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SUBVARIETIES OF $\mathcal{SU}_C(2)$ AND 2θ-DIVISORS IN THE JACOBIAN

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Abstract. We explore some of the interplay between Brill-Noether subvarieties of the moduli space $\mathcal{SU}_C(2,K)$ of rank 2 bundles with canonical determinant on a smooth projective curve and 2θ-divisors, via the inclusion of the moduli space into $|2\Theta|$, singular along the Kummer variety. In particular we show that the moduli space contains all the trisecants of the Kummer and deduce that there are quadrisecant lines only if the curve is hyperelliptic; we show that for generic curves of genus < 6, though no higher, bundles with > 2 sections are cut out by Γ_{00}; and that for genus 4 this locus is precisely the Donagi-Izadi nodal cubic threefold associated to the curve.

Let $\mathcal{SU}_C(2,L)$ denote the projective moduli variety of semistable rank 2 vector bundles with determinant $L \in \mathrm{Pic}(C)$ on a smooth curve C of genus $g > 2$; and suppose that $\deg L$ is even. It is well-known that, on the one hand, the singular locus of $\mathcal{SU}_C(2,L)$ is isomorphic to the Kummer variety of the Jacobian; and, on the other hand, that when C is nonhyperelliptic $\mathcal{SU}_C(2,\mathcal{O})$ has an injective morphism into the linear series $|2\Theta|$ on the Jacobian J_C^{g-1} which restricts to the Kummer embedding $a \mapsto \Theta_a + \Theta_{-a}$ on the singular locus. Dually $\mathcal{SU}_C(2,K)$ injects into the linear series $|\mathcal{L}|$ on J_C^0, where $\mathcal{L} = \mathcal{O}(2\Theta_\kappa)$ for any theta characteristic κ, and again this map restricts to the Kummer map $J_C^{g-1} \to |2\Theta|^\vee = |\mathcal{L}|$ on the singular locus. This map to projective space (the two cases are of course isomorphic) comes from the complete series on the ample generator of the Picard group, and (at least for a generic curve) is an embedding of the moduli space. Moreover, its image contains much of the geometry studied in connection with the Schottky problem; notably the configuration of Prym-Kummer varieties.

In this paper we explore a little of the interplay, via this embedding, between the geometry of vector bundles and the geometry of 2θ-divisors. On the vector bundle side we are principally concerned with the Brill-Noether loci $\mathcal{W}^r \subset \mathcal{SU}_C(2,K)$ defined by the condition $h^0(E) > r$ on stable bundles E. These are analogous to the very classical varieties $W_{g-1}^r \subset J_C^{g-1}$. Unlike the line bundle theory, however, general results—connectedness, dimension, smoothness and so on—are not known for the varieties $\mathcal{W}^r$ (see [6]).

On the 2θ side we shall consider the Fay trisecants of the 2θ-embedded Kummer variety, and the subseries $\mathbf{P}\Gamma_{00} \subset |\mathcal{L}|$ consisting of divisors having multiplicity ≥ 4 at the origin. This subseries is known to be important in the study of principally polarised abelian varieties (ppav's) [10]: in the Jacobian of a curve its base locus is the surface $C - C \subset J_C^0$ (plus a pair of isolated points in the case $g = 4$) [22],

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whereas for a ppav which is not a Jacobian it is conjectured that the origin is the only base point (but see [3]).

The organisation and main results of the paper are as follows. In the first two sections we study two families of lines on $\mathcal{SU}_C(2,K) \subset |\mathcal{L}|$ (or equivalently $\mathcal{SU}_C(2,\mathcal{O}) \subset |2\Theta|$), each of dimension $3g-2$. These are the Hecke lines, coming from vector bundles of odd degree, on the one hand, and lines lying inside g-dimensional linearly embedded extension spaces (generating the lowest stratum of the Segre stratification), on the other. We prove (**1.3** and **1.4**) that a line in $\mathcal{SU}_C(2,K)$ lies in both of these families if and only if it intersects the Kummer variety. Moreover, we show (4) that every trisecant of the Kummer variety is such a line, and in particular lies on the moduli space. (This fact is certainly well-known to the experts, but we are not aware of a reference in the literature.) As a corollary we show (5) that the Kummer variety has quadrisecant lines if and only if the curve is hyperelliptic.

In section 3 we introduce the subschemes $\mathcal{W}^r \subset \mathcal{SU}_C(2,K)$, and as a first step in their study we examine the stratification by h^0 of spaces of extensions, which will then map rationally into $\mathcal{SU}_C(2,K)$. The natural formulation of this stratification turns out to involve the Clifford index, and as an easy by-product we obtain the inequality (**3.7**)

$$h^0(E) \leq g+1-\mathrm{Cliff}(C)$$

for any semistable rank 2 bundle E with $\det E = K$.

In section 4 we prove, using a spectral curve construction, that (**4.1**)

$$\mathcal{W}^2 = \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2,K)$$

provided C is nonhyperelliptic of genus 3 or 4, or nontrigonal of genus 5. On the other hand, we show later on (**8.3**) that the equality fails for all curves of genus 6.

The remaining four sections of the paper are devoted to examining some of the geometry in detail for each of the cases $g = 3,4,5,6$. For genus 3 the moduli space $\mathcal{SU}_C(2,K)$ is embedded in $\mathbf{P}^7$ as the unique Heisenberg-invariant quartic singular along the Kummer variety—the so-called *Coble quartic*. We examine the configuration $\mathcal{W}^1 \subset \mathcal{W} \subset \mathbf{P}^6$, where $\mathbf{P}^6$ is the hyperplane spanned by the 'generalised theta divisor' $\mathcal{W} = \mathcal{W}^0$. It is known that $\mathcal{W}$ has a unique triple point $\mathcal{W}^2 = \mathbf{P}\Gamma_{00}$; we show (**5.3**) that $\mathcal{W}^1$ is a Veronese cone (already to be found in the classical literature [7]) with vertex $\mathcal{W}^2$, and whose generators are trisecants of the Kummer variety corresponding to a natural embedding of $|K|$ in the parameter space of all trisecants. In addition, we identify the tangent cone of $\mathcal{W}$ at the triple point (**5.5**): this is nothing but the secant variety of the Veronese surface, with equation $\det A = 0$, where A is a symmetric 3×3 matrix.

To each nonhyperelliptic curve of genus 4 one can associate a nodal cubic threefold $\mathcal{T} \subset \mathbf{P}^4$, which can be described in various ways. The view we adopt here is that it is the rational image of $\mathbf{P}^3$ via the linear system of cubics containing the canonical curve. There is an identification $\mathbf{P}^4 \xrightarrow{\sim} \mathbf{P}\Gamma_{00}$, due to Izadi [13], and we prove (**6.4**) that this restricts to an isomorphism $\mathcal{T} \xrightarrow{\sim} \mathcal{W}^2$, with the node mapping to $\mathcal{W}^3 = \{g^1_3 \oplus h^1_3\}$, the direct sum of the two trigonal line bundles on the curve. For the proof of this we make use of Izadi's description of the lines in $\mathcal{T}$ as pencils of 2θ-divisors. Note also that by passing to the tangent cone at the origin, $\mathbf{P}\Gamma_{00}$ may be viewed as a linear system of quartics in canonical space $\mathbf{P}^3$. We observe (**6.11**) that projection of the cubic $\mathcal{W}^2$ away from the node can be naturally identified with the quotient of Γ_{00} by q^2, where q is the unique quadric vanishing on the canonical curve.

For genus 5 we show (**7.2**) that $\mathcal{W}^3$ is a Veronese surface cutting the Kummer variety in the image of a plane quintic. If the curve is nontrigonal this quintic is the discriminant of the net of quadrics containing the canonical curve, while in the trigonal case it is isomorphic to the projection of the canonical curve from a trisecant.

Finally we show (**8.1**) that for a nontrigonal curve of genus 6 the locus $\mathcal{W}^4$ is a single point, stable if C is not a plane quintic (this case was also observed by Mukai in [18]), while $\mathcal{W}^4$ is a line not meeting the Kummer variety in the trigonal case. In the generic case $\mathcal{W}^4$ is the vertex of a configuration of five $\mathbf{P}^6$s which form the intersection of $\mathcal{SU}_C(2,K)$ with $\mathbf{P}\Gamma_{00}$ residual to $\mathcal{W}^2$.

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I. Lines

1. The g-plane ruling

For a line bundle L on the curve C we denote by $\mathcal{SU}_C(2,L)$ the projective moduli variety of semistable rank 2 vector bundles with determinant L; and in particular we shall be concerned with $\mathcal{SU}_C(2,K)$. The semistable boundary of this space is the image of $J_C^{g-1} \to \mathcal{SU}_C(2,K)$ mapping $L \mapsto L \oplus KL^{-1}$; this is the singular locus when $g > 2$. Throughout the paper we shall view both the moduli space and the Kummer variety—when C is nonhyperelliptic—as lying in the projective space $|2\Theta|^\vee = \mathbf{P}^{2^g-1}$ in the standard way: by the complete linear series $|\mathcal{L}|$ on $\mathcal{SU}_C(2,K)$, where $\mathcal{L}$ is the ample generator of Pic $\mathcal{SU}_C(2,K) \cong \mathbf{Z}$, restricting to $\mathcal{O}(2\Theta)$ on the Jacobian. A *line* on $\mathcal{SU}_C(2,K)$ is then an embedded $\mathbf{P}^1$ on which the restriction of $\mathcal{L}$ has degree one.

We shall consider the following subvariety of $\mathcal{SU}_C(2,K)$ ruled by g-planes. For $x \in \mathrm{Pic}^{g-2}(C)$ let $\mathbf{P}(x) = \mathbf{P}H^1(C, K^{-1}x^2) \cong \mathbf{P}^g$. This parametrises isomorphism classes of extensions

$$0 \to x \to E \to Kx^{-1} \to 0,$$

and thus has a moduli map to $\mathcal{SU}_C(2,K)$, which is linear (with respect to $\mathcal{L}$) and injective (see also remark 3.3 below). Globally we have a ruling:

$$\begin{array}{ccc}\mathbf{P}U & \xrightarrow{\varepsilon} & \mathcal{SU}_C(2,K)\\ \big\downarrow{\scriptstyle \mathbf{P}^g} & & \\ \mathrm{Pic}^{g-2}(C) & & \end{array}$$

where $U = R^1\pi_* K^{-1}\mathcal{N}^2$, with $\mathcal{N} \to C \times \mathrm{Pic}^{g-2}(C)$ a Poincaré bundle and $\pi : C \times \mathrm{Pic}^{g-2}(C) \to \mathrm{Pic}^{g-2}(C)$ the natural projection.

We shall need to make repeated use, in what follows, of the following result of Lange–Narasimhan [14]. Consider any extension

$$0 \to n_0 \to F \to n_0^{-1} \otimes \det F \to 0,$$

where $n_0 \subset F$ is a line subbundle of maximal degree. This is represented by a point f of the extension space $\mathbf{P}H^1(C, n_0^2 \otimes \det F^\vee) = \mathbf{P}H^0(C, Kn_0^{-2} \otimes \det F)^\vee$, into which the curve C maps via the linear series $|Kn_0^{-2} \otimes \det F|$. For an effective divisor D on C, we shall denote by $\overline{D}$ the linear span in $\mathbf{P}H^1(C, n_0^2 \otimes \det F^\vee)$ of the image of this divisor. Then the following is proposition 2.4 of [14]:

1.1. Lemma. *With the above notation there is a bijection, given by* $\mathcal{O}(D) = n^{-1}n_0^{-1} \otimes \det F$, *between:*

1. *line subbundles* $n \subset F$, $n \neq n_0$, *of maximal degree; and*
2. *line bundles* $\mathcal{O}(D)$ *with degree* $\deg D = \deg F - 2\deg n_0$ *and such that* $f \in \overline{D}$.

Let us return now to the g-planes $\mathbf{P}(x) \hookrightarrow \mathcal{SU}_C(2, K)$, where $x \in \mathrm{Pic}^{g-2}(C)$, and the following well-known facts. The curve C maps into $\mathbf{P}(x)$ via $|K^2x^{-2}|$, as a special case of the Lange–Narasimhan picture. Moreover, a point of $\mathbf{P}(x)$ represents a stable bundle (with x as maximal line subbundle) precisely away from the image of C; while a point $q \in C \subset \mathbf{P}(x)$ represents the equivalence class of the semistable bundle $x(q) \oplus Kx^{-1}(-q)$. In other words there is a commutative diagram

(1)
$$\begin{tikzcd} C \arrow[r, "t_x"] \arrow[d, "|K^2x^{-2}|"'] & J_C^{g-1} \arrow[d] \\ \mathbf{P}(x) \arrow[r, "\varepsilon"] & \mathcal{SU}_C(2,K) \end{tikzcd}$$

where $t_x : q \mapsto x(q)$ and the second vertical arrow maps $L \mapsto L \oplus KL^{-1}$.

The incidence relations between g-planes of this ruling can be given as follows.

