

QUANTUM COHOMOLOGY OF PROJECTIVE BUNDLES OVER $\mathbb{P}^n$

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ABSTRACT. In this paper we study the quantum cohomology ring of certain projective bundles over the complex projective space $\mathbb{P}^n$. Using excessive intersection theory, we compute the leading coefficients in the relations among the generators of the quantum cohomology ring structure. In particular, Batyrev's conjectural formula for quantum cohomology of projective bundles associated to direct sum of line bundles over $\mathbb{P}^n$ is partially verified. Moreover, relations between the quantum cohomology ring structure and Mori's theory of extremal rays are observed. The results could shed some light on the quantum cohomology for general projective bundles.

1. INTRODUCTION

Quantum cohomology, proposed by Witten's study [16] of two dimensional non-linear sigma models, plays a fundamental role in understanding the phenomenon of mirror symmetry for Calabi-Yau manifolds. This phenomenon was first observed by physicists motivated by topological field theory. A topological field theory starts with correlation functions. The correlation functions of the sigma models are linked with the intersection numbers of cycles in the moduli space of holomorphic maps from Riemann surfaces to manifolds. For some years, the mathematical construction of these correlation functions has remained a difficult problem because the moduli spaces of holomorphic maps usually are not compact and may have the wrong dimension. The quantum cohomology theory was first put on a firm mathematical footing in [12], [13] for semi-positive symplectic manifolds (including Fano and Calabi-Yau manifolds), using the method of symplectic topology. Recently, an algebro-geometric approach has been taken by [8], [9]. The results of [12], [13] have been redone in the algebraic geometric setting for the case of homogeneous spaces. The advantage of homogeneous spaces is that the moduli spaces of holomorphic maps always have the expected dimension and their compactifications are nice. Beyond the homogeneous spaces, one cannot expect such nice properties for the moduli spaces. The projective bundles are perhaps the simplest examples. However, by developing sophisticated excessive intersection theory, it is possible that the algebro-geometric method can work for any projective manifold. In turn, it may shed new light on removing the semi-positivity condition in the symplectic setting.

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Although we have a solid foundation for the quantum cohomology theory at least for semi-positive symplectic manifolds, the calculation has remained a difficult task. So far, there are only a few examples which have been computed, e.g., Grassmannian [14], some rational surfaces [6], flag varieties [4], some complete intersections [3], and the moduli space of stable bundles over Riemann surfaces [15]. One common feature for these examples is that the relevant moduli spaces of rational curves have the expected dimension. This feature enables one to use the intersection theory. We should mention that there are many predications based on mathematically unjustified mirror symmetry (for Calabi-Yau 3-folds) and the linear sigma model (for toric varieties). In this paper, we attempt to determine the quantum cohomology of projective bundles over projective space $\mathbb{P}^n$. In contrast to previous examples, the relevant moduli spaces in our case frequently do not have the expected dimensions. These moduli spaces make the calculations more difficult. We overcome this difficulty by using the excessive intersection theory.

There are two main ingredients in our arguments. The first one is a result of Siebert and Tian (Theorem 2.2 in [14]), which says that if the ordinary cohomology $H^*(X; \mathbb{Z})$ of a symplectic manifold X with the symplectic form ω is the ring generated by $\alpha_1, \dots, \alpha_s$ with the relations $f^1, \dots, f^t$, then the quantum cohomology $H_\omega^*(X; \mathbb{Z})$ of X is the ring generated by $\alpha_1, \dots, \alpha_s$ with t new relations $f_\omega^1, \dots, f_\omega^t$, where each new relation f_ω^i is just the relation f^i evaluated in the quantum cohomology ring structure. It was known that the quantum product $\alpha \cdot \beta$ is the deformation of the ordinary cup product by the lower order terms, called quantum corrections. The second ingredient is that under certain numerical conditions, most of the quantum corrections vanish. Moreover, the nontrivial quantum corrections seem to come from Mori's extremal rays.

Let V be a rank- r bundle over $\mathbb{P}^n$, and $\mathbb{P}(V)$ be the corresponding projective bundle. Let h and ξ be the cohomology classes of a hyperplane in $\mathbb{P}^n$ and the tautological line bundle in $\mathbb{P}(V)$ respectively. For simplicity, we make no distinction between h and π^*h , where $\pi: \mathbb{P}(V) \rightarrow \mathbb{P}^n$ is the natural projection. Denote the product of i copies of h and j copies of ξ in the ordinary cohomology ring by $h_i \xi_j$, and the product of i copies of h and j copies of ξ in the quantum cohomology ring by $h^i \cdot \xi^j$. For $i = 0, \dots, r$, put $c_i(V) = c_i \cdot h_i$ for some integer c_i . It is well known that $-K_{\mathbb{P}(V)} = (n+1-c_1)h + r\xi$, and the ordinary cohomology ring $H^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations

$$(1.1) \quad h_{n+1} = 0 \quad \text{and} \quad \sum_{i=0}^r (-1)^i c_i \cdot h_i \xi_{r-i} = 0.$$

In particular, $H^{2(n+r-2)}(\mathbb{P}(V); \mathbb{Z})$ is generated by $h_{n-1}\xi_{r-1}$ and $h_n\xi_{r-2}$, and its Poincaré dual $H_2(\mathbb{P}(V); \mathbb{Z})$ is generated by $(h_{n-1}\xi_{r-1})_*$ and $(h_n\xi_{r-2})_*$, where for $\alpha \in H^*(\mathbb{P}(V); \mathbb{Z})$, α_* stands for its Poincaré dual. We have

$$(1.2) \quad -K_{\mathbb{P}(V)}(A) = a(n+1-c_1) + r \cdot \xi(A) = a(n+1-c_1) + r(ac_1 + b)$$

for $A = (ah_{n-1}\xi_{r-1} + bh_n\xi_{r-2})_* \in H_2(\mathbb{P}(V); \mathbb{Z})$.

By definition, V is an ample (respectively, nef) bundle if and only if the tautological class ξ is an ample (respectively, nef) divisor on $\mathbb{P}(V)$. Assume that V is ample such that either $c_1 \leq (n+1)$, or $c_1 \leq (n+r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef. Then both ξ and $-K_{\mathbb{P}(V)}$ are ample divisors. Thus, $\mathbb{P}(V)$ is a Fano variety, and its quantum cohomology ring is well-defined [13]. Here we choose the symplectic form ω on $\mathbb{P}(V)$ to be the Kahler form ω such that $[\omega] = -K_{\mathbb{P}(V)}$. Let f_ω^1 and f_ω^2 be

the two relations in (1.1) evaluated in the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$. Then, by Theorem 2.2 in [14], the quantum cohomology $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations f_ω^1 and f_ω^2 :

$$(1.3) \quad H_\omega^*(\mathbb{P}(V); \mathbb{Z}) = \mathbb{Z}[h, \xi] / (f_\omega^1, f_\omega^2)$$

By Mori's Cone Theorem [5], $\mathbb{P}(V)$ has exactly two extremal rays, R_1 and R_2 . Up to an order of R_1 and R_2 , the integral generator A_1 of R_1 is represented by lines in the fibers of the projection π . We shall show that under certain numerical conditions, the nontrivial homology classes $A \in H_2(\mathbb{P}(V); \mathbb{Z})$ which give nontrivial quantum corrections are A_1 and A_2 , where A_2 is represented by some smooth rational curves in $\mathbb{P}(V)$ which are isomorphic to lines in $\mathbb{P}^n$ via π . In general, it is unclear whether A_2 generates the second extremal ray R_2 . However, we shall prove that under further restrictions on V , A_2 generates the extremal ray R_2 . These analyses enable us to determine the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$.

The simplest ample bundle over $\mathbb{P}^n$ is of the form $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $m_i > 0$ for every i . Since we can twist V by $\mathcal{O}_{\mathbb{P}^n}(-1)$ without changing $\mathbb{P}(V)$, we can assume that $\min\{m_1, \dots, m_r\} = 1$. In this case, $\mathbb{P}(V)$ is a special case of toric variety. Batyrev [2] conjectured a general formula for the quantum cohomology of toric varieties. Furthermore, he computed the contributions from certain moduli spaces of holomorphic maps which have the expected dimensions. In our case, the contributions Batyrev computed are only part of the data to compute the quantum cohomology. As we explained earlier, the difficulty in our case lies precisely in computing the contributions from the moduli spaces with the wrong dimensions. Nevertheless, in our case, Batyrev's formula (see also [1]) reads as follows.

Batyrev's Conjecture. *Let $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $m_i > 0$ for every i . Then the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is generated by h and ξ with two relations:*

$$h^{n+1} = \prod_{i=1}^r (\xi - m_i h)^{m_i - 1} \cdot e^{-t(n+1+r-\sum_{i=1}^r m_i)} \quad \text{and} \quad \prod_{i=1}^r (\xi - m_i h) = e^{-tr}.$$

Our first result partially verifies Batyrev's conjecture.

Theorem A. *Batyrev's conjecture holds if*

$$\sum_{i=1}^r m_i < \min(2r, (n+1+2r)/2, (2n+2+r)/2).$$

Note that under the numerical condition of Theorem A, only extremal rational curves with fundamental classes A_1 and A_2 give contributions to the two relations in the quantum cohomology. The moduli space $\mathfrak{M}(A_2, 0)$ of rational curves with fundamental class A_2 does not have the expected dimension in general, but it is compact. This fact greatly simplifies the excessive intersection theory involved. If the numerical condition is removed, then we have to consider other moduli spaces (for example $\mathfrak{M}(kA_2, 0)$ with $k > 1$ and its excessive intersection theory). These moduli spaces are not compact in general. So we have extra difficulties in handling the compactification of these moduli spaces and the appropriate excessive intersection theory. These seem to be difficult problems, which we shall not pursue here.

In general, ample bundles over $\mathbb{P}^n$ are not direct sums of line bundles. We can say much less about their quantum cohomology. However, we obtain some results about its general form and compute the leading coefficient.

Theorem B. (i) *Let V be a rank- r ample bundle over $\mathbb{P}^n$. Assume either $c_1 \leq n$, or $c_1 \leq (n+r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef, so that $\mathbb{P}(V)$ is Fano. Then the quantum cohomology $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations*

$$h^{n+1} = \sum_{i+j \leq (c_1-r)} a_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(n+1-i-j)},$$

$$\sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr} + \sum_{i+j \leq (c_1-n-1)} b_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(r-i-j)},$$

where the coefficients $a_{i,j}$ and $b_{i,j}$ are integers depending on V .

(ii) *If we further assume that $c_1 < 2r$, then the leading coefficient $a_{0,c_1-r} = 1$.*

It is understood that when $c_1 \leq n$, then the summation $\sum_{i+j \leq (c_1-n-1)}$ in the second relation in Theorem B (i) does not exist. In general, it is not easy to determine all the integers $a_{i,j}$ and $b_{i,j}$ in Theorem B (i). However, it is possible to compute these numbers when $(c_1 - r)$ is relatively small. For instance, when $(c_1 - r) = 0$, then necessarily $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ and it is well-known that the quantum cohomology $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations $h^{n+1} = e^{-t(n+1)}$ and $\sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr}$. When $(c_1 - r) = 1$ and $r < n$, then necessarily $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$. When $(c_1 - r) = 1$ and $r = n$, then $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$ or $V = T_{\mathbb{P}^n}$, the tangent bundle of $\mathbb{P}^n$. In these cases, $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef. In particular, the direct sum cases have been computed by Theorem A. We shall prove the following.

Proposition C. *The quantum cohomology ring $H_\omega^*(\mathbb{P}(T_{\mathbb{P}^n}); \mathbb{Z})$ with $n \geq 2$ is the ring generated by h and ξ with the two relations*

$$h^{n+1} = \xi \cdot e^{-tn} \quad \text{and} \quad \sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + (-1)^n) \cdot e^{-tn}.$$

Recall that for an arbitrary projective bundle over a general manifold, its cohomology ring is a module over the cohomology ring of the base with the generator ξ and the second relation of (1.1). Naively, one may think that the quantum cohomology of a projective bundle is a module over the quantum cohomology of the base with the generator ξ and the quantized second relation. Our calculation shows that one cannot expect such simplicity for its quantum cohomology ring. We hope that our results could shed some light on the quantum cohomology for general projective bundles, which we shall leave for future research.

Our paper is organized as follows. In section 2, we discuss the extremal rays and extremal rational curves. In section 3, we review the definition of quantum product and compute some Gromov-Witten invariants. In the remaining three sections, we prove Theorem B, Theorem A, and Proposition C respectively.

