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MORITA EQUIVALENCE FOR CROSSED PRODUCTS BY HILBERT C^*-BIMODULES

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Abstract. We introduce the notion of the crossed product $A \rtimes_X \mathbb{Z}$ of a C^*-algebra A by a Hilbert C^*-bimodule X. It is shown that given a C^*-algebra B which carries a semi-saturated action of the circle group (in the sense that B is generated by the spectral subspaces B_0 and B_1), then B is isomorphic to the crossed product $B_0 \rtimes_{B_1} \mathbb{Z}$. We then present our main result, in which we show that the crossed products $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are strongly Morita equivalent to each other, provided that A and B are strongly Morita equivalent under an imprimitivity bimodule M satisfying $X \otimes_A M \simeq M \otimes_B Y$ as $A - B$ Hilbert C^*-bimodules. We also present a six-term exact sequence for K-groups of crossed products by Hilbert C^*-bimodules.

1. Introduction

If A is a C^*-algebra and α is an automorphism of A, let B be the crossed product C^*-algebra $B = A \rtimes_\alpha \mathbb{Z}$ ([17]). It is well known that B carries a natural action of the circle group, called the dual action, such that the spectral subspaces B_n ([12]) are naturally identified with Au^n, where u is the canonical unitary implementing the automorphism. The B_n then form a $\mathbb{Z}$-grading of B, which has the property that B is generated by (B_0 and) B_1.

Now, suppose that one is given a circle action on a C^*-algebra C, which is semi-saturated in the sense that C is generated by the spectral subspaces C_0 and C_1. A natural question to ask is whether C is isomorphic to a crossed product of C_0 by some automorphism.

The answer to this question is obviously no, as one could be dealing with the trivial action of the circle, a situation that could not possibly arise as a dual action. An affirmative answer can be given, however, when the action is regular ([12]), although this requires an extension of the concept of crossed products to the realm of partial automorphisms. The goal of the present work is, precisely, to study the case in which the regularity property is absent. The important ingredient one must rely upon is the Hilbert C^*-bimodule ([6]) over the fixed-point algebra, which is provided by the first spectral subspace.

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Starting with an arbitrary Hilbert C^*-bimodule X over a C^*-algebra A, we define a crossed product $B = A \rtimes_X \mathbb{Z}$. Here, the bimodule X replaces, in a sense, the automorphism α, as above. We then prove that, given a semi-saturated circle action on a C^*algebra B, one must have $B = B_0 \rtimes_{B_1} \mathbb{Z}$. Examples of this situation are given for the case of partial automorphisms, and also for quantum Heisenberg manifolds, where application of our techniques is crucial, as there are no partial automorphisms available.

The regularity property, which we do not assume, is granted once one tensors everything by the algebra of compact operators. We may, therefore, use the generalized Pimsner-Voiculescu sequence of [12] to prove a similar result for the situation considered here, provided that we have separability.

The main result of this work, however, is related to the concept of strong Morita equivalence ([19]). That is, suppose we are considering two crossed products by Hilbert C^*-bimodules as above, say $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$, in which the underlying algebras A and B are known to be strongly Morita equivalent to each other, under an imprimitivity bimodule M. The question whether $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are strongly Morita equivalent to each other is our main concern, and we prove this provided that one has that $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic as $A - B$ Hilbert C^*-bimodules. This should be thought of as a generalization of the result obtained independently by Curto, Muhly and Williams ([9]) on one hand, and Combes ([8]) on the other, in which a necessary condition is given for two strongly Morita equivalent C^*-algebras to remain strongly Morita equivalent after one takes their crossed product by a locally compact group. Related results were obtained by Bui ([7]), Kaliszewski ([15]), and Echterhoff ([10]).

The construction described by Pimsner in [18] is, in a sense, a generalization of our crossed products by Hilbert C^*-bimodules. In fact, one can show that the algebra $\tilde{\mathcal{O}}_X$, constructed by Pimsner, is isomorphic to the algebra $A \rtimes_X \mathbb{Z}$ which we define here. Conversely, since Pimsner's algebra $\tilde{\mathcal{O}}_X$ carries a semi-saturated circle action, it also follows that his construction is a special case of ours. The drawback is, of course, that no good characterization of the fixed-point algebra for this action is available.

Pimsner also provides a six-term exact sequence for K-groups of the algebras arising from his construction; but, due to the fact that the main results in [18] are obtained under the hypothesis that the algebra A is generated by the scalar products, there is no immediate relationship between Pimsner's results and ours. It seems reasonable, nevertheless, to expect that a common generalization could be found.

Last, but not least, we would like to mention that crossed products by Hilbert C^*-bimodules have also been studied by Larry Brown, although no written version of his work is available at the moment.

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2. Crossed products by Hilbert C^*-bimodules

Definition 2.1. Let a C^*-algebra A and an $A - A$ Hilbert C^*-bimodule X (in the sense of [6, 1.8]) be given. A *covariant representation* of (A, X) is a pair (π_A, π_X)

of representations into some $\mathbb{B}(\mathfrak{H})$ such that *both* module actions and *both* inner products become the ones inherited from $\mathbb{B}(\mathfrak{H})$, i.e.