1.2. Proposition. *Suppose that C is nonhyperelliptic. For $x, y \in \mathrm{Pic}^{g-2}(C)$ the intersection $\mathbf{P}(x) \cap \mathbf{P}(y)$ is either empty, or:*

1. *the secant line $\overline{pq}$ of the curve (in either of $\mathbf{P}(x)$ or $\mathbf{P}(y)$) if $x \otimes y = K(-p-q)$;*
2. *the point $x(p) \oplus Kx^{-1}(-p) \in \mathrm{Kum}(J_C^{g-1})$, if $H^0(C, Kx^{-1}y^{-1}) = 0$ and $x \otimes y^{-1} = \mathcal{O}(q-p)$.*

Proof. First we note that at any point $E \in \mathbf{P}(x)$ away from the curve the residual g-planes $\mathbf{P}(y)$ through E can be identified, via lemma 1.1, with the set of effective divisors $p + q$ such that E lies on the secant line $\overline{pq}$; and the line bundles x, y are then related by

(2)
$$x \otimes y = K(-p-q).$$

Note that the point p on the curve in $\mathbf{P}(x)$ represents the bundle

$$x(p) \oplus Kx^{-1}(-p) = y(q) \oplus Ky^{-1}(-q),$$

i.e. it coincides with the image of q on the curve in $\mathbf{P}(y)$; and similarly $q \in C \subset \mathbf{P}(x)$ coincides with $p \in C \subset \mathbf{P}(y)$. This shows that condition (2) is equivalent to $\overline{pq} \subset \mathbf{P}(x) \cap \mathbf{P}(y)$.

On the other hand, when C is nonhyperelliptic $\mathbf{P}(x)$ and $\mathbf{P}(y)$ cannot intersect in dimension greater than one: for then a generic point E of the intersection would lie on distinct secant lines $\overline{pq}$ and $\overline{rs}$, both satisfying (2), and hence $\mathcal{O}(p+q) = \mathcal{O}(r+s)$, a contradiction.

The only other possibility for nonempty intersection $\mathbf{P}(x) \cap \mathbf{P}(y)$ is that this intersection is a single point, in which case it must be a point of the Kummer variety, and we easily find case 2. $\square$

Next we recall the Hecke correspondence between $\mathcal{SU}_C(2, K)$ and $\mathcal{SU}_C(2, K(p))$, where $p \in C$ is a point of the curve. For a stable bundle $F \in \mathcal{SU}_C(2, K(p))$ we shall write $l_F \cong \mathbf{P}^1 \subset \mathcal{SU}_C(2, K)$ for the image of

$$\begin{array}{rcl} \mathbf{P}\,\mathrm{Hom}(F, \mathcal{C}_p) & \to & \mathcal{SU}_C(2, K), \\ \phi & \mapsto & \ker \phi. \end{array}$$

This is called the *Hecke line* associated to the bundle F. Our aim in the remainder of this section will be to compare Hecke lines in $\mathcal{SU}_C(2, K)$ with lines contained in the g-planes $\mathbf{P}(x)$.

1.3. Theorem. *Let $x \in \mathrm{Pic}^{g-2}(C)$. A projective line $l \subset \mathbf{P}(x)$ is Hecke if and only if it meets the image of the curve $C \subset \mathbf{P}(x)$.*

Proof. Suppose that a Hecke line l_F is contained in $\mathbf{P}(x)$. This means that for all extensions of sheaves of the form

$$0 \to E \to F \to \mathcal{C}_p \to 0,$$

where $K(p)$ is the determinant of F, the kernel E contains x as a line subbundle. This means we have a pencil of homomorphisms $x \hookrightarrow F$, for which we have two possibilities.

Either the image subsheaf is constant and is in the kernel of every homomorphism $F \to \mathcal{C}_p$. Then there is an inclusion of sheaves $x(p) \subset F$, which by stability of F is a line subbundle, i.e. F is an extension

$$0 \to x(p) \to F \to Kx^{-1} \to 0.$$

But the space $\mathbf{P}H^1(C, K^{-1}x^2(p))$ of such extensions is the image of the projection of $\mathbf{P}(x) = \mathbf{P}H^1(C, K^{-1}x^2)$ from the point $p \in C \subset \mathbf{P}(x)$, i.e. the set of lines in $\mathbf{P}(x)$ passing through the point p. It is not hard to see that the line l corresponding to F in this manner is precisely l_F—and we note that *every* line meeting the curve arises in this way.

Or—the second possibility—the image sheaf is non-constant, in which case we have a subsheaf $x \oplus x \subset F$, with torsion quotient supported on some effective divisor D. But then $\det F = K(p)$ implies that $\mathcal{O}(D) = Kx^{-2}(p)$. So $\deg D = 3$ and we observe that

$$h^0(C, K^2x^{-2}(-D)) = h^0(C, K(-p)) = g - 1,$$

i.e. that $\dim \overline{D} = 1$. In this case the Hecke line l_F is just $\overline{D}$ and is trisecant to the curve. $\square$

It follows by a dimension count that for $g > 2$ a generic Hecke line in $\mathcal{SU}_C(2, K)$ is not contained in any g-plane. We shall show next that those that are are precisely the Hecke lines that meet the Kummer variety (i.e. the singular locus).

1.4. Theorem. *Let $l_F \subset \mathcal{SU}_C(2, K)$ be any Hecke line.*

1. *There is a canonical surjection from line subbundles $n \subset F$ with $\deg n = g-1$ to points of intersection $l_F \cap \mathrm{Kum}(J_C^{g-1})$ with the Kummer variety, which is bijective if l_F is not a tangent line of the Kummer variety.*
2. *The intersection $l_F \cap \mathrm{Kum}(J_C^{g-1})$ is nonempty if and only if l_F is contained in a g-plane $\mathbf{P}(x)$ for some $x \in \mathrm{Pic}^{g-2}(C)$. If C is nonhyperelliptic and $l_F \cap \mathrm{Kum}(J_C^{g-1})$ has cardinality k, then the number of such g-planes is $1+\binom{k}{2}$.*

1.5. *Remarks.* (i) We have two irreducible families of lines in $\mathcal{SU}_C(2,K)$: the Hecke lines and the lines contained in g-planes of the ruling. These families have the same dimension $3g-2$, and we expect that each is a component of the Hilbert scheme of all lines. Theorems 1.3 and 1.4 would then say that *the intersection of these two components consists of the members of each family which meet the Kummer variety.*

(ii) The cardinality of $l_F \cap \mathrm{Kum}(J_C^{g-1})$ satisfies $k \le 4$ if C is hyperelliptic and $k \le 3$ otherwise; this follows from part 1 and proposition 5.1 of [16]. See also corollary 2.2 below.

Proof of Theorem 1.4. For $n \subset F$ consider the diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & Kn^{-1} \arrow[d] & \\
0 \arrow[r] & n \arrow[r] & F \arrow[r] \arrow[dr, "\phi"'] & Kn^{-1}(p) \arrow[r] \arrow[d] & 0 \\
 & & & \mathcal{C}_p \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Then if $E = \ker\phi$ we have a surjective sheaf map $E \to Kn^{-1} \to 0$, and hence $n \subset E$. So E is S-equivalent to $n \oplus Kn^{-1}$ and defines a point of intersection $l_F \cap \mathrm{Kum}(J_C^{g-1})$. To see that this is surjective, let $n \oplus Kn^{-1} \in \mathrm{Kum}(J_C^{g-1})$ be a point of intersection with the Hecke line l_F. This means there is an exact sequence

$$0 \to E \to F \to \mathcal{C}_p \to 0$$

where E is S-equivalent to $n \oplus Kn^{-1}$; i.e. at least one of n or Kn^{-1} is a line subbundle of E, and hence of F.

In a moment we shall verify that in this construction we have

$$n \oplus Kn^{-1} \subset F \iff l_F \text{ is tangent to } \mathrm{Kum}(J_C^{g-1}) \text{ at } n \oplus Kn^{-1}. \tag{3}$$

This will show that the correspondence is bijective when l_F is not a tangent line.

For part 2, we note first that if l_F is contained in a g-plane $\mathbf{P}(x)$, then by theorem 1.3 it meets the curve $C \subset \mathbf{P}(x)$ and hence the Kummer variety. For the converse it suffices, by the preceding construction, to suppose that there is a degree $g-1$ line subbundle $n \subset F$. Then, letting $x = n(-p)$, it follows that F is represented by a point of the extension space $\mathbf{P}H^1(C, K^{-1}n^2(-p)) = \mathbf{P}H^1(C, K^{-1}x^2(p))$ and hence determines—as in the proof of theorem 1.3—a line $l \subset \mathbf{P}(x)$ through $p \in C$, which coincides with the Hecke line l_F.

Notice that by lemma 1.1 any residual degree $g-1$ line subbundles $m \subset F$ correspond to points $q \in C$ by the relation $K(p-q) = n \otimes m$; and in this case l_F must be the secant $\overline{pq} \subset \mathbf{P}(x)$. In particular, Kn^{-1} is a subbundle of F if and only if l_F is the tangent line to $C \subset \mathbf{P}(x)$ at p—this proves (3).

Finally, if we fix any g-plane containing l_F, then by proposition 1.2 the residual such g-planes correspond bijectively to the effective divisors $p+q$ such that $l_F = \overline{pq}$, i.e. to pairs of intersection points of l_F with the Kummer. And so we obtain $1+\binom{k}{2}$ for the number of such g-planes. □

2. Trisecants of the Kummer variety

Recall that the quotient $\mathrm{Kum}(J_C^{g-1})$ of J_C^{g-1} by the Serre involution is embedded in $\mathbf{P}^{2^g-1}$ by the linear system $|2\Theta|$, and that this embedding extends to the moduli space $\mathcal{SU}_C(2,K)$ containing the Kummer as its singular locus (when $g > 2$). In this embedding the Kummer possesses a unique irreducible 4-dimensional family of trisecant lines, which characterise Jacobians amongst principally polarised abelian varieties. We shall briefly describe this family of trisecants (see [19] or [8]).

The base $\mathcal{F}$ of the family is the fibre product:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & S^4C \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{Abel-Jacobi}} \\ \mathrm{Pic}^{g-3}(C) & \longrightarrow & \mathrm{Pic}^4(C) \end{array}$$

where the bottom map sends $a \mapsto Ka^{-2}$. An element of $\mathcal{F}$, in other words, is a pair $(a, D) \in \mathrm{Pic}^{g-3}(C) \times S^4C$ such that $a^2 = \mathcal{O}(K-D)$. Writing $D = p+q+r+s$, one shows that the following three points of $\mathbf{P}^{2^g-1}$ are collinear:

$$\begin{array}{lll} \phi(a(q+r)) & \phi(a(p+r)) & \phi(a(p+q)) \\ = \phi(a(p+s)), & = \phi(a(q+s)), & = \phi(a(r+s)). \end{array} \tag{4}$$

We shall refer to the lines of $\mathbf{P}^{2^g-1}$ parametrised by $\mathcal{F}$ in this way as the *Fay trisecants*.

2.1. Theorem. *The Fay trisecants are precisely the Hecke lines which are trisecant to the Kummer variety. In particular they all lie on $\mathcal{SU}_C(2,K)$.*

Proof. We ask for the condition on a Hecke line $l_F \subset \mathcal{SU}_C(2,K)$ for it to be trisecant to the Kummer variety. Let $\det F = K(p)$. For l_F to meet the Kummer variety, F must have a line subbundle $n \subset F$ of degree $g-1$. Then by theorem 1.4, l_F is a trisecant if and only if F has two further degree $g-1$ line subbundles. By stability these are maximal, and so by lemma 1.1 the residual subbundles correspond bijectively to points of C mapping to the extension class of F under

$$C \xrightarrow{|K^2n^{-2}(p)|} \mathbf{P}H^1(C, K^{-1}n^2(-p)).$$

Thus trisecants l_F correspond to *nodes* of the image curve under the linear series $|K^2n^{-2}(p)|$; and the condition for such a node is that for points $q, r \in C$,

$$\begin{array}{rcl} h^0(C, K^2n^{-2}(p-q-r) & \geq & h^0(C, K^2n^{-2}(p)) - 1 \\ & = & g-1; \end{array}$$

or equivalently $h^0(C, K^{-1}n^2(-p+q+r)) \geq 1$. This in turn is equivalent to $K^{-1}n^2(-p+q+r) = \mathcal{O}(s)$ for some $s \in C$. We conclude that the necessary and sufficient condition for l_F to be a trisecant $\overline{pqr}$ of $C \subset \mathbf{P}(x)$, $x = n(-p)$, is

$$n^2 = K(p-q-r+s) \quad \text{or equivalently} \quad x^2 = K(-p-q-r+s). \tag{5}$$

One can now verify, using (1), that the points of intersection of l_F with the Kummer—i.e. with the curve $C \subset \mathbf{P}(x)$—are the three points (4), where $a = x(-s)$. □

2.2. Corollary. 1. *If C is nonhyperelliptic, then no Fay trisecant has more than three intersection points with the Kummer variety.*
2. *If C is hyperelliptic, then all Fay lines are exactly quadrisecant.*

Proof. By theorems 1.4 and 2.1 every Fay line lies in $\mathbf{P}(x)$ for some $x \in \mathrm{Pic}^{g-2}(C)$. Let D be an effective divisor on C with $\dim \overline{D} = 1$ in $\mathbf{P}(x)$. This is equivalent to $h^0(C, K^2x^{-2}(-D)) = g-1$, or, by Riemann-Roch,

$$h^0(C, K^{-1}x^2(D)) = \deg D - 2.$$

If $\deg D = 5$ then this says that $|K^{-1}x^2(D)|$ maps C birationally to a plane cubic, which is impossible; while if $\deg D = 4$ then it is equivalent to

$$Kx^{-2} = \mathcal{O}(D-H), \tag{6}$$

where H is a hyperelliptic divisor. This proves part 1; for part 2 let $\overline{pqr} \subset \mathbf{P}(x)$ be the trisecant constructed in the proof of theorem 2.1, and consider $D = p+q+r+\tau(s)$, where $\tau : C \leftrightarrow C$ is the hyperelliptic involution. Then (6) follows from (5), and we see that $\overline{pqr} = \overline{D}$ is a quadrisecant. □

II. Loci

3. Brill-Noether loci in $\mathcal{SU}_2(2, K)$

Let $\mathcal{W} \subset \mathcal{SU}_C(2, K)$ be the closure of the locus of stable bundles E for which $H^0(C, E) \neq 0$, i.e. the 'theta divisor' for rank 2 bundles. In terms of the map $\phi : \mathcal{SU}_C(2, K) \to |2\Theta|^\vee$, $\mathcal{W}$ is the unique hyperplane section tangent to the Kummer variety $\mathrm{Kum}(J_C^{g-1})$ along the image of the theta divisor $\mathrm{Kum}(\Theta)$.