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2. EXTREMAL RATIONAL CURVES

Assume that V is ample such that either $c_1 \leq (n+1)$, or $c_1 \leq (n+r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef. In this section, we study the extremal rays and extremal rational curves in the Fano variety $\mathbb{P}(V)$. By Mori's Cone Theorem (p. 25 in [5]), $\mathbb{P}(V)$ has precisely two extremal rays $R_1 = \mathbb{R}_{\geq 0} \cdot A_1$ and $R_2 = \mathbb{R}_{\geq 0} \cdot A_2$ such that the cone $\text{NE}(\mathbb{P}(V))$ of curves in $\mathbb{P}(V)$ is equal to $R_1 + R_2$ and A_1 and A_2 are the homology classes of two rational curves E_1 and E_2 in $\mathbb{P}(V)$ with $0 < -K_{\mathbb{P}(V)}(A_i) \leq \dim(\mathbb{P}(V)) + 1$. Up to orders of A_1 and A_2 , we have $A_1 = (h_n \xi_{r-2})_*$, that is, A_1 is represented by lines in the fibers of π . It is also well-known that if $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$ with $m_1 \leq \dots \leq m_r$, then $A_2 = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_*$, which is represented by a smooth rational curve in $\mathbb{P}(V)$ isomorphic to a line in $\mathbb{P}^n$ via π . However, in general, it is not easy to determine the homology class A_2 and the extremal rational curves representing A_2 . Assume that

$$(2.1) \quad V|_{\ell} = \bigoplus_{i=1}^r \mathcal{O}_{\ell}(m_i)$$

for generic lines $\ell \subset \mathbb{P}^n$, where we let $m_1 \leq \dots \leq m_r$. Since V is ample, $m_1 \geq 1$.

Lemma 2.2. *Let $A = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_*$. Then,*

- (i) *A is represented by a smooth rational curve isomorphic to a line in $\mathbb{P}^n$;*
- (ii) *$A_2 = A$ if and only if $(\xi - m_1 h)$ is nef;*
- (iii) *$A_2 = A$ if $2c_1 \leq (n+1)$;*
- (iv) *A cannot be represented by reducible or nonreduced curves if $m_1 = 1$.*

Proof. (i) Let $\ell \subset \mathbb{P}^n$ be a generic line. Then we have a natural projection $V|_{\ell} = \bigoplus_{i=1}^r \mathcal{O}_{\ell}(m_i) \rightarrow \mathcal{O}_{\ell}(m_1)$. By Proposition 7.12 in Chapter II of [7], this surjective map $V|_{\ell} \rightarrow \mathcal{O}_{\ell}(m_1) \rightarrow 0$ induces a morphism $g : \ell \rightarrow \mathbb{P}(V)$. Then $g(\ell)$ is isomorphic to ℓ via the projection π . Since $h([g(\ell)]) = 1$ and $\xi([g(\ell)]) = m_1$, we have

$$[g(\ell)] = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_* = A.$$

(ii) First of all, if $A_2 = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_*$, then for any curve E , $[E] = a(h_n \xi_{r-2})_* + b[h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_*$ for some nonnegative numbers a and b ; so $(\xi - m_1 h)([E]) = a \geq 0$; therefore $(\xi - m_1 h)$ is nef. Conversely, if $(\xi - m_1 h)$ is nef, then $0 \leq (\xi - m_1 h)([E]) = ac_1 + b - am_1$ where $[E] = (ah_{n-1} \xi_{r-1} + bh_n \xi_{r-2})_*$ for some curve E ; thus $[E] = (ac_1 + b - am_1)(h_n \xi_{r-2})_* + a[h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_*$; it follows that $A_2 = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_* = A$.

(iii) Let $A_2 = (ah_{n-1} \xi_{r-1} + bh_n \xi_{r-2})_*$. Since $A_1 = (h_n \xi_{r-2})_*$ and $a = h(A_2) \geq 0$, $a \geq 1$. If $a > 1$, then since $2c_1 \leq (n+1)$, we see that

$$\begin{aligned} -K_{\mathbb{P}(V)}(A_2) &= (n+1 - c_1)a + r \cdot \xi(A_2) \geq 2(n+1 - c_1) + r \\ &> n + r = \dim(\mathbb{P}(V)) + 1; \end{aligned}$$

but this contradicts with $-K_{\mathbb{P}(V)}(A_2) \leq \dim(\mathbb{P}(V)) + 1$. Thus $a = 1$ and $A_2 = (h_{n-1} \xi_{r-1} + bh_n \xi_{r-2})_*$. Now $[\pi(E_2)] = \pi_*(A_2) = (h_{n-1})_*$. So $\pi(E_2)$ is a line in $\mathbb{P}^n$. Since $V|_{\ell} = \bigoplus_{i=1}^r \mathcal{O}_{\ell}(m_i)$ for a generic line $\ell \subset \mathbb{P}^n$, it follows that $V|_{\pi(E_2)} = \bigoplus_{i=1}^r \mathcal{O}_{\pi(E_2)}(m'_i)$, where $m'_i \geq m_1$ for every i . Thus, $\xi(A_2) \geq m_1$, and so $c_1 + b \geq m_1$. It follows that

$$A_2 = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_* + (c_1 + b - m_1) \cdot (h_n \xi_{r-2})_*.$$

Therefore, $A_2 = [h_{n-1} \xi_{r-1} + (m_1 - c_1) h_n \xi_{r-2}]_* = A$.

(iv) Since $\xi(A) = m_1 = 1$ and ξ is ample, the conclusion follows. $\square$

Next, let $\mathfrak{M}(A, 0)$ be the moduli space of morphisms $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$ with $[\mathrm{Im}(f)] = A$. In the lemma below, we study the morphisms in $\mathfrak{M}(A, 0)$ when $A = [h_{n-1}\xi_{r-1} + (m - c_1)h_n\xi_{r-2}]_*$. Note that $\xi(A) = m$.

Lemma 2.3. *Let $A = [h_{n-1}\xi_{r-1} + (m - c_1)h_n\xi_{r-2}]_*$.*

- (i) *If $\mathfrak{M}(A, 0) \neq \emptyset$, then $m \geq m_1$ and $\mathfrak{M}(A, 0)$ consists of embeddings $f : \ell \rightarrow \mathbb{P}(V)$ induced by surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m) \rightarrow 0$ where ℓ are lines in $\mathbb{P}^n$;*
- (ii) *If $m = m_1$ and $m_1 = \dots = m_k < m_{k+1} \leq \dots \leq m_r$, then the moduli space $\mathfrak{M}(A, 0)$ has (complex) dimension $(2n + k)$;*
- (iii) *If $m \geq m_r$, then $\mathfrak{M}(A, 0)$ has dimension $(2n + r + rm - c_1)$.*

Proof. (i) Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$ be a morphism in $\mathfrak{M}(A, 0)$. Then $[\mathrm{Im}(f)] = A = [h_{n-1}\xi_{r-1} + (m - c_1)h_n\xi_{r-2}]_*$. Since $h(A) = 1$, $\pi^*H \cap f(\mathbb{P}^1)$ consists of a single point for any hyperplane H in $\mathbb{P}^n$. Thus, $\pi|_{f(\mathbb{P}^1)} : f(\mathbb{P}^1) \rightarrow (\pi \circ f)(\mathbb{P}^1)$ is an isomorphism and $\ell = (\pi \circ f)(\mathbb{P}^1)$ is a line in $\mathbb{P}^n$. Since $h([\ell]) = 1$, $(\pi \circ f) : \mathbb{P}^1 \rightarrow \ell = (\pi \circ f)(\mathbb{P}^1)$ is also an isomorphism, and so is $f : \mathbb{P}^1 \rightarrow f(\mathbb{P}^1)$. Replacing $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$ by $f \circ (\pi \circ f)^{-1} : \ell \rightarrow \mathbb{P}(V)$, we conclude that $\mathfrak{M}(A, 0)$ consists of embeddings $f : \ell \rightarrow \mathbb{P}(V)$ such that $[\mathrm{Im}(f)] = A$, ℓ are lines in $\mathbb{P}^n$, and $\pi|_{f(\ell)} : f(\ell) \rightarrow \ell$ are isomorphisms. In particular, these embeddings $f : \ell \rightarrow \mathbb{P}(V)$ are sections to the natural projection $\pi|_{\mathbb{P}(V|_\ell)} : \mathbb{P}(V|_\ell) \rightarrow \ell$. Thus, by Proposition 7.12 in Chapter II of [7], these embeddings are induced by surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m) \rightarrow 0$. By (2.1), the splitting type of the restrictions of V to generic lines in $\mathbb{P}^n$ is $(m_1, \dots, m_r)$ with $m_1 \leq \dots \leq m_r$; thus we must have $V|_\ell = \bigoplus_{i=1}^r \mathcal{O}_\ell(m'_i)$, where $m'_i \geq m_1$ for every i . It follows that $m \geq \min\{m'_1, \dots, m'_r\} \geq m_1$.

(ii) Note that all lines in $\mathbb{P}^n$ are parameterized by the Grassmannian $G(2n + 1)$, which has dimension $2(n - 1)$. For a fixed generic line $\ell \subset \mathbb{P}^n$, the surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m_1) \rightarrow 0$ are parameterized by

$$\mathbb{P}(\mathrm{Hom}(V|_\ell, \mathcal{O}_\ell(m_1))) \cong \mathbb{P}\left(\bigoplus_{i=1}^r H^0(\ell, \mathcal{O}_\ell(m_1 - m_i))\right) \cong \mathbb{P}^{k-1}.$$

It follows from (i) that as the generic line ℓ varies, the morphisms $f : \ell \rightarrow \mathbb{P}(V)$ induced by these surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m_1) \rightarrow 0$ form an open dense subset of $\mathfrak{M}(A, 0)$. Thus, $\dim \mathfrak{M}(A, 0) = 3 + 2(n - 1) + (k - 1) = 2n + k$.

(iii) As in the proof of (ii), for a fixed generic line $\ell \subset \mathbb{P}^n$, the surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m) \rightarrow 0$ are parameterized by a nonempty open subset of

$$\mathbb{P}(\mathrm{Hom}(V|_\ell, \mathcal{O}_\ell(m))) \cong \mathbb{P}\left(\bigoplus_{i=1}^r H^0(\ell, \mathcal{O}_\ell(m - m_i))\right) \cong \mathbb{P}^{(rm - c_1 + r) - 1}.$$

As the generic line ℓ varies, the morphisms $f : \ell \rightarrow \mathbb{P}(V)$ induced by these surjective maps $V|_\ell \rightarrow \mathcal{O}_\ell(m) \rightarrow 0$ form an open dense subset of $\mathfrak{M}(A, 0)$. It follows that $\mathfrak{M}(A, 0)$ has dimension $(2n + r + rm - c_1)$. $\square$

3. CALCULATION OF GROMOV-WITTEN INVARIANTS

In this section, we shall compute some Gromov-Witten invariants of $\mathbb{P}(V)$. First of all, we recall that for two homogeneous elements α and β in $H^*(\mathbb{P}(V); \mathbb{Z})$, the quantum product $\alpha \cdot \beta \in H^*(\mathbb{P}(V); \mathbb{Z})$ can be written as

$$(3.1) \quad \alpha \cdot \beta = \sum_{A \in H_2(\mathbb{P}(V); \mathbb{Z})} (\alpha \cdot \beta)_A \cdot e^{t \cdot K_{\mathbb{P}(V)}(A)},$$

where $(\alpha \cdot \beta)_A$ has degree $\deg(\alpha) + \deg(\beta) + 2K_{\mathbb{P}(V)}(A)$ and is defined by

$$(\alpha \cdot \beta)_A(\gamma_*) = \Phi_{(A,0)}(\alpha, \beta, \gamma)$$

for a homogeneous cohomology class $\gamma \in H^*(\mathbb{P}(V); \mathbb{Z})$ with

$$(3.2) \quad \deg(\gamma) = -2K_{\mathbb{P}(V)}(A) + 2(n + r - 1) - \deg(\alpha) - \deg(\beta).$$

Furthermore, for higher quantum products, we have

$$(3.3) \quad \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k = \sum_{A \in H_2(\mathbb{P}(V); \mathbb{Z})} (\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k)_A \cdot e^{t \cdot K_{\mathbb{P}(V)}(A)},$$

where $(\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k)_A$ is defined as $(\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k)_A(\gamma_*) = \Phi_{(A,0)}(\alpha_1, \alpha_2, \dots, \alpha_k, \gamma)$. Thus, $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k = \alpha_1 \alpha_2 \dots \alpha_k + (\text{lower order terms})$, where $\alpha_1 \alpha_2 \dots \alpha_k$ stands for the ordinary cohomology product of $\alpha_1, \alpha_2, \dots, \alpha_k$, and the degree of a lower order term is dropped by $2K_{\mathbb{P}(V)}(A)$ for some $A \in H_2(\mathbb{P}(V); \mathbb{Z})$ which is represented by a nonconstant effective rational curve.

There are two explanations for the Gromov-Witten invariant $\Phi_{(A,0)}(\alpha, \beta, \gamma)$ defined by the second author [12]. Recall that the Gromov-Witten invariant is only defined for a generic almost complex structure and that $\mathfrak{M}(A, 0)$ is the moduli space of morphisms $f : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$ with $[\text{Im}(f)] = A$. Assume the genericity conditions:

- (i) $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is smooth in the sense that $h^1(N_f) = 0$ for every $f \in \mathfrak{M}(A, 0)$, where N_f is the normal bundle, and
- (ii) the homology class A is only represented by irreducible and reduced curves.