(i) $\pi_X(ax) = \pi_A(a)\pi_X(x)$,
(ii) $\pi_X(xa) = \pi_X(x)\pi_A(a)$,
(iii) $\langle x, y\rangle_L = \pi_X(x)\pi_X(y)^*$,
(iv) $\langle x, y\rangle_R = \pi_X(x)^*\pi_X(y)$,

where $a \in A$, $x, y \in X$; cf. [11, 4.5].

We start out by noting that faithful covariant representations always exist. By a *restriction* $\pi|B$ of a representation π into $\mathbb{B}(\mathfrak{H})$ of a C^*-algebra A to a subalgebra B we will always mean the representation on the essential Hilbert space $\overline{\pi(B)\mathfrak{H}}$. We denote by $(\pi_A^a, \mathfrak{H}_A^a)$ the reduced atomic representation of A ([17, 4.3.7]). If X and Y are subsets of a C^*-algebra, we denote by XY the closed span of the set $\{xy|x \in X, y \in Y\}$. If X and Y are, respectively, $A-B$ and $B-C$ Hilbert C^*-bimodules, we denote by $X^\sim$ the dual $B-A$ Hilbert C^*-bimodule, and by $X \otimes_B Y$ the tensor product, with structure as in [19, 6.17,5.9]. We indicate also by the "$\sim$" that an element of the set X is being considered as an element of the Hilbert C^*-bimodule $X^\sim$.

Lemma 2.2. *Let B be a hereditary subalgebra of the C^*-algebra A. There exists a cardinal $\mathfrak{c}_0$ such that whenever $\mathfrak{c} \geq \mathfrak{c}_0$,*

$$\mathfrak{c}\pi_A^a|_B \simeq \mathfrak{c}\pi_B^a,$$

where "$\simeq$" denotes unitary equivalence.

Proof. An irreducible representation of B extends uniquely to one of A by [17, 4.1.8]. On the other hand, the restriction of any irreducible representation of A to B is also an irreducible representation (or the trivial map) by [17, 4.1.5]. We put $\mathfrak{c}_0 = \max\{\aleph_0, \operatorname{card}\hat{A}\}$, and assume that $\mathfrak{c} \geq \mathfrak{c}_0$. As every irreducible representation of B occurs at least once in π_A^a, it occurs exactly $\mathfrak{c}$ times in $\mathfrak{c}\pi_A^a$ up to unitary equivalence. □

Proposition 2.3. *Given (A, X) as above, for a large enough cardinal $\mathfrak{c}$ there exists a covariant representation (π_A, π_X) on $\mathfrak{c}\mathfrak{H}_A^a$ with π_A faithful (and hence π_X isometric by* [11, 4.6]*).*

Proof. Let L denote the linking algebra $\left[\begin{smallmatrix} A & X \\ X^\sim & A \end{smallmatrix}\right]$; cf. [6, 2.2]. We denote by A_1 (resp. A_2) the copy of A in the upper left (resp. lower right) corner. By the above, there exists a cardinal $\mathfrak{c}$ such that

$$\mathfrak{c}\pi_L^a|_{A_1} \simeq \mathfrak{c}\pi_A^a \simeq \mathfrak{c}\pi_L^a|_{A_2}.$$

Let $\pi' = \mathfrak{c}\pi_L^a$, and denote the unitary implementing the equivalence between $\pi'|A_1$ and $\pi'|A_2$ by u. By [11, 4.8], we get a representation $(\pi_A^L, \pi_A^R, \pi'_X)$ on $\mathfrak{c}\mathfrak{H}_A^a$ of X as an $A-A$ C^*-bimodule, where actually $\pi_A^L = \pi'|A_1$ and $\pi_A^R = \pi'|A_2$ by construction. Setting $\pi_A = \pi_A^L$ and $\pi_X = \pi'_X u$, we get a covariant representation, e.g. since

$$\pi_X(x)^*\pi_X(y) = u^*\pi'_X(x)^*\pi'_X(y)u = u^*\pi_A^R(\langle x, y\rangle_R)u = \pi_A(\langle x, y\rangle_R). \quad \square$$

Definition 2.4. Let a C^*-algebra A and an $A-A$ Hilbert C^*-bimodule X be given. A *crossed product* of A by X is a C^*-algebra B and (structure-preserving) maps

$\iota_A : A \to B$, $\iota_X : X \to B$ such that for every covariant representation (π_A, π_X) there is a unique $*$-homomorphism φ making the following diagram commute:

$$\begin{tikzcd} (A,X) \arrow[d, "(\iota_A,\iota_X)"'] \arrow[dr, "(\pi_A,\pi_X)"] & \\ B \arrow[r, "\varphi"'] & \mathbb{B}(\mathfrak{H}) \end{tikzcd}$$

Existence of crossed products can be obtained by making use of the results on enveloping C^*-algebras defined by generators and relations (see [3], [16]). That A and X are embedded in this crossed product is essentially a consequence of Proposition 2.3 above. To get a less abstract characterization of the crossed product, we carry out an explicit construction below.

We can see immediately that the crossed product must be unique. For if both B_1 and B_2 satisfy the conditions of Definition 2.4, assume that $B_i \subseteq \mathbb{B}(\mathfrak{H}_i)$ and apply the existence part of the definition to get maps $\varphi_1 : B_1 \to B_2$ and $\varphi_2 : B_2 \to B_1$ making the diagram commute. Then $\varphi_1 \circ \varphi_2$ must intertwine the embeddings of A and X; and since id_{B_1} has the same property, $\varphi_1 \circ \varphi_2 = \mathrm{id}_{B_1}$ by uniqueness. By symmetry, the φ_i are isomorphisms. We will denote this unique object by $A \rtimes_X \mathbb{Z}$. We will show later that this generalizes the crossed product by partial isomorphisms introduced in [12].