The Brill-Noether loci are the subschemes

$$\mathcal{W} \supset \mathcal{W}^1 \supset \cdots \supset \mathcal{W}^{g-1} \supset \mathcal{W}^g,$$

where $\mathcal{W}^r$ is the closure of the set of stable bundles E for which $h^0(C, E) \geq r+1$. (We shall see in a moment that $\mathcal{W}^{g+1} = \emptyset$—see proposition 3.7.)

3.1. *Remark.* The local structure of $\mathcal{W}^r$ is governed by a symmetric Petri map

$$S^2H^0(C, E) \to H^0(C, K \otimes \mathrm{ad}\, E),$$

where $\mathrm{ad}\, E$ is the bundle of trace-free endomorphisms; as a consequence $\mathcal{W}^r$ has expected codimension $\binom{r+2}{2}$ in $\mathcal{SU}_C(2, K)$. (See for example [6].)

In addition, it is not hard to show that $\mathcal{W}^1$ *is the union of all Hecke lines meeting* $\mathcal{W}^2$. We shall see this illustrated for curves of genus 3 in theorem 5.3.

In order to study the Brill-Noether loci $\mathcal{W}^r$ we shall analyse them first in spaces $\mathbf{P}\,\mathrm{Ext}^1(K-D, D)$ of extensions

$$0 \to \mathcal{O}(D) \to E \to \mathcal{O}(K-D) \to 0 \tag{7}$$

for some line bundle $\mathcal{O}(D) \in \mathrm{Pic}^d(C)$. Usually, though not always, we shall think of D as an effective divisor; indeed E has sections if and only if it can be expressed as such an extension with D effective.

3.2. *Remarks.* (i) Note that by semistability $d \leq g-1$, with equality if and only if E is S-equivalent to $\mathcal{O}(D) \oplus \mathcal{O}(K-D)$. Moreover, every $E \in \mathcal{SU}_C(2,K)$ is such an extension for some $D \in \mathrm{Pic}^d(C)$ with

$$d \geq \left[\frac{g-1}{2}\right].$$

This follows from a classical result of Segre and Nagata (see [15]), which says that every ruled surface of genus g has a section with self-intersection at most g.

(ii) It will be convenient below to introduce the Clifford index $\mathrm{Cliff}(D) = \deg D - 2r(D)$, where $r(D) = h^0(D) - 1$, into our notation. Recall that the Clifford index $\mathrm{Cliff}(C)$ of the curve is defined to be the minimum value of $\mathrm{Cliff}(D)$ for which $h^i(D) \geq 2$ for $i = 0, 1$ (see [12]).

Recall also that

$$\mathrm{Cliff}(C) \leq \left[\frac{g-1}{2}\right],$$

with equality for generic C.

As in section 1, the curve C maps into the space of such extensions; and we shall denote the rational coarse moduli map of this space by ε_D:

$$C \xrightarrow{|2K-2D|} \mathbf{P}\,\mathrm{Ext}^1(K-D,D) \cong \mathbf{P}^{3g-4-2d} \xrightarrow{\varepsilon_D} \mathcal{SU}_C(2,K).$$

We shall denote by $\mathcal{I}_C$ the ideal sheaf of the image curve in $\mathbf{P}^{3g-4-2d}$; and we shall write Bl_C for the blow-up along this curve and $\mathrm{Sec}^n C$ for the variety of its n-secant $(n-1)$-planes; although of course the map $C \to \mathbf{P}\,\mathrm{Ext}^1(K-D,D)$ is not necessarily an embedding or even birational.

We shall write

$$\mathcal{W}_D = \varepsilon_D(\mathbf{P}\,\mathrm{Ext}^1(K-D,D)).$$

Note here that by $\varepsilon_D(\Omega)$, where $\Omega \subset \mathbf{P}\,\mathrm{Ext}^1(K-D,D)$, we shall always mean the proper transform of Ω, i.e. the closure in $\mathcal{SU}_C(2,K)$ of the image of the domain of definition of ε_D.

3.3. *Remark.* The rational map ε_D has been studied in detail by Bertram and others (see [5]), and resolves to a morphism $\widetilde{\varepsilon}_D$ of the blow-up:

$$\begin{array}{ccccccccc}\mathbf{P}\,\mathrm{Ext}^1(K-D,D) & \longleftarrow & \mathrm{Bl}_C & \longleftarrow & \mathrm{Bl}_{\widetilde{\mathrm{Sec}_2 C}} & \longleftarrow & \cdots & \longleftarrow & \mathrm{Bl}_{\widetilde{\mathrm{Sec}_{g-2-d} C}} \\ &&&&&&&& \downarrow{\scriptstyle \widetilde{\varepsilon}_D} \\ &&&&&&&& \mathcal{SU}_C(2,K)\end{array}$$

Moreover, hyperplanes of $|\mathcal{L}| = |2\Theta|^\vee$ pull back to divisors of $|\mathcal{I}_C^{g-2-d}(g-1-d)|$ on $\mathbf{P}\,\mathrm{Ext}^1(K-D,D)$. We shall need these facts only in the cases $d = g-2$ (already discussed in section 1) and $d = g-3$; in both of these cases ε_D comes from the complete series $|\mathcal{I}_C^{g-2-d}(g-1-d)|$.

It is easy to analyse the filtration of each $\mathbf{P}\,\mathrm{Ext}^1(K-D,D)$ by the dimension $h^0(E)$. For any such extension we have

$$h^0(E) = h^0(D) + h^0(K-D) - \operatorname{rank}\delta(E) = g+1-\mathrm{Cliff}(D) - \operatorname{rank}\delta(E) \tag{8}$$

where $\delta(E) : H^0(K-D) \to H^1(D)$ is the coboundary homomorphism in the cohomology sequence of (7). (Note that (8) gives an upper bound $g+1-\mathrm{Cliff}(C)$ on $h^0(E)$; see proposition 3.7 below.)

By Serre duality $\delta(E)$ is an element of $\otimes^2 H^1(D)$, while its transpose $\delta(E)^t$ is the coboundary map for the dual sequence tensored with K: $0 \to \mathcal{O}(D) \to K \otimes E^\vee \to \mathcal{O}(K-D) \to 0$. But $K \otimes E^\vee = E$, and so $\delta(E) = \delta(E)^t$. We have therefore constructed a linear homomorphism

$$\delta : \mathrm{Ext}^1(K-D,D) \to S^2H^1(D), \tag{9}$$

with respect to which $h^0(E)$ satisfies (8). But the *rank* stratification of $S^2H^1(D)$ coincides with the *secant* stratification of its embedded Veronese variety:

$$\begin{array}{rcl} \mathrm{Ver} : \mathbf{P}H & \hookrightarrow & \mathbf{P}S^2H, \\ \xi & \mapsto & \xi \otimes \xi, \end{array}$$

where $H = H^1(D)$. In other words,

$$\mathrm{Sec}^n(\mathrm{Ver}\,\mathbf{P}H) = \{\ a \in \mathbf{P}S^2H \mid \operatorname{rank} a \le n\ \}$$

for $n = 1, \dots, \dim H = r(D) - d + g$. On the other hand, the homomorphism δ is dual to the multiplication map $S^2H^0(K-D) \to H^0(2K-2D)$, and so the above Veronese embedding fits into the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & \mathbf{P}H^1(D) \arrow[d, "\mathrm{Ver}"] \\ \mathbf{P}\,\mathrm{Ext}^1(K-D,D) \arrow[r, "\delta"] \arrow[d, "\varepsilon_D"'] & \mathbf{P}S^2H^1(D) \\ \mathcal{W}_D & \end{tikzcd} \tag{10}$$

We now define

$$\begin{array}{rcl} \Omega^0_D & = & \mathbf{P}\ker\delta, \\ \Omega^n_D & = & \delta^{-1}(\mathrm{Sec}^n(\mathrm{Ver}\,\mathbf{P}H^1(D))), \quad n = 1, \dots, g-d+r(D). \end{array}$$

(When it is convenient we shall drop the subscript and write $\Omega^n = \Omega^n_D$.) Thus if Ω^0 is nonempty, then $\Omega^0 \subset \Omega^1 \subset \cdots \subset \Omega^{g-d+r(D)} \subset \mathbf{P}\,\mathrm{Ext}^1(K-D,D)$ is a sequence of *cones* with vertex Ω^0.

We can therefore state the main conclusion of this section as follows:

$$h^0(E) = g+1-\mathrm{Cliff}(D) - n \qquad \text{for } E \in \Omega^n_D \backslash \Omega^{n-1}_D. \tag{11}$$

3.4. Example. $\mathbf{d} = g-2$. If $D \in \mathrm{Pic}^{g-2}(C)$, then $\mathbf{P}(D) = \mathbf{P}\,\mathrm{Ext}^1(K-D,D)$ is a g-plane of the ruling of section 1. The map $\varepsilon_D : \mathbf{P}(D) \hookrightarrow \mathcal{SU}_C(2,K)$ is a linear embedding, and we shall not distinguish between $\mathbf{P}(D)$ and its image. In this case (11) says:

$$\mathcal{W}_D \cap \mathcal{W}^r = \Omega^{2r(D)+2-r} \subset \mathbf{P}(D), \qquad r = r(D), \dots, 2r(D)+2.$$

The cones Ω^n are constructed using $\delta : \mathrm{Ext}^1(K-D, D) \to S^2H^1(D)$ where $\dim \mathrm{Ext}^1(K-D, D) = g+1$ and $\dim H^1(D) = h^0(D)+1$.

If $h^0(D) = 0$, then $\dim S^2H^1(D) = 1$ and $\Omega^0 \subset \Omega^1 = \mathbf{P}(D)$ is a hyperplane, on which $h^0(E) = 1$. In other words, $\Omega^0 = \mathbf{P}(D) \cap \mathcal{W}$.

If $h^0(D) = 1$, then $\dim S^2H^1(D) = 3$. Since $|K-D|$ is a pencil, $S^2H^0(K-D) \to H^0(2K-2D)$ is necessarily injective and so again δ is surjective. In this case, therefore,

$$\begin{array}{lcll} \Omega^0 & \cong & \mathbf{P}^{g-3} & \text{on which } h^0(E) = 3, \\ \Omega^1 & = & \text{quadric of rank 3,} & h^0(E) = 2, \\ \Omega^2 & = & \mathbf{P}(D), & h^0(E) = 1. \end{array}$$

If $h^0(D) = 2$ then the series $|K-D|$ maps $f : C \to \mathbf{P}^2$ with degree g; and the homomorphism δ is no longer surjective in general. In fact surjectivity fails precisely when f maps C onto a conic, which cannot happen if g is odd, but can occur for a trigonal curve of genus 6, for example: if $|L| = g^1_3$ then take $D = K - 2L$. In case δ *is* surjective we have:

$$\begin{array}{lcll} \Omega^0 & \cong & \mathbf{P}^{g-6} & \text{on which } h^0(E) = 5, \\ \Omega^1 & = & \text{cone over a Veronese} & \\ & & \text{surface in } \mathbf{P}^5, & h^0(E) = 4, \\ \Omega^2 & = & \text{cone over cubic} & \\ & & \text{hypersurface } S^2\mathbf{P}^2 \hookrightarrow \mathbf{P}^5, & h^0(E) = 3, \\ \Omega^3 & = & \mathbf{P}(D), & h^0(E) = 2. \end{array}$$

And so on.