Then the complex structure is already generic, and one can use algebraic geometry to calculate the Gromov-Witten invariants. Moreover, $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is compact with the expected complex dimension

$$(3.4) \quad -K_{\mathbb{P}(V)}(A) + (n + r - 1) - 3.$$

The first explanation for $\Phi_{(A,0)}(\alpha, \beta, \gamma)$ is that when α, β, γ are classes of subvarieties B, C, D of $\mathbb{P}(V)$ in general position, $\Phi_{(A,0)}(\alpha, \beta, \gamma)$ is the number of rational curves E in $\mathbb{P}(V)$ such that $[E] = A$ and E intersects with B, C, D (counted with suitable multiplicity). The second explanation for $\Phi_{(A,0)}(\alpha, \beta, \gamma)$ is that

$$\Phi_{(A,0)}(\alpha, \beta, \gamma) = \int_{\mathfrak{M}(A,0)} e_0^*(\alpha) \cdot e_1^*(\beta) \cdot e_2^*(\gamma),$$

where the evaluation map $e_i : \mathfrak{M}(A, 0) \rightarrow \mathbb{P}(V)$ is defined by $e_i(f) = f(i)$.

Assume that the genericity condition (i) is not satisfied but $h^1(N_f)$ is independent of $f \in \mathfrak{M}(A, 0)$ and $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is smooth with dimension

$$-K_{\mathbb{P}(V)}(A) + (n + r - 1) - 3 + h^1(N_f).$$

Then one can form an obstruction bundle COB of rank $h^1(N_f)$ over the moduli space $\mathfrak{M}(A, 0)$. Moreover, if the genericity condition (ii) is satisfied, then by Proposition 5.7 in [11], we have

$$(3.5) \quad \Phi_{(A,0)}(\alpha, \beta, \gamma) = \int_{\mathfrak{M}(A,0)} e_0^*(\alpha) \cdot e_1^*(\beta) \cdot e_2^*(\gamma) \cdot e(COB)$$

where $e(COB)$ stands for the Euler class of the bundle COB .

We remark that in general, the cohomology class $h_i \xi_j$ may not be representable by a subvariety of $\mathbb{P}(V)$. However, since ξ is ample, $s\xi$ is very ample for $s \gg 0$. Thus, the multiple $th_i \xi_j$ with $t \gg 0$ can be represented by a subvariety of $\mathbb{P}(V)$ whose image in $\mathbb{P}^n$ is a linear subspace of codimension i . Since $\Phi_{(A,0)}(\alpha, \beta, h_i \xi_j) =$

$1/t \cdot \Phi_{(A,0)}(\alpha, \beta, t \cdot h_i \xi_j)$ for α and β in $H^*(\mathbb{P}(V); \mathbb{Z})$, it follows that to compute $\Phi_{(A,0)}(\alpha, \beta, h_i \xi_j)$, it suffices to compute $\Phi_{(A,0)}(\alpha, \beta, t \cdot h_i \xi_j)$. In the proofs below, we shall assume implicitly that $t = 1$ for simplicity.

Now we compute the Gromov-Witten invariant $\Phi_{((h_n \xi_{r-2})_*, 0)}(\xi, \xi_{r-1}, h_n \xi_{r-1})$.

Lemma 3.6. $\Phi_{((h_n \xi_{r-2})_*, 0)}(\xi, \xi_{r-1}, h_n \xi_{r-1}) = 1$.

Proof. First of all, we notice that $A = (h_n \xi_{r-2})_*$ can only be represented by lines ℓ in the fibers of π . In particular, there is no reducible or nonreduced effective curve representing A . Thus, $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is compact and has dimension

$$\dim(\mathbb{P}^n) + \dim G(2, r) = n + 2(r - 2) = n + 2r - 4,$$

which is the expected dimension by (3.4) (here we use $G(2, r)$ to stand for the Grassmannian of lines in $\mathbb{P}^{r-1}$). Next, we want to show that $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is smooth. Let $p = \pi(\ell)$. Then from the two inclusions $\ell \subset \pi^{-1}(p) \subset \mathbb{P}(V)$, we obtain an exact sequence relating normal bundles:

$$0 \rightarrow N_{\ell|\pi^{-1}(p)} \rightarrow N_{\ell|\mathbb{P}(V)} \rightarrow (N_{\pi^{-1}(p)|\mathbb{P}(V)})|_{\ell} \rightarrow 0.$$

Since $N_{\ell|\pi^{-1}(p)} = N_{\ell|\mathbb{P}^{r-1}} = \mathcal{O}_{\ell}(1)^{\oplus(r-2)}$ and $N_{\pi^{-1}(p)|\mathbb{P}(V)} = (\pi|_{\pi^{-1}(p)})^* T_{p, \mathbb{P}^n}$, the previous exact sequence is simplified into the exact sequence

$$0 \rightarrow \mathcal{O}_{\ell}(1)^{\oplus(r-2)} \rightarrow N_{\ell|\mathbb{P}(V)} \rightarrow (\pi|_{\ell})^* T_{p, \mathbb{P}^n} \rightarrow 0.$$

It follows that $H^1(\ell, N_{\ell|\mathbb{P}(V)}) = 0$. Thus, $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is smooth.

Finally, the Poincaré dual of $h_n \xi_{r-1}$ is represented by a point $q_0 \in \mathbb{P}(V)$. If a line $\ell \in \mathfrak{M}(A, 0)$ intersects q_0 , then $\ell \subset \pi^{-1}(\pi(q_0))$. Since the restriction of ξ to the fiber $\pi^{-1}(\pi(q_0)) \cong \mathbb{P}^{r-1}$ is the cohomology class of a hyperplane in $\mathbb{P}^{r-1}$, we conclude that $\Phi_{((h_n \xi_{r-2})_*, 0)}(\xi, \xi_{r-1}, h_n \xi_{r-1}) = 1$. $\square$

Next we show the vanishing of some Gromov-Witten invariant.

Lemma 3.7. Let $A = b(h_n \xi_{r-2})_*$ with $b \geq 1$ and $\alpha \in H^*(\mathbb{P}(V); \mathbb{Z})$. Then

$$\Phi_{(A,0)}(h_{p_1} \xi_{q_1}, h_{p_2} \xi_{q_2}, \alpha) = 0,$$

if p_1, q_1, p_2, q_2 are nonnegative integers with $(q_1 + q_2) < r$.

Proof. We may assume that α is a homogeneous class in $H^*(\mathbb{P}(V); \mathbb{Z})$. By (3.2),

$$\begin{aligned} \frac{1}{2} \cdot \deg(\alpha) &= (n + r - 1) - K_{\mathbb{P}(V)}(A) - (p_1 + p_2 + q_1 + q_2) \\ &= (n + r + br - 1) - (p_1 + p_2 + q_1 + q_2). \end{aligned}$$

Let $\alpha = h_{(n+r+br-1)-(p_1+p_2+q_1+q_2+q_3)} \xi_{q_3}$ with $0 \leq q_3 \leq (r-1)$. Let B, C, D be the subvarieties of $\mathbb{P}(V)$ in general position, whose homology classes are Poincaré dual to $h_{p_1} \xi_{q_1}, h_{p_2} \xi_{q_2}, \alpha$ respectively. Then the homology classes of $\pi(B), \pi(C), \pi(D)$ in $\mathbb{P}^n$ are Poincaré dual to $h_{p_1}, h_{p_2}, h_{(n+r+br-1)-(p_1+p_2+q_1+q_2+q_3)}$ respectively. Since $(q_1 + q_2 + q_3) < (2r - 1)$, we have $p_1 + p_2 + [(n + r + br - 1) - (p_1 + p_2 + q_1 + q_2 + q_3)] = (n + r + br - 1) - (q_1 + q_2 + q_3) > n$. Thus, $\pi(B) \cap \pi(C) \cap \pi(D) = \emptyset$. Notice that the genericity conditions (i) and (ii) mentioned earlier in this section are not satisfied for $b \geq 2$. However, we observe that these conditions can be relaxed by assuming:

- (i') $h^1(N_f) = 0$ for every $f \in \mathfrak{M}(A, 0)$ such that $\text{Im}(f)$ intersects B, C, D , and
- (ii') there is no reducible or nonreduced effective (connected) curve E such that $[E] = A$ and E intersects B, C, D .

In fact, we will show that there is no effective connected curve E at all representing A and intersecting B, C, D . This obviously implies (i'), (ii') and

$$\Phi_{(A,0)}(h_{p_1}\xi_{q_1}, h_{p_2}\xi_{q_2}, \alpha) = 0.$$

Suppose that $E = \sum a_i E_i$ is such an effective connected curve, where $a_i > 0$ and E_i is irreducible and reduced. Then, $\sum a_i [E_i] = [E] = A$. Since $(h_n \xi_{r-2})_*$ generates an extremal ray for $\mathbb{P}(V)$, $[E_i] = b_i (h_n \xi_{r-2})_*$ for $0 < b_i \leq b$. Thus the curves E_i are contained in the fibers of π . Since E is connected, all the curves E_i must be contained in the same fiber of π . So $\pi(E)$ is a single point. Since E intersects B, C, D , $\pi(E)$ intersects with $\pi(B), \pi(C), \pi(D)$. It follows that $\pi(B) \cap \pi(C) \cap \pi(D)$ contains $\pi(E)$ and is nonempty. Therefore we obtain a contradiction. $\square$

Finally, we show that if $c_1 < 2r$ and $A = [h_{n-1}\xi_{r-1} + (1 - c_1)h_n\xi_{r-2}]_*$, then $\Phi_{(A,0)}(h, h_n, h_n\xi_{2r-c_1-1}) = 1$. Since $c_1 < 2r$, we see that for a generic line $\ell \subset \mathbb{P}^n$,

$$V|_\ell = \mathcal{O}_\ell(1)^{\oplus k} \oplus \mathcal{O}_\ell(m_{k+1}) \oplus \dots \oplus \mathcal{O}_\ell(m_r),$$

where $k \geq 1$ and $2 \leq m_{k+1} \leq \dots \leq m_r$. We remark that even though the moduli space $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is compact by Lemma 2.2 (iv), it may not have the correct dimension by Lemma 2.3 (ii). The proof is lengthy, but the basic idea is that we shall determine the obstruction bundle and use the formula (3.5).

Lemma 3.8. *Let V be a rank- r ample vector bundle over $\mathbb{P}^n$ satisfying $c_1 < 2r$ and the assumption of Theorem B(i). If $A = [h_{n-1}\xi_{r-1} + (1 - c_1)h_n\xi_{r-2}]_*$, then*

$$\Phi_{(A,0)}(h, h_n, h_n\xi_{2r-c_1-1}) = 1.$$

Proof. Note that by Lemma 2.2 (iv), the moduli space $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ is compact. Let B, C, D be the subvarieties of $\mathbb{P}(V)$ in general position, whose homology classes are Poincaré dual to $h, h_n, h_n\xi_{2r-c_1-1}$ respectively. Then the homology classes of $\pi(B), \pi(C), \pi(D)$ in $\mathbb{P}^n$ are Poincaré dual to h, h_n, h_n respectively. Thus $\pi(C)$ and $\pi(D)$ are two different points in $\mathbb{P}^n$. Let ℓ_0 be the unique line passing through $\pi(C)$ and $\pi(D)$. Let $V|_{\ell_0} = \mathcal{O}_{\ell_0}(1)^{\oplus k} \oplus \mathcal{O}_{\ell_0}(m_{k+1}) \oplus \dots \oplus \mathcal{O}_{\ell_0}(m_r)$, where $2 \leq m_{k+1} \leq \dots \leq m_r$. Since $c_1 < 2r$, $k \geq 1$. Let $f : \ell \rightarrow \mathbb{P}(V)$ be a morphism in $\mathfrak{M}(A, 0)$ for some line $\ell \in \mathbb{P}^n$. If $\text{Im}(f)$ intersects with B, C , and D , then $\ell = \ell_0$. As in the proof of Lemma 2.3 (ii), the morphisms $f : \ell_0 \rightarrow \mathbb{P}(V)$ in $\mathfrak{M}(A, 0)$ are parameterized by $\mathbb{P}(\text{Hom}(V|_{\ell_0}, \mathcal{O}_{\ell_0}(1))) \cong \mathbb{P}^{k-1}$; moreover, $\text{Im}(f)$ are of the form

$$(3.9) \quad \ell_0 \times \{q\} \subset \ell_0 \times \mathbb{P}^{k-1} = \mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k}) \subset \mathbb{P}(V|_{\ell_0}) \subset \mathbb{P}(V),$$

where q stands for points in $\mathbb{P}^{k-1} \subset \mathbb{P}^{r-1} \cong \pi^{-1}(\pi(D))$. Note that $\ell_0 \times \{q\}$ always intersects with B and C , and that D is a dimension- $(c_1 - r)$ linear subspace in $\mathbb{P}^{r-1} \cong \pi^{-1}(\pi(D))$. Thus, $\ell_0 \times \{q\}$ intersects with B, C, D simultaneously if and only if $\ell_0 \times \{q\}$ intersects with D , and if only if

$$(3.10) \quad q \in \mathbb{P}^{c_1+k-2r} \stackrel{\text{def}}{=} \mathbb{P}^{k-1} \cap D \subset \mathbb{P}^{r-1} \cong \pi^{-1}(\pi(D)).$$

It follows that $\mathfrak{M}/PSL(2; \mathbb{C}) \cong \mathbb{P}^{c_1+k-2r}$, where $\mathfrak{M}$ consists of morphisms $f \in \mathfrak{M}(A, 0)$ such that $\text{Im}(f)$ intersects with B, C, D simultaneously.