Lemma 2.5. *Let X be an $A - B$ Hilbert C^*-bimodule, Y a $B - C$ Hilbert C^*-bimodule. Assume that $A, B, C, X,$ and Y are all faithfully represented in $\mathbb{B}(\mathfrak{H})$ so that the module actions and inner products become the ones inherited from $\mathbb{B}(\mathfrak{H})$. Then, as an $A - C$ Hilbert C^*-bimodule, $X \otimes_B Y$ is isomorphic to XY.*

Proof. One checks that the map given by

$$x \otimes y \mapsto \pi_X(x)\pi_Y(y)$$

is well-defined and a Hilbert C^*-bimodule isomorphism. □

Corollary 2.6. *When (π_A, π_X) is a faithful covariant representation of (A, X),*

$$\bigotimes_1^N X \simeq \prod_1^N \pi_X(X).$$

We construct a $\mathbb{Z}$-bundle with fibers given by

$$X_n = \begin{cases} \overbrace{X \otimes_A X \otimes_A \cdots \otimes_A X}^{n} & \text{if } n > 0, \\ A & \text{if } n = 0, \\ \underbrace{X^\sim \otimes_A X^\sim \otimes_A \cdots \otimes_A X^\sim}_{-n} & \text{if } n < 0. \end{cases}$$

Given $m, n \in \mathbb{Z}$, we will build maps $\cdot : X_m \times X_n \to X_{m+n}$, and must do this in different manners according to the signs of m and n.

1°: $m = n = 0$: We employ the multiplication in A.

2°: $m, n > 0$: Concatenation.

3°: $m, n < 0$: Concatenation.

4°: $m = 0, n > 0$: We define maps

$$\cdot : X_0 \times X_n \to X_n, \quad a\cdot(x_1 \otimes \cdots \otimes x_n) = ax_1 \otimes \cdots \otimes x_n$$

and

$$\cdot : X_n \times X_0 \to X_n, \quad (x_1 \otimes \cdots \otimes x_n)\cdot a = x_1 \otimes \cdots \otimes x_n a.$$

5°: $m = 0, n < 0$: We set

$$\cdot : X_0 \times X_n \to X_n, \quad a\cdot(\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n}) = (x_1 a^*)^\sim \otimes \cdots \otimes \tilde{x}_{-n}$$

and

$$\cdot : X_n \times X_0 \to X_n, \quad (\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n})\cdot a = \tilde{x}_1 \otimes \cdots \otimes (a^* x_{-n})^\sim.$$

6°: $m > 0, n < 0$: We define the product recursively by

$$(x_1 \otimes \cdots \otimes x_m)\cdot(\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n}) = (x_1 \otimes \cdots \otimes x_{m-1}\langle x_m, y_1\rangle_L)\cdot(\tilde{y}_2 \otimes \cdots \otimes \tilde{y}_{-n})$$

and

$$(\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n})\cdot(x_1 \otimes \cdots \otimes x_m) = (\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-n-1})\cdot(\langle y_{-n}, x_1\rangle_R x_2 \otimes \cdots \otimes x_m).$$

7°: $m < 0, n > 0$: as in 6° above.

To make sense of the last two definitions, one must first define maps from, e.g., $X_m \otimes X_{-n}$ to $X_{m-1} \otimes X_{-n-1}$. The existence of such maps follows by facts such as $\langle xa, y\rangle_L = \langle x, ya^*\rangle_L$ (cf. [6, 1.9]).

We also define maps $^* : X_n \to X_{-n}$ by

1°: $n = 0$: The involution on A.

2°: $n > 0$: We set $(x_1 \otimes \cdots \otimes x_n)^* = \tilde{x}_n \otimes \cdots \otimes \tilde{x}_1$.

3°: $n < 0$: We set $(\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n})^* = x_{-n} \otimes \cdots \otimes x_1$.

We claim that with these operations, the $\{X_n\}$ form a C^*-algebraic bundle over $\mathbb{Z}$ in the sense of [14]. To avoid tedious computations with simple tensors, we find a family of faithful representations of the X_n onto the same Hilbert space which respect the structure imposed on the X_n by these operations. With this in hand, we can reduce the proofs of the relevant axioms to corresponding axioms of operators on a Hilbert space, all of which are well-known.

Let (π_A, π_X) denote a representation of (X, A). We define maps $\rho_n : X_n \to \mathbb{B}(\mathfrak{H})$ by $\rho_0(a) = \pi_A(a)$,

$$\rho_n(x_1 \otimes \cdots \otimes x_n) = \pi_X(x_1) \cdots \pi_X(x_n)$$

for $n > 0$, and

$$\rho_n(\tilde{x}_1 \otimes \cdots \otimes \tilde{x}_{-n}) = \pi_X(x_1)^* \cdots \pi_X(x_{-n})^*$$

We now prove

Lemma 2.7. *For all $k, l \in \mathbb{Z}$ and all $a \in X_k, b \in X_l$,*

(i) $\rho_{k+l}(a\cdot b) = \rho_k(a)\rho_l(b)$,

(ii) $\rho_{-k}(a^*) = \rho_k(a)^*$.

