POINCARÉ EMBEDDING OF THE DIAGONAL

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ABSTRACT. There is a Poincaré embedding structure on the diagonal $X \to X \times X$ under the conditions: i) X is formed by gluing two compact smooth manifolds along their boundaries using a homotopy equivalence and ii) a *square-root closed* condition is satisfied by the fundamental groupoid of the boundary.

1. INTRODUCTION

Let M be a smooth manifold and $\Delta : M \to M \times M$, the diagonal embedding. Then there is a tubular neighborhood U of ΔM, with a retraction $r : U \to \Delta M = M$, so that the tangent sphere bundle of M is isomorphic to the one whose projection is $r|_{\partial U}$. This provides a Poincaré embedding structure, as described in *1.1* below, on the diagonal, which suggests that, for a Poincaré complex in general, a Poincaré embedding structure on the diagonal might give rise to something like a *tangent fibration*.

In this paper, a *Poincaré pair* is in the sense of C.T.C. Wall ([12]). A CW-complex X is referred to as a *Poincaré complex* if $(X, \emptyset)$ is a Poincaré pair. For any sphere fibration ξ over a topological space, $S\xi$ denotes the total space and $D\xi$, the mapping cylinder of the projection. The following definition is essentially due to Wall (p.113, [13]):

Definition 1.1. Given Poincaré pairs, (A, B), (X, Y), respectively of dimension n and $n + q$, and a continuous map $f : (A, B) \to (X, Y)$, a Poincaré embedding structure on f consists of:

a $(q - 1)$-sphere fibration ν with projection $p : S\nu \to A$,

a Poincaré pair $(X', S\nu \cup_{S\nu|_B} Y')$ of dimension $n + q$,

a homotopy equivalence $h : (D\nu \cup_{S\nu} X', D\nu|_B \cup_{S_{\nu|_B}} Y') \to (X, Y)$,

so that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} (A, B) & \xrightarrow{\quad f \quad} & (X, Y) \\ \downarrow & & h \uparrow \\ (D\nu, D\nu|_B) & \longrightarrow & (D\nu \cup_{S\nu} X', D\nu|_B \cup_{S_{\nu|_B}} Y') \end{array}$$

where the unmarked arrows are the inclusions.

In particular, we shall call ν and $(X', S\nu \cup_{S\nu|_B} Y')$, respectively, the *normal fibration* and the *complement* of the Poincaré embedding structure on f.

One might be tempted to define the *tangent fibration* of a Poincaré complex X as the normal fibration of a Poincaré embedding structure on the diagonal $\Delta : X \to$

$X \times X$. However, it is not yet known whether the diagonal always admits a Poincaré embedding structure. Even if Poincaré embedding structures on Δ do exist, it is not clear whether the normal fibrations of two Poincaré embedding structures are necessarily fiber homotopy equivalent to each other, since the sphere fibrations are not in the stable range (see, for example, p.115, [13]). In fact, the author has shown that this is indeed so, which consists of his Ph.D. thesis together with this paper. (The nature of the argument, however, is significantly different from that of the current paper; the former is more algebraic than geometric while the latter is exactly the opposite. Therefore, it seems appropriate to regard each of the two as an independent paper.)

This paper will concentrate on the problem of whether the diagonal $\Delta : X \to X \times X$ admits a Poincaré embedding structure.

In fact, there are results such as [5], [7], [8] which assert that: *given any continuous map $f : Y \to X$ between finite Poincaré complexes of formal dimension i and n, respectively, with $n \geq 2i + 1$, there exists a Poincaré embedding structure on f, in particular, if X is simply connected.* Unfortunately they do not apply to our situation, because of their assumptions on the fundamental group and on the dimension.

