal:*

$$G\text{-}Sig(D_E) = G\text{-}Sig(D_{h^*(E)}) \in K_0^G(\mathcal{A}).$$

In other words, the equivariant signature of M with coefficients in a unitary flat G-$\mathcal{A}$-bundle is a homotopy invariant in the sense of Theorem 1. Here $K_0^G(\mathcal{A})$ is the equivariant K-theory group of a G-C^*-algebra $\mathcal{A}$. By a G-$\mathcal{A}$-bundle E over M, we mean that E is a bundle over M with fibers modeled on a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0. The following equivariant Connes-Gromov-Moscovici theorem extends Theorem 1 to the case of almost flat G-$\mathcal{A}$-bundles.

Theorem 2. *Let M, N, h and G be as in Theorem 1. Then there exists a constant $\tau > 0$ such that for every τ-flat G-$\mathcal{A}$-bundle E over M,*
$$G\text{-}Sig(D_E) = G\text{-}Sig(D_{h^*(E)}) \in K_0^G(\mathcal{A}).$$

Here we say E is τ-flat if there exists a connection ∇^E on E such that the norm of its curvature is less than or equal to τ (cf. Section 4). We will use Theorem 1 as a crucial step to prove the following equivariant Novikov conjecture.

Theorem 3. *Let X be a locally finite Euclidean building and B be its geometric realization. Assume $Y = B$ in (1). Suppose G acts on Y by type permuting isometries. Then $\varphi_*([D_M]) = \psi_*([D_N])$.*

We refer to Section 5 for the terminology in Theorem 3. Another question studied in this paper is the existence of the equivariant maps in (1). We have the following theorem on the existence of an equivariant map from $\tilde{M}$ to a metric space W of nonpositive curvature, where $\tilde{M}$ is the universal covering space of M.

Theorem 4. *Let M be a closed oriented Riemannian G-manifold, W a (simply connected) metric space of nonpositive curvature, and Γ the closed subgroup of the diffeomorphism group $Diff(\tilde{M})$ of $\tilde{M}$ which covers the G-action on M. Suppose that $\rho : \Gamma \to Isom(W)$ is a continuous homomorphism and Γ acts on W via ρ. Then there exists a Γ-equivariant map from $\tilde{M}$ to W.*

Note that Γ is an extension of the fundamental group $\pi_1(M)$ of M by G. Theorem 4 extends the Korevaar-Schoen theorem [KoS] on the existence of a $\pi_1(M)$-equivariant map from $\tilde{M}$ to W which is fundamental in the super rigidity [GrS]. Consequently, we obtain a general result on the existence of the equivariant maps in (1). We point out that the metric spaces of nonpositive curvature include the geometric realization B of the Euclidean building X and complete simply connected Riemannian manifolds of nonpositive curvature.

The paper is arranged as follows. In Section 2 we will define the so-called equivariant signature-type elements in the equivariant operator K-theory group which are modeled on the signature operator[AtS]. We will then obtain the main machinery of this paper: that is, to give sufficient conditions for two equivariant signature-type elements to be equal. In Sections 3 and 4 we will verify these sufficient conditions for the signature elements with coefficients in flat and almost flat equivariant C^*-algebra bundles, respectively. Hence, we get the equivariant Connes-Gromov-Moscovici theorem. We should point out that the method here will be used in [GoR] to prove the homotopy invariance of the family Novikov-Shubin invariants. We also expect that the Hilsum-Skandalis method could be used for a $\mathbb{Z}/k$-version of the Novikov conjecture (for $\mathbb{Z}/k$-manifolds), which will be discussed in another paper. The result in Section 3 for the flat case will provide us a substitute for the equivariant Mishchenko symmetric signature. This observation is an essential point of our approach to the equivariant Novikov conjecture. We will use this result together with injectivity of the assembly operator K-theory map β in Section 5 to prove Theorem 3 and its extended version, namely, for those Y in (1) having Euclidean buildings as the universal covering spaces. Our argument also works for Y a complete Riemannian G-manifold of nonpositive curvature. Therefore we obtain the Rosenberg-Weinberger theorem without the finite generation of $K_*^G(B\pi(Y))$. Finally, we will prove Theorem 4 in Section 6.

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2. Signature-Type Elements in Equivariant K-Theory

This section is largely a generalized version of [HiS] and [KaM] to the equivariant case. Our effort here is to deal with some technical points about group actions.

To begin with, we assume throughout this section that G is a compact group, $\mathcal{A}$ is a G-C^*-algebra over $\mathbb{C}$ and $\mathcal{E}$ is a right G-$\mathcal{A}$-module. Here G acts on $\mathcal{A}$ and $\mathcal{E}$ by continuous automorphisms which are compatible with the module structure,

i.e., $g(xa) = g(x)g(a), g \in G, x \in \mathcal{E}, a \in \mathcal{A}$. If $\mathcal{E}$ is graded with a grading operator ε ($\varepsilon^2 = 1$), then we require $g\varepsilon = \varepsilon g, \forall g \in G$. See [Bla] for the terminology.

Definition 1. Suppose that Q is a $\mathbb{C}$-sesquilinear map from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{A}$.

(i) If $Q(\xi, \eta) = Q(\eta, \xi)^*, Q(\xi, \eta a) = Q(\xi, \eta)a$ and $gQ(\xi, \eta) = Q(g\xi, g\eta)$ for $a \in \mathcal{A}, \xi, \eta \in \mathcal{E}, g \in G$, then Q is called an equivariant quadratic form on $\mathcal{E}$.

(ii) If $\mathcal{E}$ is graded and $Q(\xi, \eta) = 0$ for $\partial\xi = \partial\eta$, then Q is of degree 1.

(iii) Q is regular if there is an equivariant $\mathcal{A}$-linear bijection T on $\mathcal{E}$, $gT = Tg$, such that $\mathcal{E}$ with the scalar product $\langle . \rangle$ defined by $\langle \xi, \eta \rangle = Q(\xi, T\eta)$ is a Hilbert G-$\mathcal{A}$-module. T is then said to be compatible with Q.

(iv) A scalar product $\langle . \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ is compatible with Q if $\langle \xi, \eta \rangle = Q(\xi, T\eta)$ for some T compatible with Q.

(v) Two Hilbert G-$\mathcal{A}$-scalar products $\langle . \rangle_1$ and $\langle . \rangle_2$ on $\mathcal{E}$ are compatible if there exists an equivariant $\mathcal{A}$-linear invertible operator T on $\mathcal{E}$ such that $\langle \xi, \eta \rangle_2 = \langle \xi, T\eta \rangle_1$.

Remark 1. (*a*) Clearly, T in (v) is unique, and any T satisfying $\langle \xi, \eta \rangle_2 = \langle \xi, T\eta \rangle_1$ is injective.

(*b*) If there are adjoint operators S_1^* and S_2^* of an operator S on $\mathcal{E}$ with respect to the compatible scalar products $\langle . \rangle_1$ and $\langle . \rangle_2$, then $S_2 = T^{-1}S_1^*T$. In fact,

$$\langle S\xi, \eta \rangle_2 = \langle S\xi, T\eta \rangle_1 = \langle \xi, S_1^*T\eta \rangle_1 = \langle \xi, T^{-1}S_1^*T\eta \rangle_2 = \langle \xi, S_2^*\eta \rangle_2.$$

(*c*) T in (*v*) is selfadjoint and positive with respect to $\langle . \rangle_1$ and $\langle . \rangle_2$:

$$\langle T\xi, \eta \rangle_1 = \overline{\langle \xi, T\eta \rangle_1} = \overline{\langle \xi, \eta \rangle_2} = \langle \eta, \xi \rangle_2 = \langle \eta, T\xi \rangle_1.$$

Thus, $T_1^* = T$ and $T_2^* = T^{-1}T_1^*T = T$. Clearly, $0 < \langle \xi, \xi \rangle_2 = \langle \xi, T\xi \rangle_1$ and $\langle \xi, T\xi \rangle_2 = \langle T\xi, T\xi \rangle_1 > 0$ for $\xi \neq 0$ in $\mathcal{E}$. Hence T is positive.

(*d*) If two scalar products $\langle . \rangle_1$ and $\langle . \rangle_2$ on $\mathcal{E}$ are compatible with Q, then $\langle . \rangle_1$ and $\langle . \rangle_2$ are compatible. In fact, let $\langle \xi, \eta \rangle_i = Q(\xi, T_i\eta)$, $i = 1, 2$. Then $\langle \xi, \eta \rangle_2 = Q(\xi, T_2\eta) = Q(\xi, T_1(T_1^{-1}T_2)\eta) = \langle \xi, T_1^{-1}T_2\eta \rangle_1$.

(*e*) If T is compatible with Q, then T is selfadjoint and there exists an equivariant operator T_1 on $\mathcal{E}$ compatible with Q such that $T_1^2 = I$. In fact,

$$\langle T\xi, \eta \rangle = \overline{Q(T\eta, T\xi)} = \overline{\langle T\eta, \xi \rangle} = \langle \xi, T\eta \rangle.$$

T is selfadoint. Hence T^{-1} is also selfadjoint. Since T^2 is invertible and positive, $U = |T^2|^{\frac{1}{2}}$ is well defined, invertible and positive. Let $T_1 = TU^{-1}$. Then $T_1^2 = I$, and $Q(\xi, T_1\eta) = Q(\xi, TU^{-1}\eta) = \langle \xi, U^{-1}\eta \rangle$ is a Hilbert scalar product on $\mathcal{E}$. T_1 is compatible with Q.

(*f*) If $\mathcal{E}$ is endowed with scalar product $\langle . \rangle$ compatible with Q, $\langle \xi, \eta \rangle = Q(\xi, T\eta)$, and S is an operator on $\mathcal{E}$ such that there is an adjoint S^* with respect to $\langle . \rangle$, then $S' = TS^*T^{-1}$ is the conjugate of S with respect to Q. Indeed,

$$Q(S\xi, \eta) = \langle S\xi, T^{-1}\eta \rangle = \langle \xi, T^{-1}(TS^*T^{-1})\eta \rangle = Q(\xi, TS^*T^{-1}\eta).$$

Let $\mathcal{E}$ be a Hilbert G-$\mathcal{A}$-module with a scalar product $\langle . \rangle$. Recall that the space $\mathcal{L}(\mathcal{E})$ of bounded operators on $\mathcal{E}$ consists of all continuous $\mathcal{A}$-linear maps $S : \mathcal{E} \to \mathcal{E}$ such that its adjoint S^* exists and is $\mathcal{A}$-linear. The space $\mathcal{K}(\mathcal{E})$ of compact operators on $\mathcal{E}$ is the ideal of $\mathcal{L}(\mathcal{E})$ generated by $Q_{x,y}, x, y \in \mathcal{E}$, where $Q_{x,y}(z) = x\langle y, z \rangle$. G acts on $\mathcal{L}(\mathcal{E})$ by $g(S)(\xi) = gS(g^{-1}(\xi))$. Denote by $\mathcal{L}_G(\mathcal{E})$ (resp., $\mathcal{K}_G(\mathcal{E})$) all G-continuous operators S in $\mathcal{L}(\mathcal{E})$ (resp., $\mathcal{K}(\mathcal{E})$), i.e., $g \to g(S)$ is norm continuous. Obviously, if $S \in \mathcal{L}(\mathcal{E})$ is equivariant, then $S \in \mathcal{L}_G(\mathcal{E})$. A regular operator on $\mathcal{E}$ is a densely defined operator S on $\mathcal{E}$ with densely defined adjoint S^* such that $I + S^*S$

has a dense range in $\mathcal{E}$. $\mathcal{L}(\mathcal{E}), \mathcal{K}(\mathcal{E})$ (resp., $\mathcal{L}_G(\mathcal{E}), \mathcal{K}_G(\mathcal{E})$) and regular operators do not depend on compatible Hilbert G-$\mathcal{A}$-module scalar products.

Using the convention that 1-graded and 0-graded mean graded and trivially graded , we have the following definition:

Definition 2. (1) $\mathcal{L}_G^k(\mathcal{A})$ is the set of all triples $(\mathcal{E}, Q, D)$, where $\mathcal{E}$ is a k-graded G-$\mathcal{A}$-module, Q is a strongly nondegenerate G-quadratic form of degree k, $k = 0, 1$, and $D \in \mathcal{L}_G(\mathcal{E})$ is G-equivariant such that

(a) $D + D' \in \mathcal{K}_G(\mathcal{E})$; (b) $D^2 \in \mathcal{K}_G(\mathcal{E})$; (c) there are equivariant S_1 and S_2 in $\mathcal{L}_G(\mathcal{E})$ such that $S_1 D + D S_2 - I \in \mathcal{K}_G(\mathcal{E})$; (d) D is of degree k.

(2) $\mathcal{L}_{G,u}^k(\mathcal{A})$ is the set of all triples $(\mathcal{E}, Q, D)$, where $\mathcal{E}$ and $\mathcal{A}$ are the same as in (1), but D is an equivariant regular operator such that

(a) $D + D' \in \mathcal{L}_G(\mathcal{E})$; (b) $im(D) \subset dom(D)$ and $D^2 \in \mathcal{L}_G(\mathcal{E})$; (c) there are equivariant S_1 and S_2 in $\mathcal{K}_G(\mathcal{E})$ with $im(S_2) \subset dom(D), DS_2, S_1 D \in \mathcal{L}_G(\mathcal{E})$ and $S_1 D + D S_2 - I \in \mathcal{K}_G(\mathcal{E})$; (d) D is of degree k.

We will see in Section 3 that this definition is modeled on the signature element. The following two elementary lemmas will be used to analyze $\mathcal{L}_{G,u}^k(\mathcal{A})$.

Lemma 1 ([HiS]). (a) *If D is a densely defined adjointable equivariant operator on a Hilbert G-$\mathcal{A}$-module $\mathcal{E}$ and $S \in \mathcal{L}_G(\mathcal{E})$ is such that $im(S) \subset dom(D^*)$, then $D^* S$ is in $\mathcal{L}_G(\mathcal{E})$.*

(b) *If D is a regular adjointable equivariant operator on $\mathcal{E}$ and $S \in \mathcal{L}_G(\mathcal{E})$ is such that $SD \in \mathcal{L}_G(\mathcal{E})$, then $im(S^*) \subset dom(D^*)$ and $D^* S^* = (SD)^*$.*

(c) *If D_1 and D_2 are two regular adjointable equivariant operators on $\mathcal{E}$ such that $D_1 : dom(D_1) \to \mathcal{E}$ is bijective and $dom(D_1) \subset dom(D_2)$, then $D_2 D_1^{-1}$ is in $\mathcal{L}_G(\mathcal{E})$.*

Lemma 2 ([HiS]). *Let D be a regular equivariant operator on $\mathcal{E}$ such that $D^2 = 0$. Then $D + D^*$ is selfadjoint and regular on $\mathcal{E}$, and $dom(D + D^*) = dom(D) \cap dom(D^*)$.*

We now consider the properties of elements in $\mathcal{L}_G^k(\mathcal{A})$ and $\mathcal{L}_{G,u}^k(\mathcal{A})$.

Lemma 3. (a) *If $(\mathcal{E}, Q, D) \in \mathcal{L}_G^k(\mathcal{A})$, then $D + D^*$ is invertible modulo $\mathcal{K}_G(\mathcal{E})$, where D^* is the adjoint of D with respect to the scalar product compatible with Q.*

(b) *If $(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^k(\mathcal{A})$, then $D + D^*$ is regular and selfadjoint with domain equal to $dom(D) \cap dom(D^*)$, and has the resolvent in $\mathcal{K}_G(\mathcal{E})$.*

Proof. We first consider the trivially graded case.

(a) The proof is to find the inverse of $D + D^*$ modulo $\mathcal{K}_G(\mathcal{E})$. Let $p = S_1 D, q = D S_2$, where S_i are as in Definition 2. Since $D^2 \in \mathcal{K}_G(\mathcal{E})$,

$$p^2 = S_1 D S_1 D = S_1 D (I - D S_2 + k) = S_1 D + k_1 = p \mod (\mathcal{K}_G(\mathcal{E})),$$

and

$$q^2 = D S_2 D S_2 = (k' + I - S_1 D) D S_2 = D S_2 + k_1' = q \mod (\mathcal{K}_G(\mathcal{E}))$$

for $k_i, k_i' \in \mathcal{K}_G(\mathcal{E})$. Then $q D p = D S_2 D (I - D S_2 + k) = D S_2 D - k_2 + D S_2 D k = (I - S_1 D + k) D - k_2 + D S_2 D k = D + k_3 = D \mod (\mathcal{K}_G(\mathcal{E}))$. Take a projection $(\mod \mathcal{K}_G(\mathcal{E}))$ $p' \in \mathcal{L}_G(\mathcal{E})$ [Bla] such that $p' p = p'$ and $p p' = p \mod(\mathcal{K}_G(\mathcal{E}))$. Since

$p+q-I = S_1D+DS_2-I \in \mathcal{K}_G(\mathcal{E}), p = I-q \bmod (\mathcal{K}_G(\mathcal{E}))$, and $qDp = (I-p)Dp = D \bmod (\mathcal{K}_G(\mathcal{E}))$. We get

$$(I-p)Dp = (I-p)Dpp' = Dp' \mod (\mathcal{K}_G(\mathcal{E})),$$

and

$$(I-p')(I-p)Dp = (I-p')Dp' \mod (\mathcal{K}_G(\mathcal{E})).$$

On the other hand, $(I-p')(I-p)Dp = (I-p'-p+p'p)Dp = (I-p)Dp = D \bmod (\mathcal{K}_G(\mathcal{E}))$. Hence, $D = (I-p')Dp' \bmod(\mathcal{K}_G(\mathcal{E}))$. Let $p_1 = p'S_1(I-p'), p_2 = p'S_2(I-p')$. We have $p_1D = p'S_1D - p'S_1p'D = p'S_1D = p'p = p' \bmod (\mathcal{K}_G(\mathcal{E}))$, since $p'D = p'(I-p')Dp' = 0 \bmod (\mathcal{K}_G(\mathcal{E}))$. Then

$$
\begin{aligned}
Dp_2 &= qDpp'S_2(I-p') = qDpS_2(I-p') = DS_2(I-p') \\
&= (I-p)(I-p') = I-p' \mod (\mathcal{K}_G(\mathcal{E})),
\end{aligned}
$$

and

$$
\begin{aligned}
p_1 &= p'S_1(I-p')(I-p') = p'S_1(I-p')Dp_2 = p_1Dp_2 \\
&= p'p_2 = p'S_2(I-p') = p_2 \mod (\mathcal{K}_G(\mathcal{E})).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(p_1+p_1^*)(D+D^*) &= p' + (I-p')S_1^*p'D + p_1D^* + (Dp_1)^* \\
&= p' + (Dp_2)^* = p' + (I-p') = I \mod (\mathcal{K}_G(\mathcal{E})).
\end{aligned}
$$

Here we have used $p'D = p'(I-p')Dp' = 0 \bmod (\mathcal{K}_G(\mathcal{E}))$ and $p_1D^* = p_2D^* = p'S_2(I-p')D^* = p'S_2(I-p')p'D^*(I-p') = 0$. This implies that $(D+D^*)^{-1} = p_1+p_1^* \bmod (\mathcal{K}_G(\mathcal{E}))$.

(b) We divide the proof into three steps.

Step 1. First we assume $D^2 = 0$. By Lemma 2, $D+D^*$ is selfadjoint and regular. Write $S_1D+DS_2 = I-k, S_i, k \in \mathcal{K}_G(\mathcal{E})$. Since $D^2 = 0, (S_1D)^2 = S_1D(I-k-DS_2) = S_1D - S_1Dk = S_1D \bmod (\mathcal{K}_G(\mathcal{E}))$, so we can thus choose $p \in \mathcal{L}_G(\mathcal{E})$ such that $p = p^*, p^2 - p \in \mathcal{K}_G(\mathcal{E}), pS_1D - p \in \mathcal{K}_G(\mathcal{E})$ and $S_1Dp - S_1D \in \mathcal{K}_G(\mathcal{E})$ [Bla]. Since $I-(pS_1D+DS_2(I-p)) = I-DS_2-(pS_1D-DS_2p) = S_1D-(pS_1D-(I-S_1D)p) = S_1D(I-p)-p(S_1D-I) \in \mathcal{K}_G(\mathcal{E}), DS_2(I-p) = I-pS_1D = 1-p \bmod (\mathcal{K}_G(\mathcal{E}))$ and $p = pS_1D = p^* = D^*S_1^*p \bmod (\mathcal{K}_G(\mathcal{E}))$. Thus, $q \stackrel{\text{def}}{=} I-D^*S_1^*p-DS_2(I-p) = I-(pS_1D+DS_2(I-p)) \in \mathcal{K}_G(\mathcal{E})$. Note that $dom(D+D^*) = dom(D)\cap dom(D^*)$. By Lemma 1(c) and $dom(i+D+D^*)^* \subset dom(D), \alpha \stackrel{\text{def}}{=} D((i+D+D^*)^{-1})^* \in \mathcal{L}_G(\mathcal{E})$. This implies that $D = \alpha(i+D+D^*)^*$. Also $D^* = \beta(i+D+D^*)^*$ for some $\beta \in \mathcal{L}_G(\mathcal{E})$ by the same argument. Using the fact that $\alpha^* = (i+D+D^*)^{-1}D^*, \beta^* = (i+D+D^*)^{-1}D$, and S_1^*p, q and $S_2(I-p)$ are all in $\mathcal{K}_G(\mathcal{E})$, we obtain

$$
\begin{aligned}
(i+D+D^*)^{-1} &= (i+D+D^*)^{-1}(q+D^*S_1^*p+DS_2(I-p)) \\
&= (i+D+D^*)^{-1}q+(\alpha^*S_1^*p+\beta^*S_2(I-p)) \in \mathcal{K}_G(\mathcal{E}).
\end{aligned}
$$

This proves that if $D^2 = 0$, then $(D+D^*)$ has resolvent in $\mathcal{K}_G(\mathcal{E})$.

Step 2. Suppose $D = D_0 + a$, where D_0 is regular, $D_0^2 = 0$ and $a \in \mathcal{L}_G(\mathcal{E})$. Since $\|(\lambda i + D_0 + D_0^*)^{-1}\| < \lambda^{-1}$ for sufficiently large $\lambda > 0$,

$$(\lambda i + D + D^*)^{-1} = (I + (\lambda i + D_0 + D_0^*)^{-1}(a+a^*))^{-1}(\lambda i + D_0 + D_0^*)^{-1}.$$

This proves that $(\lambda i + D + D^*)^{-1} \in \mathcal{K}_G(\mathcal{E})$, since $(\lambda i + D_0 + D_0^*)^{-1} \in \mathcal{K}_G(\mathcal{E})$.

Step 3. In general, we define for $(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^0(\mathcal{A})$

$$D_1 = \left[\begin{array}{cc} D & D^2 \\ -I & -D \end{array} \right], \qquad D_2 = \left[\begin{array}{cc} D & 0 \\ 0 & -D \end{array} \right].$$

Then $D_1^2 = 0, im(D_1) \subset dom(D_1)$. Clearly, $(\mathcal{E} \oplus \mathcal{E}, Q \oplus (-Q), D_j) \in \mathcal{L}_{G,u}^0(\mathcal{A})$ and $D_2 = D_1 - \left[\begin{array}{cc} 0 & D^2 \\ -I & 0 \end{array} \right]$. Steps 1 and 2 show that $D_2 + D_2^*$ is regular with resolvent in $\mathcal{K}_G(\mathcal{E} \oplus \mathcal{E})$. Therefore, $D + D^*$ is regular with resolvent in $\mathcal{K}_G(\mathcal{E})$.