The 'universal' case of (11) is the case $D = 0$. This says that $\mathcal{W}^r$ is composed of the image of $\Omega^{g-r} \subset \mathbf{P}\,\mathrm{Ext}^1(K, \mathcal{O}) \cong \mathbf{P}^{3g-4}$ together with those of the corresponding cones in the exceptional divisors of the blow-up of remark 3.3:

$$\mathcal{W}^r = \bigcup_{\substack{D \geq 0 \\ \deg D \leq g-2}} \varepsilon_D \Omega^{g-r-\mathrm{Cliff}(D)}.$$

In this case diagram (10) becomes

$$\begin{tikzcd} & C \arrow[r, "|K|"] \arrow[d, "|2K|"'] & \mathbf{P}^{g-1} \arrow[d, "\mathrm{Ver}"] \\ \Omega^{g-r} \subset & \mathbf{P}^{3g-4} \arrow[r, "\delta"] \arrow[d, "\varepsilon_0"'] & \mathbf{P}S^2H^1(\mathcal{O}) \\ \mathcal{W}^r \subset & \mathcal{W} & \end{tikzcd} \tag{12}$$

3.5. *Remark.* Note that if C is nonhyperelliptic, then by Noether's theorem δ in (12) is injective. Then Ω^n is the intersection of $\mathbf{P}^{3g-4} \subset \mathbf{P}S^2H^1(\mathcal{O})$ with the secant variety $\mathrm{Sec}^n(\mathrm{Ver}\,\mathbf{P}^{g-1})$, and in particular contains $\mathrm{Sec}^n C \subset \mathbf{P}^{3g-4}$. One can show, in fact, that Green's conjecture on the syzygies of the canonical curve (see [12]) implies

$$\Omega^n = \mathrm{Sec}^n C \subset \mathbf{P}^{3g-4} \qquad \text{for } n < \mathrm{Cliff}(C).$$

One consequence of this statement is that $\Omega^{\mathrm{Cliff}(C)}$ is the smallest cone in the sequence containing semistable extensions.

We conclude this section with two inequalities. The first, which will be useful later, is due to Mukai ([18], proposition 3.1):

3.6. Lemma. *If $|D|$ is base-point-free, the n for any rank 2 vector bundle E with $\det E = K$ we have $h^0(E(-D)) \geq h^0(E) - \deg D$.*

3.7. Proposition. *For all semistable bundles E in $\mathcal{SU}_C(2,K)$ we have*

$$h^0(E) \leq g + 1 - \mathrm{Cliff}(C).$$

3.8. *Remark.* In particular, this bound becomes

$$h^0(E) \leq \begin{cases} g & \text{for nonhyperelliptic } C, \\ g-1 & \text{for } C \text{ not trigonal or a plane quintic,} \\ \cdots & \\ [g/2]+2 & \text{for generic } C. \end{cases}$$

The bound $h^0(E) \leq g$ for nonhyperelliptic curves was observed by Laszlo [17], proposition IV.2.

Proof of proposition 3.7. We may assume that E comes from an extension in $\mathbf{P}\,\mathrm{Ext}^1(K-D, D)$, where, by remark 3.2 (i),

$$\left[\frac{g-1}{2}\right] \leq \deg D \leq g-1.$$

The right-hand inequality implies $h^0(D) \leq h^1(D)$, so that if $h^0(D) \geq 2$, then $\mathrm{Cliff}(D) \geq \mathrm{Cliff}(C)$ by definition. If, on the other hand, $h^0(D) \leq 1$, then $\mathrm{Cliff}(D) \geq \deg D \geq \mathrm{Cliff}(C)$ by the left-hand inequality together with remark 3.2 (ii). In either case, therefore, the proposition follows from (8). $\square$

4. $\mathbf{P}\Gamma_{00}$

We shall as usual identify $\mathcal{SU}_C(2,K)$ with its image in $|2\Theta|^\vee = |\mathcal{L}|$, where $\mathcal{L} = \mathcal{O}(2\Theta_\kappa) \in \mathrm{Pic}(J_C)$ for any theta characteristic $\kappa \in \vartheta(C)$. Namely, a stable bundle $E \in \mathcal{SU}_C(2,K)$ is mapped to the divisor $D_E \in |\mathcal{L}|$ defined by

$$D_E = \{ \ L \in J_C \mid h^0(C, L \otimes E) > 0 \ \}.$$

On the other hand, one can consider the linear system $\mathbf{P}\Gamma_{00} \subset |\mathcal{L}|$ defined by

$$\begin{aligned} \mathbf{P}\Gamma_{00} &= \{ \ D \in |\mathcal{L}| \mid \ \mathrm{mult}_0 D \geq 4 \ \} \\ &= \{ \ D \in |\mathcal{L}| \mid C - C \subset \mathrm{supp} \ \ D \ \}. \end{aligned} \tag{13}$$

For the equivalence of these two definitions see [10] or [22]; one can show, in addition, that $\mathbf{P}\Gamma_{00}$ has codimension $1 + \frac{1}{2}g(g+1)$.

It is easy to verify that the Brill-Noether locus $\mathcal{W}^2$ is always contained in the subspace $\mathbf{P}\Gamma_{00}$. The main result of this section is a partial converse:

4.1. Theorem. $\mathcal{W}^2 \subset \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2,K) \subset \mathcal{W}^1$. *Moreover, if C is nonhyperelliptic of genus 4 or nontrigonal of genus 5, then $\mathcal{W}^2 = \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2,K)$.*

4.2. *Remark.* We shall show later that $\mathcal{W}^2 \neq \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2,K)$ for curves of genus 6 (see remark 8.3). For genus 4 the embedding $\mathcal{W}^2 \subset \mathbf{P}\Gamma_{00}$ will be described in theorems 6.4 and 6.11.

4.3. Lemma. *Suppose F is a semistable vector bundle of rank 2 and degree $2d$, where $0 \leq d \leq g-1$; and $k \geq 0$ an integer. Then $h^0(F) \geq k$ if and only if $h^0(F(D)) \geq k$ for all $D \in S^{g-1-d}C$.*

Before proving this lemma let us show how it implies theorem 4.1. We suppose that $C - C \subset D_E$ for a stable bundle $E \in \mathcal{SU}_C(2, K)$, and we show first that $h^0(E) \geq 2$: by hypothesis $h^0(E(p - q)) \geq 1$ for all $p, q \in C$, so by the lemma we deduce that $h^0(E(-q)) \geq 1$ for all $q \in C$. From this it follows that $h^0(E) \geq 2$, since $h^0(E) \geq 1$ and equality would imply that every section vanishes at arbitrary $q \in C$, a contradiction.

Now suppose that $h^0(E) = 2$ and consider the evaluation map $e_q : H^0(E) \to E_q$ for $q \in C$. Since $h^0(E(-q)) \geq 1$ we have $\operatorname{rank} e_q \leq 1$ for all $q \in C$, and hence the sections of E generate a line subbundle $L \subset E$ with $h^0(L) = 2$. But by stability of E this must satisfy $\deg L < g - 1$, so C admits a g^1_{g-2}. So if C is nonhyperelliptic of genus 4 or is nontrigonal of genus 5 we obtain a contradiction, and we conclude that $h^0(E) \geq 3$. □

For the lemma, we shall prove the following equivalent statement. Let $E \in \mathcal{SU}_C(2, \mathcal{O})$ and $\xi \in \operatorname{Pic}^d(C)$, $0 \leq d \leq g - 1$. Then

$$H^0(C, \xi \otimes E) \geq k \iff H^0(C, \xi(D) \otimes E) \geq k \ \forall\ D \in S^{g-1-d}C. \tag{14}$$

We shall introduce a *spectral curve* $q : B = B_s \to C$ (see [4]). Namely, B_s is the subscheme of the total space of the canonical line bundle $K \xrightarrow{q} C$ with equation $x^2 = s$, where $s \in H^0(C, K^2)$ is a generic section. This is a smooth double cover of C of genus $g_B = 4g - 3$, and there is a dominant rational map of finite degree of the Prym variety on to $\mathcal{SU}_C(2)$:

$$\begin{array}{rcl} Q_s = \operatorname{Nm}_q^{-1}(K) & \to & \mathcal{SU}_C(2), \\ \zeta & \mapsto & q_*\zeta. \end{array}$$

Moreover, the images of these rational maps cover the moduli space as the section s varies, and so for any $E \in \mathcal{SU}_C(2)$ we may assume that $E = q_*\zeta$ for some line bundle $\zeta \in Q_s$, for suitable $s \in H^0(C, K^2)$.

By the projection formula the left-hand side of (14) is

$$H^0(C, \xi \otimes E) = H^0(B_s, L), \qquad \text{where } L = \zeta \otimes q^*\xi.$$

Notice that for $d < g - 1$ we have $\deg L = 2g - 2 + 2d \leq 4g - 6 = g_B - 3$, and in particular the Serre dual linear series $|K_B L^{-1}|$ is base-point-free for generic $\zeta \in Q_s$. By choosing s generically we may assume, for any given $E \in \mathcal{SU}_C(2)$ and $\xi \in J^d$, that this is the case.

We shall need:

4.4. Lemma. *Suppose, for $q : B \to C$ a double cover as above, that $|N|$ is a base-point-free linear series on B. Then either $N = q^*N'$ for some $N' \in \operatorname{Pic} C$, or*

$$h^0(B, N \otimes q^*\mathcal{O}(-x)) = h^0(B, N) - 2$$

for generic $x \in C$.

Proof. Write $q^{-1}(x) = x_1 + x_2$ with $x_1 \neq x_2$. Then $h^0(B, N) - h^0(B, N(-x_1 - x_2))$ is the rank of the evaluation map $H^0(B, N) \to \mathcal{C}_{x_1} \oplus \mathcal{C}_{x_2}$; and either this rank is 2 for generic $x \in C$ or it is ≤ 1 for all $x \in C$. In the latter case, the base-point-free hypothesis ensures that the image of the evaluation map is not contained in either summand; this implies that every divisor in $|N|$ is symmetric, so $N = q^*N'$ as asserted. □

We shall want to apply the lemma to $N = K_B L^{-1}$; we begin by observing that this line bundle cannot be symmetric, as follows. Since $K_B = q^* K_C^2$ and $L = \zeta \otimes q^* \xi$, $N = q^* N'$ would imply that $\zeta = q^* \tau$ for some $\tau \in \mathrm{Pic}\ C$. But then

$$E = q_* \zeta = \tau \otimes q_* \mathcal{O}_B = \tau \oplus K_C^{-1} \tau,$$

violating semistability.

So finally, consider the right-hand side of (14). By the projection formula this space is

$$H^0(C, \xi(D) \otimes E) = H^0(B, L \otimes q^* \mathcal{O}(D)).$$

We note that $\deg L \otimes q^* \mathcal{O}(D) = 4g - 4 = g_B - 1$, so by Riemann-Roch

$$h^0(C, \xi(D) \otimes E) = h^0(B, K_B L^{-1} \otimes q^* \mathcal{O}(-D)).$$

We now apply the lemma $e = g - 1 - d$ times to $N = K_B L^{-1}$: this gives, for $D \in S^e C$ generic, $h^0(C, \xi(D) \otimes E) = h^0(B, K_B L^{-1}) - 2e$.

Proof of (14). Assuming the right-hand side we have, by the last remark and by choosing D generically, $h^0(B, K_B L^{-1}) \geq k + 2e$. Consequently

$$\begin{array}{rcl} h^0(C, \xi \otimes E) & = & h^0(B, L) \\ & = & h^0(B, K_B L^{-1}) + \deg L - g_B + 1 \\ & \geq & k + 2e + \deg L - g_B + 1 \\ & = & k. \end{array}$$

The converse is trivial. □

III. Low genera

5. Genus 3

In this section we shall take C to be a nonhyperelliptic curve of genus 3. Then $\mathcal{SU}_C(2, K)$ is the *Coble quartic* associated to the Kummer variety in $\mathbf{P}^7$ (see [20] and [7], §33). It is well-known that in this case the 3-plane ruling $\varepsilon : \mathbf{P}U \to \mathcal{SU}_C(2, K)$ of section 1 is surjective and has degree 8. This follows easily from remark 3.2 (i) and lemma 1.1.

The behaviour of $h^0(E)$ in each 3-plane of the ruling is given by example 3.4. Namely, if $h^0(x) = 0$ then $\mathcal{W} \subset \mathcal{SU}_C(2, K)$ cuts $\mathbf{P}(x)$ transversally in a 2-plane along which $h^0(E) = 1$, while:

$$\mathcal{W}^1 = \bigcup_{p \in C} \Omega_p^1, \qquad \Omega_p^1 = \text{quadric cone} \subset \mathbf{P}(p). \tag{15}$$

In a moment we shall show that the vertices Ω_p^0 of these cones all coincide at a single point of $\mathcal{SU}_C(2, K)$ (see 5.2 and 5.3).

5.1. *Remark.* Note that for each $p \in C$ the image of C in $\mathbf{P}(p)$ lies on the cone Ω_p^1; and projecting along the generators is the trigonality $f : C \to \mathbf{P}^1$ given by the series $|K(-p)|$. (Conversely, one may show that the image of C in a 3-plane $\mathbf{P}(x)$ lies on a quadric cone only if $x = \eta(p)$ for some $p \in C$ and some square root of the trivial line bundle, $\eta^2 = \mathcal{O}$.)