If $c_1 + k - 2r = 0$, then $a_0 = \Phi_{(A,0)}(h, h_n, h_n\xi_{2r-c_1-1}) = 1$. But in general, we have $c_1 + k - 2r \geq 0$. We shall use (3.5) to compute $a_0 = \Phi_{(A,0)}(h, h_n, h_n\xi_{2r-c_1-1})$. Let $N_f = N_{\ell_0 \times \{q\}|\mathbb{P}(V)}$ be the normal bundle of $\text{Im}(f) = \ell_0 \times \{q\}$ in $\mathbb{P}(V)$. If $h^1(N_f)$

is constant for every $f \in \mathfrak{M}$, then by (3.5), $\Phi_{(A,0)}(h, h_n, h_n \xi_{2r-c_1-1})$ is the Euler number $e(COB)$ of the rank- $(c_1 + k - 2r)$ obstruction bundle COB over

$$\mathfrak{M}/PSL(2; \mathbb{C}) \cong \mathbb{P}^{c_1+k-2r}.$$

Thus we need to show that $h^1(N_f)$ is constant for every $f \in \mathfrak{M}$.

First, we study the normal bundle $N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)}$. The three inclusions

$$(3.11) \quad \ell_0 \times \mathbb{P}^{c_1+k-2r} \subset \ell_0 \times \mathbb{P}^{k-1} = \mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k}) \subset \mathbb{P}(V|_{\ell_0}) \subset \mathbb{P}(V)$$

give rise to two exact sequences relating normal bundles:

$$0 \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V|_{\ell_0})} \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)} \rightarrow N_{\mathbb{P}(V|_{\ell_0}) | \mathbb{P}(V)} \rightarrow 0,$$

$$0 \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k})} \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V|_{\ell_0})} \rightarrow N_{\mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k}) | \mathbb{P}(V|_{\ell_0})} \rightarrow 0.$$

Notice that $N_{\mathbb{P}(V|_{\ell_0}) | \mathbb{P}(V)} = (\pi|_{\mathbb{P}(V|_{\ell_0})})^*(N_{\ell_0 | \mathbb{P}^n}) = \mathcal{O}_{\ell_0}(1)^{\oplus(n-1)}$ and that

$$N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k})} = N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \ell_0 \times \mathbb{P}^{k-1}} = \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)^{\oplus(2r-c_1-1)}.$$

Since $V|_{\ell_0} = \mathcal{O}_{\ell_0}(1)^{\oplus k} \oplus \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(m_i)$, it follows that $\xi|_{\ell_0 \times \mathbb{P}^{k-1}} = \mathcal{O}_{\ell_0}(1) \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(1)$ and

$$\begin{aligned} N_{\mathbb{P}(\mathcal{O}_{\ell_0}(1)^{\oplus k}) | \mathbb{P}(V|_{\ell_0})} &= \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(-m_i) \otimes \xi|_{\ell_0 \times \mathbb{P}^{k-1}} \\ &= \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(1 - m_i) \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(1). \end{aligned}$$

Thus the previous two exact sequences are simplified to

$$(3.12) \quad 0 \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V|_{\ell_0})} \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)} \rightarrow \mathcal{O}_{\ell_0}(1)^{\oplus(n-1)} \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)^{\oplus(2r-c_1-1)} \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V|_{\ell_0})}$$

$$(3.13) \quad \rightarrow \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(1 - m_i) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1) \rightarrow 0.$$

Now (3.13) splits, since for $k+1 \leq i \leq r$ we have $m_i \geq 2$ and

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_{\ell_0}(1 - m_i) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1), \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)) \\ = H^1(\ell_0 \times \mathbb{P}^{c_1+k-2r}, \mathcal{O}_{\ell_0}(m_i - 1)) = 0. \end{aligned}$$

Thus, the normal bundle $N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V|_{\ell_0})}$ is isomorphic to

$$\bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(1 - m_i) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1) \oplus \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)^{\oplus(2r-c_1-1)},$$

and the exact sequence (3.12) becomes the exact sequence

$$\begin{aligned}
 (3.14) \quad 0 \rightarrow \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(1-m_i) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1) \oplus \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)^{\oplus(2r-c_1-1)} \\
 \rightarrow N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)} \rightarrow \mathcal{O}_{\ell_0}(1)^{\oplus(n-1)} \rightarrow 0.
 \end{aligned}$$

Restricting (3.14) to $\ell_0 \times \{q\}$ and taking the long exact cohomology sequence, we get

$$\begin{aligned}
 (3.15) \quad \bigoplus_{i=k+1}^r H^1(\mathcal{O}_{\ell_0}(1-m_i)) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)|_q \\
 \rightarrow H^1((N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}}) \rightarrow 0.
 \end{aligned}$$

Next, we determine N_f and show that $h^1(N_f) \leq c_1 + k - 2r$. The two inclusions $\ell_0 \times \{q\} \subset \ell_0 \times \mathbb{P}^{c_1+k-2r} \subset \mathbb{P}(V)$ give an exact sequence

$$0 \rightarrow N_{\ell_0 \times \{q\} | \ell_0 \times \mathbb{P}^{c_1+k-2r}} \rightarrow N_{\ell_0 \times \{q\} | \mathbb{P}(V)} \rightarrow (N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}} \rightarrow 0.$$

Since $N_{\ell_0 \times \{q\} | \ell_0 \times \mathbb{P}^{c_1+k-2r}} = T_{q, \mathbb{P}^{c_1+k-2r}}$, the above exact sequence becomes

$$(3.16) \quad 0 \rightarrow T_{q, \mathbb{P}^{c_1+k-2r}} \rightarrow N_f \rightarrow (N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}} \rightarrow 0.$$

Thus, $h^1(N_f) = h^1((N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}})$. By (3.15), we obtain

$$\begin{aligned}
 h^1(N_f) &= h^1((N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}}) \leq \sum_{i=k+1}^r h^1(\mathcal{O}_{\ell_0}(1-m_i)) \\
 &= \sum_{i=k+1}^r (m_i - 2) = c_1 + k - 2r.
 \end{aligned}$$

Finally, we show that $h^1(N_f) = c_1 + k - 2r$. It suffices to prove that $h^1(N_f) \geq c_1 + k - 2r$. Since ℓ_0 is a generic line in $\mathbb{P}^n$ and $V|_{\ell_0} = \mathcal{O}_{\ell_0}(1)^{\oplus k} \oplus \bigoplus_{i=k+1}^r \mathcal{O}_{\ell_0}(m_i)$, $\dim \mathfrak{M}(A, 0) = (2n + k)$ by Lemma 2.3 (ii). Since $h^0(N_f)$ is the dimension of the Zariski tangent space of $\mathfrak{M}(A, 0)/PSL(2; \mathbb{C})$ at f , $h^0(N_f) \geq (2n + k - 3)$. Thus,

$$h^1(N_f) = h^0(N_f) - \chi(N_f) \geq (2n + k - 3) - (2n + 2r - c_1 - 3) = k + c_1 - 2r.$$

Therefore, $h^1(N_f) = c_1 + k - 2r$. In particular, $h^1(N_f)$ is independent of $f \in \mathfrak{M}$. To obtain the obstruction bundle COB over $\mathbb{P}^{c_1+k-2r}$, we notice that (3.15) gives

$$\bigoplus_{i=k+1}^r H^1(\mathcal{O}_{\ell_0}(1-m_i)) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)|_q \cong H^1((N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}}).$$

Thus by the exact sequence (3.16), we conclude that

$$\begin{aligned}
 (3.17) \quad H^1(N_f) &\cong H^1((N_{\ell_0 \times \mathbb{P}^{c_1+k-2r} | \mathbb{P}(V)})|_{\ell_0 \times \{q\}}) \\
 &\cong \bigoplus_{i=k+1}^r H^1(\mathcal{O}_{\ell_0}(1-m_i)) \otimes \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)|_q.
 \end{aligned}$$

It follows that $COB = \mathcal{O}_{\mathbb{P}^{c_1+k-2r}}(1)^{\oplus(c_1+k-2r)}$. By (3.5), we obtain

$$a_0 = \Phi_{(A,0)}(h, h_n, h_n \xi_{2r-c_1-1}) = e(COB) = 1. \quad \square$$

4. PROOF OF THEOREM B

In this section, we prove Theorem B, which we restate below.

Theorem 4.1. (i) *Let V be a rank- r ample bundle over $\mathbb{P}^n$. Assume either $c_1 \leq n$, or $c_1 \leq (n+r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef, so that $\mathbb{P}(V)$ is Fano. Then the quantum cohomology $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations*

$$(4.2) \quad h^{n+1} = \sum_{i+j \leq (c_1-r)} a_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(n+1-i-j)},$$

$$(4.3) \quad \sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr} + \sum_{i+j \leq (c_1-n-1)} b_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(r-i-j)},$$

where the coefficients $a_{i,j}$ and $b_{i,j}$ are integers depending on V .

(ii) *If we further assume that $c_1 < 2r$, then the leading coefficient $a_{0,c_1-r} = 1$.*

Proof. (i) First, we determine the first relation f_ω^1 in (1.3). By Lemma 3.7,

$$(4.4) \quad h \cdot h_p = h_{p+1} + \sum_{A \in H'_2} (h \cdot h_p)_A \cdot e^{tK_{\mathbb{P}(V)}(A)},$$

where $p \geq 1$ and H'_2 stands for $H_2(\mathbb{P}(V); \mathbb{Z}) - \mathbb{Z} \cdot (h_n \xi_{r-2})_*$. Thus,

$$h^{n-p} \cdot h_{p+1} = h^{n-p+1} \cdot h_p - \sum_{A \in H'_2} h^{n-p} \cdot (h \cdot h_p)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}.$$

If $(h \cdot h_p)_A \neq 0$, then $A = [E]$ for some effective curve E . So $a = h(A) \geq 0$. Since $A \in H'_2$, $a \geq 1$. We claim that $-K_{\mathbb{P}(V)}(A) \geq (n+1-c_1+r)$, with equality if and only if $A = [h_{n-1}\xi_{r-1} + (1-c_1)h_n\xi_{r-2}]_* \stackrel{\text{def}}{=} A_2$. Indeed, if $c_1 \leq n$, then $-K_{\mathbb{P}(V)}(A) = (n+1-c_1)a + r \cdot \xi(A) \geq (n+1-c_1+r)$, with equality if and only if $a = \xi(A) = 1$, that is, if and only if $A = A_2$; if $c_1 \leq (n+r)$ and $(\xi-h)$ is nef, then again $-K_{\mathbb{P}(V)}(A) = (n+1+r-c_1)a + r \cdot (\xi-h)(A) \geq (n+1-c_1+r)$, with equality if and only if $a = 1$ and $(\xi-h)(A) = 0$, that is, if and only if $A = A_2$. Thus, $\deg((h \cdot h_p)_A) = 1+p+K_{\mathbb{P}(V)}(A) \leq (p-n+c_1-r)$, and $\deg(h^{n-p} \cdot (h \cdot h_p)_A) \leq (c_1-r)$. Using induction on p and keeping track of the exponential $e^{tK_{\mathbb{P}(V)}(A)}$, we obtain

$$0 = h_{n+1} = h^{n+1} - \sum_{i+j \leq (c_1-r)} a_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(n+1-i-j)}.$$

Therefore, the first relation f_ω^1 for the quantum cohomology ring is

$$h^{n+1} = \sum_{i+j \leq (c_1-r)} a_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(n+1-i-j)}.$$

Next, we determine the second relation f_ω^2 in (1.3). We need to compute the quantum product $h^i \cdot \xi^{r-i}$ for $0 \leq i \leq r$. First, we calculate the quantum product ξ^r . Note that if $A = (bh_n\xi_{r-2})_*$ with $b \geq 1$, then $-K_{\mathbb{P}(V)}(A) = br \geq r$, with $-K_{\mathbb{P}(V)}(A) = r$ if and only if $A = (h_n\xi_{r-2})_* \stackrel{\text{def}}{=} A_1$. Thus, for $p \geq 1$,

$$\xi \cdot \xi_p = \begin{cases} \xi_{p+1} + \sum_{A \in H'_2} (\xi \cdot \xi_p)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } p < r-1, \\ \xi_r + (\xi \cdot \xi_{r-1})_{A_1} \cdot e^{-tr} + \sum_{A \in H'_2} (\xi \cdot \xi_{r-1})_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } p = r-1. \end{cases}$$