Now for the graded case we repeat the above proof to get (a) and (b), even though the D_i in Step 3 are not of degree 1. $\square$

Remark 2. We obtain by the proof of part (a) that if $D \in \mathcal{L}_G(\mathcal{E})$ is equivariant and $S_1, S_2 \in \mathcal{L}_G(\mathcal{E})$ are such that $D^2 = 0$ and $S_1 D + D S_2 = I$, then $D + D^*$ is invertible with the inverse $(D + D^*)^{-1} = p' S_1 (I - p') + (I - p') S_1^* p'$, where p' was given in the proof.

To define the maps from $\mathcal{L}_G^k(\mathcal{A})$ and $\mathcal{L}_{G,u}^k(\mathcal{A})$ to $K_k^G(\mathcal{A})$, the equivariant K-theory group of $\mathcal{A}$, let us recall the definition of Kasparov's KK-theory. Let $\mathcal{A}$ and $\mathcal{B}$ be two G-C^*-algebras. $(\mathcal{E}, \varphi, F)$ is called a Kasparov G-bimodule if $\mathcal{E}$ is a countably generated graded Hilbert G-$\mathcal{B}$-module, $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{E})$ is an equivariant graded $*$-homomorphism, $F \in \mathcal{L}_G(\mathcal{E})$ is of degree 1 such that $[F, \varphi(a)]$, $(F - F^*)\varphi(a), (F^2 - I)\varphi(a), (g(F) - F)\varphi(a)$ are in $\mathcal{K}_G(\mathcal{E})$ and $\varphi(a)F \in \mathcal{L}_G(\mathcal{E})$. Two G-bimodules $(\mathcal{E}_j, \varphi_j, F_j)$ are isomorphic if there exists an equivariant isomorphism $u : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ such that $F_2 = u F_1 u^{-1}$. Let $E^G(\mathcal{A}, \mathcal{B})$ be the set of all isomorphic classes of G-bimodules $(\mathcal{E}, \varphi, F)$.

A homotopy between $(\mathcal{E}_0, \varphi_0, F_0)$ and $(\mathcal{E}_1, \varphi_1, F_1)$ is an element $(\mathcal{E}, \varphi, F) \in E^G(\mathcal{A}, C([0, 1], \mathcal{B}))$ such that $(\mathcal{E}, \varphi, F)$ generates a family $\{(\mathcal{E}_t, \varphi_t, F_t) \in E^G(\mathcal{A}, \mathcal{B})\}$ given by $\mathcal{E}_t = \mathcal{E} \hat{\otimes}_{C([0,1], \mathcal{B})} \mathcal{B}, \varphi_t = \varphi|_t, F_t = F \otimes I$. The homotopy gives rise to an equivalence $\sim$. Then the Kasparov KK-theory group is $KK^G(\mathcal{A}, \mathcal{B}) = E^G(\mathcal{A}, \mathcal{B})/ \sim$ and $KK_1^G(\mathcal{A}, \mathcal{B}) = KK^G(\mathcal{A}, \mathcal{B} \otimes C_{2,0}) \simeq KK^G(\mathcal{A} \otimes C_{2,0}, \mathcal{B})$, where $C_{2,0}$ is the Clifford algebra of $\mathbb{C}^2$ associated with the quadratic form $Q(x_1, x_2) = x_1^2 + x_2^2$. In particular, $KK_i^G(\mathbb{C}, \mathcal{B}) \simeq K_i^G(\mathcal{B}), i = 0, 1$.

We can also use unbounded modules to define $KK^G(\mathcal{A}, \mathcal{B})$ [BaJ] as follows. $(\mathcal{E}, \varphi, F)$ is an unbounded G-bimodule if $\mathcal{E}$ is a graded Hilbert G-$\mathcal{B}$-module, $\varphi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{E})$ is a G-$*$-homomorphism, and F is a selfadjoint regular equivariant operator of degree 1 on $\mathcal{E}$ such that

(a) $(I + F^2)^{-1} \varphi(a) \in \mathcal{K}_G(\mathcal{E}), \forall a \in \mathcal{A}$;

(b) $\{a \in \mathcal{A} : [F, \varphi(a)]$ is densely defined and extends to an element in $\mathcal{L}_G(\mathcal{E})\}$ is dense in $\mathcal{A}$.

Let $E_u^G(\mathcal{A}, \mathcal{B})$ be the set of all unbounded G-bimodules. There is a map $\psi : E_u^G(\mathcal{A}, \mathcal{B}) \rightarrow E^G(\mathcal{A}, \mathcal{B})$ given by $\psi((\mathcal{E}, \varphi, F)) = (\mathcal{E}, \varphi, F(I + F^2)^{-\frac{1}{2}})$.

Suppose that $\mathcal{A}$ and $\mathcal{E}$ are trivially graded. Let $(\mathcal{E}, Q, F) \in \mathcal{L}_G^0(\mathcal{A})$ and $T \in \mathcal{L}_G(\mathcal{E})$ be associated with Q, $T^2 = I$. Since T is an involution, we can use it to grade $\mathcal{E}$. It follows from $D^* = T D' T$ that $T(D + D^*) = T D + D' T$ and $(D + D^*)T = D T + T D'$. Using $D' = -D \bmod (\mathcal{K}_G(\mathcal{E}))$, we get $T(D + D^*) = -(DT - TD) = -(D + D^*)T \bmod (\mathcal{K}_G(\mathcal{E}))$. Hence, $D + D^*$ is of degree 1 mod $(\mathcal{K}_G(\mathcal{E}))$. By Lemma 3, $D + D^*$ is selfadjoint and invertible mod $(\mathcal{K}_G(\mathcal{E}))$. Then $D + D^* = F|D + D^*|$ $\bmod(\mathcal{K}_G(\mathcal{E})), |D + D^*| = ((D + D^*)^2)^{\frac{1}{2}}$ and $F = (D + D^*)|D + D^*|^{-1} \bmod (\mathcal{K}_G(\mathcal{E}))$.

Since $D + D^*$ is of degree 1 mod $(\mathcal{K}_G(\mathcal{E}))$, F is of degree 1 mod$(\mathcal{K}_G(\mathcal{E}))$. We have the following:

(a) $F^* = |D + D^*|^{-1}(D + D^*) = F \mod (\mathcal{K}_G(\mathcal{E}))$;

(b) $F^2 - I = (D + D^*)^2|D + D^*|^{-2} - I = 0 \mod (\mathcal{K}_G(\mathcal{E}))$;

(c) $g(F) - F \in \mathcal{K}_G(\mathcal{E})$: since $gQ(\xi, \eta) = Q(g\xi, g\eta)$ and $gT = Tg$, we have $Q(gD'\xi, g\eta) = Q(D'g\xi, g\eta)$. Hence, $gD' = D'g$, and then $D^*g = gD^*$, since $D^* = TD'T$. It follows that $g(D + D^*) = (D + D^*)g$ and $g|D + D^*| = |D + D^*|g$. From this we get $gF = Fg \mod (\mathcal{K}_G(\mathcal{E}))$;

(d) define $\varphi : \mathbb{C} \to \mathcal{L}_G(\mathcal{E})$ by $\varphi(\lambda) = \lambda I$. φ is equivariant.

Therefore, $(\mathcal{E}, \varphi, F)$ is a G-$\mathbb{C}$-$\mathcal{A}$-module.

Let $(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^0(\mathcal{A})$, and let $T \in \mathcal{L}_G(\mathcal{E})$ be compatible with Q such that $T^2 = I$. Suppose that the conjugate D' of D with respect to Q is $D' = -D$. Then $D^* = TD'T = -TDT$ and $T(D + D^*) = -(D + D^*)T$. This implies that $D + D^*$ is of degree 1 if $\mathcal{E}$ is graded by T. We know already by Lemma 3 that $D + D^*$ is selfadjoint and regular with resolvent in $\mathcal{K}_G(\mathcal{E})$. Therefore, $(\mathcal{E}, \varphi, D + D^*)$ is an unbounded G-$\mathbb{C}$-$\mathcal{A}$-module, where $\varphi(\lambda) = \lambda I$.

Let $\mathcal{A}$ be trivially graded and $(\mathcal{E}, \varphi, D) \in \mathcal{L}_G^1(\mathcal{A})$ be such that $D' = -D$. Then $T(D + D^*) = -(D + D^*)T$, where $T \in \mathcal{L}_G(\mathcal{E})$ is compatible with Q and $T^2 = I$. Let ε be the grading operator of $\mathcal{E}$, $\varepsilon^2 = I$. Since $Q(\xi, \eta) = 0$ for $\partial\xi = \partial\eta$, we must have $\varepsilon T = -T\varepsilon$. Using $D\varepsilon = -\varepsilon D$ and $D^*\varepsilon = -\varepsilon D^*$, we see that $\varepsilon(D + D^*) = -(D + D^*)\varepsilon$. Let $P = \frac{I + iT\varepsilon}{2}$, $\mathcal{E}_1 = P\mathcal{E}$ and $D_1 = P(D + D^*)P$ on $\mathcal{E}_1$. Then $P^2 = P, P^* = P, T\varepsilon P = PT\varepsilon$ and $P(D + D^*) = (D + D^*)P$. As for $\mathcal{L}_G^0(\mathcal{A})$, we can construct F from $(\mathcal{E}_1, Q, D_1)$ such that $F^* = F, F^2 = I \mod (\mathcal{K}_G(\mathcal{E}))$. Then $(\mathcal{E}_1, F)$ determines an element in $K_1^G(\mathcal{A})$ (cf. [HiS] and [Bla], p.185).

Similarly, we can work out an unbounded G-$\mathbb{C}$-$\mathcal{A}$-module from $(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^1(\mathcal{A})$ with $D' = -D$ and $\mathcal{A}$ trivially graded, which is denoted by

$$(\mathcal{E}_1, Q, P(D + D^*)P).$$

To summarize, we have the following:

Definition 3. (1) $\psi_0 : \mathcal{L}_G^0(\mathcal{A}) \to K_0^G(\mathcal{A})$ is given by $\psi_0((\mathcal{E}, Q, D)) = [(\mathcal{E}, F)]$.

(2) Let $\mathcal{L}_{G,u}^{0'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^0(\mathcal{A}) : D' = -D\}$. $\psi_{0,u} : \mathcal{L}_{G,u}^{0'}(\mathcal{A}) \to K_0^G(\mathcal{A})$ is defined by $\psi_{0,u}((\mathcal{E}, Q, D)) = [(\mathcal{E}, F)]$, where $F = (D + D^*)(1 + (D + D^*)^2)^{-\frac{1}{2}}$.

(3) Let $\mathcal{A}$ be trivially graded and $\mathcal{L}_G^{1'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_G^1(\mathcal{A}) : D = -D'\}$. $\psi_1 : \mathcal{L}_G^{1'}(\mathcal{A}) \to K_1^G(\mathcal{A})$ is defined by $\psi_1((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)]$.

(4) Let $\mathcal{A}$ be trivially graded and $\mathcal{L}_{G,u}^{1'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^1(\mathcal{A}) : D = -D'\}$. $\psi_{1,u} : \mathcal{L}_{G,u}^{1'}(\mathcal{A}) \to K_1^G(\mathcal{A})$ is defined by $\psi_1((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)]$, where $F = P(D + D^*)P(I + (P(D + D^*))^2)^{-\frac{1}{2}}$.

Remark 3. $\psi_0 : \mathcal{L}_G^0(\mathcal{A}) \to K_0^G(\mathcal{A})$ is independent of the choice of F and T. In fact, if T_1 and T_2 are compatible with Q, $T_i^2 = I$. Let $U = T_2T_1T_2$. Then $U^2 = I$ and $Q(\xi, U\eta) = Q(T_2\xi, T_1T_2\eta) = \langle T_2\xi, T_2\eta\rangle_1$. U is compatible with Q. By the equivariant stability theorem, we can assume $\mathcal{E} = \mathcal{H}_\mathcal{A}$, the universal Hilbert G-$\mathcal{A}$-module. Using the invertibility of $U^{\frac{1}{2}}$ and contractibility of $GL(\mathcal{H}_\mathcal{A})$, we get a homotopy h_t in $GL(\mathcal{H}_\mathcal{A})$ connecting $U^{\frac{1}{2}}$ to I. Let $\tilde{h}_t = h_t^*h_t$. Then $\tilde{h}_t$ is a selfadjoint homotopy between U and I. Hence we obtain a homotopy H_t connecting T_1 and T_2 such that $H_t^2 = I$ and H_t is compatible with Q. Thus, ψ_0 is independent of the

choice of T. Clearly, ψ_0 is independent of the choice of F, since two such choices differ by an element in $\mathcal{K}_G(\mathcal{E})$.

A similar remark applies to ψ_1.

The following lemma gives a link between $\mathcal{L}^{0'}_{G,u}(\mathcal{A})$ and $\mathcal{L}^0_G(\mathcal{A})$.

Lemma 4 ([HiS]). *Let* $(\mathcal{E}, Q, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ *be such that* $D' = -D$. *Let* $U = D(I + D^*D)^{-\frac{1}{2}}$. *Then, for* $\mathcal{E}$ *with the scalar product compatible with* Q,

$$\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_0((\mathcal{E}, Q, U)).$$

Lemma 5. *Let* $\mathcal{A}$ *be trivially graded. Suppose that* $(\mathcal{E}, \varphi, D) \in \mathcal{L}^1_{G,u}(\mathcal{A})$ *is such that* $D' = -D$. *Let* $U = D(I + D^*D)^{-\frac{1}{2}}$. *Then, for* $\mathcal{E}$ *with the scalar product associated with* Q,

$$\psi_{1,u}((\mathcal{E}, Q, D)) = \psi_1((\mathcal{E}, Q, U)).$$

Proof. Note that $\psi_{1,u}((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)], F = P(D + D^*)(I + (P(D + D^*))^2)^{-\frac{1}{2}}$, where $\mathcal{E}_1 = P\mathcal{E}, P = \frac{I + iT\varepsilon}{2}$, and $\psi_1((\mathcal{E}, Q, U)) = [(\mathcal{E}_1, F_1)]$ with $F_1 \in \mathcal{L}_G(\mathcal{E}_1)$ such that $P(U + U^*)P - F_1|P(U + U^*)P| \in K_G(\mathcal{E}_1)$. The proof is the same as that of Lemma 4. $\square$

We now examine when the images under ψ_0 of two elements in $\mathcal{L}^0_G(\mathcal{A})$ are equal.

Lemma 6 ([HiS]). *Let* $(\mathcal{E}, \varphi, D) \in \mathcal{L}^0_G(\mathcal{A})$ *(resp.,* $\mathcal{L}^0_{G,u}(\mathcal{A})$*).*
   (a) If $Ker(D) = im(D)$ *and* $D' = -D$, *then* $\psi_0((\mathcal{E}, Q, D)) = 0$.
   (b) $\psi_0((\mathcal{E}, -Q, -D)) = -\psi_0((\mathcal{E}, Q, D))$.
   The same results hold for $\psi_{0,u}$.

Proof. (a) By Lemma 4, it suffices to prove the assertion for the bounded case. By definition of ψ_0, it is enough to check $F^2 = I$ and $F^* = F$, which are guaranteed by the invertibility of $D + D^*$. In fact, if $D + D^*$ is invertible, then $((D + D^*)^2)^{\frac{1}{2}}$ is invertible with the inverse $((D + D^*)^2)^{-\frac{1}{2}}$ and $F^2 = ((D + D^*)((D + D^*)^2)^{-\frac{1}{2}})^2 = I$.

To prove that $D + D^*$ is invertible, let us first note that for two Hilbert modules $\mathcal{E}$ and $\mathcal{E}_1$ and $S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_1)$, SS^* is invertible, provided S is surjective. Indeed, by the open mapping theorem, there is $k > 0$ such that $SS^* \geq k^{-2}$. We take $\mathcal{E}_1 = Ker(D) = im(D)$ and $S = D : \mathcal{E} \to im(D)$. S is surjective. Then SS^* is invertible. This implies that zero is at worst an isolated spectral point of $S^*S = D^*D$, since $Spec(SS^*) \setminus \{0\} = Spec(S^*S) \setminus \{0\}$. We can then choose a submodule $\mathcal{E}_2$ in $\mathcal{E}$ which is orthogonal to $\mathcal{E}_1$ such that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and DD^* is invertible on $\mathcal{E}_1$ and D^*D is invertible on $\mathcal{E}_2$. It follows that $DD^* + D^*D$ is invertible on $\mathcal{E}$. In view of $D^2 = 0$, $(D + D^*)^2 = DD^* + D^*D$ is invertible on $\mathcal{E}$. Hence $D + D^*$ is invertible in $\mathcal{L}(\mathcal{E})$ and then in $\mathcal{L}_G(\mathcal{E})$, since $D + D^*$ is equivariant.

(b) If we change Q to $-Q$ and D to $-D$, then T is altered to $-T$, $D^* = TD'T$ to $-D^*$, and then F to $-F$. Hence, we change $(\mathcal{E}, F)$ to $(\mathcal{E}, -F)$, i.e., $\psi_0((\mathcal{E}, -Q, -D)) = [(\mathcal{E}, -F)] = -[(\mathcal{E}, F)] = -\psi_0((\mathcal{E}, Q, D))$. $\square$

Lemma 7. *Let* $\mathcal{A}$ *be trivially graded and* $(\mathcal{E}, Q, D) \in \mathcal{L}^1_G(\mathcal{A})$ *(resp.,* $\mathcal{L}^1_{G,u}(\mathcal{A})$*).*
   (a) If $Ker(D) = im(D)$ *and* $D' = -D$, *then* $\psi_1((\mathcal{E}, Q, D)) = 0$.
   (b) $\psi_1((\mathcal{E}, -Q, -D)) = -\psi_1((\mathcal{E}, Q, D))$, *where* $-\mathcal{E}$ *is equal to* $\mathcal{E}$ *graded by* $-\varepsilon$.
   The same results hold for $\psi_{1,u}$.

Proof. (a) Note that $(D + D^*)$ commutes with $P = \frac{I + iT\varepsilon}{2}$. The proof of Lemma 6 implies that $(D + D^*)$ is invertible in $\mathcal{L}_G(\mathcal{E})$. Hence $P(D + D^*)$ is invertible in $\mathcal{L}_G(\mathcal{E}_1)$. As a result, $\psi_1((\mathcal{E}, Q, D)) = 0$.

(b) Observe that if Q and D are changed to $-Q$ and $-D$, resp., then T is changed to $-T$, ε to $-\varepsilon$, D^* to $-D^*$, P to P, and hence $P(D + D^*)$ to $-P(D + D^*)$. The result then follows easily. $\qquad\square$

3. Case of Flat Bundles

We now use the machinery in Section 2 to prove the homotopy invariance of the signature elements with coefficients in flat bundles.

We assume throughout this section that G is a compact Lie group, $\mathcal{A}$ is a G-C^*-algebra over $\mathbb{C}$, M is a closed, oriented Riemannian G-manifold and E is a smooth G-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0. Let $\mathcal{E}_E = L^2(E)$ be the completion of $C^\infty(M, \wedge_\mathbb{C} T^* M \otimes E)$ with respect to the scalar product

$$\langle \xi, \eta \rangle = \int_M \langle \xi(x), \eta(x) \rangle.$$

Here, we used a G-equivariant Riemannian structure on M to get a Hilbert G-$\mathcal{A}$-module structure on $\wedge_\mathbb{C} T_x^* M \otimes E_x, x \in M$.

Let $\xi, \eta \in \wedge_\mathbb{C} T_x^* M \otimes E_x$. Define $\xi^* \wedge \eta \in \wedge_\mathbb{C} T_x^* M \otimes \mathcal{A}$ by first letting

$$(\alpha \otimes e_1)^* \wedge (\beta \otimes e_2) = \bar\alpha \wedge \beta \otimes \langle e_1, e_2 \rangle, \qquad \alpha, \beta \in \wedge_\mathbb{C} T_x^* M, e_i \in E_x,$$

$\langle e_1, e_2 \rangle \in \mathcal{A}$, and then extending this to general elements ξ, η, where $\langle . \rangle$ is the scalar product on $E_x = E_0$.

We now define a signature element in $\mathcal{L}_{G,u}^k(\mathcal{A})$. Let Q be the quadratic form on $\mathcal{E}_E$ given by

$$Q(\xi, \eta) = i^{\partial\xi(n - \partial\xi)} \int_M \xi_x^* \wedge \eta_x$$

for $\xi \in \mathcal{E}_E$ with homogeneous degree $\partial\xi$. Here n is the dimension of M. Q is equivariant since the metric and the scalar product $\langle . \rangle$ on E_0 are equivariant. Let us check that Q is regular, i.e., there exists an equivariant $\mathcal{A}$-bijection $T \in \mathcal{L}(\mathcal{E}_E)$ such that $\langle \xi, \eta \rangle = Q(\xi, T\eta)$ is a scalar product. Let

$$(T\xi)_x = i^{-\partial\xi(n - \partial\xi)} (* \otimes I_E)(\xi_x), \ \ \xi \in \mathcal{E}_E.$$

Here $*$ is the Hodge operator, $*^2 \alpha = (-1)^{\partial\alpha(n - \partial\alpha)} \alpha, \alpha \in \wedge_\mathbb{C} T^* M$. We have $T^2 \xi = i^{-2\partial\xi(n - \partial\xi)} (*^2 \otimes I_E)\xi = \xi$. Hence $T^2 = I$. Clearly, T is equivariant and $\mathcal{A}$-bijective. The quantity

$$\langle \xi, \eta \rangle = Q(\xi, T\eta) = i^{\partial\xi(n - \partial\xi)} \int_M \xi_x^* \wedge (T\eta)_x = \int_M \xi_x^* \wedge (* \otimes I_E)(\eta_x)$$

is evidently a scalar product. Q also is a quadratic form: $Q(\xi, \eta) = Q(\eta, \xi)^*$ and $Q(\xi, \eta a) = Q(\xi, \eta)a, a \in \mathcal{A}$.

To use $\mathcal{E}_E$ and Q to define an element in $\mathcal{L}_{G,u}^k(\mathcal{A})$, we need an operator D satisfying the condition of Definition 2. Let ∇ be a metric-preserving antisymmetric (briefly, unitary) equivariant connection on E. Define an operator D_E by

$$D_E(\xi) = i^{\partial\xi} \nabla\xi, \ \ \xi \in \mathcal{E}.$$

In fact, let $\mathcal{H}_\mathcal{A} = L^2(G) \otimes_\mathbb{C} H_\mathcal{A}$ be the universal G-$\mathcal{A}$-module [Bla]. By the equivariant stability theorem, E_0 can be embedded in $\mathcal{H}_\mathcal{A}$. Then E can be constructed via a family of smooth equivariant projections $P = \{P_x\}_{x \in M}$ on $\mathcal{H}_\mathcal{A}$. Here each

P_x has a finitely generated projective range. In other words, $P : M \to \mathcal{L}(\mathcal{H}_\mathcal{A})$ is smooth and the fiber $E_x = P_x(\mathcal{H}_\mathcal{A})$. Then let

$$(\nabla \xi)_x = (I_{\wedge_\mathbb{C} T_x^* M} \otimes P_x)((d \otimes I_{\mathcal{H}_\mathcal{A}})\xi)_x, \ \ \xi \in C^\infty(M, \wedge_\mathbb{C} T^* M \otimes E).$$

The following lemma implies that D_E satisfies the condition of Definition 2.