We consider now the birational map $\varepsilon_0 : \mathbf{P}\,\mathrm{Ext}^1(K, \mathcal{O}) = \mathbf{P}^5 \to \mathcal{W} \subset \mathcal{SU}_C(2, K)$ and the diagram (12). The map δ is an isomorphism: its dual $S^2 H^0(K) \to H^0(2K)$ is surjective by Noether's theorem and injective since the canonical curve $C \subset \mathbf{P}^2$ is not contained in any quadric. Thus $\Omega^1 \subset \mathbf{P}^5$ is the Veronese surface; and it is well-known that its variety of secant lines Ω^2 is a cubic hypersurface isomorphic to $S^2\mathbf{P}^2$. Thus we have a tower of rational maps, where ε_0 is given by the complete linear series $|\mathcal{I}_C(2)|$ on $\mathbf{P}^5$ (see remark 3.3):

$$\begin{tikzcd}
\mathrm{Bl}_C(\mathbf{P}^5) \arrow[d] \arrow[dr] & \\
\mathbf{P}^5 = \Omega^3 \arrow[r, "\varepsilon_0"] \arrow[d, dash] & \mathcal{W} \arrow[d, dash] \\
S^2\mathbf{P}^2 = \Omega^2 \arrow[r] \arrow[d, dash] & \mathcal{W}^1 \arrow[d, dash] \\
C \subset \mathrm{Ver}(\mathbf{P}^2) = \Omega^1 \arrow[r] & \mathcal{W}^2
\end{tikzcd}$$

First of all, this allows us to recover the following result of Laszlo [17] and Paranjape-Ra and manan [21]. The bundle V appearing here is simply the normal bundle of C canonically embedded in its Jacobian.

5.2. Proposition. *$\mathcal{W}^2$ consists of a single point, i.e. there is a unique stable bundle $V \in \mathcal{SU}_C(2,K)$ with $h^0(V) = 3$.*

Proof. Since the canonical curve $C \subset \mathbf{P}^2$ has degree 4, any quadric of the series $|\mathcal{I}_C(2)|$ either contains the Veronese surface Ω^1 or has no further points of intersection. Thus Ω^1 contracts to a single point $V \in \mathcal{SU}_C(2,K)$ under ε_0.

On the other hand, for each $p \in C$ the 3-plane $\mathbf{P}(p) \subset \mathcal{SU}_C(2,K)$ is the image of the fibre $\mathbf{P}N_{C/\mathbf{P}^5}$ of the exceptional divisor in the blow-up, by remark 3.3. With this identification the point $\Omega^0_p \in \mathbf{P}(p)$ is the normal direction of $\Omega^1 = \mathrm{Ver}(\mathbf{P}^2) \supset C$, and is therefore contained in the closure of the image of Ω^1. In other words $\Omega^0_p = \{V\}$, and since we've seen that there are no further points of $\mathcal{W}^2$, this completes the proof. ☐

We now wish to give a geometric description of $\mathcal{W}^1 \subset \mathcal{SU}_C(2,K)$, and to this end we consider again the space $\mathcal{F}$ of Fay trisecants of the Kummer variety. Notice that for genus 3 there is a natural inclusion of the canonical series

$$|K| \hookrightarrow \mathcal{F} \xrightarrow{J[2]} S^4C$$

given by $D \mapsto (0, D)$ (see section 2). For $D \in |K|$ let us denote the corresponding trisecant by $t_D \subset \mathcal{SU}_C(2,K)$. If $D = p+q+r+s$, then, by the proof of theorem 2.1, t_D lies in the four 3-planes $\mathbf{P}(p), \mathbf{P}(q), \mathbf{P}(r), \mathbf{P}(s)$; in $\mathbf{P}(p)$, t_D is the trisecant line $\overline{qrs}$, and similarly in the other three spaces. By remark 5.1, on the other hand, this line is a generator of the cone $\Omega^1_p \subset \mathbf{P}(p)$, and conversely every generator is such a trisecant. By (1.5), therefore, we conclude that

$$\mathcal{W}^1 = \bigcup_{D \in |K|} t_D \subset \mathcal{SU}_C(2,K). \tag{16}$$

From this we obtain the following description of $\mathcal{W}^1$.

5.3. Theorem. *The subvariety $\mathcal{W}^1 \subset \mathcal{SU}_C(2,K) \subset \mathbf{P}^7$ has the following structure:*

1. *$\mathcal{W}^1$ is a cone over the Veronese surface $|K| = \mathbf{P}^2 \xrightarrow{|\mathcal{O}(2)|} \mathbf{P}^5$;*

2. $\mathcal{W}^1$ *has point vertex* $\mathcal{W}^2 \in \mathcal{SU}_C(2,K)$*;*
3. $\mathcal{W}^1$ *intersects the Kummer variety in the theta divisor* $\mathrm{Kum}(\Theta)$*, and projection along the generators of the cone coincides with the 3 to 1 Gauss map* $\mathrm{Kum}(\Theta) \to |K|$.

Proof. We have already seen that each trisecant $t_{p+q+r+s}$, where $p+q+r+s \in |K|$, passes through the point $V \in \mathcal{W}^2$. Assigning to the divisor $p+q+r+s$ the tangent direction of $t_{p+q+r+s}$ at $V \in \mathcal{SU}_C(2,K)$ therefore defines an injective map

$$\pi : |K| \to \mathbf{P}T\mathcal{SU}_C(2,K)|_V \cong \mathbf{P}^5,$$

for which $\pi^*\mathcal{O}(1) = \mathcal{O}(2)$ on the pencils $|K(-p)| \subset |K|$, and hence on the whole plane. Parts 1 and 2 of the theorem follow straightaway.

From (1) we see that the trisecant $t_{p+q+r+s}$ meets the Kummer variety in the three points

$$\begin{array}{c} \mathcal{O}(p+s) \oplus \mathcal{O}(q+r), \\ \mathcal{O}(q+s) \oplus \mathcal{O}(p+r), \\ \mathcal{O}(r+s) \oplus \mathcal{O}(p+q); \end{array}$$

which is equivalent to part 3. □

5.4. *Remark.* The Veronese cone of theorem 5.3 appears in the work of Coble ([7], §48). In particular, Coble exhibits a uniquely determined cubic hypersurface in $\mathbf{P}^6$ which cuts out $\mathrm{Kum}(\Theta)$ on the cone $\mathcal{W}^1$. It would be interesting to interpret this cubic in terms of vector bundles.

Finally, we shall sketch two proofs of the following fact.

5.5. Theorem. *At the triple point* $\mathcal{W}^2 = \{V\}$ *the theta divisor* $\mathcal{W}$ *has projectivised tangent cone* $\mathbf{P}T_V\mathcal{W} \cong \Omega^2 = S^2\mathbf{P}^2 \subset \mathbf{P}^5$.

First proof. Since V is stable we can identify $T_V\mathcal{SU}_C(2,K)$ with $H^1(C, \mathrm{ad}\, V)$. We have already remarked (remark 3.1) that since $\det V = K$ the Petri map factors through the symmetric product $S^2H^0(C,V) \to H^0(C, K \otimes \mathrm{ad}\, V)$; and this is dual to a map

$$\mu : H^1(C, \mathrm{ad}\, V) \to S^2H^0(C,V)^\vee \subset \mathrm{Hom}(H^0(V), H^1(V)).$$

By standard Brill-Noether type arguments the tangent cone $T_V\mathcal{W}$ is the pull-back under μ of the homomorphisms with nontrivial kernel (see for example [17]). On the other hand, one can show that μ is an isomorphism in the present case. The tangent cone is therefore precisely the locus of singular quadratic forms on $H^0(C,V)$, and hence isomorphic to $\Omega^2 = S^2\mathbf{P}^2$. □

Second proof (due to B. van Geemen). This exploits the fact that the hypersurface $\mathcal{W} \subset \mathbf{P}^6$ has degree 4 (since it is a hyperplane section of the Coble quartic), while V is a triple point of $\mathcal{W}$ ([17], proposition IV.7). It follows that $\mathbf{P}T_V\mathcal{W}$ is the complement in $\mathbf{P}^5$ of the (Zariski open) image of $\mathcal{W}$ under the rational projection map away from the point V.

We consider, then, the following diagram:

$$\begin{tikzcd}[column sep=small] & \mathcal{W} \subset & \mathbf{P}^6 \arrow[d, "\pi_V"] \\ \mathbf{P}^5 \arrow[ur, "\varepsilon_0 = \lambda_{|\mathcal{I}_C(2)|}"] \arrow[r, "\Delta"] & & \mathbf{P}^5 \arrow[from=1-2, dr] \end{tikzcd} \qquad (17)$$

We have seen that $\mathcal{W}$ is the (closed) image of $\mathbf{P}^5$ under the rational map ε_0 given by the complete linear series of quadrics through the bicanonical curve, contracting $\mathrm{Ver}(\mathbf{P}^2)$ down to the point V. Thus the rational map Δ is given by the complete linear series of quadrics containing $\mathrm{Ver}(\mathbf{P}^2)$. It is well-known that this can be identified with the inversion map of symmetric 3×3 matrices (geometrically, it sends a plane conic to its dual conic). Δ is a birational involution, blowing up the locus $\Omega^1 = \mathrm{Ver}(\mathbf{P}^2)$ of rank 1 conics and contracting the exceptional divisor down to the locus $\Omega^2 = S^2\mathbf{P}^2$ of rank 2 (dual) conics.

The image of Δ, and hence of $\pi_V|_W$, is therefore the complement of Ω^2, and we are done. $\square$

6. Genus 4

To any nonhyperelliptic curve of genus 4 one can associate in a canonical way a nodal cubic threefold $\mathcal{T} \subset \mathbf{P}^4$ as follows (see [9]). The canonical curve $C \subset \mathbf{P}^3$ lies on a unique quadric surface $Q \subset \mathbf{P}^3$ and is the base-locus of a 4-dimensional linear system of cubics; we define $\mathcal{T} \subset \mathbf{P}^4$ to be the image of the rational map

$$\lambda_{|\mathcal{I}_C(3)|} : \mathbf{P}^3 \to \mathbf{P}^4.$$

Let us denote by $g^1_3, h^1_3 \in \Theta \subset \mathrm{Pic}^3(C)$ the two trigonal line bundles on the curve. These satisfy $g^1_3 \otimes h^1_3 = K$, and coincide precisely when the curve has a vanishing theta-null. The quadric surface Q is ruled by trisecants $\overline{D} \subset \mathbf{P}^3$ of the curve, where D belongs to $|g^1_3|$ or $|h^1_3|$ (and Q is singular precisely when the two pencils coincide); it therefore contracts to a point $t_0 \in \mathcal{T}$ under $\lambda_{|\mathcal{I}_C(3)|}$. Any hyperplane through $t_0 \in \mathbf{P}^4$ then pulls back to Q plus a residual hyperplane, and it follows that projection away from the point t_0 is the birational inverse of $\lambda_{|\mathcal{I}_C(3)|}$:

$$\begin{array}{ccccc} & & \mathcal{T} & \subset & \mathbf{P}^4 \\ & \lambda_{|\mathcal{I}_C(3)|} \nearrow & & \searrow & \downarrow \pi_{t_0} \\ \mathbf{P}^3 & & = & & \mathbf{P}^3 \end{array} \tag{18}$$

This allows us to see that $\mathcal{T}$ is a cubic: a general hyperplane $H \subset \mathbf{P}^4 = |\mathcal{I}_C(3)|^\vee$ identifies with $\mathbf{P}^3$ under the projection π_{t_0}, and under this identification its intersection with $\mathcal{T}$ is the cubic surface corresponding to the point of $|\mathcal{I}_C(3)|$ annihilated by H.

6.1. Proposition. $\mathrm{mult}_{t_0} \mathcal{T} = 2$, *and the projectivised tangent cone at this point is* $\mathbf{P}T_{t_0}\mathcal{T} = Q \subset \mathbf{P}^3$.

Proof. That $\mathrm{mult}_{t_0} \mathcal{T} = 2$ follows at once from the fact that the projection $\pi_{t_0} : \mathcal{T} \to \mathbf{P}^3$ is birational and $\deg \mathcal{T} = 3$. On the other hand, let $H \subset \mathbf{P}^4$ be any hyperplane passing through t_0 and $H' = \pi_{t_0}(H) \subset \mathbf{P}^3$ its projection. Then $H \cap \mathcal{T}$ is the cubic surface obtained by blowing up the six points $C \cap H' \subset \mathbf{P}^2$; these six points lie on the conic $Q' = Q \cap H'$, and it is well-known that the resulting cubic surface is nodal with projectivised tangent cone $Q' \subset \mathbf{P}^2$ at the node. $\square$

6.2. *Remark.* From this we can easily write down an equation for $\mathcal{T}$: choosing a simplex of reference with $t_0 \in \mathbf{P}^4$ as one vertex, and the opposite face corresponding to a choice of cubic surface $F \in |\mathcal{I}_C(3)|$, the threefold $\mathcal{T}$ has equation

$$z_0 Q(z_1, \dots, z_4) + F(z_1, \dots, z_4) = 0.$$

This is the description given by Donagi.

We shall need next the Fano surface $F(\mathcal{T})$ of lines on $\mathcal{T}$, which is easy to describe. First note that there is an inclusion

$$\begin{array}{rcl} i : C & \hookrightarrow & F(\mathcal{T}), \\ p & \mapsto & l_p = \text{line joining } t_0 \text{ to } p \in C \subset \mathbf{P}^3. \end{array}$$

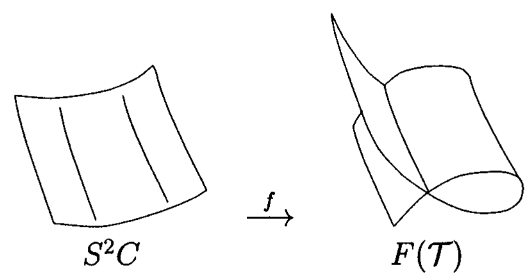
In other words, $l_p \subset \mathbf{P}^4$ is the line joining t_0 to the point p on the canonical curve via the projection π_{t_0}, and it is easy to see that these are precisely the lines through $t_0 \in \mathbf{P}^4$ which lie on $\mathcal{T}$. We now map

$$\begin{array}{rcl} f : S^2C & \to & F(\mathcal{T}), \\ p+q & \mapsto & l_{pq} = \text{residual line in } \mathcal{T} \cap \mathrm{Span}\{l_p, l_q\}. \end{array}$$

(Note that this still makes sense on the diagonal of S^2C: if $p = q$, then $\mathrm{Span}\{l_p, l_q\}$ is interpreted to mean the 2-plane spanned by t_0 and the tangent line to the canonical curve at $p \in C$.)