Proof. By totality of the set of simple tensors, we may restrict our attention to these. We obviously have, for $k > 0$,

$$\begin{aligned} \rho_{-k}((x_1 \otimes \cdots \otimes x_k)^*) &= \rho_{-k}(x_k \otimes \cdots \otimes x_1) \\ &= \pi_X(x_k)^* \cdots \pi_X(x_1)^* \\ &= (\pi_X(x_1) \cdots \pi_X(x_k))^* \\ &= \rho_k(x_1 \otimes \cdots \otimes x_k)^*, \end{aligned}$$

and similarly for negative k.

Proving (i) requires an induction argument based on the recursive definition of the multiplication operation. One first checks the statement for $k = 0$ or $l = 0$, and in the cases where k and l have the same sign. When $k > 0$ and $l < 0$, say, one writes

$$\begin{aligned}
&\rho_{k+l}((x_1 \otimes \cdots \otimes x_k)\cdot(\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-l})) \\
&\quad= \rho_{(k-1)+(l+1)}((x_1 \otimes \cdots \otimes x_{k-1}\langle x_k, y_1\rangle_L)\cdot(\tilde{y}_2 \otimes \cdots \otimes \tilde{y}_{-l})) \\
&\quad= \rho_{k-1}(x_1 \otimes \cdots \otimes x_{k-1}\langle x_k, y_1\rangle_L)\rho_{l+1}(\tilde{y}_2 \otimes \cdots \otimes \tilde{y}_{-l})) \\
&\quad= \pi_X(x_1)\cdots\pi_X(x_{k-1}\langle x_k, y_1\rangle_L)\pi_X(y_2)^*\cdots\pi_X(y_{-l})^* \\
&\quad= \pi_X(x_1)\cdots\pi_X(x_{k-1})\pi_A(\langle x_k, y_1\rangle_L)\pi_X(y_2)^*\cdots\pi_X(y_{-l})^* \\
&\quad= \pi_X(x_1)\cdots\pi_X(x_{k-1})\pi_X(x_k)\pi_X(y_1)^*\pi_X(y_2)^*\cdots\pi_X(y_{-l})^* \\
&\quad= \rho_k(x_1 \otimes \cdots \otimes x_k)\rho_l(\tilde{y}_1 \otimes \cdots \otimes \tilde{y}_{-l}),
\end{aligned}$$

where the crucial second step is the induction hypothesis. □

Proposition 2.8. $\mathcal{B} = (X_n, \cdot, ^*)$ *thus defined forms a* C^**-algebraic bundle over* $\mathbb{Z}$ *(in the sense of* [14] *or* [13, 2.2]*).*

Proof. We follow the checklist in [13, 2.2]. Axioms (i) and (v) follow by definition. To check all the other axioms, fix a faithful covariant representation (π_A, π_X) by Proposition 2.3 and define ρ_k as above. As these representations are all isometric – they are $A - A$ Hilbert C^*-bimodule isomorphisms by Corollary 2.6 – we may check all the properties in $\mathbb{B}(\mathfrak{H})$ instead. For instance, associativity follows from associativity in $\mathbb{B}(\mathfrak{H})$ as

$$\rho_{k+l+m}(a\cdot(b\cdot c)) = \rho_k(a)\rho_{l+m}(b\cdot c) = \rho_k(a)\rho_l(b)\rho_m(c) = \rho_{k+l+m}((a\cdot b)\cdot c). \quad \square$$

Theorem 2.9. *The cross-sectional algebra* $C^*(\mathcal{B})$ *over* $\mathcal{B} = (X_n, \cdot, ^*)$ *(as defined in* [14]*) is the crossed product of* A *by* X*.*

Proof. There are natural isometric maps $\iota_A : A \to X_0$ and $\iota_X : X \to X_1$. These extend into first $L^1(\mathcal{B})$, then $C^*(\mathcal{B})$. We denote all of these maps by ι_A, ι_X.

Given a covariant representation (π_A, π_X) of (X, A) onto $\mathfrak{H}$, define ρ_k as above. Because the ρ_k respect group structure and the involution by Lemma 2.7, they give rise to a $*$-homomorphism $\varphi : C_c(\mathcal{B}) \to \mathbb{B}(\mathfrak{H})$ by

$$\sum a_i\delta_i \mapsto \sum \rho_i(a_i).$$

We may extend φ first to $L^1(\mathcal{B})$, then to $C_c(\mathcal{B})$, and the map thus achieved clearly makes the diagram (2.4) commute. Also, as X_0 and X_1 generate $L^1(\mathcal{B})$, this is the only map with this property. □

Corollary 2.10. *The map* ι_A *is faithful. The map* ι_X *is isometric.*

Proof. Take (π_A, π_X) isometric, cf. Proposition 2.3, and construct φ as in the proof above. We have $\varphi \circ \iota_A = \rho_0 = \pi_A$, whence ι_A must be norm-preserving. Then so is π_X by [11, 4.6]. □

3. Applications of generalized crossed products

The following theorem describes the structure of C^*-algebras carrying a semi-saturated action of the circle. We recall that an action of the circle on a C^*-algebra B is said to be semi-saturated ([12, 4.1]) if B is generated by the fixed-point subalgebra and the first spectral subspace.