We will consider the case when the Poincaré complex X under study is *formed by gluing two smooth manifolds along their boundaries using a homotopy equivalence* (see *4.1*). Also we shall demand an extra condition:

Definition 1.2. Let $H < G$ be groupoids. We will say H satisfies the *square-root closed condition* in G if '$a^2 \in H$' implies '$a \in H$' for any $a \in G$. In particular, we will say a groupoid G satisfies the *diagonal square-root closed condition* if '$a^2 = b^2$' implies '$a = b$' for any $a, b \in G$, that is, if the diagonal subgroupoid $\Delta G < G \times G$ satisfies the square-root closed condition.

The square-root closed condition above played an important role in the study of the splitting problem by Cappell ([2]).

The following is the main theorem of the paper.

Theorem 1.3. *Let (X, Y) be a Poincaré pair, $\dim X \geq 4$, such that $(X, Y) = (M_0 \cup_f M_1, \partial^- M_0 \sqcup \partial^+ M_1)$, where $(M_i; \partial^- M_i, \partial^+ M_i; \emptyset)$, $i = 0, 1$, are manifold triads and $f : \partial^+ M_0 \to \partial^- M_1$ is a homotopy equivalence. Assume the fundamental groupoid of $\partial^+ M_0$ satisfies the diagonal square-root closed condition. Then there exists a Poincaré embedding structure on the diagonal $\Delta : (X, Y) \to (X, Y) \times (X, Y)$.*

Here the square-root closed condition is needed *not because of the Cappell's splitting theorem (see 5.1 below) but because of an embedding problem*:

Lemma 1.4. *Assume that M^n, $K^{2n-1} \subset N^{2n}$, $n \geq 3$, are connected compact smooth manifolds and that $\partial K = K \cap \partial N$, K is transversal to ∂N and the normal bundle K in N is trivial. Let $f : M \to N$ be a continuous map such that $f|_{\partial M}$ is an embedding into $\partial K \subset N$ and f factors through $f' : M \to K$ by the inclusion $K \hookrightarrow N$ up to homotopy rel ∂M. Then, if $f'_* \pi_1(M) < \pi_1(K)$ satisfies the square-root closed condition, f is homotopic to an embedding rel ∂M.*

As an immediate consequence, we have:

Corollary 1.5. *Let M^n, N^{2n}, $n \geq 3$, be connected compact smooth manifolds and $F : M \times I \to N$ be a homotopy such that F_0 and F_1 are embeddings. Assume*

$$(F_0)_* \pi_1(M) < \pi_1(N) \ \textit{satisfies the square-root closed condition.}$$

Then, the map $\bar{F} : M \times I \to N \times [-1,1] \times I$, defined by $\bar{F}(x,t) = (F(x,t), 0, t)$, is homotopic to a concordance rel $M \times \{0, 1\}$.

Here a concordance means simply an embedding.

At the moment, it is far from being clear how essential the square-root closed condition is in *1.3* above. For example, one may alter the manifolds M_0, M_1 and $\pi_1(\partial^+ M_0) = \pi_1(\partial^- M_1)$ by trading handles of index at most two along $\partial^+ M_0$ (or $\partial^- M_1$) preserving the homotopy type of the Poincaré pair: One may assume f sends a regular neighborhood of any 1-complex in $\partial^+ M_0$ to such in $\partial^- M_1$ by applying a splitting theorem,[1] and f can be assumed as a diffeomorphism when restricted to such a regular neighborhood since the fundamental group of G/O is trivial.[2]

Nevertheless this does not mean that one may alter $\partial^+ M_0$ so that its fundamental group satisfies the diagonal square-root closed condition. The fact that such as the square-root closed condition is needed in *1.3* makes it seem more plausible that: *there might be a finite Poincaré complex for which the diagonal does not admit any Poincaré embedding.*

The author would like to thank his advisor, Professor Frank Connolly at Notre Dame, for his help in writing this paper. Among other things, his suggestions have brought a substantial improvement to the style of the paper. The author is more than grateful also for the encouragement and the inspiring conversations Professor Connolly has provided throughout his graduate years.

2. Gluing spaces

We start by introducing the notion of *gluing*:

Definition 2.1. Let (X, A) be a pair and Y, a space. If $f : A \to Y$ is a continuous map, we define $X \cup_f Y$ as the quotient space of the disjoint union $X \sqcup Y$ with respect to the equivalence relation generated by $a \sim f(a)$. We will often refer to $X \cup_f Y$ as the space formed by *gluing X to Y using f*.