If the secant line $\overline{pq} \subset \mathbf{P}^3$ is not on Q, then $\lambda_{|\mathcal{I}_C(3)|}(\overline{pq}) = l_{pq}$; whilst if $\overline{pq} \subset Q$ then it contracts down to t_0, but $l_{pq} = l_r$, where $r \in \overline{pq} \cap C$ is the third point of the trisecant. Thus f is a birational morphism and is injective away from the two curves $C \hookrightarrow S^2C$ defined by $r \mapsto g^1_3(-r)$ and $r \mapsto h^1_3(-r)$, each of which it identifies with $i(C)$:

(19)



Izadi [13] makes use of the lines on $\mathcal{T}$ to identify $\mathcal{T} \subset \mathbf{P}^4 \xrightarrow{\sim} \mathbf{P}\Gamma_{00}$ (theorem 6.3 below). Namely, for $r \in C$ and for $p+q \in F(\mathcal{T})\backslash i(C)$ (which we identify with the corresponding subset of S^2C as above) she constructs pencils which we shall denote by $l'_r, l'_{pq} \subset \mathbf{P}\Gamma_{00}$ respectively. These are characterised by their base locus: for any $p, q \in C$ let

$$\Sigma_{pq} = C - C \cup W_2 - p - q \cup p + q - W_2 \subset J_C. \tag{20}$$

Then the pencil $l'_r \subset \mathbf{P}\Gamma_{00}$ has base locus $\Sigma_{pq} \cup \Sigma_{p'q'}$ where $f^{-1}(i(r)) = \{p+q, p'+q'\}$, i.e. $p+q+r \in |g^1_3|$ and $p'+q'+r \in |h^1_3|$; and the pencil $l'_{pq} \subset \mathbf{P}\Gamma_{00}$ has base locus $\Sigma_{pq} \cup \Sigma(X)$, where (in Izadi's notation—see [13], §7)

$$\begin{array}{rcll} \Sigma(X) & = & \{s+t-s'-t' \mid & s,t,s',t' \in C, \\ & & & h^0(K-p-q-s-t) > 0, \\ & & & h^0(K-p-q-s'-t') > 0\}. \end{array} \tag{21}$$

(In this notation X denotes a curve of genus 5 in the fibre of the Prym map over J_C, though this will not concern us here.)

Izadi's result, in our (nonhyperelliptic Jacobian) situation, can then be stated as follows.

6.3. Theorem. *Let $C \subset \mathbf{P}^3$ be a canonical curve of genus 4, and $\mathbf{P}^4 = |\mathcal{I}_C(3)|^\vee$ be the ambient space of its associated cubic threefold $\mathcal{T}$. Then there is a natural identification $\mathbf{P}^4 \xrightarrow{\sim} \mathbf{P}\Gamma_{00}$ under which $l_r \xrightarrow{\sim} l'_r$, $l_{pq} \xrightarrow{\sim} l'_{pq}$ and the node $t_0 \in \mathcal{T}$ maps to $\Theta - g^1_3 \cup \Theta - h^1_3$.*

We now return to consider the Brill-Noether loci in $\mathcal{SU}_C(2,K) \subset \mathbf{P}^{15}$ and to state our main result. Recall that $\mathcal{W}^2 = \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2,K)$ (by theorem 4.1).

6.4. Theorem. *If C is a nonhyperelliptic curve of genus 4, then $\mathcal{W}^2 \subset \mathcal{SU}_C(2,K) \subset \mathbf{P}^{15}$ is precisely the Donagi-Izadi cubic threefold $\mathcal{T} \subset \mathbf{P}\Gamma_{00} = \mathbf{P}^4$; with node at $g^1_3 \oplus h^1_3 \in \mathrm{Kum}(J^3_C)$.*

6.5. *Remark.* Note that (up to S-equivalence) $g^1_3 \oplus h^1_3$ is the unique semistable bundle in $\mathcal{SU}_C(2,K)$ with $h^0 = 4$, and so $\mathcal{W}^3$ is by definition empty. This is a consequence of Mukai's lemma 3.6: since $|g^1_3|$ is base-point-free, $h^0(E) \geq 4$ would imply that $h^0({g^1_3}^{-1} \otimes E) \geq 1$, and hence by semistability that $g^1_3 \subset E$. So E is S-equivalent to $g^1_3 \oplus K{g^1_3}^{-1} = g^1_3 \oplus h^1_3$.

(In fact, if $g^1_3 \neq h^1_3$ then this is an isomorphism, since $h^1_3 \subset E$ by the same argument. If, on the other hand, $g^1_3 = h^1_3$ then one can check, using the arguments of section 3, that as well as $g^1_3 \oplus g^1_3$ there is, up to isomorphism, a unique nonsplit extension E with $h^0(E) = 4$: the space of all such extensions is $\mathbf{P}(g^1_3) = \mathbf{P}H^0(K)^\vee$, in which the canonical curve lies on a quadric cone. E is then the extension corresponding to the vertex of the cone.)

Thus it makes sense to view $\mathcal{W}^3 = \{g^1_3 \oplus h^1_3\}$.

6.6. Lemma. *Suppose that C is nonhyperelliptic of genus 4 or nontrigonal of genus 5. Then for every stable $E \in \mathcal{SU}_C(2,K)$ with $h^0(E) = 3$ the exterior multiplication map $\phi_E : \bigwedge^2 H^0(E) \to H^0(K)$ is injective.*

Proof. Since every element of $\bigwedge^2 H^0(E)$ is decomposable, i.e. of the form $s \wedge t$, a nontrivial element of $\ker \phi_E$ would give two independent sections s, t generating a line subbundle $L \subset E$. Then $r(L) \geq 1$ while by stability $\deg L \leq g-2$, contrary to the hypotheses on C. $\square$

In the genus 4 case the lemma determines a rational map (defined away from the single point $g^1_3 \oplus h^1_3$) $\pi : \mathcal{W}^2 \to \mathbf{P}^3 = |K|^\vee$ sending $E \mapsto \mathrm{im}\, \phi_E \subset |K|$. In proving theorem 6.4 we shall in fact prove slightly more, namely that the following diagram commutes (and we shall also extend this diagram in theorem 6.11 below):

$$\begin{array}{ccc} \mathcal{T} & = & \mathcal{W}^2 \subset \mathbf{P}\Gamma_{00} \\ \pi_{t_0} \searrow & & \swarrow \pi \\ & \mathbf{P}^3 & \end{array} \tag{22}$$

6.7. Proposition. *For each $p \in C \subset \mathbf{P}^3$ the closure of the fibre $\pi^{-1}(p) \subset \mathcal{W}^2$ is a Hecke line l_F with $\det F = K(p)$. Moreover, l_F is the unique such Hecke line contained in $\mathcal{W}^2$ and passing through the point $g^1_3 \oplus h^1_3$.*

Proof. Consider a stable bundle $E \in \pi^{-1}(p)$. By definition the sections of E fail to generate E at the point p, and we denote by $D_p \subset E_p$ the line in the fibre at p which is generated by global sections. This line uniquely determines an extension

$$0 \to E \to F \to \mathcal{C}_p \to 0,$$

and by construction the coboundary map $\delta : \mathcal{C} \to H^1(E)$ vanishes, so that $h^0(F) = 4$ and $H^0(E) \subset H^0(F)$ coincides with the subspace $H^0(F(-p))$. Finally, since E is stable, F is stable. This proves the first part of the proposition, with $l_F \subset \mathcal{W}^2$.

Now choose a section $s \in H^0(F)$ not lying in $H^0(E)$. Then $s(p) \neq 0$ and spans a line in the fibre F_p; we consider a nonzero homomorphism $u : F \to \mathcal{C}_p$ such that this line coincides with $\ker u_p$. Then by construction $\ker u \subset F$ is a semistable bundle with $h^0(\ker u) = 4$, and hence by remark 6.5 $\ker u = g^1_3 \oplus h^1_3$ up to S-equivalence; and this point therefore lies on l_F.

It remains to show that a Hecke line with these properties is unique. If $g^1_3 \neq h^1_3$ then by remark 6.5 F has subsheaf $g^1_3 \oplus h^1_3$, and by [2], lemma 3.2, this determines the Hecke line l_F uniquely. Alternatively one can argue similarly to the vanishing theta-null case, to which we shall now restrict.

So assume that $g^1_3 = h^1_3$. By theorems 1.3 and 1.4 l_F, lies in some 4-plane $\mathbf{P}(x)$, meeting the curve at the image of a point $q \in C$: thus $x = g^1_3(-q)$. By the proof of 1.3, *either* $p = q$ *or* l_F is a trisecant $\overline{D}$ where $\mathcal{O}(D) = Kx^{-2}(p)$. On the other hand, the second case does not occur for the following reason: by (10) and (11) in section 3, $h^0 \geq 3$ in $\mathbf{P}(x)$ precisely along a line Ω^0_x, projection away from which maps C onto a plane conic via the linear series $|g^1_3(q)|$ with the single base point q. Thus (since $h^0 \geq 3$ along l_F) $l_F = \Omega^0_x$ and meets the curve only at one point (with multiplicity 2). Thus $l_F = \Omega^0_{g^1_3 - p}$ and is uniquely determined. □

6.8. *Remark.* In the case $g^1_3 \neq h^1_3$ one can show that l_F is the intersection of the two 4-planes $\mathbf{P}(g^1_3(-p))$ and $\mathbf{P}(h^1_3(-p))$ (in the notation of section 1), and in each space is the tangent line to the curve at the image of $p \in C$. In the case $g^1_3 = h^1_3$ just discussed in the above proof, the curve $C \subset \mathbf{P}(g^1_3(-p))$ has a cusp at $p \in C$. The line l_F, passing through p, is not the tangent line but is the vertex of a rank 3 quadric containing the curve.

6.9. Proposition. *For each $p \in C$ the Hecke line of the previous proposition coincides with Izadi's pencil $l'_p = l_p = \pi^{-1}_{t_0}(p)$.*

Proof. Consider a stable bundle $E \in \pi^{-1}(p) = l_F$. We shall show that $E \in l'_p$; since both sets are lines the result will follow.

So we have to show that the divisor $D_E = \{L \in J_C | h^0(C, L \otimes E) > 0\}$ contains the surfaces Σ_{st} and $\Sigma_{s't'}$, where $p + s + t \in |g^1_3|$ and $p + s' + t' \in |h^1_3|$. (Note that for $E \in \mathcal{W}^2$ we have $D_E \in \mathbf{P}\Gamma_{00}$, and hence $C - C \subset D_E$ a priori—see section 4.) Since D_E is symmetric it is enough to prove that $W_2 - s - t \subset D_E$, i.e. that $h^0(E(p + q - s - t)) > 0$ for all $p, q \in C$. We shall show that $h^0(E(-s - t)) \geq 1$ (and note that by proposition 4.3 this is actually equivalent); and similarly that $h^0(E(-s' - t')) \geq 1$.

By hypothesis $\operatorname{im} \phi_E = H^0(K(-p))$; and we have a natural 2-dimensional subspace $V \subset H^0(K(-p))$, namely

$$V = H^0(K(-p - s)) = H^0(K(-p - t)) = H^0(K(-p - s - t)).$$

So consider the subspace $\phi^{-1}_E(V) \subset \bigwedge^2 H^0(E)$ and choose sections $u, v, w \in H^0(E)$ such that $u \wedge v, u \wedge w$ form a basis of $\phi^{-1}_E(V)$. Since $v \wedge w \notin \phi^{-1}_E(V)$, the effective divisor $(u \wedge w) \in |K|$ is not supported at s or t; this implies that the sections v, w generate E at the points $s, t \in C$. However, by construction $s + t \leq (u \wedge v)$ and $s + t \leq (u \wedge w)$; and we claim that this can only occur if $u(s) = u(t) = 0$. For if $u(s) \neq 0$, for example, then $\mathcal{C}u(s) = \mathcal{C}v(s)$ and $\mathcal{C}u(s) = \mathcal{C}w(s)$, and hence $s \in$

$\operatorname{supp}(v \wedge w)$, a contradiction. Hence we obtain a nonzero section $u \in H^0(E(-s-t))$; and similarly we can do the same for $H^0(E(-s'-t'))$. $\square$

Let us return to the proof of theorem 6.4. In seeking stable bundles with three sections we may consider extensions $E \in \mathbf{P}\operatorname{Ext}^1(K-D, D)$ with $\deg D = 1$ or 2 (using remark 3.2). If $E \in \Omega^n \backslash \Omega^{n-1}$, then by (11)

$$h^0(E) = 5 - \operatorname{Cliff}(D) - n.$$

Thus either $D = p \in C$, and $E \in \Omega^1$; or $D = p+q \in S^2C$, and $E \in \Omega^0_{p+q} \cong \mathbf{P}^1$. The second case is that of example 3.4; we shall show next that this case exhausts all such bundles.