Lemma 8. *With the above notation,*

(1) *$\forall \omega \in C^\infty(M, \wedge_\mathbb{C} T^* M)$ and $\xi \in dom(D_E)$ we have $\omega \wedge \xi \in dom(D_E)$ and $D_E(\omega \wedge \xi) = i^{\partial \xi + \partial \omega} d\omega \wedge \xi + i^{-\partial \omega} \omega \wedge D_E \xi$;*

(2) *if D'_E is the conjugate of D_E with respect to Q, then $D'_E = -D_E$;*

(3) *$im(D_E) \subset dom(D_E)$ and $D_E^2 \in \mathcal{L}_G(\mathcal{E}_E)$;*

(4) *there are equivariant S_1 and S_2 in $\mathcal{K}_G(\mathcal{E}_E)$ such that $im(S_2) \subset dom(D_E)$, $D_E S_2, S_1 D_E \in \mathcal{L}_G(\mathcal{E}_E)$ and $S_1 D_E + D_E S_2 - I \in \mathcal{K}_G(\mathcal{E}_E)$.*

Proof. (1) Let $\omega \in C^\infty(M, \wedge_\mathbb{C} T^* M), \xi \in dom(D_E)$. Assume first that $\xi = \alpha \otimes a, \alpha \in C^\infty(M, \wedge_\mathbb{C} T^* M)$ and $a \in E_0, P_x a = a$. Then

$$\begin{aligned}
D_E(\omega \wedge \xi)_x &= i^{\partial \omega + \partial \alpha} \nabla(\omega \wedge \alpha \otimes a)_x \\
&= i^{\partial \omega + \partial \alpha}[(I \otimes P_x)((d\omega \wedge \alpha + (-1)^{\partial \omega} \omega \wedge d\alpha) \otimes a)_x] \\
&= i^{\partial \omega + \partial \alpha}[(d\omega \wedge \alpha \otimes P_x a) + (-1)^{\partial \omega} \omega \wedge (I \otimes P_x)(d\alpha \otimes a)_x] \\
&= (i^{\partial \omega + \partial \alpha} d\omega \wedge \xi + i^{-\partial \omega} \omega \wedge D_E \xi)_x.
\end{aligned}$$

Similarly, we can treat the general case.

(2) To show $D'_E = -D_E$, it is enough to check that $Q(D_E \xi, \eta) = -Q(\xi, D_E \eta)$. Let $\xi = \alpha \otimes a, \eta = \beta \otimes b, \alpha, \beta \in C^\infty(M, \wedge_\mathbb{C} T^* M), a, b \in E_0, P_x a = a, P_x b = b, D_E \xi = i^{\partial \alpha}(d\alpha \otimes Pa), D_E \eta = i^{\partial \beta}(d\beta \otimes Pb)$. Then

$$\begin{aligned}
Q(D_E \xi, \eta) &= i^{3\partial \xi + (\partial \alpha + 1)\partial \beta} \int_M (\overline{d\alpha} \wedge \beta) \otimes \langle Pa, b \rangle \\
&= -i^{(\partial \beta + 1)\partial \alpha} i^{\partial \beta} \int_M (\bar\alpha \wedge d\beta) \otimes \langle a, Pb \rangle \\
&= -Q(\xi, D_E \eta).
\end{aligned}$$

Similarly, one can check that $Q(D_E \xi, \eta) = -Q(\xi, D_E \eta)$ for the general case.

(3) Since the projection map $x \to P_x$ is smooth, $im(D_E) \subset dom(D_E)$. By the fact that the curvature ∇^2 is an $\mathcal{L}_G(E_0)$-valued 2-form on M, D_E^2 is in $\mathcal{L}_G(\mathcal{E}_E)$.

(4) Let E be constructed by a family of equivariant smooth $\mathcal{A}$-projections $P = \{P_x\}_{x \in M}$ with finitely generated projective ranges: $E_x = P_x F_0$, where F_0 is a finitely generated G-$\mathcal{A}$-projective Hilbert module. Then $D_E = PD_F P, D_F$ is the flat exterior differential on the trivial G-$\mathcal{A}$-bundle $M \times F_0 = F$. As in the ordinary case [Hil], $(I + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$ and $D_F(I + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$ for $\mathcal{F} = L^2(F)$. Since the orthogonal complement $E^\perp$ of E in F is given by $(I - P)F$, and its connection $D_{E^\perp}$ is $(I - P)D_F(I - P)$, we have

$$\begin{aligned}
D_E \oplus D_{E^\perp} &= PD_F - PD_F(I - P) + (I - P)D_F(I - P) \\
&= D_F - (I - P)D_F - (2P - I)D_F(I - P) = D_F + A,
\end{aligned}$$

where $A = (P - I)D_F P + (P)D_F(I - P)$. Let us first check that

$$B_1 = (D_F + A)(\lambda + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$$

and

$$B_2 = (\lambda + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1} \in \mathcal{K}_G(\mathcal{F}), \qquad \lambda \to \infty.$$

Now

$$(\lambda + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1}$$
$$= (I + (\lambda + D_F^* D_F + D_F D_F^*)^{-1}$$
$$\times (D_F^* A + A D_F^* + A^* D_F + D_F A^* + A^* A + A A^*))^{-1}$$
$$\times (\lambda + D_F^* D_F + D_F D_F^*)^{-1},$$

since the products of A, A^* with D_F and D_F^* are bounded operators (by means of the argument of Sobolev spaces) and since the norm of the operator $B_3 = (\lambda + D_F^* D_F + D_F D_F^*)^{-1}(D_F^* A + A D_F^* + A^* D_F + D_F A^* + A^* A + A A^*)$ is less than 1 for sufficiently large λ. It follows that $B_2 \in \mathcal{K}_G(\mathcal{F})$. Since A is bounded, $A B_2 \in \mathcal{K}_G(\mathcal{F})$ and $D_F B_2 = D_F (I + B_3)^{-1}(\lambda + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$.

Now let $S_1 = (\lambda + D_E^* D_E + D_E D_E^*)^{-1} D_E^*$ and $S_2 = D_E^*(\lambda + D_E^* D_E + D_E D_E^*)^{-1}$. Then S_1 extends to be in $\mathcal{K}_G(\mathcal{E}_E)$. Also S_2 is in $\mathcal{K}_G(\mathcal{E}_E)$. We have

$$S_1 D_E + D_E S_2 - I = (\lambda + D_E^* D_E + D_E D_E^*)^{-1} D_E^* D_E$$
$$+ D_E D_E^*(\lambda + D_E^* D_E + D_E D_E^*)^{-1} - I$$
$$= -\lambda(\lambda + D_E^* D_E + D_E D_E^*)^{-1} + [(\lambda + D_E^* D_E + D_E D_E^*)^{-1}, D_E^* D_E]$$
$$= -\lambda(\lambda + D_E^* D_E + D_E D_E^*)^{-1} + (\lambda + D_E^* D_E + D_E D_E^*)^{-1}$$
$$\times (D_E^* D_E^2 D_E^* - D_E (D_E^*)^2 D_E)(\lambda + D_E^* D_E + D_E D_E^*)^{-1} \in \mathcal{K}_G(\mathcal{E}_E).$$

$\square$

Therefore, we have constructed an element $(\mathcal{E}_E, Q, D_E) \in \mathcal{L}_{G,u}^{0'}(\mathcal{A})$ for $dim(M) = 2k$. If $dim(M) = 2k + 1$, we proceed as follows.

Let $\varepsilon(\xi) = (-1)^{\partial \xi} \xi$ for a homogeneous $\xi \in \mathcal{E}_E$. Then ε extends to a grading on $\mathcal{E}_E$. Clearly, $\varepsilon T = -T\varepsilon$ (this is not true for $dim(M) = 2k$), $\varepsilon D_E = -D_E \varepsilon$. We get $(\mathcal{E}_E, Q, D_E) \in \mathcal{L}_{G,u}^{1'}(\mathcal{A})$. Hence, we can apply the maps $\psi_{i,u}$ in Definition 3 to the element $(\mathcal{E}_E, Q, D_E)$. Denote

$$G\text{-}Sig(D_E) = \psi_{i,u}((\mathcal{E}_E, Q, D_E)) \in K_i^G(\mathcal{A}), \quad dim(M) = 2k + i, \ i = 0, 1.$$

We call $G\text{-}Sig(D_E)$ the G-signature of M with coefficients in E.

We now consider the central problem of this section. Let E be a smooth G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0. As usual, E defines an element $[E] \in K_0^G(\mathcal{A} \otimes C(M))$, where $C(M)$ is the space of all continuous functions on M. Let $[D_M]$ be the equivariant signature element in $K_G^*(C(M))$. Then

$$\psi(E, M) \overset{\text{def}}{=} [E] \otimes_{C(M)} [D_M] \in K_0^G(\mathcal{A} \otimes C(M)) \otimes_{C(M)} KK_*^G(C(M), \mathbb{C})$$
$$\to KK_*^G(\mathbb{C}, \mathcal{A}) = K_*^G(\mathcal{A}), \quad * = 0, 1.$$

Here we need $\mathcal{A}$ to be σ-unital in order to use the Kasparov product.

Question. *When is $\psi(E, M)$ a G-pseudo-equivalence invariant? Namely, let $h : N \to M$ be an orientation-preserving G-pseudo-equivalence of closed, connected oriented Riemannian G-manifolds. When is $\psi(h^*(E), N)$ equal to $\psi(E, M)$?*

Recall that h is a G-pseudo-equivalence if h is a homotopy equivalence and G-equivariant. The following observation is important: if h is equivariantly homotopic to an orientation-preserving equivariant map $h_1 : N \to M$, then $h^*(E)$ is isomorphic to $h_1^*(E)$. Hence $[h^*(E)] = [h_1^*(E)] \in K_0^G(\mathcal{A} \otimes C(N))$ (cf. [Hus], Thm. 4.7, p. 29).

According to the G-smooth approximation theorem [Bre, Thm. 4.2, p. 317], h is G-equivariantly homotopy equivalent to a smooth orientation-preserving G-pseudo-equivalence $h_1 : N \to M$. Therefore, it suffices to check $\psi(h_1^*(E), N) = \psi(E, M)$.

The following lemma then reduces the question for the even dimensional case to the G-pseudo-equivalence invariance of the G-signature $G\text{-}Sig(D_E)$ in $K_0^G(\mathcal{A})$.

Lemma 9. *Let $\mathcal{A}$ be a unital G-C^*-algebra and M be a closed oriented Riemannian G-manifold of even dimension. Suppose that E is a smooth G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0. Let $(\mathcal{E}_E, Q, D_E)$ be as in Lemma 8. Then*

$$G\text{-}Sig(D_E) = [E] \otimes_{C(M)} [D_M] \in K_0^G(\mathcal{A}).$$

Proof. Recall that $[E] \in KK_0^G(\mathbb{C}, \mathcal{A} \otimes C(M))$ is defined by the Kasparov module $(\Gamma(E) \oplus 0, I, 0)$ and $[D_M] \in KK_0^G(C(M), \mathbb{C})$ is given by

$$(L^2(M, T^*M), \varphi, (D_M + D_M^*)(I + (D_M + D_M^*)^2)^{-\frac{1}{2}}),$$

where $\Gamma(E)$ is the space of continuous sections of E and φ is the multiplication by elements in $C(M)$. Assume first that E is trivial, $E = M \times E_0$. Then $D_E = D_M \otimes I$ and $D_E + D_E^* = (D_M + D_M^*) \otimes I$. The Kasparov product of $[E]$ and $[D_M]$ is given by the following Kasparov bimodule:

$$(((C(M) \otimes E_0) \oplus 0) \otimes_{C(M) \otimes \mathcal{A}} (L^2(M, T^*M) \otimes \mathcal{A}),$$

$$I, (D_M + D_M^*)(I + (D_M + D_M^*)^2)^{-\frac{1}{2}} \otimes I).$$

where $((C(M) \otimes E_0) \oplus 0) \otimes_{C(M) \otimes \mathcal{A}} (L^2(M, T^*M) \otimes \mathcal{A}) \simeq L^2(M, T^*M \otimes E)$ with the inner product $\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_2, \langle x_1, y_1 \rangle_1 y_2 \rangle_2$ equivalent to the inner product on $L^2(M, T^*M \otimes E)$. This proves the assertion for the case of trivial bundles. In general, $E = PF$, where F is a trivial smooth G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$ module F_0 and $P : M \to \mathcal{L}(F_0)$ is a smooth family of projections. Then

$$D_E = PD_F P, \qquad D_E + D_E^* = P(D_F + D_F^*)P.$$

Hence, $(D_E + D_E^*)(I + (D_E + D_E^*)^2)^{-\frac{1}{2}} = P(D_F + D_F^*)(I + (D_F + D_F^*)^2)^{-\frac{1}{2}}P$ is a $(D_F + D_F^*)(I + (D_F + D_F^*)^2)^{-\frac{1}{2}}$-connection of $\mathcal{E}_E$ (cf. [Bla], Prop. 18.3.3, p. 206). The result follows easily from definition of the Kasparov product. $\square$

Remark 4. (1) If $\mathcal{A}$ is not unital, then $K_0^G(\mathcal{A} \otimes C(M)) = Ker\{K_0^G(C(M) \otimes \mathcal{A}^+) \xrightarrow{i^*} K_0^G(C(M) \otimes \mathbb{C})\}$ and $[E] = [((\Gamma(E) \oplus 0), I, 0)] \in Ker(i^*)$, since $\Gamma(E) \otimes_{C(M) \otimes \mathcal{A}} C(M) \otimes \mathbb{C} = 0$, where $\mathcal{A}^+$ is the G-C^*-algebra obtained by adjoining an identity to $\mathcal{A}$. Note also that $\mathcal{E}_E$ can be considered as an $\mathcal{A}^+$-module and $[(\mathcal{E}_E, Q, D_E)]$ formed in the same manner as $G\text{-}Sig(D_E)$ is in $Ker(K_0^G(\mathcal{A}^+) \to K_0^G(\mathbb{C}))$, i.e., this element is $G\text{-}Sig(D_E)$. With this in mind, we see that Lemma 9 holds for a σ-unital $\mathcal{A}$, namely,

$$G\text{-}Sig(D_E) = [(\mathcal{E}_E, Q, D_E)] = [E] \otimes_{C(M)} [D_M].$$

(2) Let M be a closed oriented Riemannian G-manifold of odd dimension. As usual, the equivariant signature element $[D_M] \in K_G^1(C(M))$ can be defined as the restriction of $[D_{M \times S^1}]$ to the part $K_G^1(C(M))$ in the splitting of $K_G^0(C(M \times S^1))$, where G acts trivially on S^1. Thus the equivariant Novikov conjecture for the odd dimensional case is reduced to that for the even dimensional one. Hence, we will focus only on the even dimensional case.

Clearly, $G\text{-}Sig(D_E)$ is independent of the connection on E, since any two connections on E differ by a bounded operator in $\mathcal{L}_G(\mathcal{E}_E)$.

To consider the G-pseudo-equivalence invariance of $G\text{-}Sig(D_E)$, we need the following technical proposition.

Proposition 1. *Let $(\mathcal{E}_i, Q_i, D_i) \in \mathcal{L}^0_{G,u}(\mathcal{A}), i = 1, 2$, and $R \in \mathcal{L}_G(\mathcal{E}_1, \mathcal{E}_2)$ satisfy*

(a) $D'_i = -D_i, D_i^2 = 0$;

(b) $R(dom(D_1)) \subset dom(D_2), RD_1 = D_2R$;

(c) $R : \frac{Ker(D_1)}{im(D_1)} \to \frac{Ker(D_2)}{im(D_2)}$ *is an isomorphism;*

(d) *there is an equivariant $S \in \mathcal{L}(\mathcal{E}_1)$ such that $S(dom(D_1)) \subset dom(D_1)$ and $I - R'R = D_1S + SD_1$;*

(e) *there is an equivariant $\varepsilon \in \mathcal{L}(\mathcal{E}_1)$ such that $\varepsilon(dom(D_1)) \subset dom(D_1), \varepsilon D_1 = -D_1\varepsilon, \varepsilon^2 = I, \varepsilon' = \varepsilon$ and $\varepsilon(I - R'R) = (I - R'R)\varepsilon$.*

Then $\psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{0,u}((\mathcal{E}_2, Q_2, D_2))$.

Proof. We can assume that $S' = -S$. In fact, for $\xi, \eta \in dom(D_1)$,

$$Q_1(S'\xi, D_1\eta) = Q_1(\xi, SD_1\eta) = Q_1(\xi, (I - R'R)\eta) + Q_1(D_1\xi, S\eta).$$

It follows that $S'\xi \in dom(D_1), S'(dom(D_1)) \subset dom(D_1)$ and

$$I - R'R = -(D_1S' + S'D_1).$$

Take $\tilde{S} = \frac{S - S'}{2}$. We have $\tilde{S}(dom(D_1)) \subset dom(D_1)$ and $I - R'R = D_1\tilde{S} + \tilde{S}D_1$. $\tilde{S}$ is obviously equivariant. Thus we can consider $\tilde{S}$ with $\tilde{S}' = -\tilde{S}$.

Let $(\mathcal{E}, Q, D) = (\mathcal{E}_1 \oplus \mathcal{E}_2, Q_1 \oplus (-Q_2), D_1 \oplus (-D_2))$. By Lemma 6,

$$\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) + \psi_{0,u}((\mathcal{E}_2, -Q_2, -D_2))$$
$$= \psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) - \psi_{0,u}((\mathcal{E}_2, Q_2, D_2)).$$

Thus it suffices to prove $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$.

For $t \in [0, 1]$ we define $B_t(\xi, \eta) = Q(T_t\xi, T_t\eta), C_t(\xi, \eta) = Q(L_t\xi, \eta)$, where

$$T_t = \begin{bmatrix} I & 0 \\ itR\varepsilon & I \end{bmatrix}, L_t = \begin{bmatrix} I - R'R & (i\varepsilon + tS)R' \\ R(i\varepsilon + tS) & I \end{bmatrix}, \nabla_t = \begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix}.$$

We have the following:

(i) $DT_t = T_tD$, since $\varepsilon D_1 = -D_1\varepsilon$ and $RD_1 = D_2R$.

(ii) $L_t\nabla_t = -\nabla'_tL_t$. This follows from $I - R'R = D_1S + SD_1, D_1^2 = 0$ and $D_1R' = R'D_2$. Note that by $D'_i = -D_i$,

$$\nabla'_t = \begin{bmatrix} -D_1 & 0 \\ -tR & D_2 \end{bmatrix}.$$

(iii) $C_t(\nabla_t\xi, \eta) = Q(-\nabla'_tL_t\xi, \eta) = -C_t(\xi, \nabla_t\eta)$. Hence ∇_t is antiselfadjoint with respect to $C_t(., .)$.

(iv) Since ε commutes with $R'R$, $L_0 = T'_1T_1$ and

$$T'_t = \begin{bmatrix} I & it\varepsilon R' \\ 0 & I \end{bmatrix}.$$

Let us check that $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

(a) D is antisymmetric with respect to B_t. Indeed, by (i), $DT_t = T_tD$ and $D' = -D$ we get $B_t(D\xi, \eta) = Q(DT_t\xi, T_t\eta) = Q(T_t\xi, -DT_t\eta) = B_t(\xi, -D\eta)$.

(b) Clearly, $D^2 \in \mathcal{L}_G(\mathcal{E})$.

(c) Obviously, there are equivariant R_1 and R_2 in $\mathcal{L}_G(\mathcal{E})$ such that $R_1D + DR_2 - I \in \mathcal{K}_G(\mathcal{E})$, since D_i has the corresponding property.

(d) B_t is associated with the invertible operator $T_t \in \mathcal{L}_G(\mathcal{E})$, and hence is a strongly nondegenerate quadratic form. In fact, the inverse of T_t is

$$T_t^{-1} = \begin{bmatrix} I & 0 \\ -itR\varepsilon & I \end{bmatrix}.$$

B_t is also regular with a compatible operator $T_t^* T T_t$, where T is the operator compatible with Q, $T^2 = I$. Indeed,

$$\langle \xi, \eta \rangle_{B_t} = B_t(\xi, T_t^* T T_t \eta) = Q(T_t \xi, T_t(T_t^* T T_t)\eta) = \langle T_t^* T T_t \xi, T_t^* T T_t \eta \rangle.$$

Therefore, $(\mathcal{E}, B_t, D) \in \mathcal{L}_{G,u}^0(\mathcal{A})$ for $0 \leq t \leq 1$. In other words, $(\mathcal{E}, B_t, D) \in \mathcal{L}_{G,u}^0(C([0,1], \mathcal{A}))$.

Let us now check that $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}_{G,u}^0(\mathcal{A})$ for t near 0. Since T_t is invertible and $L_0 = T_1' T_1$, L_0 is also invertible. It follows that L_t is invertible for t near 0, say $0 \leq t \leq t_0$. By (iii), ∇_t is antisymmetric with respect to C_t and $\nabla_t^2 = 0$, since $D_1 R' = R' D_2$ and $D_i^2 = 0$. Let $S_i, \tilde{S}_i \in \mathcal{K}_G(\mathcal{E}_i)$ be such that $S_i D_i + D_i \tilde{S}_i - I \in \mathcal{K}_G(\mathcal{E}_i), S_i D_i \in \mathcal{L}_G(\mathcal{E}_i), im(\tilde{S}_i) \subset dom(D_i), D_i \tilde{S}_i \in \mathcal{L}_G(\mathcal{E}_i), i = 0, 1$. Then

$$\begin{bmatrix} S_1 & 0 \\ 0 & -S_2 \end{bmatrix}\begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix} + \begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix}\begin{bmatrix} \tilde{S}_1 & 0 \\ 0 & -\tilde{S}_2 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} S_1 D_1 + D_1 \tilde{S}_1 - I & tS_1 R' - tR'\tilde{S}_2 \\ 0 & S_2 D_2 + D_2 \tilde{S}_2 - I \end{bmatrix} \in \mathcal{K}_G(\mathcal{E}),$$

since $S_1 \in \mathcal{K}_G(\mathcal{E}_1), \tilde{S}_2 \in \mathcal{K}_G(\mathcal{E}_2)$ and $R' \in \mathcal{L}_G(\mathcal{E}_1, \mathcal{E}_2)$. Note that $\langle \xi, \eta \rangle_{C_t} \overset{\text{def}}{=} C_t(\xi, TL_t\eta) = Q(L_t\xi, TL_t\eta) = \langle L_t\xi, L_t\eta \rangle$. We have that C_t is a strongly nondegenerate and regular quadratic form. Thus we conclude that $(\mathcal{E}_t, C_t, \nabla_t)$ is in $\mathcal{L}_{G,u}^0(\mathcal{A})$ for $0 \leq t \leq t_0$.