The following is well-known (for example, see [9]) and we omit the proof.

Lemma 2.2. *Let (X, A), (Y, B) be NDR-pairs and V, W, spaces. Assume the following diagram commutes up to homotopy:*

$$
\begin{array}{ccccc}
X & \xleftarrow{\iota_A} & A & \xrightarrow{f} & V \\
\downarrow h_1 & & \downarrow h_0 & & \downarrow h \\
Y & \xleftarrow{\iota_B} & B & \xrightarrow{g} & W
\end{array}
$$

in which vertical arrows are homotopy equivalences and ι_A, ι_B are the inclusions. Then $X \cup_f V$ and $Y \cup_g W$ are homotopy equivalent to each other.

The following is also well known (for example, see *2.5, 6*, [4], also *2.1*, [12]):

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[1] Wall's result will be enough if we assume f is a simple homotopy equivalence and in general one may use Cappell's result.

[2] The author would like to thank the anonymous referee for his kind explanation of this technique of altering $\pi_1(\partial^+ M_0)$.

Lemma 2.3. *Let $(X_i, A_i^- \cup A_i^+)$, $i = 0, 1$, be n-dimensional Poincaré pairs. Furthermore, writing $B_i = A_i^- \cap A_i^+$, assume $(A_i^\pm, B_i)$ are Poincaré pairs. If $f : (A_0^+, B_0) \to (A_1^-, B_1)$ is a homotopy equivalence, then $(X_0 \cup_f X_1, q(A_0^- \sqcup A_1^+))$ is an n-dimensional Poincaré pair in which q is the quotient map.*

Let A, A' be respectively subspaces of X and X'. We will say $f : X \to X'$ is a *homotopy equivalence rel A* if (a) f restricts to a homeomorphism $f|_A : A \to A'$, (b) there is a homotopy inverse $g : X' \to X$ such that $g|_{A'} : A' \to A$ is the inverse of $f|_A$, and, furthermore, (c) there are homotopies H from gf to 1_X and H' from fg to $1_{X'}$ such that $H(\cdot, t)$ and $H'(\cdot, t)$ are the identity maps on A and A' respectively for any $t \in I$.

Let X, A be spaces such that $A \subset X$. We will call a *closed* neighborhood N of A in X a *collar* of A if (N, A) and $(A \times I, A \times \{0\})$ are homeomorphic to each other as pairs.

The following will be useful for the later discussions, in particular, because of its 'relative' clause. We omit the proof.

Proposition 2.4. *Let (X, A) be a pair and Y, a space. Assume A has a collar in X identified with $A \times I$, where A is identified with $A \times \{0\}$. Assume f, g are continuous maps from A to Y homotopic to each other. Then, $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent to each other rel $(X - A \times [0, 1)) \sqcup Y$.*

The following and the subsequent remark will be needed for the proof of *Claim 4.A* below.

Lemma 2.5. *Let $X = A \cup (B \times I) \cup C$ be a space for which $A \cap (B \times I) = B \times \{1\}$ and $(A \cup (B \times I)) \cap C = B \times \{0\}$. Let Y be another space. Assume $f, g : X \to Y$ are continuous maps homotopic to each other rel $A \cup (B \times I)$. Regard f, g as maps from $X \times \{0\}$ up to the identification $X \times \{0\} = X$. Then $(X \times I) \cup_f Y$ is homotopy equivalent to $(X \times I) \cup_g Y$ rel $X \times \{1\} \cup q((A \times I) \sqcup Y)$, where q denotes the quotient map.*

We only sketch the proof of this lemma by providing:

A construction of a homotopy equivalence $\alpha : (X \times I) \cup_f Y \to (X \times I) \cup_g Y$: Assume a homotopy $H : X \times I \to Y$ is given so that $H_0 = f$, $H_1 = g$, $H(x, t) \equiv f(x, t) = g(x, t)$, for all $x \in A \cup (B \times I)$, $t \in I$.