As for classical crossed products and crossed products by partial automorphisms ([12]), the crossed products introduced in the previous section carry a natural dual action. This is gotten by extending the maps

$$\sum a_i\delta_i \mapsto \sum t^i a_i \delta_i$$

from $C_c(\mathcal{B})$ to maps $\beta_t : A \rtimes_X \mathbb{Z} \to A \rtimes_X \mathbb{Z}$.

Theorem 3.1. *Suppose B is a C^*-algebra with a circle action α_t. Let B_0 and B_1 denote the fixed-point algebra and the first spectral subspace for α_t, respectively. Then α_t is semi-saturated if and only if B is isomorphic to $B_0 \rtimes_{B_1} \mathbb{Z}$ by an isomorphism sending α_t to the dual action.*

Proof. Let β_t be the dual action on $B_0 \rtimes_{B_1} \mathbb{Z}$. It is clear from our construction that $\iota_{B_0}(B_0)$ and $\iota_{B_1}(B_1)$ generate $B_0 \rtimes_{B_1} \mathbb{Z}$. These are contained in the fixed-point algebra and the first spectral subspace for β_t, respectively. To prove the other implication, we must prove that $\iota_{B_0}(B_0)$ is in fact the fixed-point space for β_t. Consider the conditional expectation onto β's fixed-point algebra

$$Q_0(x) = \int_{\mathbb{T}} \beta_t(x)dt$$

as well as the fixed-point algebra $(B_0 \rtimes_{B_1} \mathbb{Z})_0$. We have

$$\iota_{B_0}(B_0) \subseteq (B_0 \rtimes_{B_1} \mathbb{Z})_0 = Q_0(B_0 \rtimes_{B_1} \mathbb{Z}) = \overline{Q_0(C_c(\mathcal{B}))} \subseteq \overline{\iota_{B_0}(B_0)}.$$

However, as ι_{B_0} is an isometry by Corollary 2.10, its image is closed, and $\iota_{B_0}(B_0)$ is exactly the fixed-point algebra.

Consider now the diagram

$$\begin{tikzcd}
(B_0, B_1) \arrow[d, "(\iota_{B_0}, \iota_{B_1})"'] \arrow[dr] & \\
C^*(\mathcal{B}) \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

where the diagonal maps are the inclusions. The induced map φ is $\beta - \alpha$ covariant as on $C_c(\mathcal{B})$,

$$\begin{aligned}
\varphi(\beta_t(\sum a_i\delta_i)) &= \varphi(\sum t^i a_i\delta_i) = \varphi(\sum \alpha_t(a_i)\delta_i) = \sum \alpha_t(a_i) \\
&= \alpha_t(\sum a_i) = \alpha_t(\varphi(\sum a_i\delta_i))
\end{aligned}$$

It is clear that φ is injective on the fixed-point algebra of β, $\iota_{B_0}(B_0)$, as it sends $a_0\delta_0$ to a_0. Then φ must be injective according to [12, 2.9]. Finally, φ is onto as a consequence of semisaturation. $\square$

Example 3.2 (Crossed products by partial automorphisms)**.** Let (I, J, θ) be a partial automorphism of a C^*-algebra A, as in [12]. The dual action of the circle on the crossed product $A \rtimes_\theta \mathbb{Z}$ (cf. the paragraph preceding Theorem 3.1) is semi-saturated ([12, 4.7]) and its fixed-point subalgebra is isomorphic to A ([12, 3.9]). It follows from Theorem 3.1 that $A \rtimes_\theta \mathbb{Z}$ is isomorphic to the crossed product of A by the first spectral subspace. The first spectral subspace J_θ is the $A - A$ Hilbert C^*-bimodule whose underlying vector space is the ideal J with the usual left action and the right action

$$x \cdot a = \theta(\theta^{-1}(x)a)$$

and the inner products

$$\langle x,y\rangle_L = xy^* \quad \text{and} \quad \langle x,y\rangle_R = \theta^{-1}(x^*y).$$

Note that in the case of an automorphism, i.e. when $I = J = A$ and $\theta \in \mathrm{Aut}(A)$, the Hilbert C^*-bimodule J_θ is the element of the Picard group of A corresponding to θ^{-1} by the map considered in [5, 3].

Returning to the context of the partial automorphisms, we see that two homomorphisms from $K_i(J)$ to $K_i(I)$ arise naturally in this context. One of them is the map $(\theta^{-1})_*$ induced by θ^{-1}. On the other hand, since J_θ is a left-full $J-I$ Hilbert C^*-bimodule, it induces, as in [11, 5], homomorphisms

$$(J_\theta)_* : K_i(J) \longrightarrow K_i(I).$$

We next show that these two maps agree. As in [11, 3.14], for $\alpha \in K_0(J)$, let $p,q \in M_n(J^\sim)$ be such that $\alpha = [p]-[q]$ and $p-q \in M_n(J)$. Consider the Fredholm operator

$$T : pJ^n \longrightarrow qJ^n$$

defined by $T((x_i)) = q(x_i)$. Then $(J_\theta)_*$ is, by definition, the index of the operator $T\otimes \mathrm{id} : pJ^n \otimes_J J_\theta \longrightarrow qJ^n \otimes_J J_\theta$. In general, if p is any projection in $M_n(J^\sim)$, the map

$$\Phi_p : pJ^n \otimes_J J_\theta \longrightarrow \theta^{-1}(p)I^n$$

given by $\Phi_p(x_n \otimes x) = \theta^{-1}(x_n x)$, where θ^{-1} is extended componentwise to J^n, is a right A-Hilbert module isomorphism.