Now $C_0(\xi, \eta) = Q(L_0\xi, \eta) = Q(T_1' T_1 \xi, \eta) = B_1(\xi, \eta)$ and $\nabla_0 = D$. We have $(\mathcal{E}, B_1, D) = (\mathcal{E}, C_0, \nabla_0)$. Hence,

$$\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}, B_0, D)) = \psi_{0,u}((\mathcal{E}, B_1, D)) = \psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})).$$

Hence it suffices to show that $\psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})) = 0$. By Lemma 6, we need only to check that $im(\nabla_{t_0}) = ker(\nabla_{t_0})$.

Obviously, $\nabla_{t_0}^2 = 0, im(\nabla_{t_0}) \subset ker(\nabla_{t_0})$. To show that $Ker(\nabla_{t_0}) \subset im(\nabla_{t_0})$, let $(\xi_1, \xi_2) \in \mathcal{E}$ be such that $\nabla_{t_0}(\xi_1, \xi_2) = 0$. By condition (c), $R' : \frac{Ker(D_2)}{im(D_2)} \to \frac{Ker(D_1)}{im(D_1)}$ is bijective. Then there exists $\xi_2' \in \mathcal{E}_2$ such that $\xi_2 = -D_2\xi_2'$, since $-t_0 R'\xi_2 = D_1\xi_1 = 0$ in $\frac{Ker(D_1)}{im(D_1)}$. This implies that $D_1\xi_1 = -t_0 R'\xi_2 = t_0 R' D_2\xi_2' = t_0 D_1 R'\xi_2'$. Hence $\xi_1 - t_0 R'\xi_2' \in Ker D_1$. The bijectivity of R' implies that there exists $[\xi_2''] \in \frac{Ker(D_2)}{im(D_2)}$ such that $t_0 R'(\xi_2'') = \xi_1 - t_0 R'\xi_2'$. Thus there exists $\xi_1' \in \mathcal{E}_1$ such that $\xi_1 - t_0 R'(\xi_2' + \xi_2'') = D_1\xi_1'$. Note that $D_1\xi_2'' = 0$. We get $\xi_1 = t_0 R'(\xi_2' + \xi_2'') + D_1\xi_1'$ and $\xi_2 = -D_2(\xi_2' + \xi_2'')$. We have thus found $(\xi_1', \xi_2''') \in \mathcal{E}$ such that $\nabla_{t_0}(\xi_1', \xi_2''') = (\xi_1, \xi_2)$. Therefore, $im(\nabla_{t_0}) = ker(\nabla_{t_0})$. $\square$

The idea of the above proof is first to transform the equality of two elements $\psi_{0,u}(\mathcal{E}_i, Q_i, D_i)$ to the vanishing of $\psi_{0,u}(\mathcal{E}, Q, D)$, where $(\mathcal{E}, Q, D)$ is the direct sum formed in the proof. Then we construct a path of $(\mathcal{E}, C_t, \nabla_t)$ in $\mathcal{L}_{G,u}^0(\mathcal{A}), 0 \leq t \leq t_0$, such that $(\mathcal{E}, C_0, \nabla_0) = (\mathcal{E}, Q, D)$ and $(\mathcal{E}, C_{t_0}, \nabla_{t_0})$ satisfies $im(\nabla_{t_0}) = Ker(\nabla_{t_0})$. By homotopy invariance of $\psi_{0,u}(\mathcal{E}, Q, D)$ and Lemma 6, we prove that $\psi_{0,u}(\mathcal{E}, Q, D) = \psi_{0,u}(\mathcal{E}, C_{t_0}, \nabla_{t_0}) = 0$.

We now use Proposition 1 to prove the G-pseudo-equivalence invariance of G-$Sig(D_E)$ for a flat G-$\mathcal{A}$-bundle E. Our goal in the rest of this section is to check the condition of Proposition 1.

Let $\mathcal{A}$ be a G-C^*-algebra, N and M two closed, oriented Riemannian G-manifolds and E a G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0. E is said to be G-unitary flat if E is furnished with an equivariant scalar product and an equivariant unitary flat connection. Let $f : N \to M$ be an equivariant smooth map. It is clear that if E is a G-unitary flat bundle, then so is $f^*(E)$. If f and f_1 are two equivariant smooth maps from N into M which are G-equivariantly homotopic via a G-homotopy H, then $H^* : f^*(E) \to f_1^*(E)$ induces an isomorphism. Suppose that ∇ is an equivariant unitary flat connection on E and $D_E \xi = i^{\partial \xi} \nabla \xi$; then $D_{f^*(E)} = D_{f_1^*(E)}$. Observe that the map $f^* : C^\infty(M, \wedge_{\mathbb{C}} T^*M \otimes E) \to C^\infty(N, \wedge_{\mathbb{C}} T^*N \otimes f^*(E))$ need not be extendable to a bounded operator from $\mathcal{E}_E = L^2(E)$ to $\mathcal{E}_{f^*(E)} = L^2(f^*(E))$, since f^* is not closed in general. To get around this problem, we follow [HiS] and use embeddings.

Let $\omega \in \wedge_{\mathbb{C}} T^*N$. Define $e_\omega : \wedge_{\mathbb{C}} T^*N \to \wedge_{\mathbb{C}} T^*N$ by $e_\omega(\alpha) = \alpha \wedge \omega$.

Lemma 10 ([HiS]). *Let M and N be closed, oriented Riemannian G-manifolds.*

(a) *Suppose $h : N \to M$ is a smooth equivariant map and $\omega \in \wedge_{\mathbb{C}} T^*N$ is equivariant. Let N_0 be the support of ω in N and h be a G-submersion from a neighborhood of N_0 into M. Then $e_\omega h^* : \mathcal{E}_E \to \mathcal{E}_{h^*(E)}$ is an equivariant operator in $\mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$, where E is a G-$\mathcal{A}$-unitary flat bundle over M, and the norm of $e_\omega h^*$ is independent of E.*

(b) *Assume $h : N \times [0,1] \to M$ is a smooth equivariant map and $\omega \in \wedge_{\mathbb{C}} T^*N$ is a closed equivariant form on N, where G acts trivially on $[0,1]$. Suppose h (resp., h_0, h_1) is a G-submersion from a neighborhood of $N_0 \times [0,1]$ (resp., N_0) into M. If ∇ is an equivariant unitary flat connection on the G-$\mathcal{A}$-unitary flat bundle E over M, then there exists an equivariant operator $S \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$ such that*

$$e_\omega h_1^* - e_\omega h_0^* = h_0^*(\nabla)S + S\nabla.$$

Proof. (a) Since $e_\omega h^*$ is equivariant, it suffices to show the boundedness of $e_\omega h^*$. The proof is the same as that of Lemma 3.2 in [HiS]. We provide more details below.

Using a partition of unity, we see that the assertion is a local statement. Thus let $N = O \times M$ and E be trivial. $h : N \to M$ is then a projection. Let $\alpha dy_I \otimes \xi \in \wedge_{\mathbb{C}} T^*M \otimes E_0$. We have

$$h^*(\alpha dy_I \otimes \xi)_{(x,y)} = \alpha(h(x,y))dh_I \otimes h^*(\xi)_{(x,y)} = \alpha(y)dy_I \otimes \xi,$$
$$e_\omega h^*(\alpha dy_I \xi)_{(x,y)} = \alpha(y)dy_I \wedge \omega(x,y) \otimes \xi.$$

Moreover,

$$\|e_\omega h^*(\alpha dy_I \otimes \xi)\|^2 = \| \int_N \overline{(\alpha(y)dy_I \wedge \omega(x,y))} \wedge *(\alpha(y)dy_I \wedge \omega(x,y)) \otimes \langle \xi, \xi \rangle \|^2$$
$$= \|\alpha dy_I \wedge \omega\|^2 \|\langle \xi, \xi \rangle\|^2 \le \|\omega\|^2 \|\alpha dy_I \otimes \xi\|^2.$$

Hence, $\|e_\omega h^*\| \le \|\omega\|$.

(b) Recall that the contraction $i_{\frac{\partial}{\partial t}}$ is defined by

$$i_{\frac{\partial}{\partial t}}(\alpha dx_I \wedge dt) = (-1)^{|I|} \alpha dx_I.$$

$i_{\frac{\partial}{\partial t}}$ is equivariant since G acts trivially on $[0,1]$. Define, for $\omega_1 \in \mathcal{E}_E$,

$$S(\omega_1) = \int_0^1 i_{\frac{\partial}{\partial t}}(e_\omega h^*(\omega_1))dt \in L^2(N, \wedge_{\mathbb{C}} T^* N \otimes h_0^*(E)).$$

Here ω is considered as a smooth form on $N \times [0,1]$. By part (a), $e_\omega h^*$ is bounded, and $i_{\frac{\partial}{\partial t}}$ is also bounded. S is thus bounded. Clearly, S is equivariant.

Since $e_\omega h_1^* - e_\omega h_0^* = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*)dt$, it is enough to show locally that

$$h_0^*(\nabla)S + S\nabla = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*)dt.$$

Let $\omega_1 = \sum_I \alpha_I dy_I \otimes \xi$, $dy_I = dy_{i_1} \wedge \ldots \wedge dy_{i_I}$. We have

$$e_\omega h^*(\omega_1) = h^*(\omega_1) \wedge \omega = h^*(\omega_1) \wedge \omega(z, y, t)$$
$$= \sum_I \alpha_I(h(z, y, t)) \wedge_{i \in I} (\sum_k h_{i, z_k} dz_k + \sum_j h_{i, y_j} dy_j + h_{i, t} dt) \wedge \omega(z, y, t) \otimes \xi,$$

where $h : O \times M \times [0,1] \to M$, z and y are the variables of O and M, resp., and h_{i, z_k}, h_{i, y_j} and $h_{i, t}$ are the derivatives of h_i with respect to z_k, y_j and t, respectively. Then

$$\frac{\partial}{\partial t}(e_\omega h^*(\omega_1))dt = \frac{\partial}{\partial t}[\sum_I \alpha_I(h) \wedge_{i \in I} (\sum_k h_{i, z_k} dz_k + \sum_j h_{i, y_j} dy_j) \wedge \omega]dt \otimes \xi$$

(2)

$$= \sum_I \{\sum_s \alpha_{I, \alpha}(h) h_{s, t} \wedge_{i \in I} (\sum_k h_{i, z_k} dz_k + \sum_j h_{i, y_j} dy_j) \wedge \omega \wedge dt \otimes \xi$$
$$+ \alpha_I(h) \wedge_{l=1}^I (-1)^{l-1}(\sum_k h_{i_l, z_k, t} dz_k + \sum_j h_{i_l, y_j, t} dy_j)$$
$$\times \wedge_{i \in I, i \neq i_l} (\sum_k h_{i, z_k} dz_k + \sum_j h_{i, y_j} dy_j) \wedge \omega dt \otimes \xi\}.$$

Since ∇ is flat, we can assume locally that $\nabla = d \otimes p$, where p is locally a constant projection. We have

$$\nabla(\omega_1) = d(\sum_I \alpha_I dy_I) \otimes \xi = \sum_{I, s} \alpha_{I, s} dy_s \wedge dy_I \otimes \xi,$$

$$h^* \nabla(\omega_1) = \sum_{I, s} \alpha_{I, s}(h(z, y, t))dh_s \wedge_{i \in I} dh_i \otimes \xi,$$

$$e_\omega h^* \nabla(\omega_1) = \sum_I [\sum_s \alpha_{I, s}(h(z, y, t))(\sum_k h_{s, z_k} dz_k + \sum_j h_{s, y_j} dy_j + h_{s, t} dt)$$
$$\times \wedge_{i \in I}(\sum_k h_{i, z_k} dz_k + \sum_j h_{i, y_j} dy_j + h_{i, t} dt) \wedge \omega(z, y, t) \otimes \xi],$$

$$\int_0^1 [i_{\frac{\partial}{\partial t}}(e_\omega h^* \nabla(\omega_1))]dt$$

(3)
$$= \int_0^1 \sum_I \sum_s \{\sum_s \alpha_{I,s}(h)[-(\sum_k h_{s,z_k}dz_k + \sum_j h_{s,y_j}dy_j)$$

$$\wedge_{l=1}^I (-1)^{l-1} \wedge_{i\in I, i\neq i_l} (\sum_k h_{i,z_k}h_{i_l,t}dz_k + \sum_j h_{i,y_j}h_{i_l,t}dy_j)$$

$$+ \wedge_{i\in I}(\sum_k h_{i,z_k}h_{s,t}dz_k + \sum_j h_{i,y_j}h_{s,t}dy_j)]\} \wedge \omega \wedge dt \otimes \xi.$$

Now since $d\omega = 0$,

$$h_0^*(\nabla)(\int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega h^*(\omega_1)))dt)$$

$$= \int_0^1 \nabla(\sum_I \alpha_I(h) \wedge_{l=1}^I (-1)^{l-1} \wedge_{i\in I, i\neq i_l} (\sum_k h_{i,z_k}h_{i_l,t}dz_k$$

$$+ \sum_j h_{i,y_j}h_{i_l,t}dy_j) \wedge \omega \otimes \xi)dt$$

$$= \int_0^1 \{\sum_I (\sum_s \alpha_{I,s}(h)dh_s \wedge_{l=1}^I (-1)^{l-1}$$

(4)

$$\times \wedge_{i\in I, i\neq i_l}(\sum_k h_{i,z_k}h_{i_l,t}dz_k + \sum_j h_{i,y_j}h_{i_l,t}dy_j) \wedge \omega$$

$$+ \alpha_I(h) \wedge_{l=1}^I (-1)^{l-1}(\sum_k h_{i_l,t,z_k}dz_k + \sum_j h_{i_l,t,y_j}dy_j)$$

$$\times \wedge_{i\in I, i\neq i_l}(\sum_k h_{i,z_k}dz_k + \sum_j h_{i,y_j}dy_j) \wedge \omega)\}dt \otimes \xi.$$

Adding (3) and (4) together and using (2), we get

$$(h_0^*(\nabla)S + S\nabla)(\omega_1) = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*(\omega_1))dt.$$

$\square$

Let $h_{!,\omega}$ be the conjugate of $e_\omega h^*$ with respect to the quadratic form Q, where $h : N \to M$ is a submersion in a neighborhood of the support of ω. Locally, we can find a formula for $h_{!,\omega}$. Let $h : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ be the projection. If ω is a continuous function with compact support in $\mathbb{R}^m \times \mathbb{R}^k$, then for $\alpha dx_I \otimes \xi \in \wedge_{\mathbb{C}}T^*\mathbb{R}^m$ we have $e_\omega h^*(\alpha dx_I \otimes \xi) = \alpha(x)dx_I \wedge \omega \otimes \xi$. For $|J_1| = k$ we can calculate

$$Q(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta, e_\omega h^*(\alpha dx_I \otimes \xi))$$

$$= i^{(m-|I_1|)k}Q(h_{!,\omega}(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta), \alpha dx_I \otimes \xi),$$

with

$$h_{!,\omega}(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta) = i^{(m-|I_1|)|J_1|}dx_{I_1} \int \bar{\omega}(x,y)\beta(x,y)dy_{J_1} \otimes \eta,$$

and $h_{!,\omega} = 0$ for $|J_1| \neq k$.

Let G act on $\mathbb{R}^k$ by isometries, and let B^k be the equivariant unit open ball in $\mathbb{R}^k$. Suppose that $P : N \times B^k \to M$ is an equivariant submersion and ν_k is an equivariant volume form of mass 1 on B^k. Let $q : N \times B^k \to N$ and $\pi_k : N \times B^k \to B^k$ be the projections. $\omega = \pi_k^*(\nu_k)$ and $P_0 = P|_{N \times \{0\}}$. Define

$$R_{p,\nu_k} = q_! e_\omega P^* \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{P_0^*(E)}),$$

where $q_! : \mathcal{E}_{P^*(E)} \to \mathcal{E}_{P_0^*(E)}$ is the pushforward. The following lemma is crucial in verifying the condition of Proposition 1.

Lemma 11. (a) Let $P' : N \times B^l \to M$ be another equivariant submersion and ν_l be an equivariant volume form of mass 1 on B^l. If the maps $P(x, o)$ and $P'(x, o)$ are equivariantly homotopic, then there exists an $S \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{P_0^*(E)})$ such that

$$D_E S + S D_E = R_{p,\nu_k} - R_{p'\nu_l}.$$

(b) Let $P' : O \times B^l \to N$ be an equivariant submersion and ν_l an equivariant volume form of mass 1 on B^l, where O is a compact oriented Riemannian G-manifold. If $P'' : O \times B^l \times B^k \to M$ is given by $P''(z, s, t) = P(P'(z, s), t)$ and $\nu = \nu_l \times \nu_k$, then $R_{p',\nu_l} R_{p,\nu_k} = R_{p'',\nu}$.

Proof. Since the volume forms ν_k, ν_l and ν have compact supports, there exists an equivariant $\varphi \in C_c^\infty(B^k)$ such that $\varphi \nu_k = \nu_k$. As a consequence, $R_{p,\nu_k} = q_{!,\varphi} e_\omega P^*$.

(a) Let $\tilde{P} : N \times B^k \times B^l \times B^r \times [0, 1] \to M$ be an equivariant submersion such that $\tilde{P}(x, s, t, u, 0) = P(x, s)$ and $\tilde{P}(x, s, t, u, 1) = P'(x, t)$. This $\tilde{P}$ exists since P and P' are submersions and homotopic. Let ν_r be a volume form of mass 1 on B^r with compact support. Define $\tilde{P}_i : N \times B^k \times B^l \times B^r \to M$ by $\tilde{P}_i(z) = \tilde{P}(z, i), z \in N \times B^k \times B^l \times B^r, i = 0, 1$. Then $R_{p,\nu_k} = R_{\tilde{p}_0, \nu_k \times \nu_l \times \nu_r} = q_!^{k,l,r} e_{\tilde{\omega}} \tilde{P}_0^*$ and $R_{p',\nu_l} = R_{\tilde{p}_1, \nu_k \times \nu_l \times \nu_r} = q_!^{k,l,r} e_{\tilde{\omega}} \tilde{P}_1^*$, where $q^{k,l,r} : N \times B^k \times B^l \times B^r \to N, \pi_{k,l,r} : N \times B^k \times B^l \times B^r \to B^k \times B^l \times B^r$ are projections and $\tilde{\omega} = (\pi_{k,l,r})^*(\nu_k \times \nu_l \times \nu_r)$. By Lemma 10, there is an equivariant $S_1 \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{\tilde{P}_0^*(E)})$ such that

$$q_!^{k,l,r}(e_{\tilde{\omega}} \tilde{P}_1^* - e_{\tilde{\omega}} \tilde{P}_0^*) = q_!^{k,l,r}(\tilde{P}_0^*(D) S_1 + S_1 D) = D(q_!^{k,l,r} S_1) + (q_!^{k,l,r} S_1) D.$$

Hence, $R_{p',\nu_l} - R_{p,\nu_k} = DS + SD$, $S = q_!^{k,l,r} S_1$.

(b) Let $\bar{P} : O \times B^k \times B^l \to N \times B^k, \bar{q} : O \times B^k \times B^l \to O \times B^l$ and $q' : O \times B^l \to O$ be defined by

$$\bar{P}(z, s, t) = (P'(z, t), s), \quad \bar{q}(z, s, t) = (z, t), \quad q'(z, t) = z.$$

Let $\pi_{k,l} : O \times B^k \times B^l \to B^k \times B^l$ and $\pi_l : O \times B^k \times B^l \to B^l$ be the projections and $\tilde{\omega}_1 = (\pi_{k,l})^*(\nu_k \times \nu_l), \omega_1 = (\pi_l)^*(\nu_l)$. Then for the projection $q^0 : O \times B^k \times B^l \to O$,

$$R_{p'',\nu} = q_!^0 e_{\tilde{\omega}_1}(P'')^* = q_!' \bar{q}_! e_{\omega_1} \bar{P}^* e_\omega P^*.$$

This identity is illustrated by the following diagrams:

$$q'_!\bar{q}_*(\overline{P}^*P^*(E)) \xleftarrow{\;q'_!\;} \bar{q}_*(\overline{P}^*P^*(E)) \xleftarrow{\;\bar{q}_!e_{\omega_1}\;} \overline{P}^*P^*(E) \xleftarrow{\;\overline{P}^*e_\omega\;} P^*(E) \xleftarrow{\;P^*\;} E$$

with vertical maps down to

$$O \xleftarrow{\;q'\;} O \times B^l \xleftarrow{\;\bar{q}\;} O \times B^k \times B^l \xrightarrow{\;\overline{P}\;} N \times B^k \xrightarrow{\;P\;} M.$$

Here the following identities were used:

$$P''(z,s,t) = P(P'(z,s),t) = P\bar{P}(z,s,t),$$

$$q^0 = q'\bar{q} : O \times B^k \times B^l \xrightarrow{\bar{q}} O \times B^l \xrightarrow{q'} O,$$

$$\tilde{\omega}_1 = (\pi_l)^*(\nu_l) \wedge (\tilde{\pi})^*(\pi_k^*(\nu_k)) = \omega_1 \wedge (\tilde{\pi})^*(\omega),$$

$$\tilde{\pi} : O \times B^k \times B^l \to O \times B^k, \ \pi_k : O \times B^k \to B^k, \ e_{(\tilde{\pi})^*(\omega)}\bar{P}^* = \bar{P}^*e_\omega.$$

For the purpose of illustration, we also used an unusual arrow notation for the pullback of vector bundles. On the other hand, $\bar{q}_!e_{\omega_1}\bar{P}^* = e_{\omega'_1}\bar{q}_!\bar{P}^* = e_{\omega'_1}(P')^*q_!$ with $\omega'_1 = (\pi^l)^*(\nu_l)$, and $\omega_1 = (\pi_l)^*(\nu_l)$, where $\pi^l : O \times B^l \to B^l$ and $\pi_l : O \times B^k \times B^l \to B^l$ are the projections. This can be seen from the following diagrams:

$$\bar{q}_*(\overline{P}^*P^*(E)) \xleftarrow{\;\bar{q}_!e_{\omega_1}\;} \overline{P}^*P^*(E) \xleftarrow{\;\overline{P}^*\;} P^*(E)$$

with vertical maps down to

$$O \times B^l \xleftarrow{\;\bar{q}\;} O \times B^k \times B^l \xrightarrow{\;\overline{P}\;} N \times B^k,$$

$$P'^*q_*P^*(E) \xleftarrow{\;e_{\omega'_1}\;} (P')^*q_*P^*(E) \xleftarrow{\;(P')^*\;} q_*P^*(E) \xleftarrow{\;q_!\;} P^*(E)$$

with maps down to

$$O \times B^l \xrightarrow{\;P'\;} N \xleftarrow{\;q\;} N \times B^k,$$

where, as before, $\bar{P}(z,s,t) = (P'(z,t),s)$ and $\bar{q}(z,s,t) = (z,t)$, and $P'(z,t) = z$. Therefore,

$$R_{p'',\nu} = q'_!\bar{q}_!e_{\omega_1}\bar{P}^*e_\omega P^* = q'_!e_{\omega'_1}(P')^*q_!e_\omega P^*$$
$$= R_{p',\nu_l}R_{p,\nu_k}.$$

$\square$

The geometric idea of the first part of the above proof is to throw two small spaces $N \times B^l$ and $N \times B^k$ into a large space $N \times B^l \times B^k$, and then to use homotopies between submersions in the large space to compare the maps R_{p,ν_k} and R_{p',ν_l}. Similarly, we used the factors B^k and B^l in the second part of the above proof to break the submersions on the large spaces into submersions on the small spaces. An essential point is that two factors B^l and B^k in $B^l \times B^k$ enable us to move around the submersions.