Note that $(\xi \cdot \xi_{r-1})_{A_1}$ is of degree zero; by Lemma 3.6, we obtain $(\xi \cdot \xi_{r-1})_{A_1} = \Phi_{(A_1,0)}(\xi, \xi_{r-1}, h_n \xi_{r-1}) = 1$. Therefore for $p \geq 1$,

$$(4.5) \quad \xi \cdot \xi_p = \begin{cases} \xi_{p+1} + \sum_{A \in H'_2} (\xi \cdot \xi_p)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } p < r-1, \\ \xi_r + e^{-tr} + \sum_{A \in H'_2} (\xi \cdot \xi_{r-1})_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } p = r-1. \end{cases}$$

Now, for $i \geq 1$ and $j \geq 1$ with $i+j \leq r$, we have

$$h_i \cdot \xi_j = \begin{cases} h_i \xi_j + \sum_{A \in H'_2} (h_i \cdot \xi_j)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } i+j < r, \\ h_i \xi_j + (h_i \xi_j)_{A_1} \cdot e^{-tr} + \sum_{A \in H'_2} (h_i \cdot \xi_j)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}, & \text{if } i+j = r; \end{cases}$$

when $i+j = r$, $(h_i \xi_j)_{A_1}$ is of degree zero; by Lemma 3.7, we have $(h_i \cdot \xi_j)_{A_1} = \Phi_{(A_1,0)}(h_i, \xi_{r-i}, h_n \xi_{r-1}) = 0$. Therefore for $i \geq 1$ and $j \geq 1$ with $i+j \leq r$,

$$(4.6) \quad h_i \cdot \xi_j = h_i \xi_j + \sum_{A \in H'_2} (h_i \cdot \xi_j)_A \cdot e^{tK_{\mathbb{P}(V)}(A)}.$$

From the proof of the first relation f_ω^1 , we see that if α and β are homogeneous elements in $H^*(\mathbb{P}(V); \mathbb{Z})$ with $\deg(\alpha) + \deg(\beta) = m \leq r$, then $\deg((\alpha \cdot \beta)_A) \leq m - (n+1-c_1+r)$ for $A \in H'_2$. Thus if γ is a homogeneous element in $H^*(\mathbb{P}(V); \mathbb{Z})$ with $\deg(\gamma) = r-m$, then $\deg(\gamma \cdot (\xi \cdot \xi_p)_A) \leq (c_1 - n - 1)$. Since $\sum_{i=0}^r (-1)^i c_i \cdot h_i \xi_{r-i} = 0$, it follows from (4.4), (4.5), and (4.6) that the second relation f_ω^2 is

$$\sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr} + \sum_{i+j \leq (c_1-n-1)} b_{i,j} \cdot h^i \cdot \xi^j \cdot e^{-t(r-i-j)}.$$

(ii) From the proof of the first relation in (i), we see that

$$-K_{\mathbb{P}(V)}(A) \geq (n+1-c_1+r)$$

with equality if and only if $A = A_2$; moreover, the term ξ^{c_1-r} can only come from the quantum correction $(h \cdot h_n)_{A_2}$. Now

$$(h \cdot h_n)_{A_2} = \left(\sum_{i=0}^{c_1-r} a'_i h_i \xi_{c_1-r-i} \right) \cdot e^{-t(n+1-c_1+r)},$$

where $a'_0 = \Phi_{(A_2,0)}(h, h_n, h_n \xi_{2r-c_1-1})$. Since $c_1 < 2r$, $(c_1 - r) < r$. By (4.4), (4.5), and (4.6), we conclude that $h_i \xi_{c_1-r-i} = h^i \cdot \xi^{c_1-r-i} + (\text{lower degree terms})$. Thus $a_{0,c_1-r} = a'_0 = \Phi_{(A_2,0)}(h, h_n, h_n \xi_{2r-c_1-1})$. By Lemma 3.8, $a_{0,c_1-r} = 1$. $\square$

It is understood that when $c_1 \leq n$, then the summations on the right-hand sides of the second relations (4.3) and (4.9) below do not exist.

Next, we shall sharpen the results in Theorem 4.1 by imposing additional conditions on V . Let V be a rank- r ample vector bundle over $\mathbb{P}^n$. Then $c_1 \geq r$. Thus if $c_1 < 2r$ and if either $2c_1 \leq (n+r)$ or $2c_1 \leq (n+2r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef, then the conditions in Theorem 4.1 are satisfied.

Corollary 4.7. (i) *Let V be a rank- r ample vector bundle over $\mathbb{P}^n$ with $c_1 < 2r$. Assume that either $2c_1 \leq (n+r)$, or $2c_1 \leq (n+2r)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef, so that $\mathbb{P}(V)$ is a Fano variety. Then the first relation (4.2) is*

$$(4.8) \quad h^{n+1} = \left(\sum_{i=0}^{c_1-r} a_i \cdot h^i \cdot \xi^{c_1-r-i} \right) \cdot e^{-t(n+1+r-c_1)},$$

where the integers a_i depend on V . Moreover, $a_0 = 1$.

(ii) Let V be a rank- r ample vector bundle over $\mathbb{P}^n$. Assume that $2c_1 \leq (2n+r+1)$ and $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is nef, so that $\mathbb{P}(V)$ is Fano. Then the second relation (4.3) is

$$(4.9) \quad \sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr} + \sum_{i=0}^{c_1-n-1} b_i \cdot h^i \cdot \xi^{c_1-n-1-i} \cdot e^{-t(n+1+r-c_1)},$$

where the integers b_i depend on V .

Proof. (i) From the proof of Theorem 4.1 (i), we notice that it suffices to show that the only homology class $A \in H'_2 = H_2(\mathbb{P}(V); \mathbb{Z}) - \mathbb{Z} \cdot (h_n \xi_{r-2})_*$ which has nonzero contributions to the quantum corrections in (4.4) is

$$A = [h_{n-1} \xi_{r-1} + (1 - c_1) h_n \xi_{r-2}]_* \stackrel{\text{def}}{=} A_2.$$

In other words, if $A = (ah_{n-1} \xi_{r-1} + bh_n \xi_{r-2})_*$ with $a \neq 0$ and if $\Phi_{(A,0)}(h, h_p, \alpha) \neq 0$ for $1 \leq p \leq n$ and $\alpha \in H^*(\mathbb{P}(V); \mathbb{Z})$, then $A = A_2$. First of all, we show that $a = 1$. Suppose $a \neq 1$. Then $a \geq 2$. By (3.2),

$$\begin{aligned} \frac{1}{2} \cdot \deg(\alpha) &= (n+r-1) - K_{\mathbb{P}(V)}(A) - 1 - p \\ &= (n+r-1) + [(n+1-c_1)a + r \cdot \xi(A)] - 1 - p \\ &\geq \dim(\mathbb{P}(V)) + [(n+1-c_1)a + r \cdot \xi(A)] - 1 - n. \end{aligned}$$

If $2c_1 \leq (n+r)$, then $c_1 \leq n$, and

$$[(n+1-c_1)a + r \cdot \xi(A)] - 1 - n \geq 2(n+1-c_1) + r - 1 - n > 0.$$

If $2c_1 \leq (n+2r)$ and $(\xi - h)$ is nef, then $c_1 \leq n+r$, and

$$\begin{aligned} [(n+1-c_1)a + r \cdot \xi(A)] - 1 - n &= [(n+1+r-c_1)a + r \cdot (\xi - h)(A)] - 1 - n \\ &\geq 2(n+1+r-c_1) - 1 - n > 0. \end{aligned}$$

Thus, $[(n+1-c_1)a + r \cdot \xi(A)] - 1 - n > 0$, and so $\deg(\alpha)/2 > \dim(\mathbb{P}(V))$. But this is absurd. Next, we prove that $b = (1 - c_1)$, or equivalently, $\xi(A) = 1$. Suppose $\xi(A) \neq 1$. Then $\xi(A) \geq 2$. By (3.2),

$$\begin{aligned} \frac{1}{2} \cdot \deg(\alpha) &= (n+r-1) + [(n+1-c_1) + r \cdot \xi(A)] - 1 - p \\ &\geq \dim(\mathbb{P}(V)) + [(n+1-c_1) + 2r] - 1 - n \\ &> \dim(\mathbb{P}(V)), \end{aligned}$$

since $c_1 < 2r$. But once again this is absurd.

(ii) We follow the previous arguments for (i). Again it suffices to show that if $A = (ah_{n-1} \xi_{r-1} + bh_n \xi_{r-2})_*$ with $a \neq 0$ and if $\Phi_{(A,0)}(\alpha_1, \alpha_2, \alpha) \neq 0$ for some $\alpha_1, \alpha_2, \alpha \in H^*(\mathbb{P}(V); \mathbb{Z})$ with $\deg(\alpha_1) + \deg(\alpha_2) \leq r$, then $A = A_2$. Indeed, if $a \neq 1$ or if $a = 1$ but $\xi(A) \neq 1$, then we must have $\deg(\alpha)/2 > \dim(\mathbb{P}(V))$. But this is impossible. Therefore, $a = 1$ and $\xi(A) = 1$. So $A = A_2$. $\square$

Now we discuss the relation between the quantum corrections and the extremal rays of the Fano variety $\mathbb{P}(V)$. Let V be a rank- r ample vector bundle over $\mathbb{P}^n$ with $c_1 < 2r$ and $2c_1 \leq (n+r)$. By (4.8) and (4.3), the quantum cohomology ring $H_{\omega}^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations

$$(4.10) \quad h^{n+1} = \left(\sum_{i=0}^{c_1-r} a_i \cdot h^i \cdot \xi^{c_1-r-i} \right) \cdot e^{-t(n+1+r-c_1)},$$

$$(4.11) \quad \sum_{i=0}^r (-1)^i c_i \cdot h^i \cdot \xi^{r-i} = e^{-tr}.$$

From the proof of Theorem 4.1 (i), we notice that the quantum correction to the second relation (4.11) comes from the homology class $A_1 = (h_n \xi_{r-2})_*$ which is represented by the lines in the fibers of $\pi : \mathbb{P}(V) \rightarrow \mathbb{P}^n$. Also, we notice from the proof of Corollary 4.7 (i) that the quantum correction to the first relation (4.10) comes from the homology class $A_2 = [h_{n-1} \xi_{r-1} + (1 - c_1) h_n \xi_{r-2}]_*$; from the proof of Lemma 3.8, A_2 can be represented by a smooth rational curve isomorphic to lines in $\mathbb{P}^n$ via π . Now A_1 generates one of the two extremal rays of $\mathbb{P}(V)$. It is unclear whether A_2 generates the other extremal ray. By Lemma 2.2 (iii), if we further assume that $2c_1 \leq (n+1)$, then indeed A_2 generates the other extremal ray of $\mathbb{P}(V)$. By Lemma 2.2 (ii), A_2 generates the other extremal ray of $\mathbb{P}(V)$ if and only if $(\xi - h)$ is nef, that is, $V \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is a nef vector bundle over $\mathbb{P}^n$.

5. DIRECT SUM OF LINE BUNDLES OVER $\mathbb{P}^n$

In this section, we partially verify Batyrev's conjecture on the quantum cohomology of projective bundles associated to direct sum of line bundles over $\mathbb{P}^n$. We shall use (3.5) to compute the necessary Gromov-Witten invariants. Our first step is to recall some standard materials for the Grassmannian $G(2, n+1)$ from [3]. Then we determine a certain obstruction bundle and its Euler class. Finally we proceed to determine the first and second relations for the quantum cohomology.