For p, q and T as above, the diagram

$$\begin{array}{ccc} pJ^n \otimes_J J_\theta & \xrightarrow{T\otimes \mathrm{id}} & qJ^n \otimes_J J_\theta \\ {\scriptstyle \Phi_p}\downarrow & & \downarrow{\scriptstyle \Phi_q} \\ \theta^{-1}(p)I^n & \xrightarrow{\tilde{T}} & \theta^{-1}(q)I^n \end{array}$$

commutes for $\tilde{T}((i_n)) = \theta^{-1}(q)((i_n))$. By [11, 3.14] we now have

$$(J_\theta)_*(\alpha) = \mathrm{Ind}(\tilde{T}) = [\theta^{-1}(p)] - [\theta^{-1}(q)] = (\theta^{-1})_*(\alpha),$$

as we wanted to show.

As for the K_1-groups, let us denote by SB the suspension of a C^*-algebra B. Consider the partial automorphism $(SI, SJ, 1\otimes\theta)$ of the C^*-algebra SA. By the result above, the diagram

$$\begin{array}{ccc} K_0(SI) & \xrightarrow{SJ_*} & K_0(SJ) \\ || & & || \\ K_0(SI) & \xrightarrow{(1\otimes\theta^{-1})_*} & K_0(SJ) \end{array}$$

commutes. On the other hand, by [4, 8.2.2.], the diagram

$$\begin{array}{ccc} K_0(SI) & \xrightarrow{(1\otimes\theta^{-1})_*} & K_0(SJ) \\ \downarrow & & \downarrow \\ K_1(I) & \xrightarrow{(\theta^{-1})_*} & K_1(J) \end{array}$$

commutes, where the vertical maps are the usual isomorphisms [4, 8.2.2]. It follows that the homomorphisms from $K_1(I)$ to $K_1(J)$ induced by θ and J_θ, respectively, agree.

Example 3.3 (Quantum Heisenberg manifolds)**.** The quantum Heisenberg manifolds provide a family of examples of crossed products by C^*-Hilbert bimodules not coming from partial automorphisms. For real numbers μ, ν and a positive integer c, the quantum Heisenberg manifold $D^c_{\mu\nu}$ ([20], [1]) consists of the closure in the multiplier algebra of $C_0(\mathbb{R}\times\mathbb{T})\rtimes_\alpha\mathbb{Z}$ of the *-subalgebra

$$C^c_{\mu\nu} = \{\Phi\in C_c(\mathbb{Z}, C_b(\mathbb{R}\times\mathbb{T})) | \Phi(p, x+1, y) = e(-cp(y-p\nu))\Phi(p,x,y)\},$$

where $(\alpha f)(x,y) = f(x-2\mu, y-2\nu)$, and $e(x) = e^{2\pi i x}$. The algebras $D^c_{\mu\nu}$ carry, as in [20, 5.7], an action of the circle $\mathbb{T}$ defined by

$$(\gamma_z\Phi)(p,x,y) = z^p\Phi(p,x,y),$$

for $\Phi\in C^c_{\mu\nu}$. The nth spectral subspace $(D^c_{\mu\nu})_n$ consists of the δ_n-maps, i.e.

$$(D^c_{\mu\nu})_n = \{f\delta_n | f\in C_b(\mathbb{R}\times\mathbb{T}), f(x+1,y) = e(-cn(y-n\nu))f(x,y)\}.$$

Thus the fixed-point subalgebra can be identified with $C(\mathbb{T}^2)$.

There is no $\mathbb{T}$-covariant isomorphism between a quantum Heisenberg manifold and a crossed product by a partial automorphism θ. For in that case, since (proof of [2, 2.1])

$$(D^c_{\mu\nu})_1((D^c_{\mu\nu})_1)^* = ((D^c_{\mu\nu})_1)^*(D^c_{\mu\nu})_1 = C(\mathbb{T}^2),$$

θ would be an automorphism of $C(\mathbb{T}^2)$, and there would be a unitary element $u\in(D^c_{\mu\nu})_1$ implementing θ. If such a unitary existed, it would be a continuous function $u:\mathbb{R}\times\mathbb{T}\to\mathbb{T}$ satisfying

$$u(x+1,y) = e(-c(y-\nu))u(x,y). \tag{1}$$

But the maps $y\mapsto u(x,y)$ and $y\mapsto u(x+1,y)$ are homotopic and therefore have the same winding number, which contradicts (1).