Define a continuous map $h : I \times I \to I \times I$ as follows: $h(s, t) = (s, 0)$ if $t \le \frac{1}{2} - \frac{s}{2}$ and $h(s, t) = (s, \frac{2t+s-1}{1+s})$ if $t \ge \frac{1}{2} - \frac{s}{2}$. Then h is well-defined. Also note that h restricted to $(I \times \{0\}) \cup (\{1\} \times I) \cup (I \times \{1\})$ is the identity.

Define $\alpha : (X \times I) \cup_f Y \to (X \times I) \cup_g Y$ as follows: $\alpha[y] = [y]$ for any $y \in Y$, $\alpha[c, t] = [H(c, 2t)]$ for any $(c, t) \in C \times [0, \frac{1}{2}]$, $\alpha[c, t] = [c, 2t - 1]$ for any $(c, t) \in C \times [\frac{1}{2}, 1]$, $\alpha[b, s, t] = [b, h(s, t)]$ for any $(b, s, t) \in (B \times I) \times I$, and finally $\alpha[a, t] = [a, t]$ for any $(a, t) \in A \times I$.

Then α is well-defined. In particular, for any $(b, 0) \in (B \times I) \cap C$, we have that

$$[b, h(0, t)] = [b, 0, 0] = [f(b, 0)] = [g(b, 0)] = [H((b, 0), 2t)] \ \text{if } t \in [0, \frac{1}{2}],$$

$$[b, h(0, t)] = [b, 0, 2t - 1] = [(b, 0), 2t - 1] \ \text{if } t \in [\frac{1}{2}, 1].$$

Also note that α restricted to $q(X \times \{1\} \sqcup (A \times I) \sqcup Y) \subset (X \times I) \cup_f Y$ is the homeomorphism induced by the identity.

Then the proof of *2.5* can be completed by defining (i) an inverse $\hat{\alpha} : (X \times I) \cup_g Y \to (X \times I) \cup_f Y$ in the same manner as α, replacing H with $\bar{H}$, which is defined by $\bar{H}(x, t) = H(x, 1 - t)$, and using the same h; and (ii) the homotopies between $\hat{\alpha}\alpha$, $\alpha\hat{\alpha}$ and the identities which respectively fix the images of $X \times \{1\} \sqcup (A \times I) \sqcup Y$ by the quotient maps.

Remark 1. In particular, we will use the fact that, in the construction above, we have:

$$\alpha[c, t] = [H(c, 2t)] \text{ for any } (c, t) \in C \times [0, \frac{1}{2}],$$

$$\alpha[c, t] = [c, 2t - 1] \text{ for any } (c, t) \in C \times [\frac{1}{2}, 1].$$

3. Proof of 1.4

First of all, we need to recall a result by Haefliger (*4.1 Théorème*, [3]):

Theorem 3.1. [A. Haefliger] *Assume V, M are smooth manifolds of dimension n, m, respectively, and assume V is compact. Suppose $2m \geq 3(n+1)$. Let $f : V \to M$ be a continuous map such that f is an embedding in a neighborhood of ∂V and $f(\partial V) \cap f(V - \partial V) = \emptyset$. Assume $\pi_i(f) = 0$ for $0 \leq i \leq 2n - m + 1$. Then f is homotopic to an embedding relative to a neighborhood of ∂V.*

Proof of 1.4. Since $n \geq 3$, by a result of Whitney ([15]), up to homotopy rel ∂M, we may assume $f' : M \to K$ is a smooth map such that

i) f' is self-transverse,

ii) f' has no triple point, and

iii) f' is an embedding at the complement of the union of some finite disjoint circles embedded in M.

To be more precise, according to [15], there are three ways in which f' may fail to be an embedding at those circles:

Type A: f' restricts to $f'|_C : C \to l$, where C is an embedded circle in M and l is a line segment embedded in N' so that each interior point of l is a double point and $df'_x : T_xM \to T_{f'(x)}N$ is injective for all $x \in f'^{-1}Int\, l$ and $df'_x(T_xC) = 0$ if $f'(x) \in \partial l$.