We can now prove Theorem 1 stated in the Introduction.

Proof of Theorem 1. We need only check that $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$ satisfy the condition of Proposition 1. By the observation preceding Lemma 9, we can assume that h is smooth. Let $J : M \to \mathbb{R}^k$ be an equivariant embedding $(k \equiv 0(4))$, O an equivariant tubular neighborhood of $J(M)$ in $\mathbb{R}^k$ and $\pi : O \to M$ the associated equivariant projection (cf. [Bre]). Suppose $J(M) + B^k \subset O$. Define a submersion $P : N \times B^k \to M$ by $P(x,t) = \pi(h(x) + t), x \in N, t \in B^k$. Let ν_k be

an equivariant volume form of mass 1 on B^k and φ be an equivariant smooth map such that $\varphi\nu_k = \nu_k$. With the notation as in the paragraph preceding Lemma 11, we divide the proof into three steps.

Step 1. Condition (a) of Proposition 1 is valid already since $D^2 = 0$ by the flatness. To check condition (b) of Proposition 1, we define R to be $R_{p,\nu_k} \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$, i.e., $R = q_{!,\varphi} e_\omega P^*$. Note that $P_0(x) = P(x,o) = \pi(h(x)) = h(x)$. Clearly, $q_{!,\varphi}, e_\omega$ and P^* preserve the domain of D and commute with D. In fact, since P is equivariantly homotopic to hq,

$$
\begin{aligned}
(e_\varphi q^*)' e_\omega P^* \nabla &= (e_\varphi q^*)' e_\omega P^* (\nabla) P^* = (e_\varphi q^*)' P^* (\nabla) e_\omega P^* \\
&= h^*(\nabla)(e_\varphi q^*)' e_\omega P^*.
\end{aligned}
$$

Hence, $R(dom(D)) \subset dom(D)$ and $RD_E = D_{h^*(E)}R$.

Let $\varepsilon \in \mathcal{L}_G(\mathcal{E}_E)$ be defined by $\varepsilon(\xi) = (-1)^{\partial\xi}\xi$. Then $\varepsilon^2 = I, \varepsilon' = \varepsilon, \varepsilon D = -D\varepsilon, \varepsilon P^* = P^*\varepsilon, \varepsilon e_\omega = (-1)^k e_\omega\varepsilon$ and $\varepsilon(e_\varphi h)' = (-1)^k(e_\varphi h^*)'\varepsilon$. Hence we get $\varepsilon(I - R'R) = (I - R'R)\varepsilon$. This verifies condition (e) of Proposition 1.

Step 2. We use Lemma 11 to verify condition (c) of Proposition 1, i.e., $R : \frac{Ker(D_E)}{im(D_E)} \to \frac{Ker(D_{h^*(E)})}{im(D_{h^*(E)})}$ is an isomorphism. By Lemma 11(a), the map $R = R_{p,\nu}$ from $\frac{Ker(D_E)}{im(D_E)}$ to $\frac{Ker(D_{h^*(E)})}{im(D_{h^*(E)})}$ is independent of the submersion P and the volume form ν, i.e., $R_{p,\nu} = R_{p',\nu'}$ on $\frac{Ker(D_E)}{im(D_E)}$ as long as $P'(*,o)$ is equivariantly homotopic to $P(*,o)$. Let $h' : M \to N$ be the smooth map such that hh' and $h'h$ are (non-equivariantly) homotopic to I_M and I_N, resp. As we construct the submersion $P : N \times B^k \to M$ from h, we can find a submersion $P' : M \times B^l \to N$ from h', namely, $P'(x,t) = \pi'(h'(x) + t)$ and $\pi' : O' \to N$ is a projection with O' a tubular neighborhood of $J'(N) \subset \mathbb{R}^l$ for an equivariant embedding J' of N into $\mathbb{R}^l$. Then define $P'' : M \times B^l \times B^k \xrightarrow{P'} N \times B^k \xrightarrow{P} M$ by $P''(x,s,t) = P(P'(x,s),t)$. Clearly, $P''(x,o,o) = P(h'(x),o) = \pi(h(h'(x)) + o) = h(h'(x)) \xrightarrow{\text{homotopic}} I_M(x)$. Hence $R_{p'',\nu''} = I$ on $\frac{Ker(D_E)}{im(D_E)}$ by a non-equivariant version of Lemma 11(a). Furthermore, using a non-equivariant counterpart of Lemma 11(b), we obtain $R_{p',\nu'}R_{p,\nu} = R_{p'',\nu''} = I$ on $\frac{Ker(D_E)}{im(D_E)}$. This proves that $R_{p,\nu}$ is injective. The same reasoning shows, switching the role of $R_{p',\nu'}$ and $R_{p,\nu}$, that $R_{p,\nu}$ is surjective in $\frac{Ker(D_E)}{im(D_E)}$. Therefore, $R = R_{p,\nu}$ is an isomorphism. Note that this step does not require the equivariance of the above maps.

Step 3. We now check condition (d) of Proposition 1, i.e., there is an equivariant $S \in \mathcal{L}_G(\mathcal{E}_E)$ such that $S(dom(D_E)) \subset dom(D_E)$ and $I - R'R = D_E S + SD_E$.

Let $q_i : N \times B^k \times B^k \to N \times B^k$ be the projections, $q_i(x,t_1,t_2) = (x,t_i), i = 1, 2$. Also let $q : N \times B^k \to N$ and $\pi_k : N \times B^k \to B^k$ be the projections, and let $\omega = \pi_k^*(\nu_k)$. To find the conjugate of $e_{q_1^*(\omega)}q_2^*$, for $\tilde\eta = \alpha dx_I \wedge dt_{J_2} \otimes \eta$ and $\tilde\xi = \beta dx_{I_1} \wedge dt_{J_1} \wedge dt_{J_2'} \otimes \xi$ we compute

$$
Q(\tilde\xi, (e_{q_1^*(\omega)}q_2^*)(\tilde\eta)) = i^{k(|I_1|+|J_2'|)}Q((e_{q_1^*(\omega)}q_2^*)'(\tilde\xi), \tilde\eta)
$$

with

$$
(e_{q_1^*(\omega)}q_2^*)'(\tilde\xi) = i^{k(|I_1|+|J_2'|)}\left(\int_{B^k} \beta\overline{q_1^*(\omega)}\right)dx_{I_1} \wedge dt_{J_2'}
$$

for $|J_1| = 0$, and 0 otherwise. In particular, if $\tilde{\xi} = e_{q_2^*(\omega)}q_1^*(\tilde{\eta}) = \alpha(q_1)dx_{I_1} \wedge dt_{J_1} \wedge q_2^*(\omega)$, then

$$(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_1^*)(\alpha dx_I \wedge dt_{J_1} \otimes \eta)$$

(5)
$$= i^{k(|I_1|+k)} \int_{B^k} \alpha(q_1)\overline{q_1^*(\omega)}dx_{I_1} \wedge q_2^*(\omega) \otimes \eta$$

$$= i^{k|I_1|+k^2}(\int_{B^k} \alpha(q_1)\overline{q_1^*(\omega)})dx_{I_1} \wedge q_2^*(\omega) \otimes \eta, \quad |J_1| = 0,$$

and it is zero for $|J_1| \neq 0$. On the other hand,

$$(q_!e_\omega)(\alpha dx_{I_1} \wedge dt_{J_1}) \otimes \eta = q_!(\alpha dx_{I_1} \wedge dt_{J_1} \wedge \omega \otimes \eta)$$
$$= i^{k(n-|I_1|)}(\int_{B^k} \varphi \alpha \omega)dx_{I_1} \otimes \eta.$$

If $\tilde{\xi} = \beta dx_{I_1'} \otimes \xi$, then

$$Q(\tilde{\xi}, q_!e_\omega(\alpha dx_{I_1} \wedge dt_{J_1}))$$

$$= i^{-kn+2k|I_1'|}i^{(|I_1'|+k)(n+k-|I_1'|-k)} \int_{N \times B^k} (\bar{\beta}\varphi\omega \wedge dx_{I_1'}) \wedge \alpha dx_{I_1} \otimes \langle \xi, \eta \rangle$$

$$= i^{-kn+2k|I_1'|}Q((q_!e_\omega)'(\tilde{\xi}), \tilde{\eta}).$$

Consequently, $(q_!e_\omega)'(\tilde{\xi}) = i^{-kn+2k|I_1'|}\beta\omega \wedge dx_{I_1'} \otimes \xi$. Using this identity, we have

$$(q_!e_\omega)'(q_!e_\omega)(\alpha dx_{I_1} \otimes \eta) = i^{-kn+2k|I_1'|}i^{k(n-|I_1|)}(\int_{B^k} \alpha\omega)\omega \wedge dx_{I_1} \otimes \eta$$
$$= i^{-k|I_1|}(\int_{B^k} \alpha\omega)dx_{I_1} \wedge \omega \otimes \eta.$$

Comparing this with (6), we get

(6) $(q_!e_\omega)'(q_!e_\omega)(\alpha dx_{I_1} \otimes \eta) = i^{-2k|I_1|-k^2}(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_1^*)(\alpha dx_{I_1} \otimes \eta).$

Furthermore,

$$(e_{q_2^*(\omega)}q_2^*)(\alpha dx_I \wedge dt_{J_2} \otimes \eta) = \alpha(q_2)dx_I \wedge dt_{J_2} \wedge q_2^*(\omega) \otimes \eta$$
$$= \alpha dx_I \wedge q_2^*(\omega) \otimes \eta, |J_2| = 0,$$

and

$$(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_2^*)(\tilde{\eta}) = i^{k(|I|+k)}(\int_{B^k} \alpha\overline{q_1^*(\omega)})dx_I \wedge q_2^*(\omega) \otimes \eta$$

$$= i^{k^2+k|I|}(\int_{B^k} \overline{q_1^*(\omega)})\alpha dx_I \wedge q_2^*(\omega) \otimes \eta.$$

Clearly, $e_\omega(\alpha dx_I \otimes \eta) = \alpha dx_I \wedge \omega \otimes \eta$. Therefore,

$$e_\omega = i^{-k^2-k|I|}(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_2^*).$$

This together with (6) proves that $e_\omega = i^{k|I|}(q_!e_\omega)'(q_!e_\omega)$ modulo the boundary $D_E S + S D_E$ by Lemma 10. This implies that

$$R'R = (P^*)'(q_!e_\omega)'(q_!e_\omega)P^* = P_!(q_!e_\omega)'(q_!e_\omega)P^* = P_!e_\omega P^*$$

modulo the boundary $D_E S + S D_E$. Hence it suffices to show that $P_!e_\omega P^* = I$ modulo the boundary $D_E S + S D_E$.

Let $P_1 : M \times B^k \to M, P_2 : M \times B^k \to B^k$ be the projections and $\tilde{\pi} : M \times B^k \to M$ be the equivariant submersion given by $\tilde{\pi}(x, t) = \pi(x + t)$ and $\omega' = P_2^*(\nu_k)$. Let $P : N \times B^k \to M$ be the equivariant submersion, $P(x, t) = \pi(h(x) + t)$.

(a) $(P_1)_! e_{\omega'} P_1^*$ is the identity:

$$
\begin{aligned}
(P_1)_! e_{\omega'} P_1^*(\alpha dx_I \otimes \eta) &= (P_1)_! e_{\omega'} \alpha(P_1) dx_I \otimes \eta = (P_1)_{!,\varphi}(\alpha(P_1) dx_I \wedge \omega' \otimes \eta) \\
&= i^{(n-|I|)k}\left(\int \varphi \omega'\right) \alpha dx_I \otimes \eta = \alpha dx_I \otimes \eta, k \equiv 0(4).
\end{aligned}
$$

(b) $P_! e_\omega P^* = \tilde{\pi}_! e_{\omega'} \tilde{\pi}^*$:

$$
\begin{aligned}
P_! e_\omega P^*(\alpha dx_I \otimes \eta) &= P_!(\alpha(P) dP_I \wedge \omega \otimes \eta) \\
&= i^{(n-|I|)k} \int_{N \times B^k} \alpha(P) \omega dP_I \otimes \eta \\
&= i^{k(n-|I|)} \int_{N \times B^k} \alpha(\pi(h(x) + t))) \nu_k(\pi_k(x, t)) d(\pi(h(x) + t))_I \otimes \eta \\
&= i^{k(n-|I|)} \int_{M \times B^k} \alpha(\pi(y + t)) \nu_k(t) d\pi_I \otimes \eta \\
&= \tilde{\pi}_!(\alpha(\tilde{\pi}) d\pi_I \omega' \otimes \eta) = (\tilde{\pi}_! e_{\omega'} \tilde{\pi}^*)(\alpha dx_I \otimes \eta).
\end{aligned}
$$

Finally, $\tilde{\pi}_! e_{\omega'} \tilde{\pi}^* = (P_1)_! e_{\omega'} P_1^*$ modulo the boundary $D_E S + S D_E$ by Lemma 11. Therefore, $R'R = I$ modulo the boundary $D_E S + S D_E$.   □

Theorem 1 will play a crucial role in proving the equivariant Novikov conjecture for groups acting on Euclidean buildings. It provides a substitute for the equivariant Mishchenko symmetric signature, as we pointed out earlier.

4. Equivariant Connes-Gromov-Moscovici Theorem

Let $\mathcal{A}$ be a G-C^*-algebra, E_0 a finitely generated projective Hilbert G-$\mathcal{A}$-module, and M a closed, oriented Riemannian G-manifold. Let E be a G-$\mathcal{A}$-bundle over M whose fiber is E_0. Suppose ∇ is a unitary connection of E and $\Theta = \nabla^2$ is its curvature. Let $\tau > 0$. ∇ is said to be τ-flat if

$$
\|\Theta\| = \max_{x \in M}\{\|\Theta_x\|\} < \tau,
$$

where $\|\Theta_x\|$ is the norm of the operator Θ_x on $\wedge_{\mathbb{C}}^2 T_x M \otimes E_x$,

$$
\|\Theta_x\| = \max_{X, Y \in T_x M, \|X\|, \|Y\| \leq 1} \|\Theta_x(X, Y)\|.
$$

The norm of $\|\Theta\|$ is equal to the norm of Θ as an operator on the Hilbert G-$\mathcal{A}$-module $\mathcal{E}_E$. E is called almost flat if for every $\tau > 0$ there is a τ-flat unitary connection on E. An example of τ-flat bundles will be given at the end of this section.

To prove our main theorem of this section, we need a proposition which is the generalization of Proposition 1. We first have the following lemma.

Lemma 12. (a) Let $\tau, k > 0$ be such that $6\sqrt{2}\tau k^2 < 1$. Suppose that U is an equivariant regular operator on a Hilbert G-$\mathcal{A}$-module $\mathcal{E}$ such that $im(U) \subset dom(U)$ and $\|U^2\| < \tau^2$. If there are equivariant $R, S \in \mathcal{L}_G(\mathcal{E})$, S invertible, such that

$R(dom(U)) \subset dom(U), RU + UR = S, \|R\| \le k$ and $\|S^{-1}\| \le k$, then $U + U^*$ has a bounded inverse with the norm

$$\|(U + U^*)^{-1}\| \le \frac{2k^2}{1 - 6\sqrt{2}\tau k^2}.$$

(b) Let $\tau, k > 0$ be such that $2\sqrt{2}\tau k^4(k + 3\sqrt{k}) < 1$. Suppose $(\mathcal{E}, Q, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$, and $T \in \mathcal{L}_G(\mathcal{E})$ is invertible and equivariant such that $Q(\xi, \eta) = \langle \xi, T\eta \rangle$ for the scalar product $\langle ., . \rangle$ on $\mathcal{E}$. If $\|D^2\| \le \tau^2, \|D' + D\| \le \tau$ and there are $R, S \in \mathcal{L}_G(\mathcal{E})$ equivariant, with S invertible, such that $R(dom(D)) \subset dom(D)$ and $DR \in \mathcal{L}_G(\mathcal{E})$, $DR + RD = S, \|R\| \le k, \|S^{-1}\| \le k$ and $\|T\|\|T^{-1}\| \le k^2$, then $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$.

Proof. We first prove part (a). Let

$$U_1 = \begin{bmatrix} U & U^2/\tau \\ -\tau & -U \end{bmatrix}, \qquad R_1 = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}, \qquad S_1 = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}.$$

Then $U_1^2 = 0$, and

$$R_1 U_1 + U_1 R_1 - S_1 = \begin{bmatrix} 0 & (RU^2 - U^2 R)/\tau \\ 0 & 0 \end{bmatrix}.$$

Since

$$\|R_1 U_1 + U_1 R_1 - S_1\| = \|(RU^2 - U^2 R)/\tau\| \le \frac{2}{\tau}\|R\|\|U^2\| \le 2k\tau < 1,$$

and

$$\|S_1^{-1}(R_1 U_1 + U_1 R_1 - S_1)\| \le \|S_1^{-1}\|\|(R_1 U_1 + U_1 R_1 - S_1)\|$$
$$\le \sqrt{2}k(2k\tau) = 2\sqrt{2}k^2\tau < 1,$$

it follows that $R_1 U_1 + U_1 R_1$ has the inverse

$$(R_1 U_1 + U_1 R_1)^{-1} = (I + S_1^{-1}(R_1 U_1 + U_1 R_1 - S_1))^{-1}S_1^{-1},$$

and

$$\|(R_1 U_1 + U_1 R_1)^{-1}\| \le \|(I + S_1^{-1}(R_1 U_1 + U_1 R_1 - S_1))^{-1}S_1^{-1}\| \le \frac{1}{1 - 2\sqrt{2}k^2\tau}\sqrt{2}k.$$

Now
$$U_1^* = \begin{bmatrix} U^* & -\tau \\ U^{2*}/\tau & -U^* \end{bmatrix}, \quad U_1 + U_1^* = \begin{bmatrix} U + U^* & U^2/\tau - \tau \\ -\tau + U^*/\tau & -(U + U^*) \end{bmatrix}.$$

We show that $U_1 + U_1^*$ is invertible and $\|(U_1 + U_1^*)^{-1}\| \le \frac{2k^2}{1 - 2\sqrt{2}k^2\tau}$. In fact, U_1 commutes with $U_1 R_1 + R_1 U_1$ since $U_1^2 = 0$. Let $W = (R_1 U_1 + U_1 R_1)^{-1}$. Then U_1 also commutes with W, and $U_1 W R_1 + W R_1 U_1 = W(R_1 U_1 + U_1 R_1) = I$. Since

$$U_1 R_1 = (R_1 U_1 + U_1 R_1)W U_1 R_1 = U_1 R_1 W U_1 R_1$$

and

$$R_1 U_1 = R_1 U_1 W(R_1 U_1 + U_1 R_1) = R_1 U_1 W R_1 U_1,$$

it follows that $W U_1 R_1 = W U_1 R_1 W U_1 R_1$ and $W R_1 U_1 = W R_1 U_1 W R_1 U_1$, i.e., $W U_1 R_1$ and $W R_1 U_1$ are idempotents. Thus, we can find an equivariant projection P such that $P(W U_1 R_1) = P$ and $(W U_1 R_1)P = W U_1 R_1$. By Remark 2, $(U_1 + U_1^*)$ has the inverse $(U_1 + U_1^*)^{-1} = P(W R_1)(I - P) + (I - P)(W R_1)^* P$. Hence,

$$\|(U_1 + U_1^*)^{-1}\| \le \|W R_1\| \le \frac{\sqrt{2}k}{1 - 2\sqrt{2}k^2\tau}\sqrt{2}k = \frac{2k^2}{1 - 2\sqrt{2}k^2\tau} \stackrel{\text{def}}{=} k_1.$$

From this estimate, setting

$$Z \stackrel{\mathrm{def}}{=} (U \oplus (-U)) + (U \oplus (-U))^* - (U_1 + U_1^*) = \begin{bmatrix} 0 & \tau - U^2/\tau \\ \tau - U^2/\tau & 0 \end{bmatrix},$$

we get $\|Z\| \leq \sqrt{2}\|\tau - U^2/\tau\| \leq 2\sqrt{2}\tau$ and

$$\begin{aligned} ((U \oplus (-U)) + (U \oplus (-U))^*)^{-1} &= (Z + (U_1 + U_1^*))^{-1} \\ &= (I + ((U_1 + U_1^*))^{-1}Z)^{-1}(U_1 + U_1^*))^{-1}, \end{aligned}$$

since $\|(U_1 + U_1^*)^{-1}Z\| \leq \frac{2k^2}{1-2\sqrt{2}k^2\tau}2\sqrt{2}\tau < 1$, i.e., $6\sqrt{2}\tau k^2 < 1$. Then

$$\begin{aligned} \|((U &\oplus (-U)) + (U \oplus (-U))^*)^{-1}\| \\ &\leq \|(I + ((U_1 + U_1^*))^{-1}Z)^{-1}\|\|(U_1 + U_1^*))^{-1}\| \\ &\leq \frac{1}{1 - \frac{2k^2}{1-2\sqrt{2}k^2\tau}2\sqrt{2}\tau}\frac{2k^2}{1-2\sqrt{2}k^2\tau} = \frac{2k^2}{1 - 6\sqrt{2}\tau k^2}. \end{aligned}$$

We now prove part (2). Define a new scalar product on $\mathcal{E}$ by $(\xi, \eta) = \langle \xi, |T|\eta\rangle$. If $Z \in \mathcal{L}(\mathcal{E})$, then

$$\begin{aligned} \|Z\|_{(.)}^2 &= \max_{\|\xi\|_{(.)} \leq 1} \|(Z\xi, Z\xi)\| \\ &= \max_{\||T|^{\frac{1}{2}}\xi\| \leq 1} \|\langle Z|T|^{-\frac{1}{2}}(|T|^{\frac{1}{2}}\xi), |T|Z|T|^{-\frac{1}{2}}(|T|^{\frac{1}{2}}\xi)\rangle\| \\ &\leq \max_{\||T|^{\frac{1}{2}}\xi\| \leq 1} \||T|\|\|Z\|^2\||T|^{-\frac{1}{2}}\|^2\||T|^{\frac{1}{2}}\xi\|^2 \\ &= \|Z\|^2\||T|^{-\frac{1}{2}}\|^2\||T|\|, \end{aligned}$$

namely, $\|Z\|_{(.)} \leq k\|Z\|$. With the scalar product (.) on $\mathcal{E}$, we have $\|D^2\|_{(.)} \leq k\|D\| \leq k\tau^2, \|D' + D\|_{(.)} \leq k\|D' + D\| \leq k\tau, \|R\|_{(.)} \leq k\|R\| \leq k^2$ and $\|S\|_{(.)} \leq k\|S\| \leq k^2$. Using part (a) with k and τ replaced by k^2 and $\sqrt{k}\tau$, we obtain that $D + D^*$ is invertible and $\|(D + D^*)^{-1}\|_{(.)} \leq \frac{2(k^2)^2}{1-6\sqrt{2}\sqrt{k}\tau(k^2)^2}$, since $6\sqrt{2}\sqrt{k}\tau(k^2)^2 = 6\sqrt{2}\tau\sqrt{k}k^4 < 1$ by assumption. Thus we are done for $D' = -D$.