6.10. Proposition. $\mathcal{W}^2 = \bigcup_{p+q \in S^2C} \Omega^0_{p+q}$. *Moreover,* Ω^0_{p+q} *maps under* $\pi : \mathcal{W}^2 \to \mathbf{P}^3$ *onto the secant line* $\overline{pq}$ *if* $p+q \notin i(C)$, *while* $\Omega^0_{p+q} = \pi^{-1}(r)$ *if* $p+q+r \in |g^1_3|$ *or* $|h^1_3|$.

Proof. We first observe (by considering diagram (10)) that the line $\Omega^0_{p+q} \subset \mathbf{P}(p+q)$ meets the image of the curve if and only if $f(p+q) \in i(C)$ (see (19)); and in this case meets the curve at a point $r \in C$ representing the bundle $g^1_3 \oplus h^1_3$. By theorem 1.3, Ω^0_{p+q} is a Hecke line l_F, where one easily checks that $\det F = K(r)$. So by the uniqueness statement in 6.7, $\Omega^0_{p+q} = \pi^{-1}(r)$.

We may now assume, then, that $E \in \mathcal{W}^2$ is a stable bundle for which $\pi(E)$ does not lie on the canonical curve; $\pi(E)$ then lies on some secant line $\overline{pq} \subset \mathbf{P}^3$. This means that $\operatorname{im}\phi_E \subset H^0(K)$ is a hyperplane, distinct from $H^0(K(-p))$ and $H^0(K(-q))$ but containing the 2-dimensional subspace $H^0(K(-p-q))$. As in the proof of the previous proposition we can find a basis $u, v, w \in H^0(E)$ such that $u \wedge v, u \wedge w$ are a basis of $\phi_E^{-1} H^0(K(-p-q))$ which $v \wedge w$ completes to a basis of $\bigwedge^2 H^0(E)$. Then we have

$$p+q \leq (u \wedge v), \quad p+q \leq (u \wedge w),$$

while $p+q \not\leq (v \wedge w)$. As before, it follows from this that $u(p) = u(q) = 0$, i.e. $h^0(E(-p-q)) > 0$, and so $E \in \Omega^0_{p+q}$. $\square$

Proof of theorem 6.4. By propositions 6.7, 6.9 and theorem 6.3 it suffices to check that the line $\Omega^0_{p+q} \subset \mathcal{W}^2 \subset \mathbf{P}\Gamma_{00}$ coincides with the pencil l'_{pq} if $p+q \in F(\mathcal{T}) \backslash i(C)$, i.e. when $h^0(g^1_3(-p-q)) = h^0(h^1_3(-p-q)) = 0$.

Consider a stable bundle $E \in \Omega^0_{p+q}$. Since $h^0(E(-p-q)) > 0$, the symmetric divisor D_E trivially contains the surfaces $W_2 - p - q$ and $p+q-W_2$, while $C - C \subset D_E$ since $E \in \mathcal{W}^2$. We will show that D_E also contains the surface $\Sigma(X)$ (see (21)). Let $\lambda = \mathcal{O}(s+t-s'-t') \in \Sigma(X)$. By definition we have an exact sequence

$$0 \to \lambda(p+q) \to \lambda \otimes E \to K\lambda(-p-q) \to 0$$

with, say, extension class $f \in \operatorname{Ext}^1(K-p-q, p+q) = H^0(C, K^2(-2p-2q))^\vee$.

We have to show that $h^0(\lambda \otimes E) > 0$. We can suppose that $h^0(\lambda(p+q)) = 0$ (otherwise there is nothing to prove); so by Riemann-Roch $h^1(\lambda(p+q)) = 1$. If $h^0(K\lambda(-p-q)) > 1$, then $h^0(\lambda \otimes E) > 0$ and we are done; so we assume that $h^0(K\lambda(-p-q)) = 1$. In this case $h^0(\lambda \otimes E) > 0$ if and only if the coboundary map

$$\delta : H^0(K\lambda(-p-q)) \to H^1(\lambda(p+q))$$

vanishes, which in turn is equivalent to $\ker f$ containing the image of the multiplication map:

$$H^0(K\lambda(-p-q)) \otimes H^0(K\lambda^{-1}(-p-q)) \to \ker f \subset H^0(K^2(-2p-2q)).$$

In fact we shall check that the image is in the subspace $S^2H^0(K(-p-q)) \subset \ker f$ (see example 3.4).

This last assertion results from the definition (21): we can write

$$K = \mathcal{O}(p+q+s+t+u+v) = \mathcal{O}(p+q+s'+t'+u'+v'),$$

for some $u, v, u', v' \in C$, and hence

$$\begin{array}{rcccl} K\lambda(-p-q) & = & K(-p-q-s'-t'+s+t) & = & \mathcal{O}(u'+v'+s+t), \\ K\lambda^{-1}(-p-q) & = & K(-p-q-s-t+s'+t') & = & \mathcal{O}(u+v+s'+t'). \end{array}$$

By hypothesis these divisors are unique in their linear equivalence classes, and we can write their sum as

$$(u'+v'+s+t) + (u+v+s'+t') = (s+t+u+v) + (s'+t'+u'+v'),$$

where $s+t+u+v$ and $s'+t'+u'+v' \in |K(-p-q)|$. □

We shall conclude this section by giving another interpretation of diagram (22), as follows. First, we shall view $\Gamma_{00} \hookrightarrow S^4H^0(C,K)$ by assigning to each element the leading terms of its Taylor expansion at $0 \in J_C$, or equivalently by assigning to a divisor its tangent cone at the origin. This map is injective by [13], lemma 2.1.1.

Next we note that there is a distinguished element $q^2 \in S^4H^0(C,K)$, where $q \in S^2H^0(C,K)$ is the equation of the quadric $Q \subset \mathbf{P}^3$ containing the canonical curve. Under the above inclusion this comes from the split divisor $\Theta - g^1_3 \cup \Theta - h^1_3 \in \mathbf{P}\Gamma_{00}$.

Third, we identify $\mathbf{P}^3 = \mathbf{P}T_0J_C$ with the space of translation-invariant vector fields on the Jacobian. One can then map

$$\begin{array}{rcccl} \alpha & : & \mathbf{P}T_0J_C & \to & \mathbf{P}\big(S^4H^0(K)/\mathcal{C}q^2\big), \\ & & D & \mapsto & qDf - fDq, \end{array}$$

where $f \in H^0(\mathcal{I}_C(3))$ is any cubic through the canonical curve. It is easy to check that this construction is independent of the choice of f; moreover α is an isomorphism onto the subspace $\mathbf{P}(\Gamma_{00}/\mathcal{C}q^2)$, as observed by Beauville and Debarre [3], pages 32–33.

6.11. Theorem. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{W}^2 & \subset & \mathbf{P}\Gamma_{00} & \subset & \mathbf{P}S^4H^0(K) \\ \pi\downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^3 & \xrightarrow{\alpha} & \mathbf{P}\big(\Gamma_{00}/\mathcal{C}q^2\big) & \subset & \mathbf{P}\big(S^4H^0(K)/\mathcal{C}q^2\big) \end{array}$$

Proof. We have to check commutativity of the left-hand square, and since both vertical arrows are linear projections it is sufficient to check commutativity over points $p \in C$ of the canonical curve. For such a point denote by $D_p \in \mathbf{P}T_0J_C$ the associated constant vector field. By propositions 6.7 and 6.9 the line $\pi^{-1}(p)$ corresponds to the pencil l'_p with base locus $\Sigma_{st} = \bigcup \Sigma_{s't'}$, where the points $s, t, s', t' \in C$ are defined by $p+s+t \in |g^1_3|$ and $p+s'+t' \in |h^1_3|$. By tangent cones at the origin, the pencil l'_p corresponds to a pencil of quartics spanned by $qDf - fdq$ and q^2, for some $D \in T_0J_C$ and $f \in H^0(\mathcal{I}_C(3))$. We have to show that $D = D_p$.

Since the pencil is uniquely determined by (the tangent cone of) its base locus, it is enough to check that the two quartics $qD_pf - fD_pq$ and q^2 contain the tangent cones of $C - C$, $W_2 - s - t$ and $W_2 - s' - t'$. These tangent cones are the canonical curve $C \subset \mathbf{P}^3$ and the two trisecants $\overline{pst}, \overline{ps't'} \subset \mathbf{P}^3$ respectively. The result now follows easily: q vanishes on all three curves; while f (and hence fD_pq) vanishes on C, and—since the two trisecants span the tangent plane to Q at p—the derivative D_pq vanishes on the two lines. $\square$

7. Genus 5

Let C be a curve of genus 5. If C is nontrigonal then the canonical curve $C \hookrightarrow \mathbf{P}^4$ is the complete intersection of a net of quadrics $|\mathcal{I}_C(2)| = \mathbf{P}^2$, in which the locus $\Gamma \subset \mathbf{P}^2$ of singular quadrics is a plane quintic curve, smooth if C has no vanishing theta-nulls, otherwise having ordinary double points corresponding to quadrics of rank 3 (see [1], page 270).

Let $\Theta_{\text{sing}} = W_4^1$ be the singular locus of the theta divisor. This is a curve, and by assigning to each point $x \in \Theta_{\text{sing}}$ its projectivised tangent cone $\mathbf{P}T_x\Theta = Q_x$ we have a double cover

$$\begin{array}{rccl} f: & \Theta_{\text{sing}} & \to & \Gamma \subset \mathbf{P}^2, \\ & x & \mapsto & Q_x \\ & & & = \bigcup\limits_{D\in|x|} \overline{D} \subset \mathbf{P}^4. \end{array}$$

The sheet interchange of Θ_{sing} with respect to this double cover is induced by the Serre involution of J_C^4.

7.1. Lemma. $f^*\mathcal{O}_\Gamma(1) = \mathcal{O}_{\Theta_{\text{sing}}}(\Theta)$. *Moreover, the induced restriction map*

$$H^0(J_C^{g-1}, 2\Theta) \to H^0(\Gamma, \mathcal{O}(2))$$

is surjective.

Proof. The first part follows from [11]. To prove that the pull-back of hyperplane sections is surjective, it is sufficient to show this on the image of the Kummer map $J_C \to |2\Theta|$, $a \mapsto \Theta_a + \Theta_{-a}$. In other words, we consider the rational map

$$\alpha : J_C \to |\mathcal{O}_\Gamma(2)| \cong \mathbf{P}^5$$

sending $a \in J_C$ to the divisor whose pull-back to Θ_{sing} is $(\Theta_a + \Theta_{-a}) \cap \Theta_{\text{sing}}$. (Note that α is defined away from $C - C \subset J_C$: this follows from [22], theorem 2.4.) One can show that the map

$$\begin{array}{rcl} \beta : J_C & \to & S^{10}(\Theta_{\text{sing}}), \\ a & \mapsto & \Theta_a \cap \Theta_{\text{sing}} \end{array}$$

is injective (see, for example, [1], pages 265–268); this implies that α is a finite map, and so we are done. $\square$

It follows from this that the image of Θ_{sing} under the Kummer map is a Veronese embedding of $\Gamma \subset \mathbf{P}^2$:

$$\begin{array}{ccc} \Theta_{\text{sing}} & \xrightarrow{\text{Kum}} & \mathbf{P}^{31} \\ f\downarrow & & \uparrow v \\ \Gamma & \subset & \mathbf{P}^2 \end{array} \tag{23}$$

where $v(\mathbf{P}^2) \subset \mathbf{P}^5 \subset \mathbf{P}^{31}$ is a Veronese surface.

7.2. Theorem. *For any curve C of genus 5 the Brill-Noether locus $\mathcal{W}^3 \subset \mathcal{SU}_C(2,K)$ is a Veronese surface intersecting the Kummer variety in the Veronese image of a plane quintic $\Gamma \subset \mathbf{P}^2$. In particular:*

1. *If C is nontrigonal then $\mathcal{W}^3 = v(\mathbf{P}^2)$, where Γ is as in (23) and $v(\Gamma) = \mathrm{Kum}(\Theta_{\mathrm{sing}})$.*
2. *If C is trigonal then $\Gamma \subset \mathbf{P}H^1(g^1_3)$, where g^1_3 is the (unique) trigonal line bundle, is the projection of the canonical curve away from a trisecant; and its Veronese image cuts the Kummer in the component $C + g^1_3$ of Θ_{sing}.*

Proof of part 2. This is easily dispatched. We first remark that it is well-known that on a curve of genus ≥ 5 a g^1_3 is unique if it exists; while for a curve of genus 5 the following argument will give another proof of this fact.

Let $|D| = g^1_3$; by lemma 3.6 any stable bundle $E \in \mathcal{SU}_C(2,K)$ with $h^0(E) \geq 4$ has line subbundle $\mathcal{O}(D) \subset E$, so E belongs to the 5-plane $\mathbf{P}(g^1_3)$ of section 1. By example 3.4 we have seen that $h^0(E) = 4$ precisely along a Veronese surface in $\mathbf{P}(g^1_3)$. This intersects the Kummer precisely in the image of the curve—that is, in the Kummer image of $C + g^1_3$—and from diagram (10) this is the projection of the canonical curve, as asserted. $\square$

From now on we shall assume that the curve C is nontrigonal. Before proving part 1 of the theorem we shall need to make some further observations about the curve Γ; we consider the map

$$\begin{array}{rccl} (24) \qquad & l : S^2C & \to & (\mathbf{P}^2)^\vee, \\ & D & \mapsto & |\mathcal{I}_{C\cup\overline{D}}(2)|. \end{array}$$

In other words $l(D)$ is the pencil of quadrics containing C and the line $\overline{D}$. Note that the base locus of such a pencil is a quartic del Pezzo surface containing sixteen lines, and so $\deg l = 16$.