Type B: f' restricts to $f'|_{\tilde{C}} : \tilde{C} \to C$, which is a double covering.

Type C: f' restricts to $f'|_{C \sqcup C'} : C \sqcup C' \to \bar{C}$, a double covering, where C and C' are embedded disjoint circles in M and $\bar{C}$ is an embedded circle in N.

We start by arranging that $f^{-1}\partial K = \partial M$, which can be done using some collar neighborhoods of ∂M and ∂K. Then, step by step, we will replace f, up to homotopy rel ∂M, with another which has fewer circles on which f fails to embed:

Step 1. Let C, l be as in *Type A*. Take a regular neighborhood $U \subset N$ of l so that U is disjoint from those other circles or the other line segments in N as described above, and also from ∂N.

Note that U is diffeomorphic to D^{2n} and, in particular, simply connected.

Let V_1 be the path component of $f^{-1}U$ containing C. And let V_i, $i = 2, 3, \cdots, k$, be the path components other than V_1. Then f restricted to $\bigsqcup_{i=2}^k V_i$ is an embedding. Remove $f(V_i)$ from U for each $i = 2, 3, \cdots, k$. Denote the resulting smooth neighborhood of l by U'. Note that the fundamental group of U' is the same as U's, since $\dim U - \dim V_i \geq 3$. Then the restriction of f, say, $f_{U'} : V_1 \to U'$, is a continuous map which is an embedding on a neighborhood of ∂V_1. Since U' is
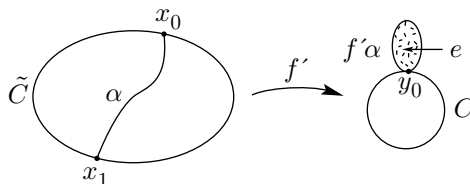
FIGURE 1

simply connected, $f_{U'}$ is homotopic to an embedding, relative to a neighborhood of ∂V_1, by *3.1*.

Thus we may assume f has one fewer circle of Type A.

Note that, even after performing this processes, all of those line segments or circles in N as described in Type A, B, C above lie in $K \subset N$. Furthermore, we can still find a regular neighborhood in N of a line segment, if any remain, of Type A which is disjoint from other lines or circles as described in Types A, B, C and, in particular, *disjoint from the closure of $fM - K$ in N and also from ∂N*

Therefore, we may repeatedly apply the above process to obtain a new f which does not have any circle of Type A.

Step 2. Now let $\tilde{C} \subset M$, $C \subset K$ as in *Type B*. Choose $y_0 \in C$ and write $f^{-1}y_0 = \{x_0, x_1\} \subset M$. Then, first of all, we observe:

Claim. *We may choose a path* $\alpha : [0,1] \to M$, $\alpha(0) = x_0$, $\alpha(1) = x_1$, *so that* $[f'\alpha] \in \pi_1(K, y_0)$ *is the identity.*

Proof. Let γ be the path from x_1 to x_0 along one of the semicircles of $\tilde{C}$. Then $f'\gamma$ is a loop in K based on y_0.

In particular, $[f'\gamma] \in f'_*\pi_1(M) < \pi_1(K)$ *since* $f'_*\pi_1(M) < \pi_1(K)$ *satisfies the square-root closed condition.* Therefore, we choose α as a path from x_0 to x_1 so that $[\gamma * \alpha] \in \pi_1(M, x_0)$ is such that $f'_*[\gamma * \alpha] \in \pi_1(K, y_0)$ is $[f'\gamma]$ itself.

We may assume $\alpha : [0,1] \to M$ is smoothly embedded and that Im α is disjoint from circles of Types B, C other than $\tilde{C}$. Furthermore, arrange so that $\alpha_* \frac{d}{dt}|_{t=i} \notin T_{x_i}\tilde{C}$, for $i = 0,1$.