Note that $(\xi, \eta) = \langle \xi, |T|\eta\rangle = Q(\xi, T^{-1}|T|\eta)$. Let $T_1 = T^{-1}|T|, T_1^2 = I$. T_1 is compatible with Q. Let $P_1 = \frac{I+T_1}{2}, P_1^2 = P_1 = P_1T_1$ and $(I - P_1)T_1 = T_1 - P_1 = -(I - P_1)$. Then

$$\begin{aligned} (D + D^*)^{(0)} &= P_1(D + D^*)P_1 + (I - P_1)(D + D^*)(I - P_1) \\ &= P_1(D + D')P_1 + (I - P_1)(D + D')(I - P_1). \end{aligned}$$

This yields $\|(D + D^*)^{(0)}\|_{(.)} \leq \sqrt{2}\|D + D'\|_{(.)}$, and then

$$\begin{aligned} \|(D + D^*) &- (D + D^*)^{(1)}\|_{(.)}\|(D + D^*)^{-1}\|_{(.)} \\ &= \|(D + D^*)^{(0)}\|_{(.)}\|(D + D^*)^{-1}\|_{(.)} \\ &\leq \sqrt{2}k\tau\frac{2(k^2)^2}{1 - 6\sqrt{2}\sqrt{k}\tau(k^2)^2} < 1, \end{aligned}$$

since $2\sqrt{2}\tau k^4(k + 3\sqrt{k}) < 1$. Therefore, $(D + D^*)^{-1}$ is invertible:

$$\begin{aligned} ((D + D^*)^{(1)})^{-1} &= ((D + D^*) - ((D + D^*) - (D + D^*)^{(1)}))^{-1} \\ &= (I - (D + D^*)^{-1}((D + D^*) - (D + D^*)^{(1)}))^{-1}(D + D^*)^{-1}. \end{aligned}$$

Observe that the map $\psi_{u,0}$ in Definition 3 can be defined for $D' \neq -D$, in which one replaces $(D + D^*)$ by $(D + D^*)^{(1)}$ in the formula for F. We have shown that $(D + D^*)^{(1)}$ is invertible. Hence, $\psi_{u,0}((\mathcal{E}, Q, D)) = 0$. □

The above proof is based on the elementary fact that for a bounded operator A with $\|A\| < 1$, $I + A$ is invertible. Indeed, this is the main reason why Theorem 1 can be extended to the case of almost flat bundles. Once the curvature is small, the norms of relevant operators can be estimated by a small constant. We will also use this fact in the following proposition, which is an extension of Proposition 1 to the almost flat case.

Proposition 2. *Let* $(\mathcal{E}_i, Q_i, D_i) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ *be such that* $D'_i = -D_i$ *and* $\mathcal{E}_i$ *have the scalar products compatible with* $Q_i, i = 1, 2$. *Suppose that there are equivariant* $R \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ *and* $\tau, k > 0$ *satisfying*

(a) $\|D_i^2\| \leq \tau^2, \|R\| \leq k, R(dom(D_1)) \subset dom(D_1)$ *and* $\|RD_1 - D_2R\| \leq \tau^2$;

(b) there is an equivariant $S \in \mathcal{L}(\mathcal{E}_1)$ *with* $\|S\| \leq k, S(dom(D_1)) \subset dom(D_1)$, *and* $\|I - R'R - D_1S - SD_1\| \leq \tau^2$;

(c) there is an equivariant $\varepsilon \in \mathcal{L}(\mathcal{E}_1)$ *with* $\varepsilon(dom(D_1)) \subset dom(D_1), \varepsilon D_1 = -D_1\varepsilon, \varepsilon^2 = I, \varepsilon' = \varepsilon$ *and* $\varepsilon(I - R'R) = (I - R'R)\varepsilon$;

(d) there is an equivariant $Z \in \mathcal{L}(\mathcal{E}_2)$ *with* $\|Z\| \leq k, Z(dom(D_2)) \subset dom(D_2)$ *and* $\|I - RR' - D_2Z - ZD_2\| \leq \tau^2$.

Then for $64\tau(k + 4\sqrt{2}k^3(1 + k)^2)^5 < 1, \tau < \frac{1}{2\sqrt{2}2^{\frac{1}{4}}}$ *and* $k \geq 1$,

$$\psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{0,u}((\mathcal{E}_2, Q_2, D_2)).$$

Furthermore, suppose (d) *is replaced by the following condition:*

(d') there are equivariant $W \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$ *and* $Z \in \mathcal{L}_G(\mathcal{E}_2)$ *such that* $\|Z\| \leq k$, $\|W\| \leq k, W(dom(D_2)) \subset dom(D_1), Z(dom(D_2)) \subset dom(D_2), \|WD_2 - D_1W\| \leq \tau^2$ *and* $\|I - RW - D_2Z - ZD_2\| \leq \tau^2$.

Then for $64\tau\sqrt{1 + 6k^2}\{k(1 + 2k^2) + 4\sqrt{2}k^3(1 + 2k^2)^3(1 + k(1 + 2k^2))^2\}^5 < 1$, $\tau < \frac{1}{2\sqrt{2}2^{\frac{1}{4}}\sqrt{1+6k^2}}$ *and* $k \geq 1$,

$$\psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{0,u}((\mathcal{E}_2, Q_2, D_2)).$$

Proof. The proof is divided into four steps.

Step 1. As in the proof of Proposition 1, we can assume $S = -S'$ and $Z = -Z'$ by considering $\frac{S-S'}{2}$ and $\frac{Z-Z'}{2}$ that satisfy (b) and (d).

Let $(\mathcal{E}, Q, D) = (\mathcal{E}_1 \oplus \mathcal{E}_2, Q_1 \oplus (-Q_2), D_1 \oplus (-D_2))$. By Lemma 7, it suffices to show that $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$. The proof is similar to that of Proposition 1.

For $t \in [0, 1]$ define

$$T_t = \left[\begin{array}{cc} I & 0 \\ itR\varepsilon & I \end{array} \right], \quad L_t = \left[\begin{array}{cc} I - R'R & (i\varepsilon + tS)R' \\ R(i\varepsilon + tS) & I \end{array} \right], \quad \nabla_t = \left[\begin{array}{cc} D_1 & tR' \\ 0 & -D_2 \end{array} \right].$$

These operators are all equivariant. Let $B_t(\xi, \eta) = Q(T_t\xi, T_t\eta)$ and $C_t(\xi, \eta) = Q(L_t\xi, \eta)$. We check that $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

If D'_t denotes the conjugate of D with respect to B_t, then, as before, $D'_t = (T'_tT_t)^{-1}D'(T'_tT_t) = -(T'_tT_t)^{-1}D(T'_tT_t)$. Since

$$T_tD - DT_t = \left[\begin{array}{cc} 0 & 0 \\ -it(RD_1 - D_2R)\varepsilon & 0 \end{array} \right],$$

$T_t D - D T_t$ is in $\mathcal{L}_G(\mathcal{E})$ by condition (a). This implies that $D + D'_t \in \mathcal{L}_G(\mathcal{E})$, since $D + D'_t = (T'_t T_t)^{-1}(T'_t(T_t D - D T_t) + (T'_t D - D T'_t)T_t)$. Now

$$\|[T_t, D]\| \le t\|RD_1 - D_2 R\| \le t\tau^2,$$

$$\|[T'_t, D]\| \le t\|R'D_2 - D_1 R'\| \le t\tau^2,$$

and $\|T_t\| \le (\|R\|t+1)\sqrt{2} \le \sqrt{2}(1+tk), \|T'_t\| \le \sqrt{2}(1+tk)$. Here we used the norm $\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ on $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Clearly,

$$T_t^{-1} = \begin{bmatrix} I & 0 \\ -itR\varepsilon & I \end{bmatrix}, \qquad (T'_t)^{-1} = \begin{bmatrix} I & -it\varepsilon R' \\ 0 & I \end{bmatrix},$$

$$(T'_t T_t)^{-1} = \begin{bmatrix} I & -it\varepsilon R' \\ -itR\varepsilon & t^2 RR' + I \end{bmatrix}.$$

We have $\|(T'_t T_t)^{-1}\| \le \sqrt{2}(1 + t^2\|RR'\| + t\|R\|) \le \sqrt{2}(1 + tk^2 + tk)$, and

$$\begin{aligned} \|D_1 + D'_t\| &\le \|(T'_t T_t)^{-1}\|(\|T_t\|\|[T_t, D]\| + \|[T'_t, D]\|\|T_t\|) \\ &\le 4t\tau^2(1+tk)(1 + t^2 k + tk). \end{aligned}$$

This proves $D + D'_t \in \mathcal{L}_G(\mathcal{E})$. Obviously, $D^2 \in \mathcal{L}_G(\mathcal{E})$, and, as in the proof of Proposition 1, we can find equivariant R_1 and $S_1 \in \mathcal{L}_G(\mathcal{E})$ such that $DS_1 + R_1 D - I \in \mathcal{K}_G(\mathcal{E})$. We have thus verified that $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

Step 2. We now check that $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ for t near 0. Since $L_0 = T'_1 T_1$,

$$\|L_0^{-1}\| \le \|T_1^{-1}\|\|(T'_1)^{-1}\| \le (\sqrt{2}(1+k))^2 = 2(1+k)^2.$$

Note that

$$L_t = \begin{bmatrix} I - R'R & i\varepsilon R' \\ iR\varepsilon & I \end{bmatrix} + \begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix} = L_0\left(I + L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix}\right).$$

We see that if

$$\left\|L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix}\right\| \le 2(1+k)^2\sqrt{2}\|S\|\|R\|t \le 2\sqrt{2}(1+k)^2 k^2 t < 1,$$

then L_t is invertible. Let $t_0 = \frac{1}{4\sqrt{2}(1+k)^2 k^2}$. We have

$$\|L_t^{-1}\| \le \left\|\left(I + L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix}\right)^{-1}\right\|\|L_0^{-1}\| \le \frac{1}{1 - 2\sqrt{2}(1+k^2)^2 k^2 t}(2(1+k)^2).$$

In particular, for $t = t_0$, $\|L_{t_0}^{-1}\| \le 4(1+k)^2$. Since $C_t(\xi, \nabla_t \eta) = Q(L_t\xi, \nabla_t \eta) = C_t(L_t^{-1}\nabla'_t L_t\xi, \eta)$, the conjugate ∇'_{t,C_t} of ∇_t with respect to C_t is $\nabla'_{t,C_t} = L_t^{-1}\nabla'_t L_t$. We have

$$L_t\nabla_t = \begin{bmatrix} (I - R'R)D_1 & t(I - R'R)R' - (i\varepsilon + tS)R'D_2 \\ R(i\varepsilon + tS)D_1 & tR(i\varepsilon + tS)R' - D_2 \end{bmatrix},$$

$$\nabla'_t L_t = \begin{bmatrix} D'_1(I - R'R) & D'_1(i\varepsilon + tS)R' \\ -tR(I - R'R) - D'_2 R(i\varepsilon + tS) & -tR(i\varepsilon + tS)R' - D'_2 \end{bmatrix}.$$

This implies that

$$L_t\nabla_t + \nabla'_t L_t = \begin{bmatrix} b & u_t \\ v_t & 0 \end{bmatrix},$$

where

$$
\begin{aligned}
b &= (I - R'R)D_1 - D_1(I - R'R) = (D_1R' - R'D_2)R + R'(D_2R - RD_1), \\
v_t &= R(i\varepsilon + tS)D_1 - tR(I - R'R) + D_2R(i\varepsilon + tS) \\
&= (D_2R - RD_1)(i\varepsilon + tS) + tR(D_1S + SD_1 - (I - R'R)),
\end{aligned}
$$

and

$$
\begin{aligned}
u_t &= -D_1(i\varepsilon + tS)R' + t(I - R'R)R' - (i\varepsilon + tS)R'D_2 \\
&= -(i\varepsilon + tS)(R'D_2 - D_1R') - (D_1S + SD_1 - (I - R'R))tR'.
\end{aligned}
$$

By assumptions (a) and (b), b, u_t and v_t are in $\mathcal{L}_G(\mathcal{E})$. Thus

$$
\nabla_t + \nabla'_{t,C_t} = L_t^{-1}(L_t\nabla_t + \nabla'_t L_t) \in \mathcal{L}_G(\mathcal{E}), \qquad t \in [0, t_0].
$$

One can easily check that $(\mathcal{E}, C_t, \nabla_t)$ satisfies the other conditions. Hence $(\mathcal{E}, C_t, \nabla_t)$ $\in \mathcal{L}^0_{G,u}(\mathcal{A})$.

Step 3. We have shown that

$$
(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A}[0,1]) \quad \text{and} \quad (\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A}[0,t_0]).
$$

Note that $\nabla_0 = D$ and $C_0 = B_1, (\mathcal{E}, B_1, D) = (\mathcal{E}, C_0, \nabla_0)$. This implies that

$$
\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}, B_0, D)) = \psi_{0,u}((\mathcal{E}, B_1, D)) = \psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})).
$$

Consequently, it suffices to prove $\psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})) = 0$.

We now use Lemma 12(b).

(i) We have

$$
\|\nabla^2_{t_0}\| \leq \tau_1^2 : \nabla_t^2 = \begin{bmatrix} D_1^2 & t(D_1R' - R'D_2) \\ 0 & D_2^2 \end{bmatrix},
$$

$$
\|\nabla^2_{t_0}\| \leq \|D_1^2\| + t_0\|D_1R' - R'D_2\| + \|D_2^2\| \leq (2\tau)^2, \ \tau_1 = 2\tau.
$$

(ii) Since $2kt_0 = 2k\frac{1}{4\sqrt{2}(1+k)^2k^2} < 1$ and $12\tau(1+k)^3 < 1$,

$$
\begin{aligned}
\|\nabla_{t_0} + \nabla'_{t,C_{t_0}}\| &= \|L_{t_0}^{-1}\|\|L_{t_0}\nabla_{t_0} + \nabla'_{t,C_{t_0}}L_{t_0}\| \leq \|L_{t_0}^{-1}\|(\|b\| + \|u_{t_0}\| + \|v_{t_0}\|) \\
&= 4(1+k)^2((\tau^2k + k\tau^2) + 2\tau^2\|i\varepsilon + tS\| + 2t_0k\tau^2) \\
&= \ \leq 8(1+k)^2\tau^2(3+k) < \tau_1.
\end{aligned}
$$

(iii) Take $X = \begin{bmatrix} S & 0 \\ R/t_0 & -Z \end{bmatrix}$. Then

$$
\nabla_{t_0}X + X\nabla_{t_0} = \begin{bmatrix} D_1S + SD_1 + R'R & t_0(SR' - R'Z) \\ -\frac{1}{t_0}(D_2R - RD_1) & D_2Z + ZD_2 + RR' \end{bmatrix}.
$$

We obtain

$$
\begin{aligned}
\|\nabla_{t_0}X + X\nabla_{t_0} - I\| &\leq \sqrt{2}\max\{\|D_1S + SD_1 + R'R - I\| + \|t_0(SR' - R'Z)\|, \\
&\qquad \|\frac{1}{t_0}(D_2R - RD_1)\| + \|D_2Z + ZD_2 + RR' - I\|\} \\
&\leq \sqrt{2}\max\{\tau^2 + 2t_0k^2, \frac{\tau^2}{t_0} + \tau^2\} < \frac{1}{2},
\end{aligned}
$$

since $\tau^2 + \frac{1}{2\sqrt{2}(1+k)^2} < \frac{1}{2\sqrt{2}}$ and $\tau^2(4\sqrt{2}(1+k)^2k^2 + 1) < \frac{1}{2\sqrt{2}}$. Therefore, $\nabla_{t_0}X + X\nabla_{t_0}$ is invertible and

$$
\|(\nabla_{t_0}X + X\nabla_{t_0})^{-1}\| \leq \frac{1}{1 - \|\nabla_{t_0}X + X\nabla_{t_0} - I\|} \leq 2.
$$

(iv) The norm of X can be estimated by

$$
\begin{aligned}
\|X\| &\leq \sqrt{2}\max\{\|S\|, \|Z\| + \|R/t_0\|\} \leq \sqrt{2}(k + \frac{k}{t_0}) \\
&= \sqrt{2}(k + 4\sqrt{2}(1 + k)^2 k^3) \overset{\text{def}}{=} k_1.
\end{aligned}
$$

(v) $\mathcal{E}$ already has the scalar product $\langle \xi, \eta \rangle = Q(\xi, T\eta)$ with $T^2 = I$. Now $C_{t_0}(\xi, \eta) = Q(\xi, L'_{t_0}\eta) = \langle \xi, TL'_{t_0}\eta \rangle$. Clearly,

$$
\begin{aligned}
\|L'_{t_0}\| &\leq \sqrt{2}\max\{\|I - R'R\| + \| - i\varepsilon + t_0 S'\|\|R'\|, \|R'(-i\varepsilon + t_0 S')\| + 1\} \\
&\leq \sqrt{2}(1 + k + k^2 + \frac{1}{4\sqrt{2}(1 + k)^2}).
\end{aligned}
$$

Also,

$$
\|(TL'_{t_0})\|\|(TL'_{t_0})^{-1}\| \leq 4\sqrt{2}((1 + k)^3 + k^2(1 + k)^2 + \frac{1}{4\sqrt{2}}) \leq k_1^2.
$$

(vi) Finally, we check that

$$
2\sqrt{2}\tau_1 k_1^4(k_1 + 3k_1^{\frac{1}{2}}) < 1,
$$

where $\tau_1 = 2\tau, k_1 = \sqrt{2}(k + 4\sqrt{2}(1 + k)^2 k^3)$. In fact,

$$
\begin{aligned}
2\sqrt{2}\tau_1 k_1^4(k_1 + 3k_1^{\frac{1}{2}}) &= 32\tau(k + 4\sqrt{2}(1 + k)^2 k^3)^5(1 + \frac{3}{\sqrt{\sqrt{2}k + 8(1 + k)^2 k^3}}) \\
&\leq 64\tau(k + 4\sqrt{2}(1 + k)^2 k^3)^5 < 1.
\end{aligned}
$$

Hence, the condition of Lemma 12 is satisfied. The proof of the first part of the proposition is completed.

Step 4. We now prove the assertion with (d) replaced by (d'). Let $Z_1 = Z - RR'Z + RSW, X_1 = I - RW - D_2 Z - ZD_2, X_2 = I - R'R - D_1 S - SD_1$. Then

$$
\|Z_1\| \leq \|Z\| + \|R\|\|R'\|\|Z\| + \|R\|\|S\|\|W\| \leq k + 2k^3,
$$

$$
\begin{aligned}
RSWD_2 + &D_2 RSW \\
&= (D_2 R - RD_1)SW + R(D_1 S + SD_1)W - RS(D_1 W - WD_2) \\
&= (D_2 R - RD_1)SW + R(I - R'R)W - RX_2 W - RS(D_1 W - WD_2).
\end{aligned}
$$

It follows that $\|RSWD_2 + D_2 RSW\| \leq \tau^2 k^2 + k(1 + k^2)k + k\tau^2 k + k^2\tau^2$. Also,

$$
\begin{aligned}
RR'ZD_2 + &D_2 RR'Z \\
&= (D_2 R - RD_1)R'Z + R(D_1 R' - R'D_2)Z + RR'(D_2 Z + ZD_2).
\end{aligned}
$$

This yields $\|RR'ZD_2 + D_2 RR'Z\| \leq k^2\tau^2 + k^2\tau^2 + k^2(1 + k^2 + \tau^2)$. Now

$$
\begin{aligned}
RR' - I + &Z_1 D_2 + D_2 Z_1 \\
&= RR' - I + D_2 Z + ZD_2 + (RSW - RR'Z)D_2 + D_2(RSW - RR'Z) \\
&= (RR' - I)X_1 - (D_2 R - RD_1)(R'Z - SW) - R(D_1 R' - R'D_2)Z \\
&\quad - RX_2 W - RS(D_1 W - WD_2).
\end{aligned}
$$

We get

$$
\|RR' - I + Z_1 D_2 + D_2 Z_1\| \leq (1 + k^2)\tau^2 + \tau^2(2k^2) + 3k^2\tau^2 = \tau^2(1 + 6k^2).
$$

Therefore, replacing τ and k in the first part of this proposition by $\tau\sqrt{1+6k^2}$ and $k(1+2k^2)$, we obtain that if

$$64\tau\sqrt{1+6k^2}\{k(1+2k^2)+4\sqrt{2}k^3(1+2k^2)^3(1+k(1+2k^2))^2\}^5 < 1,$$

then the conclusion holds. $\square$

Let M and N be two closed, oriented Riemannian G-manifolds and $h : N \to M$ be a G-pseudo-equivalence. Let E be a τ-flat G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0, where $\tau > 0$ is a constant. Let ∇ be a τ-flat connection on E. Then we have two elements $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$ in $\mathcal{L}^0_{G,u}(\mathcal{A})$. Our goal in the rest of this section is to prove that for τ small, these elements satisfy the condition of Proposition 2. Hence, $G\text{-}Sig(D_E) = G\text{-}Sig(D_{h^*(E)})$.

Lemma 13 ([HiS]). *With the above notation, if $f : N \times [0,1] \to M$ is an equivariant smooth map, then there is a constant $k > 0$ such that for every equivariant unitary connection ∇ on E, $\|f_0^*(\nabla) - f_1^*(\nabla)\| \leq k\|\nabla^2\|$.*

Proof. This lemma is independent of the group action, since we are concerned only with the norms. See the proof of Lemma 4.3 in [HiS]. $\square$

The following lemma extends Lemmas 10 and 11 to the nonflat case.

Lemma 14 ([HiS]). *Let M and N be two closed, oriented Riemannian G-manifolds and E be a G-$\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert G-$\mathcal{A}$-module E_0.*

(a) Let $h : N \to M$ be an equivariant smooth map and ω an equivariant smooth form on N. Denote by N_0 the support of ω and suppose that h is a submersion near a G-neighborhood of N_0. Then $e_\omega h^* : \mathcal{E}_E \to \mathcal{E}_{h^*(E)}$ is a bounded equivariant operator. Moreover, the norm of $e_\omega h^*$ can be estimated by a number which is independent of E. Let ∇ and ∇' be equivariant unitary connections of E and $h^*(E)$, respectively. Then $e_\omega h^*(dom(D_E)) \subset dom(\nabla')$ and $\nabla'(e_\omega h^*) - (e_\omega h^*)\nabla \in \mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$.

(b) Let $h : N \times [0,1] \to M$ be an equivariant smooth map and ω an equivariant smooth form on N. Let N_0 be the support of ω. Suppose that h (resp., h_0, h_1) is a G-submersion near a neighborhood of $N_0 \times [0,1]$ (resp., N_0). Then there is a constant $k > 0$ such that for any connection ∇ of E there exists $R \in \mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h_0^*(E)})$ with $\|R\| \leq k$ and

$$\|e_\omega h_1^* - e_\omega h_0^* - h_0^*(\nabla)R - R\nabla\| \leq k\|\nabla^2\|.$$

Proof. (a) We refer to [HiS].