We now show that $D^c_{\mu\nu}\cong C(\mathbb{T}^2)\rtimes_{(D^c_{\mu\nu})_1}\mathbb{Z}$, by proving that the circle action is semi-saturated. Let A denote the C^*-algebra generated by $C(\mathbb{T}^2)$ and $(D^c_{\mu\nu})_1$; we show by induction on n that the nth spectral subspace is contained in A, which will end the proof. So we assume that $(D^c_{\mu\nu})_n\subset A$. As in [2, 2.1], let $\Delta_i\delta_1\in(D^c_{\mu\nu})_1, i=1,2$, be such that

$$|\Delta_1(x,y)|^2 + |\Delta_2(x,y)|^2 = 1,$$

for all $(x,y)\in\mathbb{R}\times\mathbb{T}$. Given $f\delta_{n+1}\in(D^c_{\mu\nu})_{n+1}$, set

$$F_i(x,y) = f(x,y)\overline{\Delta_i}(x-2n\mu, y-2n\nu).$$

Then $F_i\delta_n\in(D^c_{\mu\nu})_n\subset A$, and

$$f\delta_{n+1} = F_1\delta_n * \Delta_1\delta_1 + F_2\delta_n * \Delta_2\delta_1,$$

where $*$ denotes the multiplication in $D^c_{\mu\nu}$.

Remark 3.4 (A Pimsner-Voiculescu sequence). In case A and X are separable, the circle action γ defined in the proof of Theorem 3.1 can be stabilized so that the generalized PV sequence in [12, 7] applies. That is, the action $\gamma\otimes\mathrm{id}$ of the circle on $(A\rtimes_X\mathbb{Z})\otimes\mathbb{K}$ is semi-saturated and regular, and therefore it is the crossed product of the fixed-point algebra $A\otimes\mathbb{K}$ by a partial automorphism $(\theta, I_R\otimes\mathbb{K}, I_L\otimes\mathbb{K},)$, where I_R and I_L denote the closed spans of $\langle X,X\rangle_R$ and $\langle X,X\rangle_L$ respectively.

Thus the generalized PV sequence in [12, 7] applies, and we get

$$\begin{array}{ccccc} K_0(I_L) & \longrightarrow & K_0(A) & \xrightarrow{i_*} & K_0(A \rtimes_X \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_X \mathbb{Z}) & \xleftarrow{i_*} & K_1(A) & \longleftarrow & K_1(I_L) \end{array}$$

4. Morita equivalence

Let X and Y denote $A-A$ and $B-B$ Hilbert C^*-bimodules, respectively, and let M be an $A-B$ Morita equivalence bimodule. Our purpose is to show that if $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic as $A-B$ Hilbert C^*-bimodules, then the crossed products $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are Morita equivalent. When $X = A_\alpha$ and $Y = B_\beta$, for some $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$, the condition $X \otimes_A M \simeq M \otimes_B Y$ is equivalent to the conditions stated in [9] and [8] to get Morita equivalent crossed products. In order to prove the result mentioned above we construct a Hilbert C^*-bimodule W over the linking algebra L of M and then prove that $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are complementary full corners of $L \rtimes_W \mathbb{Z}$.

The next representation lemma will help us avoid computations.

Lemma 4.1. *Let $A, B, L, M, X,$ and Y be as above. Then there are faithful representations π, π_X and π_Y of $L, X,$ and Y, respectively, on the same Hilbert space $\mathfrak{H}$ such that $(\pi_{|A}, \pi_X)$ and $(\pi_{|B}, \pi_Y)$ are covariant.*

Besides, if the $A-B$ Hilbert C^*-bimodules $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic, then π_X can be chosen so that $\pi_X(X)\pi(M) = \pi(M)\pi_Y(Y)$.

Proof. By Lemma 2.2, for a large enough cardinal $\mathfrak{c}$, L can be faithfully represented on $\mathfrak{H} = \mathfrak{c}\mathfrak{H}_A^a \oplus \mathfrak{c}\mathfrak{H}_B^a$ by a representation π such that $\pi|A$ and $\pi|B$ are $\mathfrak{c}\pi_A^a$ and $\mathfrak{c}\pi_B^a$, respectively. Now, by Proposition 2.3, there are faithful representations π'_X and π'_Y on $\mathfrak{c}\mathfrak{H}_A^a$ and $\mathfrak{c}\mathfrak{H}_B^a$, respectively, such that (π_A^a, π'_X) and (π_B^a, π'_Y) are covariant. To represent X and Y on $\mathfrak{H}$, set

$$\pi_X(x) = \begin{bmatrix} \pi'_X(x) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_Y(y) = \begin{bmatrix} 0 & 0 \\ 0 & \pi'_Y(y) \end{bmatrix}.$$

Assume now that $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic. Set $\mathfrak{H}_0 = \mathfrak{c}\mathfrak{H}_A^a$ and $\mathfrak{H}_1 = \mathfrak{c}\mathfrak{H}_B^a$. To simplify notation we view $X, A \subset \mathbb{B}(\mathfrak{H}_0)$, $Y, B \in \mathbb{B}(\mathfrak{H}_1)$ and $M \subset \mathbb{B}(\mathfrak{H}_1, \mathfrak{H}_0)$, by identifying those sets with their images under the representations described above, restricted to their essential subspaces. Then, by Lemma 2.5 this implies there is an isomorphism $\tau : XM \to MY$.