Note that, even if we may not assume $f(M) \subset K$ any longer, still we may choose a smooth embedding $e : D^2 \to K$, and furthermore we may choose it so that

$$\text{Im } e \cap \text{Im } f = \text{Im } f\alpha \quad \text{and} \quad e_* T_{x_i} D^2 \cap T_{y_0}C = \{0\}, \qquad i = 0,1.$$

See Figure 1.

Then we construct a regular neighborhood U of $C \cup D^2$ in N so that it is disjoint from other circles or lines in N as described in Types B, C and also disjoint from the closure of $fM - K$ in N and from ∂N.

As before, let V_1 be the path component of $f^{-1}U$ containing C and let V_i, $i = 2,3,\cdots,k$, be the path components other than V_1. Remove fV_i's, $i > 1$, from U and call the resulting space U'. Then clearly the restriction of f, say, $f_{U'} : V_1 \to U'$, induces a surjection between the fundamental groups. Therefore, we may apply *3.1*.

As in *Step 1*, we may repeat this process to remove all the circles of Type *B*.

Step 3. Assume C, C', $\bar{C}$ are as in Type C. Choose a Riemannian metric on M and take disjoint normal neighborhoods U and U' of C and C', respectively, whose

points are identified with the normal vectors to $C, C' \subset M$ with length $\leq \delta$ for some $\delta > 0$ by the exponential map. By choosing δ small enough we may assume fU, fU' are disjoint from other circles of Type C and also from the closure of $fM - K$ in N.

Now identify a neighborhood of K with $K \times [-1, 1]$ using a diffeomorphism, which is possible by the assumption. Then there is a positive number ϵ so that both $fU \times [-\epsilon, \epsilon]$, $fU' \times [-\epsilon, \epsilon] \subset K \times [-1, 1]$ are disjoint from the closure of $fM - K$ in N.

Let $\varphi : [0, \delta] \to [0, \epsilon]$ be a decreasing smooth function whose value is constantly ϵ in a neighborhood of 0 and constantly 0 in a neighborhood of δ. Then we define $g : M \to N$ by the rule:

$$g(v) = (f(v), \varphi(|v|)) \text{ for any } v \in U,$$
$$g(v) = (f(v), -\varphi(|v|)) \text{ for any } v \in U',$$

and $g \equiv f$ elsewhere. Clearly g is homotopic to f. We take this g as our new f.

We repeat this process until we exhaust all the circles of Type C to make f an embedding.

4. Constructions

In this section, we prove *1.3*. Also other results regarding the existence of Poincaré embedding structures (in particular, *4.8, 9*) are obtained. The main tools are provided by a splitting theorem by S.E. Cappell (*5.1* below), an embedding theorem by A. Haefliger (*3.1* above), and *1.5* above.

Definition 4.1. Let $(X, \partial X)$ be an n-dimensional Poincaré pair. We will say $(X, \partial X)$ admits a *decomposition by two smooth manifolds* if

i) there are smooth manifold triads $(P; P_0, P_1; \partial P_0)$, $(Q; Q_0, Q_1; \partial Q_0)$ together with a homotopy equivalence $\varphi : (P_0, \partial P_0) \to (Q_0, \partial Q_0)$ so that

ii) $(X, \partial X)$ is homotopy equivalent to $(P \cup_\varphi Q, q(P_1 \sqcup Q_1))$, where q is the quotient map (note that $\partial P_0 = \partial P_1$ and $\partial Q_0 = \partial Q_1$).

In the above, P, Q are allowed to have more than one path component. By *2.3*, any pair which admits a decomposition by two smooth manifolds is a Poincaré pair. In fact, it is a *patch space* as defined by L. Jones ([6]).