To prove part (b), let

$$R(\tilde{\xi}) = \int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega h^*(\tilde{\xi})))dt, \ \tilde{\xi} \in \mathcal{E}_E.$$

Then the norm of R can be estimated by the number k, which is independent of h and the structures on M and N. As the proof of Lemma 10,

$$(h_0^*(\nabla)R + R\nabla)(\tilde{\xi}) = (e_\omega h_1^* - e_\omega h_0^*)(\tilde{\xi}) + \int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega(h^*(\nabla) - h^{'*}(\nabla))(\tilde{\xi})))dt,$$

where $h'(x,t) = h(x,0) : N \times [0,1] \to M$. Let R_1 be defined by the second term of the above identity. Then

$$e_\omega h_1^* - e_\omega h_0^* = h_0^*(\nabla)R + R\nabla + R_1.$$

To estimate the norm of R_1, let $H(x,t,\lambda) = h(x,\lambda t)$. Then $H(x,t,0) = h'(x,t)$ and $H(x,t,1) = h(x,t)$. Using Lemma 13, we get $\|R_1\| \le k\|\nabla^2\|$. $\square$

Let $P : N \times B^k \to M$ be a G-submersion, τ a G-smooth form of mass 1 on B^k. Suppose that $q : N \times B^k \to N$ and $r : N \times B^k \to B^k$ are the projections, $\omega = r^*(\tau)$, and $h : N \to M$ is the restriction to $N \times \{o\}$ of P. Then $hq : N \times B^k \to M$ is equivariantly homotopic to P via the homotopy $H(x,t,\lambda) = P(x,\lambda t)$. For this reason, we identify $P^*(E)$ with $q^* h^*(E)$. Let $R_{P,\tau} = R_{P,\tau}(E, \nabla) = q_! e_\omega P^* \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$.

Lemma 15 ([HiS]). (a) *Let* $P' : N \times B^l \to M$ *be another G-submersion and τ' a G-smooth form of mass 1 on B^l. Suppose $h(x) = P(x,o)$ and $h'(x) = P'(x,o)$ are equivariantly homotopic. Then there exists a constant $k > 0$ such that for each pair (E, ∇) as above there is $R \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$ satisfying $\|R\| \le k$ and*

$$\|R\nabla + \nabla R + R_{P,\tau}(E, \nabla) - R_{P',\tau'}(E, \nabla)\| \le k\|\nabla^2\|.$$

(b) *Suppose $\tilde{P}' : O \times B^l \to N$ is a G-submersion and τ' a G-smooth form of mass 1 on B^l. Let $P'' : O \times B^l \times B^k \to M$ be defined by $P''(x,s,t) = P(\tilde{P}'(x,s),t)$ and $\tau'' = \tau' \times \tau$. Then there exists a constant $k > 0$ such that for each pair (E, ∇),*

$$\|R_{P',\tau'}(h^*(E), h^*(\nabla))R_{P,\tau}(E, \nabla) - R_{P'',\tau''}(E, \nabla)\| \le k\|\nabla^2\|.$$

Proof. (a) By Lemma 14, the proof is the same as that of Lemma 11.

(b) Define $f : O \to M$ by $f(x) = P''(x,o,o)$ and let $q'' : O \times B^k \times B^l \to O$ be the projection. Define two homotopies H and $\bar{H}$ between P'' and fq'' from $O \times B^l \times B^k \times [0,1]$ to M by

$$H(x,s,t,\lambda) = P''(x,\lambda s, \lambda t),$$

and

$$\bar{H}(x,s,t,\lambda) = P''(x, \sup(0, 2\lambda - 1)s, \inf(1, 2\lambda)t).$$

We have $H(x,s,t,o) = (fq'')(x,s,t), H(x,s,t,1) = P''(x,s,t)$. We can pull back the metric-preserving connection ∇ on E to a metric-preserving connection $H(\nabla)$ on $H^*(E)$ via the smooth map H, and then by the parallel transport along $[0,1]$ we define a unitary operator $U \in \mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{(fq'')^*(E)})$. The same reasoning shows that $\bar{H}$ produces unitary operators in $\mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{H_{\frac{1}{2}}^*(E)})$ and $\mathcal{L}_G(\mathcal{E}_{H_{\frac{1}{2}}^*(E)}, \mathcal{E}_{(fq'')^*(E)})$, hence a unitary operator $\bar{U} \in \mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{(fq'')^*(E)})$, where $H_{\frac{1}{2}} = H(x,s,t,\frac{1}{2})$. By definition, $R_{P'',\tau''} = q_!'' U e_{\omega''}(P'')^*$ for $\omega'' = \pi_{l,k}^*(\tau'')$, where $\pi_{l,k} : O \times B^l \times B^k \to B^l \times B^k$ is the projection. Let $\bar{R}_{P'',\tau''} = q_!'' \bar{U} e_{\omega''}(P'')^*$. Then, as in the proof of Lemma 11(b), $\bar{R}_{P'',\tau''} = R_{P',\tau'} R_{P,\tau}$. The result follows easily from [HiS]:

$$\|\bar{R}_{P'',\tau''} - R_{P'',\tau''}\| \le \|q_!''(\bar{U} - U)e_{\omega''}(P'')^*\| \le K_1\|\bar{U} - U\| \le k_1 m\|\nabla^2\|,$$

where k_1 and m are constants independent of ∇ and E. $\square$

Finally, we prove the equivariant Connes-Gromov-Moscovici theorem.

Proof of Theorem 2. We use Proposition 2 for $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$. Let $J : M \to \mathbb{R}^k$ be a G-embedding, O a G-tubular neighborhood of $J(M)$ in $\mathbb{R}^k$ and $\pi : O \to M$ the corresponding G-projection. Suppose $J(M) + B^k \subset O$. Let $P : N \times B^k \to M$ be the G-submersion given by $P(x,t) = \pi(h(x) + t)$, and τ a G-volume form of mass 1 on B^k. Take $R = R_{P,\tau} \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$. We now verify the condition of Proposition 2. To check condition (a), let $\tau_1^2 \geq \|\nabla^2\|$. Then $\|D_E^2\| = \|\nabla^2\| \leq \tau_1^2$, and $\|D_{h^*(E)}\| = \|\nabla^2\| \leq \tau_1^2$. Take a G-smooth function φ with compact support such that $\varphi\omega = \omega$ on $N \times B^k$. By Lemma 14, we can estimate $\|e_\omega P^*\|$ and $\|q_{!,\varphi}\| = \|(e_\omega P^*)'\| = \|e_\omega P^*\|$. Clearly,

$$P^*\nabla = P^*(\nabla)P^*, \qquad q^* h^*(\nabla) = (hq)^*(\nabla)q^*$$

and $e_\omega P^*(\nabla) = P^*(\nabla)e_\omega$ for the closed form ω. Then

$$(e_\varphi q^*)' e_\omega P^* \nabla = (e_\varphi q^*)' P^*(\nabla) e_\omega P^*,$$

and

$$\begin{aligned}
\|(e_\varphi q^*)' e_\omega P^* \nabla &- h^*(\nabla)(e_\varphi q^*)' e_\omega P^*\| \\
&= \|(e_\varphi q^*)' P^*(\nabla) e_\omega P^* - h^*(\nabla)(e_\varphi q^*)' e_\omega P^*\| \\
&\leq \|(e_\varphi q^*)'(P^*(\nabla) - (hq)^*(\nabla))e_\omega P^*\| \\
&\quad + \|(h^*(\nabla)(e_\varphi q^*)' - (e_\omega q^*)'(hq)^*(\nabla))e_\omega P^*\| \\
&\leq m\|\nabla^2\|,
\end{aligned}$$

by Lemmas 13 and 14 and the fact that $h^*(\nabla)(e_\varphi q^*)' = (e_\varphi q^*)'(hq)^*(\nabla)$. We have $R(dom(D_E)) \subset dom(D_{h^*(E)})$ and $\|RD_E - D_{h^*(E)}R\| \leq m_1\|\nabla^2\|$ for some $m_1 > 0$ independent of (E, ∇).

To check condition (b), let $q_i : N \times B^k \times B^k \to N \times B^k$ be given by $q_i(x, t_1, t_2) = (x, t_i)$, $P_1 : M \times B^k \to M$ be the projection and $\tilde{\pi} : M \times B^k \to M$ be the G-submersion given by $\tilde{\pi}(x, t) = \pi(x + t)$. Then we can argue as the proof of Theorem 1 by using Proposition 2 to get $S \in \mathcal{L}_G(\mathcal{E})$ such that $\|S\| \leq m_2, S(dom(D_E)) \subset dom(D_E)$ and

$$\|I - R'R - D_E S - S D_E\| \leq m_3\|\nabla^2\|$$

for constants $m_i > 0$ independent of (E, ∇). The rest of the proof is the same as that of Theorem 1.

To check condition (c), take $\varepsilon(\tilde{\xi}) = (-1)^{\partial \tilde{\xi}}\tilde{\xi}$. Then clearly $\varepsilon(dom(D_E)) \subset dom(D_E), \varepsilon D_E = -D_E \varepsilon, \varepsilon = \varepsilon', \varepsilon^2 = I$ and $\varepsilon(I - R'R) = (I - R'R)\varepsilon$. See the corresponding part of Theorem 1.

Finally we verify condition (d') of Proposition 2. Indeed, let $P_1 : M \times B^l \to N$ be a submersion such that $q_1 : M \to N$ given by $q_1(x) = P_1(x, o)$ is a homotopic inverse of h, i.e., $q_1 h$ and $h q_1$ are homotopic to the identities, respectively. Note that P_1 and q_1 may not be equivariant, and homotopy between $q_1 h$ and the identity may not be equivariant either. However, we can still identify E with $(hq_1)^*(E)$ non-equivariantly. Let τ' be a volume form of mass 1 on B^l and $W_1 = R_{P_1, \tau'} \in \mathcal{L}_G(\mathcal{E}_{h^*(E)}, \mathcal{E}_E)$. Then, as before, there are constants $m_4, m_5 > 0$ such that $\|W_1\| \leq m_5, W_1(dom(D_E)) \subset dom(D_E)$ and

$$\|W_1 D_{h^*(E)} - D_E W_1\| \leq m_4\|\nabla^2\|.$$

By Lemma 15 and Step 3 of the proof of Theorem 1, there exists $Z_1 \in \mathcal{L}_G(\mathcal{E}_{h^*(E)})$ satisfying $\|Z_1\| \leq m_6, Z_1(dom(D_{h^*(E)})) \subset dom(D_{h^*(E)})$ and

$$\|I - RW_1 - D_{h^*(E)}Z_1 - Z_1 D_{h^*(E)}\| \leq m_7 \|\nabla^2\|,$$

where m_6, m_7 are positive constants independent of (E, ∇). Let $W = \int_G g(W_1)dg$, $Z = \int_G g(Z_1)dg$. We get the required operators. Let

$$\tau^2 = \max\{\|\nabla^2\|, m_1\|\nabla^2\|, m_3\|\nabla^2\|, m_4\|\nabla^2\|, m_7\|\nabla^2\|\}$$

and $k = \max\{m_0, m_2, m_5, m_6\}$. Then we have verified the condition of Proposition 2 as long as $\|\nabla^2\|$ is sufficiently small. □

We refer to Section 3 for the main idea of the above proof. We should point out that the argument in Section 3 and this section closely follows [HiS], except that our estimates are slightly different. We called this the equivariant Hilsum-Skandalis method in the Introduction.

The following is an example of τ-flat G-vector bundles (see also [Ska]).

Example 1. For simplicity, we take $G = S^1$ and $M_1 = S^1$ with a trivial G-action. Let $E_1 = S^1 \times \mathbb{R}^m$ be a trivial vector bundle over M_1 with the G-action on a fiber given by the multiplication $e^{2\pi it} \in S^1$. Choose a basis $\{e_1, \cdots, e_m\}$ for $\mathbb{R}^m$. Let $l > 0$ be an integer. Define a parallel transport $\sigma_{[r,s]}$ along the oriented segment $[r, s] \subset \mathbb{R}^1$ by the following formula:

$$\sigma_{[r,s]}(e_j) = e^{\frac{2\pi i(s-r)(j-1)l}{m}} e_j, \quad j = 1, 2, \cdots, m.$$

It is easy to check that the connection ∇ associated with this parallel transport is G-invariant and flat. Let $u : S^1 \to U(E_1)$ be the section of the bundle $U(E_1)$ of unitary endomorphisms on E_1 given by $u(t)e_i = e_{i-1}, i \neq 1$, and $u(t)e_1 = e^{2\pi itl}e_m$. Clearly, u is G-equivariant. Hence it determines an element $[u] \in K_G^1(M_1)$. It is easy to check that

$$\sigma_{[t_0,t]}u(t_0)\sigma_{[t,t_0]} = e^{-\frac{2\pi i(t-t_0)l}{m}}u(t).$$

Let us estimate $\|\nabla(u)\|$:

$$
\begin{aligned}
\|\nabla(u)\| &= \max_{x_0 \in M_1, X \in T_{x_0}M_1, \|X\| \leq 1} \|\nabla_X(u)\|_{(E_1)_{x_0}} \\
&= \max_{\substack{x_0 \in M_1, X \in T_{x_0}M_1, \|X\| \leq 1, \\ p_0 \in (E_1)_{x_0}, \|p_0\| \leq 1}} \| \lim_{t \to t_0} \frac{\sigma_{[t,t_0]}u(t)\sigma_{[t_0,t]}(p_0) - u(t_0)(p_0)}{t - t_0} \|_{(E_1)_{x_0}} \\
&= \max \| \lim_{t \to t_0} \frac{e^{2\pi i(t-t_0)l/m} - 1}{t - t_0} u(t_0)(p_0) \|_{(E_1)_{x_0}} \\
&\leq 2\pi l/m.
\end{aligned}
$$

We may use $[u] \in K_G^1(M_1)$ to produce an element in $K_G^0(S^2)$. Indeed, let S^2 be identified with $\mathbb{C} \cup \infty$. Let E_2 and E_3 be the trivial vector bundles over the interior and exterior of the unit circle S^1 with the fiber $\mathbb{R}^m$. Using $u : S^1 \to U(\mathbb{R}^m)$ as a gluing map, we may glue the bundles E_2 and E_3 together along S^1 and get a new bundle E over S^2. This procedure also yields a connection on E which is τ-flat for $m \to \infty$, because $\|\nabla(u)\| \leq 2\pi l/m$.

In general, one may construct τ-flat G-bundles from those in the non-equivariant cases as long as the constructions in the latter cases can be made equivariantly.

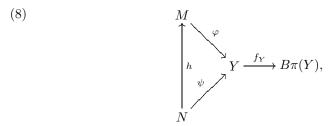## 5. Equivariant Novikov Conjecture for Euclidean Buildings

We begin with a definition of the equivariant Novikov conjecture. Let M be a closed, oriented Riemannian manifold and G a compact Lie group acting on M by isometries. Denote by $\pi(M)$ the fundamental groupoid of M, which is defined to be the set of homotopy classes (rel endpoints) of all paths in M. $\pi(M)$ can be given by $\pi(M) = \frac{\tilde{M} \times \tilde{M}}{\pi_1(M)}$, where $\tilde{M}$ is the universal covering space of M. G acts naturally on $\pi(M)$. Let $B\pi(M)$ be the equivariant classifying space of $\pi(M)$ (cf. May's appendix to [RoW2]) and $f_M : M \to B\pi(M)$ be the equivariant classifying map. If $h : N \to M$ is a G-pseudo-equivalence from another closed, oriented Riemannian G-manifold N, i.e., h is G-invariant and is a homotopy equivalence, then h induces the maps $h_* : \pi(N) \to \pi(M)$ and $h_* : B\pi(N) \to B\pi(M)$. Hence the following diagram of equivariant K-homology groups is commutative:

$$(7) \qquad \begin{array}{ccc} K_*^G(M) & \xrightarrow{(f_M)_*} & K_*^G(B\pi(M)) \\ \uparrow{\scriptstyle h_*} & & \uparrow{\scriptstyle h_*} \\ K_*^G(N) & \xrightarrow{(f_N)_*} & K_*^G(B\pi(N)). \end{array}$$

Let D_M be the equivariant signature operator on M. Then D_M defines an element $[D_M]$ in $K_*^G(M)$, which is in $K_0^G(M)$ for the even dimensional M and in $K_1^G(M)$ otherwise. Although the identity $h_*(f_N)_*([D_N]) = (f_M)_*([D_M])$ is not true in general, in view of (7), one may still expect that this identity holds for some special cases. More generally, given a G-equivariant commutative diagram

$$(8) \qquad \begin{array}{c} M \\ \\ \\ \end{array}$$

$$\begin{array}{c} M \\ \uparrow \\ h \Big| \quad {\scriptstyle \varphi} \searrow \\ \Big| \qquad\qquad Y \xrightarrow{\ f_Y\ } B\pi(Y), \\ \quad {\scriptstyle \psi} \nearrow \\ N \end{array}$$

where Y is a G-space, h is a G-pseudo-equivalence, and φ and ψ are G-equivariant maps, one conjectures the following:

$\underline{ENC_Y}$ ([RoW2]). $(f_Y)_*\varphi_*([D_M]) = (f_Y)_*\psi_*([D_N])$ in $K_*^G(B\pi(Y))$, provided $K_*^G(B\pi(Y))$ is finitely generated over the representation ring $R(G)$ of G.

Rosenberg and Weinberger have proved this conjecture for Y a complete Riemannian manifold of nonpositive curvature, provided $K_*^G(Y)$ is a finitely generated module over $R(G)$. The condition on $K_*^G(Y)$ plays an important role in [RoW2], due to the lack of the equivariant Mishchenko symmetric signature for general compact Lie group actions and other technical difficulties. The unsolved problem in this case is to remove the condition on $K_*^G(Y)$ and to prove the existence of the equivariant maps φ and ψ in (8) for a general manifold M. On the other hand, it is desirable to verify ENC_Y for those Y whose universal coverings are equivariantly isomorphic to the geometric realizations of Euclidean buildings, since Euclidean buildings are natural analogues of complete Riemannian manifolds of nonpositive curvature. The recent work of Kasparov-Skandalis [KaS] on Euclidean buildings enables us to verify ENC_Y for such Y. The goal of this section is to carry out the

proof of the equivariant Novikov conjecture for the above mentioned Y by using Theorem 1 in Section 3. Meanwhile, we will also get rid of the assumption on $K_*^G(Y)$. The existence of the G-maps φ and ψ in (8) will be considered in Section 6.

We now recall the definition of Euclidean buildings (cf. [Bro] [KaS] [Tits] for more details). Let X be a simplicial complex of dimension n and B its geometric realization. X can be considered as a set of its faces, $X = \bigcup_{0 \le k \le n} X^k$. X is said to be typed if there is a map $\theta : X^0 \to \{0, 1, \ldots, n\}$ such that for every simplex $x \in X$, the images under θ of the vertices of x are pairwise different. θ is called a type of X. There is a typed simplicial complex X_1 associated with a given simplicial complex X of dimension n such that X_1 and X have the same geometric realization. We use the notation that chambers are the simplices of dimension n; walls are the simplices of dimension $n-1$, and apartments are subcomplexes of X determined by the Weyl system.

Definition 4. (X, B) is called a Euclidean building if

(i) B has a metric such that the apartments are affinely isometric to the Euclidean space $\mathbb{R}^n$;

(ii) any pair of simplices of X is contained in an apartment;

(iii) the intersection $S \cap S'$ of any two apartments S and S' is convex and there is a simplicial isometry $j : S \to S'$ such that j is the identity map on $S \cap S'$ and preserves the type, i.e., $\theta(j(x)) = \theta(x), \forall x \in S$;

(iv) for any two chambers σ and σ' of an apartment S there is a type-preserving simplicial isometry $j : S \to S$ mapping σ to σ'.

This definition of a Euclidean building is slightly different from the one in [Tits]. In fact, every Bruhat-Tits Euclidean building is a Euclidean building in the above sense.

Example 2. (*a*) Let F be a field with a discrete valuation and $SL_n(F)$ the group of $n \times n$ matrices over F with determinant 1. Then we can associate with $SL_n(F)$ a Euclidean building. In particular, this building for $SL_2(\mathbb{Q})$ is a tree [Bro].

(*b*) The universal covering space of a complete Riemannian manifold of nonpositive curvature can be considered as a geometric realization of some topological building [BuS]. For more examples of Euclidean buildings we refer to [Bro].

Note that the geometric realization B of a Euclidean building X is contractible [Bro]. Let Γ_0 be a discrete group which acts properly and freely on the building (X, B) by type-permuting isometries. Here the action of Γ_0 is called type-permuting if there is a group homomorphism $\tilde{\pi} : \Gamma_0 \to S_n$, the permutation group of $\{0, 1, \ldots, n\}$, such that $\theta(g(x)) = \tilde{\pi}(g)\theta(x), \forall g \in \Gamma_0, x \in X$. The action is called type-preserving if $\tilde{\pi}(g) = I, \forall g \in \Gamma_0$. Then the universal covering space of $Y = B/\Gamma_0$ is B. We assume that G acts on Y by isometries via a homomorphism of G into $Isom(Y)$, the isometry group of Y, such that the lifting of the G-action to B is type-permuting. More precisely, there is a locally compact group Γ in $Isom(B)$ such that Γ is a group extension of Γ_0 by G, $1 \to \Gamma_0 \to \Gamma \to G \to 1$, and Γ acts on B by type-permuting isometries. The importance of Γ is that $C_r^*(\Gamma)$ is strongly Morita equivalent to the semidirect product $C_r^*(\pi(Y)) \rtimes G$ of $C_r^*(\pi(Y))$ and G [RoW2], where $C_r^*(\Gamma)$ (resp., $C_r^*(\pi(Y))$) is the reduced group (resp., groupoid) C^*-algebra

of Γ (resp., $\pi(Y)$). Hence

$$K_*^G(C_r^*(\pi(Y))) \simeq K_*(C_r^*(\pi(Y)) \rtimes G) \simeq K_*(C_r^*(\Gamma)).$$

To consider the ENC_Y for $Y = B/\Gamma_0$, we need a non-Hausdorff smooth manifold M_X associated with (X, B). Let us first give the following general definition.

Definition 5. We call $(U_i, U_{ij}, \varphi_{ij})_{i,j \in J}$ a Γ-atlas if it satisfies the following:

(i) Γ acts on the index set J and U_i such that $g(U_i) = U_{g(i)}, i \in J, g \in \Gamma$;

(ii) each U_i is a Hausdorff smooth manifold;

(iii) $U_{ii} = U_i$, and for $i, j \in J, U_{ij}$ is an open subset in U_i;

(iv) $\varphi_{ij} : U_{ji} \to U_{ij}$ is a diffeomorphism such that $\varphi_{ii} = I_{U_i}$ and $\varphi_{ij}(gx) = g\varphi_{ij}(x), g \in \Gamma$.