On the Grassmannian $G(2, n+1)$, there exists a tautological exact sequence

$$(5.1) \quad 0 \rightarrow S \rightarrow (\mathcal{O}_{G(2, n+1)})^{\oplus(n+1)} \rightarrow Q \rightarrow 0,$$

where the sub- and quotient bundles S and Q are of rank 2 and $(n-1)$ respectively. Let α and β be the virtual classes such that $\alpha + \beta = -c_1(S)$ and $\alpha\beta = c_2(S)$. Then

$$(5.2) \quad \text{cl}(\{\ell \in G(2, n+1) | \ell \cap h_p \neq \emptyset\}) = \frac{\alpha^p - \beta^p}{\alpha - \beta},$$

where $\text{cl}(\cdot)$ denotes the fundamental class and h_p stands for a fixed linear subspace of $\mathbb{P}^n$ of codimension p . If $P(\alpha, \beta)$ is a symmetric homogeneous polynomial of degree $(2n-2)$ (so that $P(\alpha, \beta)$ can be written as a polynomial of maximal degree in the Chern classes of the bundle S), then we have

$$(5.3) \quad \int_{G(2, n+1)} P(\alpha, \beta) = \left(\text{the coefficient of } \alpha^n \beta^n \text{ in } -\frac{1}{2}(\alpha - \beta)^2 P(\alpha, \beta) \right).$$

Let $F_n = \{(x, \ell) \in \mathbb{P}^n \times G(2, n+1) | x \in \ell\}$, and π_1 and π_2 be the two natural projections from F_n to $\mathbb{P}^n$ and $G(2, n+1)$ respectively. Then $F_n = \mathbb{P}(S^*)$, where S^* is the dual bundle of S , and $(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(1))|_{F_n}$ is the tautological line bundle over F_n . Let $\text{Sym}^m(S^*)$ be the m -th symmetric product of S^* . Then for $m \geq 0$,

$$(5.4) \quad \pi_{2*}(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(m)|_{F_n}) \cong \text{Sym}^m(S^*).$$

By the duality theorem for higher direct image sheaves (see p. 253 in [7]),

$$(5.5) \quad \begin{aligned} R^1 \pi_{2*}(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(-m)|_{F_n}) &\cong (\pi_{2*}(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(m-2)|_{F_n}))^* \otimes (\det S^*)^* \\ &\cong \text{Sym}^{m-2}(S) \otimes (\det S). \end{aligned}$$

Now, let $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $1 = m_1 = \dots = m_k < m_{k+1} \leq \dots \leq m_r$. Assume that $k \geq 1$ and $\mathbb{P}(V)$ is Fano. Then the two extremal rays of $\mathbb{P}(V)$ are

generated by the two classes $A_1 = (h_n \xi_{r-2})_*$ and $A_2 = [h_{n-1} \xi_{r-1} + (1 - c_1) h_n \xi_{r-2}]_*$. From the proof of Lemma 2.3 (ii), we see that

$$(5.6) \quad \mathfrak{M}(A_2, 0)/PSL(2; \mathbb{C}) = G(2, n+1) \times \mathbb{P}^{k-1}.$$

Let a morphism $f \in \mathfrak{M}(A_2, 0)$ be induced by some surjective map $V|_\ell \rightarrow \mathcal{O}_\ell(1) \rightarrow 0$ such that the image $\text{Im}(f)$ of f is of the form

$$\text{Im}(f) = \ell \times \{q\} \subset \ell \times \mathbb{P}^{k-1} \subset \mathbb{P}^n \times \mathbb{P}^{k-1}.$$

Then by arguments similar to the proof of (3.17), we have

$$(5.7) \quad H^1(N_f) \cong \bigoplus_{u=k+1}^r H^1(\mathcal{O}_\ell(1 - m_u)) \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(1)|_q.$$

It follows that the obstruction bundle COB over $\mathfrak{M}(A_2, 0)/PSL(2; \mathbb{C})$ is

$$(5.8) \quad COB \cong \bigoplus_{u=k+1}^r R^1 \pi_{2*}(\pi_1^* \mathcal{O}_{\mathbb{P}^n}(1 - m_u)|_{F_n}) \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(1).$$

Since $c_1(S) = -(\alpha + \beta)$ and $c_2(S) = \alpha\beta$, we obtain from (5.5) the following.

Lemma 5.9. *The Euler class of the obstruction bundle COB is*

$$(5.10) \quad e(COB) = \prod_{u=k+1}^r \prod_{v=0}^{m_u-3} [(1+v)(-\alpha) + (m_u - 2 - v)(-\beta) + \tilde{h}],$$

where $\tilde{h}$ stands for the hyperplane class in $\mathbb{P}^{k-1}$.

Next assuming $c_1 < 2r$, we shall compute the Gromov-Witten invariant

$$(5.11) \quad W_i \stackrel{\text{def}}{=} \Phi_{(A_2, 0)}(h_{\tilde{n}}, h_{n+1-\tilde{n}}, h_{n-i} \xi_{2r-c_1-1+i}),$$

where $0 \leq i \leq (c_1 - r)$ and $\tilde{n} = \lfloor \frac{n+1}{2} \rfloor$ is the largest integer $\leq (n+1)/2$.

Lemma 5.12. *Assume $c_1 < \min(2r, (n+1+2r)/2)$ and $0 \leq i \leq (c_1 - r)$. Then W_i is the coefficient of t^i in the power series expansion of*

$$\prod_{u=1}^r (1 - m_u t)^{m_u - 2}.$$

Proof. Note that the restriction of ξ to $\mathbb{P}^n \times \mathbb{P}^{k-1} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus k})$ is $(h + \tilde{h})$. Thus,

$$\begin{aligned} h_{n-i} \xi_{2r-c_1-1+i}|_{\mathbb{P}^n \times \mathbb{P}^{k-1}} &= \sum_{j=0}^{2r-c_1-1+i} \binom{2r-c_1-1+i}{j} h_{n-i+j} \tilde{h}_{2r-c_1-1+i-j} \\ &= \sum_{j=0}^i \binom{2r-c_1-1+i}{j} h_{n-i+j} \tilde{h}_{2r-c_1-1+i-j}. \end{aligned}$$

So by (3.5) (replacing $\mathfrak{M}(A_2, 0)$ by $\mathfrak{M}(A_2, 0)/PSL(2; \mathbb{C})$), (5.2), and Lemma 5.9,

$$(5.13) \quad W_i = \int_{G(2, n+1) \times \mathbb{P}^{k-1}} \tilde{P}(\alpha, \beta),$$

where $\tilde{P}(\alpha, \beta)$ is the symmetric homogeneous polynomial of degree $(2n-2)+(k-1)$:

$$\begin{aligned} \tilde{P}(\alpha, \beta) &= \frac{\alpha^{\tilde{n}} - \beta^{\tilde{n}}}{\alpha - \beta} \cdot \frac{\alpha^{n+1-\tilde{n}} - \beta^{n+1-\tilde{n}}}{\alpha - \beta} \\ &\quad \cdot \sum_{j=0}^i \binom{2r-c_1-1+i}{j} \frac{\alpha^{n-i+j} - \beta^{n-i+j}}{\alpha - \beta} \cdot \tilde{h}_{2r-c_1-1+i-j} \\ &\quad \cdot \prod_{u=k+1}^r \prod_{v=0}^{m_u-3} [(1+v)(-\alpha) + (m_u-2-v)(-\beta) + \tilde{h}] \\ &= \sum_{j=0}^i \binom{2r-c_1-1+i}{j} \frac{\alpha^{n+1} - \alpha^{n+1-\tilde{n}}\beta^{\tilde{n}} - \alpha^{\tilde{n}}\beta^{n+1-\tilde{n}} + \beta^{n+1}}{(\alpha - \beta)^2} \\ &\quad \cdot \sum_{t=0}^{n-i+j-1} \alpha^t \beta^{n-i+j-1-t} \cdot \tilde{h}_{2r-c_1-1+i-j} \\ &\quad \cdot \prod_{u=k+1}^r \prod_{v=0}^{m_u-3} [(1+v)(-\alpha) + (m_u-2-v)(-\beta) + \tilde{h}]. \end{aligned}$$

By (5.3) and (5.13), we conclude from straightforward manipulations that

$$\begin{aligned} W_i &= \sum_{j=0}^i \binom{2r-c_1-1+i}{j} \cdot (-1)^{i-j} \\ &\quad \cdot \sum_{j_{k+1}+\dots+j_r=i-j} \prod_{u=k+1}^r \binom{m_u-2}{j_u} (m_u-1)^{j_u} \\ &= \sum_{j=0}^i \binom{2r-c_1-1+i}{i-j} \cdot (-1)^j \\ &\quad \cdot \sum_{j_{k+1}+\dots+j_r=j} \prod_{u=k+1}^r \binom{m_u-2}{j_u} (m_u-1)^{j_u}. \end{aligned}$$

Thus W_i is the coefficient of t^i in the polynomial

$$\begin{aligned} &(1+t)^{2r-c_1-1+i} \cdot \prod_{u=k+1}^r [1 - (m_u-1)t]^{m_u-2} \\ &= (1+t)^{2r-c_1-1+i} \cdot \prod_{u=k+1}^r [(1+t) - m_u t]^{m_u-2} \\ &= (1+t)^{2r-c_1-1+i} \cdot \sum_{j=0}^{c_1-2r+k} \sum_{j_{k+1}+\dots+j_r=j} \\ &\quad \cdot \prod_{u=k+1}^r \binom{m_u-2}{j_u} (-m_u t)^{j_u} \cdot (1+t)^{m_u-2-j_u} \\ &= \sum_{j=0}^{c_1-2r+k} \sum_{j_{k+1}+\dots+j_r=j} \prod_{u=k+1}^r \binom{m_u-2}{j_u} (-m_u t)^{j_u} \cdot (1+t)^{i+k-1-j} \end{aligned}$$

since $\sum_{u=k+1}^r (m_u - 2 - j_u) = c_1 - 2r + k - j$. So W_i is the coefficient of t^i in

$$\begin{aligned} \prod_{u=k+1}^r (1 - m_u t)^{m_u - 2} \cdot \sum_{j=0}^{+\infty} \binom{j+k-1}{k-1} t^j &= \prod_{u=k+1}^r (1 - m_u t)^{m_u - 2} \cdot \frac{1}{(1-t)^k} \\ &= \prod_{u=1}^r (1 - m_u t)^{m_u - 2}. \end{aligned}$$

□

Proposition 5.14. *Let $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $m_i \geq 1$ for each i and*

$$\sum_{i=1}^r m_i < \min(2r, (n+1+2r)/2).$$

Then the first relation f_ω^1 for the quantum cohomology ring $H_\omega^(\mathbb{P}(V); \mathbb{Z})$ is*

$$(5.15) \quad h^{n+1} = \prod_{u=1}^r (\xi - m_u h)^{m_u - 1} \cdot e^{-t(n+1+r-\sum_{i=1}^r m_i)}.$$

Proof. We may assume that $1 = m_1 = \dots = m_k < m_{k+1} \leq \dots \leq m_r$. Since the conclusion clearly holds when $k = r$, we also assume that $k < r$. Let $c_1 = \sum_{i=1}^r m_i$. Notice that the conditions in Corollary 4.7 (i) are satisfied. Thus,

$$h^{n+1} = \left(\sum_{i=0}^{c_1-r} a_i \cdot h^i \cdot \xi^{c_1-r-i} \right) \cdot e^{-t(n+1+r-c_1)}.$$

More directly, putting $\tilde{n} = \lfloor \frac{n+1}{2} \rfloor$, then $\tilde{n} < -K_{\mathbb{P}(V)}(A_2) = (n+1+r-c_1)$, and $(n+1-\tilde{n}) < -K_{\mathbb{P}(V)}(A_2)$ unless n is even and $c_1 = (n+2r)/2$. From the proofs in Theorem 4.1 and Corollary 4.7 (i) for the first relation f_ω^1 , we have $h^{\tilde{n}} = h_{\tilde{n}}$, and $h^{n+1-\tilde{n}} = h_{n+1-\tilde{n}}$ unless n is even and $c_1 = (n+2r)/2$. Moreover, if n is even and $c_1 = (n+2r)/2$, then $h^{n+1-\tilde{n}} = h \cdot h^{n-\tilde{n}} = h \cdot h_{n-\tilde{n}} = h_{n+1-\tilde{n}} + (h \cdot h_{n-\tilde{n}})_{A_2} \cdot e^{-t(n+1+r-c_1)}$. Since $(h \cdot h_{n-\tilde{n}})_{A_2}$ is of degree zero, $(h \cdot h_{n-\tilde{n}})_{A_2} = \Phi_{(A_2,0)}(h, h_{n-\tilde{n}}, h_n \xi_{r-1})$. Since $1 \leq k < r$, we can choose a point q_0 in $\mathbb{P}(V)$ representing the homology class $(h_n \xi_{r-1})_*$ such that the point q_0 is not contained in the $(k-1)$ -dimensional linear subspace

$$\mathbb{P}^{k-1} = \mathbb{P}((\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus k})|_{\pi(q_0)}) \subset \mathbb{P}(V|_{\pi(q_0)}) \cong \mathbb{P}^{r-1}.$$

Note that for every $f \in \mathfrak{M}(A_2, 0)$, $\text{Im}(f) = \ell \times \{q\}$ for some line $\ell \subset \mathbb{P}^n$ and some point $q \in \mathbb{P}^{k-1}$. Thus $\text{Im}(f)$ cannot pass q_0 . As in the proof of Lemma 3.7, we conclude that $\Phi_{(A_2,0)}(h, h_{n-\tilde{n}}, h_n \xi_{r-1}) = 0$. Therefore, $h^{n+1-\tilde{n}} = h_{n+1-\tilde{n}}$. So

$$h^{n+1} = h^{\tilde{n}} \cdot h^{n+1-\tilde{n}} = h_{\tilde{n}} \cdot h_{n+1-\tilde{n}}.$$

By similar arguments in the proofs of Theorem 4.1 and Corollary 4.7 (i) for the first relation f_ω^1 , we see that if $(h_{\tilde{n}} \cdot h_{n+1-\tilde{n}})_A \neq 0$, then $A = 0, A_2$. Thus

$$h^{n+1} = h_{n+1} + (h_{\tilde{n}} \cdot h_{n+1-\tilde{n}})_{A_2} \cdot e^{-t(n+1+r-c_1)} = (h_{\tilde{n}} \cdot h_{n+1-\tilde{n}})_{A_2} \cdot e^{-t(n+1+r-c_1)}.$$

So it suffices to show that $(h_{\tilde{n}} \cdot h_{n+1-\tilde{n}})_{A_2} = \prod_{u=1}^r (\xi - m_u h)^{m_u - 1}$. Note that

$$\prod_{u=1}^r (\xi - m_u h)^{m_u - 1} = \prod_{u=1}^r (\xi - m_u h)_{m_u - 1},$$

where the right-hand side stands for the product in the ordinary cohomology. Thus we need to show that $(h_{\tilde{n}} \cdot h_{n+1-\tilde{n}})_{A_2} = \prod_{u=1}^r (\xi - m_u h)_{m_u-1}$, or equivalently,

$$(5.16) \quad \Phi_{(A_2,0)}(h_{\tilde{n}}, h_{n+1-\tilde{n}}, h_{n-i}\xi_{2r-c_1-1+i}) = \prod_{u=1}^r (\xi - m_u h)_{m_u-1} h_{n-i}\xi_{2r-c_1-1+i}$$

for $0 \leq i \leq (c_1 - i)$. The left-hand side of (5.16) is computed in Lemma 5.12.