We will make use of a more general form of the idea '*regular neighborhood*' and a slightly altered notion of the usual *splitting* ([2] or [13]):

Definition 4.2. *i)* Let $B \subset N$ be compact smooth manifolds such that $B \subset \text{Int } N$. $U_B \subset \text{Int } N$ is an *h-regular neighborhood of B* if $(U_B, \partial U_B)$ is a compact smooth neighborhood of B such that B is a strong deformation retract of U_B and the inclusion $\partial U_B \hookrightarrow U_B$ has a spherical homotopy fiber.

ii) Let $(M, \partial M)$, $(N, \partial N)$ be compact smooth manifolds and $f : (M, \partial M) \to (N, \partial N)$, a homotopy equivalence. Also let $B \subset N$ be a submanifold which lies in Int N. We will say that f is *h-split along B* if there is an h-regular neighborhood $U_B \subset \text{Int } N$ of B so that, writing $f^{-1}U_B \equiv U_A$, $U_A \subset M$ is a compact smooth manifold and the restrictions of f to

$$(U_A, \partial U_A) \to (U_B, \partial U_B), \quad (M - \text{Int } U_A, \partial M \sqcup \partial U_A) \to (N - \text{Int } U_B, \partial N \sqcup \partial U_B)$$

are homotopy equivalences. We shall refer to U_B as the *h-regular neighborhood of B associated* to the h-splitting of f.
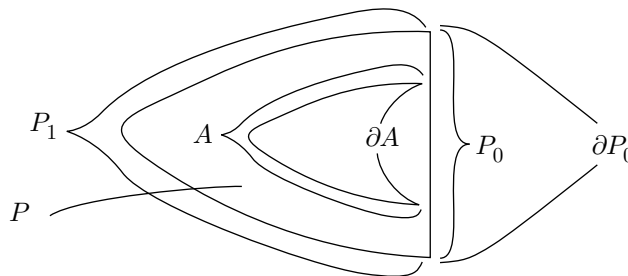
FIGURE 2

iii) If f is homotopic rel ∂M to a map which is h-split along B, then we will say f is *h-splittable* along B.

Standing Hypotheses. (See Figure 2.) Throughout the following, from *4.3* to *4.9*,

(*a*) $(P; P_0, P_1; \partial P_0)$, $(Q; Q_0, Q_1; \partial Q_0)$ are fixed compact smooth manifold triads.

(*b*) $\varphi : (P_0, \partial P_0) \to (Q_0, \partial Q_0)$ is a fixed homotopy equivalence. Also,

(*c*) $(A, \partial A)$, $(B, \partial B)$ are fixed submanifolds of (P, P_0), (Q, Q_0) such that $A \subset P - P_1$, $B \subset Q - Q_1$. (Note that $\partial A \subset \text{Int } P_0$ and $\partial B \subset \text{Int } Q_0$.)

(*d*) Furthermore, we assume that φ restricts to a homotopy equivalence $\varphi' : \partial A \to \partial B$.

We will write $(X, \partial X) \equiv (P \cup_\varphi Q, q(P_1 \sqcup Q_1))$ and $Y \equiv A \cup_{\varphi'} B$, where q is the quotient map. We will denote the canonical *injection* $Y \to (X, \partial X)$ by ι.

Goal. We will try to construct a Poincaré embedding structure (see *1.1*) on $\iota : (Y, \emptyset) \to (X, \partial X)$.

Construction 4.3. As a first step, we consider the case in which

$$\varphi : (P_0, \partial P_0) \to (Q_0, \partial Q_0)$$

is h-split along $\partial B \subset Q_0$. Under this assumption, we exhibit a Poincaré embedding structure on $\iota : Y \to (X, \partial X)$ as follows.

Let $U_{\partial B}$ be an h-regular neighborhood of ∂B associated to the h-splitting of φ, and write $U_{\partial A} \equiv \varphi^{-1} U_{\partial B}$. Extend them to h-regular neighborhoods $(U_A, U_{\partial A}) \subset (P, P_0)$ and $(U_B, U_{\partial B}) \subset (Q, Q_0)$ respectively of $(A, \partial A)$ and $(B, \partial B)$ so that $U_A \cap P_0 = U_{\partial A}$, $U_B \cap Q_0 = U_{\partial B}$ and $U_A \cap P_1 = U_B \cap Q_1 = \emptyset$.