We may associate a manifold M_0 to a Γ-atlas $(U_i, U_{ij}, \varphi_{ij})_{i,j \in J}$ by taking M_0 to be the quotient of the disjoint union of $U_i, i \in J$, by the equivalent relation "$x \sim \varphi_{ij}(x)$", $x \in U_{ji}$. M_0 is in general non-Hausdorff. As one can easily check, M_0 is Hausdorff iff the maps $(r, s) : U_{ij} \to U_i \times U_j$ given by $r(x) = x$ and $s(x) = r(\varphi_{ji}(x))$ for $x \in U_{ij}$ are proper.

The non-Hausdorff manifold M_X associated to the building (X, B) can be defined as follows. Let E be the affine Euclidean space $E = \{t = (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1\}$ and $\Sigma = \{t \in E : t = (t_0, \dots, t_n), t_i \geq 0, \forall i\}$. Since Σ is a convex set in E, we can define a continuous map $q : E \to E$ by the formula $\|q(t) - t\| = \inf\{\|t - s\| : s \in \Sigma\}$. For a subset O of $\{0, 1, \dots, n\}$ let F_O be the face of Σ defined by $F_O = \{t \in \Sigma : t = (t_0, \dots, t_n) : i \in O \text{ if } t_i \neq 0\}$. Denote by Ω_O the interior in E of $q^{-1}(F_O)$. Obviously, $F_\emptyset = \Omega_\emptyset = \emptyset$. Let $U_x = E, x \in X, U_{x,y} = \Omega_{\theta(x \cap y)}$ for $x \neq y$ in X and $\varphi_{x,y} = I$. Then $\{U_x, U_{x,y}, \varphi_{x,y}\}_{x,y \in X}$ is a Γ-atlas. We define $M_X = U^0/\sim$, where U^0 stands for the disjoint union of the $U_x, x \in X$. M_X is a non-Hausdorff smooth Γ-manifold endowed with a Γ-invariant Riemannian metric, since its tangent bundle is trivial. Indeed, fibers of the tangent bundle of M_X are the space of tangent vectors to E. The crucial property of M_X is that M_X is Γ-equivariantly homotopy equivalent to B (cf. [KaS], Lemma 4.8), i.e., there are Γ-equivariant maps $f_1 : M_X \to B$ and $f_2 : B \to M_X$ such that $f_1 f_2$ and $f_2 f_1$ are Γ-equivariantly homotopic to the identity maps, respectively. We have the diagram

$$(9) \qquad \begin{array}{ccc} B & \underset{f_1}{\overset{f_2}{\rightrightarrows}} & M_X \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ Y = B/\Gamma_0 & \underset{f_3}{\overset{f_4}{\rightrightarrows}} & Y_X = M_X/\Gamma_0, \end{array}$$

where P is the natural projection; f_3 and f_4 are defined by the commutativity of the diagram. Then Y is G-equivariantly homotopy equivalent to Y_X via the maps f_3 and f_4. The advantage of introducing Y_X is that one can use various KK-theoretic information about non-Hausdorff special manifolds. We now reduce ENC_Y to the following problems:

(1) to construct an injective map $\beta : K_*^G(Y) \to K_*^G(C_r^*(\pi(Y)))$;

(2) to show the G-pseudo-equivalence of $\beta(\varphi_*([D_M]))$.

To define the map β, let us assume that G has a fixed point in Y. This condition is not necessary for the construction of β, but it will be needed in Proposition 3 below. So we use it here as well to simplify the discussion. Thus G acts on the

fundamental group Γ_0 of Y, and $\Gamma = \Gamma_0 \rtimes G$, a semidirect product of Γ_0 and G. By Theorem 2.5 of [RoW2], $K_*^G(C_r^*(\pi(Y))) = K_*^G(C_r^*(\Gamma_0))$. Let V be the G-$C_r^*(\Gamma_0)$-bundle over Y whose fiber V_y at $y \in Y$ is the C^*-algebra $C_r^*(\Gamma_0)$, namely, $V = B \times_{\Gamma_0} C_r^*(\Gamma_0)$. V is a flat G-$C_r^*(\Gamma_0)$-bundle. Note that the restriction of V to a G-compact subset $Z \subset Y$ yields a G-$C_r^*(\Gamma_0)$-bundle over Z, and hence defines an element $[\beta_Z]$ in $K_0^G(C^*(Z) \otimes C_r^*(\Gamma_0))$. Let β_Z^* be the homomorphism from $K_*^G(Z)$ to $K_*^G(C_r^*(\Gamma_0))$ given by the Kasparov product with $[\beta_Z]$. The map β_Z^* respects the inductive limit and gives rise to a map

$$\beta : K_*^G(Y) = \lim_{\to Z \subset Y} K_*^G(Z) \to K_*^G(C_r^*(\Gamma_0)).$$

Furthermore, let U be the completion of the space of all continuous sections of V with compact support in Y, and $\varphi(f) = f I_U, f \in C(Y)$. Then $(U, \varphi, 0)$ is a Kasparov G-$C(Y)$-$\mathcal{A} \otimes C(Y)$-module with $\mathcal{A} = C_r^*(\Gamma_0)$, and defines an element $[\beta_Y] \in KK^G(C(Y), \mathcal{A} \otimes C(Y))$ (cf. [Kas]). We can consider $[\beta_Z]$ as a restriction of $[\beta_Y]$ to $W \subset Y_X$.

We are now going to interpret β as a map coming from a Dirac element on M_X. To this aim, we need a new G-C^*-algebra $C_\tau^*(U)$. Let $C^*(U)$ be the G-C^*-algebra of the groupoid $U = \bigcup_{x,y \in X} U_{x,y}$ associated with the covering $\{U_x, U_{x,y}, I\}_{x,y \in X}$ of M_X. $C^*(U)$ is the completion of $C_c(U) = \bigoplus_{x,y \in X} C_c(U_{x,y})$ with the norm given by $\|f\| = \sup\{\|f_s\|, s \in M_X\}$ for $f = \bigoplus_{x,y} f_{x,y}$ in $C_c(U)$, and $\|f_s\|$ is the operator norm on $H_s = l^2(\{x \in X; s \in U_x\})$, since f defines a finite rank operator f_s on H_s by $\langle e_x, f_s(e_y) \rangle = f(x, y, s) = f_{x,y}(s)$. The product of $C^*(U)$ is the convolution given by $(f_1 f_2)_{x,y} = \sum_z f_{1,x,z} f_{2,z,y}$ for $f_i = \bigoplus_{x,y} f_{i,x,y}$ in $C_c(U)$. As pointed out in [KaS], $C^*(U)$ is independent of the covering of M_X up to Morita equivalence. Let $C_\tau^*(U) = C^*(U) \hat{\otimes} Cliff(E)$, where $Cliff(E)$ is the Clifford algebra associated with the positive quadratic form of E; i.e., it is the C^*-algebra generated by E subject to the relations $\xi = \xi^*, \xi^2 = \|\xi\|^2, \xi \in E$.

Lemma 16 ([KaS])**.** *Let (X, B) be a locally finite Euclidean building and M_X its associated non-Hausdorff manifold.*

(a) Let p be a trivial map from M_X to a fixed point. Then $p_!$ defines a Gysin element $p_! \in KK^\Gamma(C_\tau^*(U), \mathbb{C})$ which is called the Dirac element D_X.

(b) There is an element $\eta_X \in KK^\Gamma(\mathbb{C}, C_\tau^*(U))$ such that

$$D_X \otimes_{\mathbb{C}} \eta_X = I_{C_\tau^*(U)} \in KK^\Gamma(C_\tau^*(U), C_\tau^*(U)).$$

With the above preparation, we can prove the following injectivity of the assembly map β.

Proposition 3. *Let (X, B) be a locally finite Euclidean building and Γ_0 a finitely generated discrete group acting on (X, B) properly and freely by type permuting isometries. Suppose G is a compact Lie group acting on $Y = B/\Gamma_0$ by isometries with a fixed point. Then $\beta : K_*^G(Y) \to K_*^G(C_r^*(\Gamma_0))$ is injective.*

Proof. As shown by Kasparov ([Kas], Theorems $4.10, 6.6$ and 6.7), in view of [KaS], we have that the map β is the composite

$$K_*^G(Y_X) \overset{\text{Poincaré duality}}{\simeq} K_*^G(C_\tau^*(U/\Gamma_0))$$

$$\overset{\text{Green-Julg Thm.}}{\simeq} K_*(C_\tau^*(U/\Gamma_0) \rtimes G)$$

$$\overset{\text{Morita equiv.}}{\simeq} K_*(C_\tau^*(U) \rtimes \Gamma)$$

$$\overset{j^\Gamma[D_X]}{\rightarrow} K_*(C_r^*(\Gamma)) \simeq K_*^G(C_r^*(\pi(Y))) \simeq K_*^G(C_r^*(\Gamma_0)),$$

where $\Gamma = \Gamma_0 \rtimes G$ and j^Γ is the reduction map ([Kas], Theorem 3.11), $j^\Gamma[D_X] \in KK(C_\tau^*(U) \rtimes \Gamma, C_r^*(\Gamma))$. This is similar to those in [RoW2] for manifolds of non-positive curvature. The Morita equivalence above follows from the fact that by Theorem 1 of [CMW] and Theorem 3.13 of [Kas],

$$K_*(C_\tau^*(U/\Gamma_0) \rtimes G) \simeq K_*(C_\tau^*(U)^{\Gamma_0} \rtimes G)$$

$$\overset{\text{Morita equiv.}}{\simeq} K_*((C_\tau^*(U) \rtimes \Gamma_0) \rtimes G) \simeq K_*(C_\tau^*(U) \rtimes \Gamma).$$

The Poincaré duality follows from Theorem 6.8 of [KaS] and the fact that, by (9), the non-Hausdorff manifold $Y_X = M_X/\Gamma_0$ and Y are G-equivariantly homotopy equivalent. By Lemma 16, $j^\Gamma[D_X] \otimes_{C_r^*(\Gamma)} j^\Gamma[\eta_X] = I_{C_\tau^*(U) \rtimes \Gamma}$. We obtain that β is injective.   □

We should point out that the condition on the fixed point of G in Proposition 3 is only to guarantee the Morita equivalence in the above proof. Another extremal case is when G acts freely on (X, B). By the proof of Proposition 3 and Theorem 3.13 of [Kas], the Morita equivalence holds also for this case.

We now prove the homotopy invariance of the higher equivariant signature for groups acting on Euclidean buildings. Note that the action of a locally compact group on a Euclidean building is given by a homomorphism from the group to the isometry group of the building.

Theorem 5. *Let (X, B) ba a locally finite Euclidean building, Γ_0 a discrete group acting on (X, B) properly and freely by type permuting isometries. Suppose that the compact Lie group G acts on $Y = B/\Gamma_0$ by isometries with a fixed point. Then for every G-equivariant commutative diagram (8), $\varphi_*([D_M]) = \psi_*([D_N])$.*

Proof. By Proposition 3, we know that $\beta : K_*^G(Y) \rightarrow K_*^G(C_r^*(\Gamma_0))$ is injective. It is thus sufficient to show that $\beta(\varphi_*([D_M])) = \beta(\psi_*([D_N]))$. Hence, the problem reduces to the G-pseudo-equivalence of the invariant $\beta(\varphi_*([D_M]))$. In view of the construction of β above, the KK-index theorem [RoW1] shows that

$$G - Sig(D_{\varphi^*(V)}) = \beta(\varphi_*([D_M])),$$

where V is the G-$C_r^*(\Gamma_0)$-bundle over Y constructed in the definition of β. Theorem 1 implies that G-$Sig(D_{\varphi^*(V)})$ is a G-pseudo-equivalence invariant.   □

Now Theorem 3 follows immediately from Theorem 5.

Proof of Theorem 3. Take $\Gamma_0 = \{1\}$. Then G acts on B. In view of the Bruhat-Tits fixed-point theorem ([Bro], p.157), G has a fixed point. Hence Theorem 5 is applicable. In fact, when $\Gamma_0 = \{1\}$, the Morita-equivalence is trivial and Proposition 3 is true without the condition on the fixed point.   □

Combining Theorem 1 with the proof above (cf. also [RoW2]), we can remove the finite generation condition on $K_*^G(Y)$ over $R(G)$ in the Rosenberg-Weinberger theorem. The finite generation condition of $K_*^G(Y)$ enables one to reduce the compact group action to the finite group action using the McClure theorem.

Theorem 6. *Let Y be a complete Riemannian G-manifold of nonpositive curvature in the commutative diagram* (8). *Suppose G has a fixed point in Y. Then $\varphi_*([D_M]) = \psi_*([D_N])$ in $K_*^G(Y)$.*

In particular, if Y is a complete simply connected Riemannian G-manifold of nonpositive curvature , then by the Cartan theorem (cf. [Bro], p. 157), G has a fixed point in Y. Hence the condition on the fixed point in Theorem 6 is satisfied in this case.

6. Existence of Equivariant Maps

In this section we will prove Theorem 4, which is about the existence of the equivariant maps in (8). In [KoS] Korevaar and Schoen proved the existence of a Γ_0-equivariant map from $\tilde{M}$ to a metric space W of nonpositive curvature, where Γ_0 is a finitely generated discrete group acting on $\tilde{M}$ and W by isometries (cf. also [GrS]). We want to extend their theorem to the Γ-equivariant case, where Γ is the extension of Γ_0 by the compact Lie group G,

$$1 \to \Gamma_0 \to \Gamma \to G \to 1.$$

In our situation, we take $\Gamma_0 = \pi_1(M)$ and

$$\Gamma = \{\beta_g = (g, \psi_g) \in G \times \mathit{Diff}(\tilde{M}) : \pi_M \psi_g = g\pi_M\},$$

where $\pi_M : \tilde{M} \to M$ is the natural projection. ψ_g is called a lifting of $g \in G$ to $\tilde{M}$. Any two liftings of g differ by an element of Γ_0. Γ is a locally compact group and acts on $\tilde{M}$ via ψ_g. Let $\rho : \Gamma \to \mathit{Isom}(W)$ be a continuous homomorphism, where the isometry group $\mathit{Isom}(W)$ of W is endowed with a compact-open topology. We thus have an action of Γ on W via ρ.

Recall that a complete metric space W is of nonpositive curvature (cf., for instance, [KoS]) if: 1) W is a length space, i.e., the distance $d(x, y)$ of any two points x and y in W can be realized as the length of a rectifiable curve connecting x to y (those distance-realizing curves are called geodesics); 2) for any three points x, y, z in W, let ν_{xy}, ν_{yz} and ν_{zx} be three geodesics connecting these three points, respectively. If $\bar{y}$ is the midpoint on ν_{yz}, i.e., $d(y, \bar{y}) = \frac{1}{2}d(y, z) = d(\bar{y}, z)$, then

$$d^2(x, \bar{y}) \le \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z).$$

A locally compact metric space of nonpositive curvature must be simply connected.

We now want to prove Theorem 4.

Proof of Theorem 4. **Step 1.** Existence. The proof is to extend the argument for Γ_0-equivariant maps [KoS] to our case. Let $\tilde{M}_0$ be a compact fundamental domain of Γ_0 in $\tilde{M}$ such that the boundary of $\tilde{M}_0$ has zero measure. $\tilde{M}$ is a union of all $\gamma \tilde{M}_0, \gamma \in \Gamma_0$, up to a zero measure set. Fix a point $q_0 \in W$ and define a map f from Γ to W as

$$f(\beta_g) = \rho(\beta_g)q_0.$$

Obviously, f is a Γ-equivariant map. Let μ be the Riemannian volume measure on $\tilde{M}$ which is the lifting of a Riemannian volume measure on M. For $x \in \tilde{M}$, denote by $B(x, 1)$ the unit ball in $\tilde{M}$ centered at x. Define a measure μ_x on Γ by setting

$$d\mu_x(\beta_g) = \frac{\mu(B(x, 1) \cap \beta_g(\tilde{M}_0))}{\mu(B(x, 1))} d\mu_G(g),$$

where μ_G is the normalized Haar measure on G. For $g \in G$, let

$$\Gamma_g = \{\beta_g \in \Gamma : \beta_g = (g, \psi_g), \psi_g \text{ is a lifting of } g \text{ to } \tilde{M}\}.$$

Then $\Gamma_g \simeq \Gamma_0$. Using the fact that, for a fixed $g \in G$,

$$\mu(B(x, 1) \cap \bigcup_{\beta_g \in \Gamma_g} \beta_g(\tilde{M}_0)) = \mu(B(x, 1)),$$

we have that $d\mu_x$ is a probability measure on Γ. Furthermore, one can easily check that $d\mu_x$ is Γ-equivariant, i.e., $d\mu_{\beta_{g'}(x)}(\beta_{g'}\beta_g) = d\mu_x(\beta_g)$. Consider the integral

$$I_x(q) = \int_\Gamma d^2(f(\beta_g), q) d\mu_x(\beta_g), \quad q \in W,$$

where $d(f(\beta_g), q)$ is the distance between $f(\beta_g)$ and q in W. Clearly, this integral is well defined (see (10), below). By Lemma 2.5.1 of [KoS], there exists a unique point $\bar{f}_x$ in W, called the center of mass of f, that minimizes the integral $I_x(q)$. We claim that $\bar{f}_x$ is Γ-equivariant in x, namely, $\bar{f}_{\beta_g(x)} = \rho(\beta_g)(\bar{f}_x)$. Indeed, by the Γ-equivariance of $d\mu_x$, we have that $I_{\beta_g(x)}(q) = I_x(\beta_g^{-1}(q))$. The uniqueness of the center of mass implies the claim.

We define a map $u : \tilde{M} \to W$ by $u(x) = \bar{f}_x$. u is Γ-equivariant.

Step 2. Continuity. Let $x \in M$ and $\Gamma_x = \{\beta_g \in \Gamma : \mu(\beta_g(\tilde{M}_0) \cap B(x, 1)) \neq 0\}$. By the Γ-equivariance of u, it is sufficient to check the continuity of u at x in an ε-neighborhood of $\tilde{M}_0$ ($\varepsilon > 0$ very small). We first prove that there exists a constant $k > 0$ such that for all $\beta_g \in \Gamma_x$,

$$(10) \qquad\qquad\qquad d(f(\beta_g), q_0) < k.$$

Indeed, let $g \in G$, $\beta_g = (g, \psi_g) \in \Gamma_x$, and $\{\gamma_1, \cdots, \gamma_r\}$ be a finite set of generators of Γ_0. Then there is a constant $k_g > 0$ such that for every lifting ψ_g' of g with $\beta_g' = (g, \psi_g') \in \Gamma_x$, ψ_g' differs from ψ_g only by a $\gamma \in \Gamma_0$ and the word length $|\gamma|$ of γ with respect to the above finite set of generators is less than k_g. By the covering property of the projection $\Gamma \to \Gamma/\Gamma_0 \simeq G$ and the continuity of ρ, for any $g' \in G$ very close to g, we can choose a lifting $\psi_{g'}$ of g' to $\tilde{M}$, $\beta_{g'} = (g', \psi_{g'}) \in \Gamma_x$, such that $d(\beta_{g'}(\tilde{M}_0), \beta_g(\tilde{M}_0)) < \varepsilon_g$ for some constant $\varepsilon_g > 0$. Thus for any other lifting $\psi_{g'}'$ of g', $\beta_{g'}' = (g', \psi_{g'}') \in \Gamma_x$, $\psi_{g'}'$ differs from $\psi_{g'}$ by a $\gamma' \in \Gamma_0$ with $|\gamma'| < k_g$. This is true for $g' \in U_g$, where U_g is a small neighborhood of g in G. We have

$$\begin{aligned}
d(f(\beta_{g'}'), q_0) &= d(\rho(\beta_{g'}')\gamma' q_0, q_0) \\
&\leq d(q_0, \rho(\beta_{g'}')^{-1} q_0) + d(\gamma' q_0, q_0) \\
&\leq d(f(\beta_{g'}), q_0) + k_g \max_{1 \leq j \leq r} d(\gamma_j q_0, q_0) \\
&\leq c_g d(f(\beta_g), q_0) + k_g \max_{1 \leq j \leq r} d(\gamma_j q_0, q_0),
\end{aligned}$$

for some constant $c_g > 0$ depending only on $g \in G$. Here we have used the fact that $\rho(\beta_{g'})$ is an isometry on W. Since G is compact, we can use a finite number of U_g's to cover G. This proves (10). Hence, it follows that for $\beta_g \in \Gamma_x$, $f(\beta_g)$

lies in a convex ball $B(q_0, k) \subset W$. Proposition 2.5.4 of [KoS] then shows that $u(x) \in B(q_0, k)$. We get

$$d(u(x), f(\beta_g)) \leq 2k. \tag{11}$$

Now for $y \in \tilde{M}$ very close to x, we get by a direct computation that

$$\int_\Gamma |d\mu_x(\gamma) - d\mu_y(\gamma)|$$

$$= \int_G \int_{\Gamma_g} |\frac{\mu(B(x,1) \cap \beta_g(\tilde{M}_0))}{\mu(B(x,1))} - \frac{\mu(B(y,1) \cap \beta_g(\tilde{M}_0))}{\mu(B(y,1))}| d\mu_G(g)$$

$$\leq c_x |x - y|,$$

for some constant $c_x > 0$ depending only on x. This together with (11) and Proposition 2.5.2 of [KoS] proves that

$$\begin{aligned} d^2(u(x), u(y)) &\leq 2d(u(x), u(y)) \int_\Gamma d(f(\gamma), u(x)) |d\mu_x(\gamma) - d\mu_y(\gamma)| \\ &\leq 4kc_x d(u(x), u(y)) |x - y|. \end{aligned}$$

Hence, u is continuous. $\qquad\square$

Let $\Gamma_0' = \rho(\Gamma_0)$ be discrete. Take $Y = W/\Gamma_0'$ in (8). Then G acts on Y via the homomorphism ρ. By pushing down the Γ-equivariant map in Theorem 4, we get

Corollary 1. *With the condition of Theorem 4, assume further that $\Gamma_0' = \rho(\Gamma_0)$ is discrete. If $Y = W/\Gamma_0'$, then there exists a G-equivariant map φ from M to Y.*

Corollary 1 is applicable to the case where W is the geometric realization of a Euclidean building. Another special case of Theorem 4 is when M is simply connected. Then $\Gamma = G$ acts on W via ρ. We conclude that there is a G-map from M to W in this case. See also [RoW2] for other special cases of Theorem 4.

Finally, we want to draw a conclusion on the equivariant Novikov conjecture of Baum-Connes-Higson [BCH]. Let H be a Lie group and W an oriented Riemannian manifold with a smooth, proper orientation-preserving H-action such that W/H is compact. Here the properness is in the sense of [BCH]. There is a universal classifying space $\underline{EH}$ for the spaces with proper H-actions. Thus, we have an H-map from W to $\underline{EH}$. The equivariant Novikov conjecture of [BCH] is that the H-equivariant signature $H\text{-}Sig(W)$ of W is an H-pseudo-equivalence invariant after injecting into $K_*^H(\underline{EH})$. In the situation of Theorem 5, we have a G-map from Y to $\underline{EG} = \{pt\}$. Taking $H = G$ and $W = M$, we get

Corollary 2. *With the condition of Theorem 5, the equivariant Novikov conjecture of [BCH] is true, namely, $G\text{-}Sig(M)$ is a G-pseudo-equivalence invariant in $K_*^G(pt)$.*

The argument of the present paper could be used to consider the above conjecture for more general cases.

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