Denote the right-hand side of (5.16) by $\tilde{W}_i$. Let s_i be the i -th Segre class of V . Then we have $s_i = (-1)^i \cdot \sum_{j_1+\dots+j_r=i} \prod_{u=1}^r m_u^{j_u}$ and

$$(5.17) \quad \sum_{i=0}^{+\infty} (-1)^i s_i t^i = \prod_{u=1}^r \frac{1}{1 - m_u t}.$$

Moreover, from the second relation in (1.1) we obtain, for $i \geq r$,

$$\xi_i = (-1)^{i-(r-1)} s_{i-(r-1)} \xi_{r-1} + (\text{terms with exponentials of } \xi \text{ less than } (r-1)).$$

It follows from the right-hand-side of (5.16) that $\tilde{W}_i$ is equal to

$$\begin{aligned} & \sum_{j=0}^{c_1-r} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} \xi_{m_u-1-j_u} (-m_u h)_{j_u} h_{n-i} \xi_{2r-c_1-1+i} \\ &= \sum_{j=0}^i \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u)^{j_u} h_{n-i+j} \xi_{r-1+i-j} \\ &= \sum_{j=0}^i (-1)^{i-j} s_{i-j} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u)^{j_u}. \end{aligned}$$

Therefore, the formal power series $\sum_{i=0}^{+\infty} \tilde{W}_i t^i$ is equal to

$$\begin{aligned} & \sum_{i=0}^{+\infty} \sum_{j=0}^i (-1)^{i-j} s_{i-j} t^{i-j} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u t)^{j_u} \\ &= \sum_{j=0}^{+\infty} \sum_{i=j}^{+\infty} (-1)^{i-j} s_{i-j} t^{i-j} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u t)^{j_u} \\ &= \sum_{j=0}^{+\infty} \sum_{i=0}^{+\infty} (-1)^i s_i t^i \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u t)^{j_u} \\ &= \sum_{j=0}^{+\infty} \prod_{u=1}^r \frac{1}{1 - m_u t} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u t)^{j_u} \\ &= \prod_{u=1}^r \frac{1}{1 - m_u t} \sum_{j=0}^{+\infty} \sum_{j_1+\dots+j_r=j} \prod_{u=1}^r \binom{m_u-1}{j_u} (-m_u t)^{j_u} \\ &= \prod_{u=1}^r \frac{1}{1 - m_u t} \prod_{u=1}^r (1 - m_u t)^{m_u-1} \\ &= \prod_{u=1}^r (1 - m_u t)^{m_u-2}, \end{aligned}$$

where we have applied (5.17) in the third equality. By Lemma 5.12, $\tilde{W}_i = W_i$ for $0 \leq i \leq (c_1 - r)$. Hence the formulae (5.16) and (5.15) hold. $\square$

It turns out that under certain conditions on the integers m_i , the second relation f_ω^2 for the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is much easier to determine. Note that the second relation f^2 in (1.1) can be rewritten as

$$(5.18) \quad \prod_{i=1}^r (\xi - m_i h) = 0,$$

where the left-hand side stands for the product in the ordinary cohomology ring.

Proposition 5.19. *Let $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $m_i \geq 1$ for each i , $m_i = 1$ for some i , and $\sum_{i=1}^r m_i < (2n+2+r)/2$. Then the second relation f_ω^2 for the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is*

$$(5.20) \quad \prod_{i=1}^r (\xi - m_i h) = e^{-tr},$$

where the left-hand side stands for the product in the quantum cohomology ring.

Proof. We may assume that $1 = m_1 = \dots = m_k < m_{k+1} \leq \dots \leq m_r$. So $k \geq 1$. We notice that the conditions in Corollary 4.7 (ii) are satisfied. From the proofs of Theorem 4.1 (i) and Corollary 4.7 (ii), we see that the quantum corrections to the second relation (5.18) can only come from the classes A_1, A_2 ; moreover, the quantum correction from A_1 is e^{-tr} . Thus it suffices to show that the quantum correction from A_2 is zero. In view of (3.3), it suffices to show that

$$\Phi_{(A_2, 0)}(\xi - m_1 h, \dots, \xi - m_r h, \alpha) = 0$$

for every $\alpha \in H^*(\mathbb{P}(V); \mathbb{Z})$. For $1 \leq i \leq r$, let V_i be the following subbundle of V :

$$V_i = \mathcal{O}_{\mathbb{P}^n}(m_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(m_{i-1}) \oplus \mathcal{O}_{\mathbb{P}^n}(m_{i+1}) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(m_r),$$

and let $B_i = \mathbb{P}(V_i)$ be the codimension-1 subvariety of $\mathbb{P}(V)$ induced by the projection $V \rightarrow V_i \rightarrow 0$. Then the fundamental class of B_i is $(\xi - m_i h)$. As in the proof of Lemma 3.7, we need only to show that if $f \in \mathfrak{M}(A_2, 0)$, then the image $\text{Im}(f)$ cannot intersect with $B_1, \dots, B_r$ simultaneously. In fact, we will show that $\text{Im}(f)$ cannot intersect with $B_1, \dots, B_k$ simultaneously. Indeed, $\text{Im}(f)$ is of the form

$$\text{Im}(f) = \ell \times \{q\} \subset \ell \times \mathbb{P}^{k-1} \subset \mathbb{P}^n \times \mathbb{P}^{k-1} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus k})$$

for some line $\ell \subset \mathbb{P}^n$, and $B_i|_{\pi^{-1}(\ell)} = \mathbb{P}(V_i|_\ell)$. Put $p = \pi(q) \in \mathbb{P}^n$, and

$$V|_p = \bigoplus_{i=1}^k \mathbb{C} \cdot e_i \oplus (\bigoplus_{i=k+1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)|_p),$$

where e_i is a global section of $\mathcal{O}_{\mathbb{P}^n}(m_i) = \mathcal{O}_{\mathbb{P}^n}(1)$ for $i \leq k$. Now the point q is dual to $\mathbb{C} \cdot v$ for some nonzero vector $v \in \bigoplus_{i=1}^k \mathbb{C} \cdot e_i$. Let $v = \sum_{i=1}^k a_i e_i$. Since $\ell \times \{q\}$ and B_i ($1 \leq i \leq k$) intersect, the one-dimensional vector space $\mathbb{C} \cdot v$ is also contained in $(V_i)|_p$. It follows that $a_i = 0$ for every i with $1 \leq i \leq k$. But this is impossible since v is a nonzero vector. $\square$

In summary, we have partially verified Batyrev's conjecture.

Theorem 5.21. *Let $V = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$, where $m_i \geq 1$ for each i and*

$$\sum_{i=1}^r m_i < \min(2r, (n+1+2r)/2, (2n+2+r)/2).$$

Then the quantum cohomology $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is generated by h and ξ with relations

$$h^{n+1} = \prod_{i=1}^r (\xi - m_i h)^{m_i-1} \cdot e^{-t(n+1+r-\sum_{i=1}^r m_i)} \quad \text{and} \quad \prod_{i=1}^r (\xi - m_i h) = e^{-tr}.$$

Proof. Follows immediately from Propositions 5.14 and 5.19. $\square$

6. EXAMPLES

In this section, we shall determine the quantum cohomology of $\mathbb{P}(V)$ for ample bundles V over $\mathbb{P}^n$ with $2 \leq r \leq n$ and $c_1 = r + 1$. In these cases, $V|_\ell = \mathcal{O}_\ell(1)^{\oplus(r-1)} \oplus \mathcal{O}_\ell(2)$ for every line $\ell \subset \mathbb{P}^n$. In particular, V is a uniform bundle. If $r < n$, then, by Theorem 3.2.3 in [10], $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$; if $r = n$, then by the results on pp.71-72 in [10], $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$ or $V = T_{\mathbb{P}^n}$, the tangent bundle of $\mathbb{P}^n$. When $V = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$ with $r \leq n$, the conditions in Theorem 5.21 are satisfied, so the quantum cohomology ring $H_\omega^*(\mathbb{P}(V); \mathbb{Z})$ is the ring generated by h and ξ with the two relations

$$h^{n+1} = (\xi - 2h) \cdot e^{-t(n+1+r-c_1)} \quad \text{and} \quad (\xi - h)^{r-1}(\xi - 2h) = e^{-tr}.$$

In the rest of this section, we compute the quantum cohomology of $\mathbb{P}(T_{\mathbb{P}^n})$. It is well-known that $(\xi - h)$ is a nef divisor on $\mathbb{P}(T_{\mathbb{P}^n})$, and the two extremal rays of $\mathbb{P}(T_{\mathbb{P}^n})$ are generated by $A_1 = (h_n \xi_{n-2})_*$ and $A_2 = (h_{n-1} \xi_{n-1} - n h_n \xi_{n-2})_*$. Moreover, A_2 is represented by smooth rational curves in $\mathbb{P}(T_{\mathbb{P}^n})$ induced by the surjective maps $T_{\mathbb{P}^n}|_\ell \rightarrow \mathcal{O}_\ell(1) \rightarrow 0$ for lines $\ell \subset \mathbb{P}^n$. Since $c_1 = n + 1$ and $n \geq 2$, the assumptions in Corollary 4.7 are satisfied, so the quantum cohomology ring $H_\omega^*(\mathbb{P}(T_{\mathbb{P}^n}); \mathbb{Z})$ is the ring generated by h and ξ with the two relations

$$(6.1) \quad h^{n+1} = (a_1 h + \xi) \cdot e^{-tn} \quad \text{and} \quad \sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + b_0) \cdot e^{-tn}.$$

More precisely, putting $H'_2 = H_2(\mathbb{P}(V); \mathbb{Z}) - \mathbb{Z} \cdot (h_n \xi_{n-2})_*$, we see from the proof of Corollary 4.7 (i) that the only homology class $A \in H'_2$ which has nonzero contributions to the quantum corrections in (4.4) is $A = A_2$. Thus by (4.4),

$$(6.2) \quad h \cdot h_p = \begin{cases} h_{p+1}, & \text{if } p \leq n-2, \\ h_n + a'_1 \cdot e^{-tn}, & \text{if } p = n-1, \\ h_{n+1} + (a'_2 h + a'_3 \xi) \cdot e^{-tn}, & \text{if } p = n, \end{cases}$$

where $a'_1 = \Phi_{(A_2, 0)}(h, h_{n-1}, h_n \xi_{n-1})$, $a'_3 = \Phi_{(A_2, 0)}(h, h_n, h_n \xi_{n-2})$, and

$$a'_2 = \Phi_{(A_2, 0)}(h, h_n, h_{n-1} \xi_{n-1}) - c_1 a'_3.$$

By Lemma 3.8, $a'_3 = 1$. Thus $a_1 = (a'_1 + a'_2)$ and the first relation f_ω^1 in (6.1) is

$$(6.3) \quad h^{n+1} = ((a'_1 + a'_2)h + \xi) \cdot e^{-tn}.$$

Similarly, from the proof of Corollary 4.7 (ii), we see that the only homology class $A \in H'_2$ which has nonzero contributions to the quantum corrections in (4.5) and (4.6) is also $A = A_2$. By (4.5), $\xi \cdot \xi_p = \xi_{p+1}$ if $p < n-1$, and $\xi \cdot \xi_{n-1} = \xi_n + e^{-tn} + b_2^{(n)} \cdot e^{-tn}$, where $b_2^{(n)} = \Phi_{(A_2, 0)}(\xi, \xi_{n-1}, h_n \xi_{n-1})$. Thus,

$$(6.4) \quad \xi^p = \begin{cases} \xi_p, & \text{if } p < n, \\ \xi_n + (1 + b_2^{(n)}) \cdot e^{-tn}, & \text{if } p = n. \end{cases}$$

By (4.2), we have $h \cdot h_p = h_{p+1}$ if $p < n-1$, and $h \cdot h_{n-1} = h_n + b_2^{(0)} \cdot e^{-tn}$, where $b_2^{(0)} = a'_1 = \Phi_{(A_2,0)}(h_{n-1}, h, h_n \xi_{n-1})$. Thus, we obtain

$$(6.5) \quad h^p = \begin{cases} h_p, & \text{if } p < n, \\ h_n + b_2^{(0)} \cdot e^{-tn}, & \text{if } p = n. \end{cases}$$

By (4.6), for $1 \leq i \leq (n-1)$ we have $h_{n-i} \cdot \xi_i = h_{n-i} \xi_i + b_2^{(i)} \cdot e^{-tn}$, where $b_2^{(i)} = \Phi_{(A_2,0)}(h_{n-i}, \xi_i, h_n \xi_{n-1})$. Thus by (6.4) and (6.5), we have

$$(6.6) \quad h^{n-i} \cdot \xi^i = h_{n-i} \cdot \xi_i = h_{n-i} \xi_i + b_2^{(i)} \cdot e^{-tn}.$$

Since $\sum_{i=0}^n (-1)^i c_i \cdot h_i \xi_{n-i} = 0$, it follows from (6.4), (6.5), (6.6) that

$$(6.7) \quad \sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + \sum_{i=0}^n (-1)^i c_i b_2^{(n-i)}) \cdot e^{-tn}.$$

Next, we compute the above integers a'_1, a'_2 , and $b_2^{(i)}$ for $0 \leq i \leq n$.