For $m \in M, x \in X$ and $\xi \in \mathfrak{H}_1$, we have

$$\begin{aligned} \langle \tau(xm)\xi, \tau(xm)\xi \rangle_{\mathfrak{H}_0} &= \langle \langle \tau(xm), \tau(xm) \rangle_R \xi, \xi \rangle_{\mathfrak{H}_0} \\ &= \langle \langle xm, xm \rangle_R \xi, \xi \rangle_{\mathfrak{H}_0} = \langle xm\xi, xm\xi \rangle_{\mathfrak{H}_0}. \end{aligned}$$

Therefore there is a partial isometry $u \in \mathbb{B}(\mathfrak{H}_0)$ with initial space $XM\mathfrak{H}_0$ and final space $MY\mathfrak{H}_1$, such that $\tau(xm) = uxm$. Notice that u commutes with A, because τ is an A-module isomorphism and $(XM\mathfrak{H}_0)^\perp$ is A-invariant. Besides, for $x_0, x_1 \in X$

$$(ux_0)^*(ux_1) = x_0^* u^* u x_1 = x_0^* x_1 = \langle x_0, x_1 \rangle_R,$$

and

$$(ux_0)(ux_1)^* = ux_0 x_1^* u^* = x_0 x_1^* u u^* = x_0 x_1^* = \langle x_0, x_1 \rangle_L.$$

Finally, we just showed that $(uX)M = MY$. It suffices now to replace the representation π_X by $u\pi_X$. ∎

Theorem 4.2. *Let X and Y be $A-A$ and $B-B$ Hilbert C^*-bimodules, respectively. Let M be an $A-B$ Morita equivalence bimodule such that $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic as $A-B$ Hilbert C^*-bimodules. Then $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are Morita equivalent.*

Proof. We view A, B, L, M, X, and Y as acting on a Hilbert space $\mathfrak{H}$ as in Lemma 4.1, so that $XM = MY$, and all module actions and inner products are the ones inherited from $\mathbb{B}(\mathfrak{H})$. We also assume that, as in Lemma 4.1, the span of $A\mathfrak{H} \cup B\mathfrak{H}$ is dense in $\mathfrak{H}$.

We then have

$$M^*X = M^*XMM^* = M^*MYM^* = YM^*$$

by [6, 1.7]. Now, the subspace $W = \left[\begin{smallmatrix} X & XM \\ YM^* & Y \end{smallmatrix}\right]$ is a right Hilbert C^*-module over L for the structure inherited from $\mathbb{B}(\mathfrak{H})$. On the other hand $W = \left[\begin{smallmatrix} X & MY \\ M^*X & Y \end{smallmatrix}\right]$, so it is a left Hilbert C^*-module over L for the structure inherited from $\mathbb{B}(\mathfrak{H})$. It follows that W is an $L-L$ Hilbert C^*-bimodule.

We next show that $A \rtimes_X \mathbb{Z}$ and $B \rtimes_Y \mathbb{Z}$ are complementary full corners of $L \rtimes_W \mathbb{Z}$, which, in view of [5, 1.1], will end the proof. By Theorem 2.9 $L \rtimes_W \mathbb{Z} \simeq C^*(\mathcal{B}_W)$ and $A \rtimes_X \mathbb{Z} \simeq C^*(\mathcal{B}_X)$, where $\mathcal{B}_W$ and $\mathcal{B}_X$ are the C^*-algebraic bundles whose fibers are given by

$$W_n = \begin{cases} W^n & \text{if } n > 0, \\ L & \text{if } n = 0, \\ (W^*)^{-n} & \text{if } n < 0, \end{cases} \qquad X_n = \begin{cases} X^n & \text{if } n > 0, \\ A & \text{if } n = 0, \\ (X^*)^{-n} & \text{if } n < 0, \end{cases}$$

with the structure inherited from $\mathbb{B}(\mathfrak{H})$.

The projection $p = \left[\begin{smallmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ in $\mathbb{B}(\mathfrak{H})$ is a multiplier of degree zero of $\mathcal{B}_W$, in the sense of [14, VIII, 2.14], and therefore can be viewed as a projection in $M(C^*(\mathcal{B}_W))$ ([14, VIII, 5.8]). We now show that $C^*(\mathcal{B}_X) \simeq p(C^*(\mathcal{B}_W))p$. It follows from the identities $XM = MY$ and $M^*X = YM^*$ that $W^n = \left[\begin{smallmatrix} X^n & X^nM \\ Y^nM^* & Y^n \end{smallmatrix}\right]$ for $n > 0$, and therefore $pW_np = X_n$.

Thus the Banach *-algebras $L^1(\mathcal{B}_X)$ and $pL^1(\mathcal{B}_W)p$ can be identified. By taking their respective enveloping C^*-algebras we get $C^*(\mathcal{B}_X) = pC^*(\mathcal{B}_W)p$. (To see that the enveloping C^*-algebra of $pL^1(\mathcal{B}_W)p$ is $p(C^*(\mathcal{B}_W))p$, notice that $pL^1(\mathcal{B}_W)p$ is invariant under the dual action of the circle on $L^1(\mathcal{B}_W)$ defined in the proof of Theorem 3.1. Besides, the obvious map from $C^*(pL^1(\mathcal{B}_W)p)$ to $p(C^*(\mathcal{B}_W))p$ is covariant under the extensions of that action, and it is one-to one when restricted to the fixed-point algebras. So [12, 2.9] applies.) Analogous reasoning proves that, for $q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{smallmatrix}\right]$, we have $B \rtimes_Y \mathbb{Z} \simeq q(L \rtimes_W \mathbb{Z})q$. Finally, a straightforward computation shows that $LpW^n = W^n = W^nqL$ for all $n \geq 0$, which proves that p and q are full projections. □

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