For each $D \in S^2C$ we shall identify the five quadrics

$$(25) \qquad l(D) \cap \Gamma = \{Q_1, \dots, Q_5\}.$$

Projection away from the line $\overline{D} \subset \mathbf{P}^4$ maps the canonical curve C to a 5-nodal plane sextic $C' \subset \mathbf{P}^2$. (Note, again, that the del Pezzo base locus of the pencil $l(D)$ is obtained by blowing up $\mathbf{P}^2$ in the five nodes of C'.) Let us denote by $D^{(1)}, \dots, D^{(5)} \in S^2C$ the divisors over the five nodes of $C' \subset \mathbf{P}^2$. Then by Riemann-Roch each $|D + D^{(i)}|$ is a g^1_4, and hence each

$$(26) \qquad Q_i = Q_{D+D^{(i)}} = \bigcup_{D' \in |D+D^{(i)}|} \overline{D'}, \qquad i = 1, \dots, 5,$$

is a quadric of rank ≤ 4 containing the line $\overline{D}$. These are therefore the points of intersection (25).

We now return to the proof of theorem 7.2. We consider stable extensions $E \in \mathbf{P}\,\mathrm{Ext}^1(K-D, D)$ where (by remark 3.2 (i)) we may take $\deg D = 2$ or 3. For such an extension, by (11),

$$h^0(E) = 6 - \mathrm{Cliff}(D) - n,$$

where $E \in \Omega^n_D \backslash \Omega^{n-1}_D$. So for $h^0(E) = 4$ we must have $n + \mathrm{Cliff}(D) = 2$; if $\deg D = 3$ then this forces $|D| = g^1_3$, contrary to the hypothesis that C is nontrigonal. So the only possibilities we need to consider are $D \in S^2C$, and then $h^0(E) = 4$ for $E \in \Omega^0_D \subset \mathbf{P}\,\mathrm{Ext}^1(K-D, D)$.

In this situation diagram (10) becomes

$$\begin{tikzcd} C \arrow[r, "|K-D|"] \arrow[d, "|2K-2D|"'] & C' \subset \mathbf{P}^2 \arrow[d, "\mathrm{Ver}"] \\ \Omega_D^0 = \mathbf{P}^1 \subset \mathbf{P}^7 \arrow[r, "\delta"] & \mathbf{P}^5 \end{tikzcd}$$

where $C' \subset \mathbf{P}^2$ is the 5-nodal sextic as above; and in particular δ is surjective.

It follows that $\mathcal{W}^3 = \bigcup_{D\in S^2C} \varepsilon_D \Omega_D^0 \subset \mathbf{P}^{31}$, where we observe that *each* $\varepsilon_D\Omega_D^0$ *is a nonsingular conic.* This is because by lemma 3.3 the rational map ε_D comes from the (complete) linear series $|\mathcal{I}_C(2)|$ on $\mathbf{P}^7$; while Ω_D^0 has no intersection with the base locus C since $|K-D|$ has no base points—i.e. the canonical curve has no trisecant lines since C is nontrigonal.

Finally, theorem 7.2 will follow directly once we observe that

$$\varepsilon_D \Omega_D^0 = v(l(D)) \qquad \text{for all } D \in S^2C, \tag{27}$$

where v and l are as defined in (23) and (24) respectively. To prove (27) it is sufficient, since both sides are nonsingular conics, to show that both contain the five points $v(Q_1), \dots, v(Q_5)$ (see (25) and (26)); and so it remains only to check this for $\varepsilon_D\Omega_D^0$.

First note that an extension $E \in \Omega_D^0$ fails to be stable (i.e. maps to the Kummer variety) if and only if it lies on $\mathrm{Sec}^2 C$; and there are precisely five such points, which are the intersections of Ω_D^0 with the secant lines $\overline{D^{(1)}}, \dots, \overline{D^{(5)}}$, where as before the $D^{(i)} \in S^2C$ are the nodal divisors over the curve $C' \subset \mathbf{P}^2$. The corresponding extensions then contain $\mathcal{O}(K-D-D^{(i)})$, respectively, as line subbundles—in other words, they map under ε_D to the points

$$\mathcal{O}(D+D^{(i)}) \oplus \mathcal{O}(K-D-D^{(i)}) \in \mathrm{Kum}(J_C^{g-1}).$$

By (26) and diagram (23) these are precisely the images $v(Q_1), \dots, v(Q_5)$, which completes the proof. □

8. Genus 6

By proposition 3.7 we have $h^0(E) \le 6$ for all semistable bundles E in $\mathcal{SU}_C(2,K)$ on a nonhyperelliptic curve C of genus 6, and $h^0(E) \le 5$ if C is not trigonal or a plane quintic.

8.1. Theorem. *Let C be a nonhyperelliptic curve of genus 6.*

1. *If C is not trigonal or a plane quintic, then there exists a unique stable bundle $E \in \mathcal{SU}_C(2,K)$ with $h^0(E) = 5$; i.e. $\mathcal{W}^4 = \{E\}$.*
2. *If C is trigonal, then $\mathcal{W}^4 \cong \mathbf{P}^1$ is a line, along which $h^0(E) = 5$, and does not meet* $\mathrm{Kum}(J_C^{g-1})$.
3. *If C is a plane quintic, then $h^0(E) \ge 5$ if and only if E is in the S-equivalence class of the point $g_5^2 \oplus g_5^2 \in \mathrm{Kum}(J_C^{g-1})$.*

Proof. First suppose that C is not a plane quintic, and observe that if $h^0(E) \ge 5$ and E is semistable then it is necessarily stable: otherwise E fits in an extension (7) with $\deg D = 5$ (and with D not necessarily effective). Then $h^0(D) = h^0(K-D) \le 2$, since C is not a plane quintic, and so $h^0(E) \le 4$.

On the other hand, for any tetragonal pencil $g_4^1 = |D|$ one can (following Mukai) apply lemma 3.6 to observe that $h^0(E(-D)) \ge 1$ for any such bundle. By stability this means that $\mathcal{O}(D) \subset E$ is a line subbundle, i.e. $E \in \mathbf{P}(D)$, the corresponding 6-plane of the ruling of section 1. By example 3.4, $h^0(E) = 5$ exactly for $E \in \Omega^0$; if C is nontrigonal this is a single point, and part 1 is proved.

If C is trigonal then we may take $D = K - 2L$, where $|L| = g^1_3$; $|K - D|$ maps $C \to \mathbf{P}^2$ with degree 3 onto a conic, so in this case $\ker \delta$ is 2-dimensional and $\Omega^0_D \subset \mathbf{P}(D)$ is a line. This line does not meet the image of C in $\mathbf{P}(D)$, since $|K - D|$ is base-point-free; so we have proved part 2.

For part 3, first note that by the reasoning of remark 6.5 the only semistable bundle with $h^0(E) = 6$ is $E = g^2_5 \oplus g^2_5$. On the other hand the reasoning of part 1 above yields extensions with $h^0(E) = 5$ in Ω^0_D for any $|D| = g^1_4$. In this case the tetragonal pencils are precisely the projections from points of the plane quintic, i.e. $D = L - p$ for $|L| = g^2_5$ and some $p \in C$. The map $C \to \mathbf{P}^2$ given by the series $|K - D|$ is projection of the canonical curve (which lies on a Veronese surface) away from the conic in $\mathbf{P}^5$ spanned by D, and hence has base-point p. The image is thus the plane quintic model of C; in particular δ is surjective and Ω^0_D is a single point. But because $p \in C$ is a base-point of $|K - D|$, the curve passes through Ω^0_D at the image of p, which is the equivalence class of $g^2_5 \oplus g^2_5$. This proves part 3. $\square$

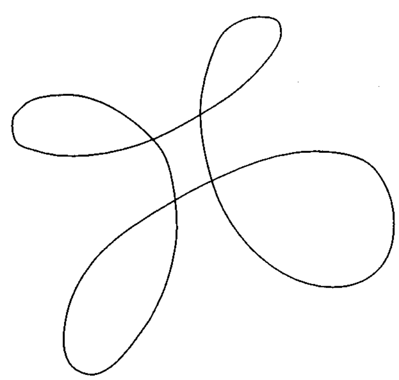
8.2. *Remark.* Part 1 of the theorem was observed by Mukai in [18]. Recall that the canonical curve lies on the (del Pezzo) transverse intersection with a $\mathbf{P}^5 \subset \mathbf{P}^9$ of the Plücker embedded Grassmannian of lines in $\mathbf{P}^4$. The bundle E in the theorem is then dual to the restriction to the curve of the tautological bundle on the Grassmannian.

It is well-known that a generic curve of genus 6 possesses five tetragonal pencils, so the point E of part 1 is common to the corresponding five 6-planes of the ruling. It is amusing to see this using the results of section 1.

Let us denote the five by $x_0, \ldots, x_4 \in \mathrm{Pic}^4(C)$; and let us recall how they are related to each other. By Riemann-Roch, $|x_0| = g^1_4$ implies that $|Kx_0^{-1}| = g^2_6$. Thus the image of

$$\lambda_{|Kx_0^{-1}|} : C \to \mathbf{P}^2$$

is a sextic with four nodes, which we shall denote by $p_1, \ldots, p_4 \in \mathbf{P}^2$:



Let $D_i \in S^2C$, $i = 1, \ldots, 4$, be the nodal divisors, i.e. $p_i = \lambda_{|Kx_0^{-1}|}(D_i)$. If we denote by $H = Kx_0^{-1}$ the hyperplane class on C, then by adjunction in the blow-up at the four nodes we have $K = 3H - D_1 - \cdots - D_4$, and hence

$$x_0 = \mathcal{O}(2H - D_1 - \cdots - D_4),$$

i.e. $|x_0|$ *is cut out by the pencil of conics through the four nodes* $p_1, \ldots, p_4$. In this model it is easy to see the remaining four g^1_4's: *for* $i = 1, \ldots, 4$ *the pencil* $|x_i|$ *is cut out by the lines through* $p_i \in \mathbf{P}^2$. Formally $x_i = \mathcal{O}(H - D_i)$, and in particular we deduce that

$$x_0 \otimes x_i = \mathcal{O}(K - D_i). \tag{28}$$

Consider again the five 6-planes $\mathbf{P}(x_0), \ldots, \mathbf{P}(x_4)$: we have just seen that the bundles E for which $h^0(E) = 5$ are the points Ω^0 in these five spaces. Let us denote these five bundles by $E_i \in \mathbf{P}(x_i)$. We claim that they all coincide:

$$E_0 = E_1 = E_2 = E_3 = E_4 \in \mathcal{SU}_C(2, K). \tag{29}$$

To see this, let us work in $\mathbf{P}(x_0)$. In example 3.4 we have seen that $\Omega^0 = \{E_0\}$ is the vertex of a Veronese cone Ω^1 containing the image of C (i.e. the intersection of $\mathbf{P}(x_0) \subset \mathcal{SU}_C(2, K)$ with $\mathrm{Kum}(J^{g-1}_C)$), and that projection away from the vertex maps C to $\mathbf{P}^2$ via the linear system $|Kx_0^{-1}|$. The image of this map is the 4-nodal sextic just noted, and it follows that the four secant lines $\overline{D_i} \subset \mathbf{P}(x_0)$, $i = 1, \ldots, 4$, where $D_i \in S^2C$ is the i-th nodal divisor as above, all pass through the vertex E_0. By proposition 1.2 together with (28) it follows that

$$\overline{D_i} = \mathbf{P}(x_0) \cap \mathbf{P}(x_i), \qquad i = 1, \ldots, 4.$$

Thus $E_0 \in \overline{D_i} \subset \mathbf{P}(x_i)$ for each i, and therefore coincides with the unique bundle $E_i \in \mathbf{P}(x_i)$ having five sections—so again we have proved (29).

8.3. *Remark.* By the proof of theorem 4.1 we have, for a curve of genus 6, $\mathcal{W}^2 \subset \mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2, K) \subset \mathcal{W}^2 \cup \bigcup_{x \in W^1_4} \mathbf{P}(x)$. We have seen that in each such $\mathbf{P}(x)$, $|x| = g^1_4$, we have $h^0(E) \geq 3$ along a cubic cone, which in particular spans $\mathbf{P}(x)$. It follows that

$$\mathbf{P}\Gamma_{00} \cap \mathcal{SU}_C(2, K) = \mathcal{W}^2 \cup \bigcup_{x \in W^1_4} \mathbf{P}(x),$$

and that this intersection properly contains $\mathcal{W}^2$, since $h^0(E) = 2$ at the generic point of each $\mathbf{P}(x)$.

For C generic, $\bigcup_{x \in W^1_4} \mathbf{P}(x)$ consists of five 6-planes meeting pairwise in ten lines concurrent at the point $\mathcal{W}^4$.

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