Lemma 6.8. *Let $V = T_{\mathbb{P}^n}$ with $n \geq 2$ and $A_2 = (h_{n-1} \xi_{n-1} - n h_n \xi_{n-2})_*$.*

- (i) $\Phi_{(A_2,0)}(h, h_n, h_{n-1} \xi_{n-1}) = n$.
- (ii) *Let $\alpha = h_j \xi_k$ and $\beta = h_s \xi_t$, where j, k, s, t are nonnegative integers such that $\max(j, k) > 0$, $\max(s, t) > 0$, and $(j + k + s + t) = n$. Then,*

$$\Phi_{(A_2,0)}(\alpha, \beta, h_n \xi_{n-1}) = 1.$$

Proof. (i) By Lemma 2.2 (iv), $\mathfrak{M}(A_2, 0)/PSL(2; \mathbb{C})$ is compact. By (3.17), we have $h^1(N_f) = 0$ for every $f \in \mathfrak{M}(A_2, 0)$. Thus, $\mathfrak{M}(A_2, 0)/PSL(2; \mathbb{C})$ is also smooth. Fix a line ℓ_0 in $\mathbb{P}^n$. Let $g : \ell_0 \rightarrow \mathbb{P}(T_{\mathbb{P}^n}|_{\ell_0}) \subset \mathbb{P}(T_{\mathbb{P}^n})$ be the embedding induced by the natural projection $T_{\mathbb{P}^n}|_{\ell_0} = \mathcal{O}_{\ell_0}(1)^{\oplus(n-1)} \oplus \mathcal{O}_{\ell_0}(2) \rightarrow \mathcal{O}_{\ell_0}(2) \rightarrow 0$. Since $h([g(\ell_0)]) = 1$ and $\xi([g(\ell_0)]) = 2$, we have $[g(\ell_0)] = [h_{n-1} \xi_{n-1} - (n-1) h_n \xi_{n-2}]_*$. So $h_{n-1} \xi_{n-1} = [g(\ell_0)]_* + (n-1) h_n \xi_{n-2}$, and

$$\Phi_{(A_2,0)}(h, h_n, h_{n-1} \xi_{n-1}) = \Phi_{(A_2,0)}(h, h_n, [g(\ell_0)]_*) + (n-1) \Phi_{(A_2,0)}(h, h_n, h_n \xi_{n-2}).$$

By Lemma 3.8, it suffices to show that $\Phi_{(A_2,0)}(h, h_n, [g(\ell_0)]_*) = 1$. Let B and C be the subvarieties of $\mathbb{P}(T_{\mathbb{P}^n})$ in general position, whose homology classes are Poincaré dual to h and h_n respectively. Then the homology classes of $\pi(B)$ and $\pi(C)$ in $\mathbb{P}^n$ are Poincaré dual to h and h_n respectively. Let $f : \ell \rightarrow \mathbb{P}(T_{\mathbb{P}^n})$ be a morphism in $\mathfrak{M}(A_2, 0)$ induced by a surjective map $T_{\mathbb{P}^n}|_{\ell} \rightarrow \mathcal{O}_{\ell}(1) \rightarrow 0$ for some line $\ell \subset \mathbb{P}^n$. If the image $\text{Im}(f)$ intersects with B, C , and $g(\ell_0)$, then ℓ intersects with $\pi(B)$, $\pi(C)$, and $\pi(g(\ell_0)) = \ell_0$. In other words, ℓ passes through the point $\pi(C)$ and intersects with ℓ_0 . Moreover, putting $p = \ell \cap \ell_0$ and noticing that every surjective map $T_{\mathbb{P}^n}|_{\ell} \rightarrow \mathcal{O}_{\ell}(1) \rightarrow 0$ factors through the natural projection $T_{\mathbb{P}^n}|_{\ell} = \mathcal{O}_{\ell}(1)^{(n-1)} \oplus \mathcal{O}_{\ell}(2) \rightarrow \mathcal{O}_{\ell}(1)^{(n-1)}$, we conclude that the $(n-1)$ -dimensional subspace $(\mathcal{O}_{\ell}(1)^{(n-1)})|_p$ in $(T_{\mathbb{P}^n}|_{\ell})|_p = T_{p, \mathbb{P}^n}$ must contain the 1-dimensional subspace $(\mathcal{O}_{\ell_0}(2))|_p$ in $(T_{\mathbb{P}^n}|_{\ell_0})|_p = T_{p, \mathbb{P}^n}$. Conversely, let $p \in \ell_0$ and let ℓ_p be the unique line connecting the two points $\pi(C)$ and p . If the $(n-1)$ -dimensional subspace $(\mathcal{O}_{\ell_p}(1)^{(n-1)})|_p$ in $(T_{\mathbb{P}^n}|_{\ell_p})|_p = T_{p, \mathbb{P}^n}$ contains the 1-dimensional subspace $(\mathcal{O}_{\ell_0}(2))|_p$ in $(T_{\mathbb{P}^n}|_{\ell_0})|_p = T_{p, \mathbb{P}^n}$, then there exists a unique surjective map $T_{\mathbb{P}^n}|_{\ell_p} \rightarrow \mathcal{O}_{\ell_p}(1) \rightarrow 0$ such that the image of the induced morphism $f : \ell_p \rightarrow \mathbb{P}(T_{\mathbb{P}^n})$ intersects $g(\ell_0)$ at the point $g(p)$. Since there exists a unique point $p \in \ell_0$ such that

the $(n-1)$ -dimensional subspace $(\mathcal{O}_{\ell_p}(1)^{(n-1)})|_p$ in $(T_{\mathbb{P}^n}|_{\ell_p})|_p = T_{p,\mathbb{P}^n}$ contains the 1-dimensional subspace $(\mathcal{O}_{\ell_0}(2))|_p$ in $(T_{\mathbb{P}^n}|_{\ell_0})|_p = T_{p,\mathbb{P}^n}$, it follows that

$$\Phi_{(A_2,0)}(h, h_n, [g(\ell_0)]_*) = 1.$$

(ii) It is well-known (see p. 176 of [7]) that there is an exact sequence

$$(6.9) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

The surjective map $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0$ induces the inclusion $\phi : \mathbb{P}(T_{\mathbb{P}^n}) \subset \mathbb{P}^n \times \mathbb{P}^n$ such that ξ is the restriction of the $(1,1)$ class in $\mathbb{P}^n \times \mathbb{P}^n$. Let B, C, q_0 be the subvarieties of $\mathbb{P}(T_{\mathbb{P}^n})$ in general position, whose homology classes are Poincaré dual to $\alpha, \beta, h_n \xi_{n-1}$ respectively. Then q_0 is a point. Put $p_0 = \pi(q_0) \in \mathbb{P}^n$. Now the morphisms in $\mathfrak{M}(A_2, 0)$ are of the forms $f : \ell \rightarrow \mathbb{P}(T_{\mathbb{P}^n})$ induced by surjective maps $T_{\mathbb{P}^n}|_{\ell} \rightarrow \mathcal{O}_{\ell}(1) \rightarrow 0$ for lines $\ell \subset \mathbb{P}^n$. If the image $\text{Im}(f)$ passes q_0 , then the line ℓ passes p_0 and q_0 is contained in the hyperplane

$$\mathbb{P}^{n-2} = \mathbb{P}((\mathcal{O}_{\ell}(1)^{\oplus(n-1)})|_{p_0}) \subset \mathbb{P}((T_{\mathbb{P}^n}|_{\ell})|_{p_0}) = \pi^{-1}(p_0) = \mathbb{P}^{n-1}.$$

Conversely, if ℓ passes p_0 and q_0 is contained in the hyperplane

$$(6.10) \quad \mathbb{P}^{n-2} = \mathbb{P}((\mathcal{O}_{\ell}(1)^{\oplus(n-1)})|_{p_0}) \subset \mathbb{P}((T_{\mathbb{P}^n}|_{\ell})|_{p_0}) = \pi^{-1}(p_0) = \mathbb{P}^{n-1},$$

then there exists a unique $f \in \mathfrak{M}(A_2, 0)$ of the form $f : \ell \rightarrow \mathbb{P}(T_{\mathbb{P}^n})$ such that $\text{Im}(f)$ passes q_0 ; moreover, putting $q_0 = (p_0, p'_0) \in \mathbb{P}^n \times \mathbb{P}^n$ such that π is the first projection of $\mathbb{P}^n \times \mathbb{P}^n$, then $\text{Im}(f) = \ell \times \{p'_0\} \subset \mathbb{P}^n \times \mathbb{P}^n$. The set of all lines ℓ passing p_0 such that q_0 is contained in the hyperplane (6.10) is parameterized by an $(n-2)$ -dimensional linear subspace $\mathbb{P}^{n-2}$ in $\mathbb{P}^n$ (the first factor in $\mathbb{P}^n \times \mathbb{P}^n$). It follows that the images $\text{Im}(f) \subset \mathbb{P}(T_{\mathbb{P}^n})$ sweep a hyperplane

$$(6.11) \quad H \stackrel{\text{def}}{=} \mathbb{P}^{n-1} \times \{p'_0\} \subset \mathbb{P}^n \times \{p'_0\}.$$

Since ξ is the restriction of the $(1,1)$ class in $\mathbb{P}^n \times \mathbb{P}^n$, $\xi|_H$ is the hyperplane class $\tilde{h}$ in $H = \mathbb{P}^{n-1} \times \{p'_0\} \cong \mathbb{P}^{n-1}$. Thus $\alpha|_H = \tilde{h}_{j+k}$ and $\beta|_H = \tilde{h}_{s+t}$. Since $(j+k+s+t) = n$ and B and C are in general position, there is a unique line in H passing $q_0 = (p_0, p'_0)$ and intersecting with B and C . Therefore,

$$\Phi_{(A_2,0)}(\alpha, \beta, h_n \xi_{n-1}) = 1.$$

□

Finally, we summarize the above computations and prove the following.

Proposition 6.12. *The quantum cohomology ring $H_{\omega}^*(\mathbb{P}(T_{\mathbb{P}^n}); \mathbb{Z})$ with $n \geq 2$ is the ring generated by h and ξ with the two relations*

$$h^{n+1} = \xi \cdot e^{-tn} \quad \text{and} \quad \sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + (-1)^n) \cdot e^{-tn}.$$

Proof. By Lemma 6.8 (ii), $a'_1 = 1$. By Lemma 3.8, $a'_3 = 1$. By Lemma 6.8 (i),

$$a'_2 = \Phi_{(A_2,0)}(h, h_n, h_{n-1} \xi_{n-1}) - c_1 a'_3 = -1.$$

Thus by (6.3), the first relation f_{ω}^1 is $h^{n+1} = \xi \cdot e^{-tn}$. By Lemma 6.8 (ii), $b_2^{(i)} = 1$ for $0 \leq i \leq n$. By (6.7), the second relation f_{ω}^2 is $\sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + \sum_{i=0}^n (-1)^i c_i) \cdot e^{-tn}$. From the exact sequence (6.9), $c_i = \binom{n+1}{i}$ for $0 \leq i \leq n$. Therefore, the relation f_{ω}^2 is $\sum_{i=0}^n (-1)^i c_i \cdot h^i \cdot \xi^{n-i} = (1 + (-1)^n) \cdot e^{-tn}$. □

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