Note that φ restricts to the homotopy equivalences:

$$\partial U_{\partial A} \to \partial U_{\partial B}, \quad U_{\partial A} \to U_{\partial B}, \quad (P_0 - \text{Int } U_{\partial A}, \partial P_0) \to (Q_0 - \text{Int } U_{\partial B}, \partial Q_0).$$

Write

$$\partial_1 U_A \equiv \partial U_A - \text{Int } U_{\partial A}, \quad \partial_1 U_B \equiv \partial U_B - \text{Int } U_{\partial B},$$
$$(U_Y, \partial U_Y) \equiv (U_A \cup_{\varphi|_{U_{\partial A}}} U_B, \partial_1 U_A \cup_{\varphi|_{\partial U_{\partial A}}} \partial_1 U_B),$$
$$K \equiv (P - \text{int } U_A) \cup_{\varphi|_{P_0 - \text{Int } U_{\partial A}}} (Q - \text{int } U_B),$$

in which Int U_A and Int U_B respectively denote the set of interior points of U_A and U_B as subspaces of P and Q.

Clearly, U_Y, $K \subset X$ define a decomposition of $(X, \partial X)$. In particular, note that $U_Y \cap K = \partial U_Y$.

Both $\partial_1 U_A \hookrightarrow U_A$, $\partial_1 U_B \hookrightarrow U_B$ have the homotopy fiber S^{k-1}, $k = \dim P - \dim A$. Furthermore, $\partial U_{\partial A} \hookrightarrow U_{\partial A}$, $\partial U_{\partial A} \hookrightarrow U_{\partial A}$ also have the homotopy fiber S^{k-1} and φ induces a homotopy equivalence $(U_{\partial A}, \partial U_{\partial A}) \to (U_{\partial B}, \partial U_{\partial B})$. Therefore we conclude that $\partial U_Y \hookrightarrow U_Y$ has the homotopy fiber S^{k-1} (see, for example, [10]).

Let ξ be the sphere fibration obtained by restricting the base space to $Y \subset U_Y$ from the Serre fibration of the inclusion $\partial U_Y \hookrightarrow U_Y$. Then $(U_Y, \partial U_Y)$ is naturally homotopic to $(D\xi, S\xi)$ by a homotopy equivalence which is the identity map when restricted to $Y \to Y \subset D\xi$. By replacing $(U_Y, \partial U_Y)$ with $(D\xi, S\xi)$, we obtain a Poincaré embedding structure on $\iota : Y \to (X, \partial X)$.

There is an obvious *equivalence relation between Poincaré embedding structures*: Let $f : X \to Y$ be a continuous map between Poincaré complexes, and let two Poincaré embedding structures $D\nu \cup_{S\nu} W$, $D\nu' \cup_{S\nu'} W'$ be given. Assume there is a homotopy equivalence $D\nu \cup_{S\nu} W \to D\nu' \cup_{S\nu'} W'$ which respects the decomposition and restricts to the identity map $X \to X$, understanding $X \subset D\nu$, $X \subset D\nu'$, with the associated homotopies also respecting the decompositions and being fixed at X. Then we say the Poincaré embedding structures are equivalent to each other.

It is not difficult to see that the Poincaré embedding structure of *4.3* does not depend on the choice of the regular neighborhoods U_A, U_B, once the embeddings $(A, \partial A) \hookrightarrow (P, P_0)$ and $(B, \partial B) \hookrightarrow (Q, Q_0)$ are given, up to the equivalence between Poincaré embedding structures.

Definition 4.4. We will refer to the Poincaré embedding structure of *4.3*, which is unique up to the equivalence, as *the Poincaré embedding structure determined by the embeddings $(A, \partial A) \hookrightarrow (P, P_0)$ and $(B, \partial B) \hookrightarrow (Q, Q_0)$*.

Now we summarize:

Proposition 4.5. *If φ is h-split along ∂B, then there is a unique Poincaré embedding structure on $\iota : Y \to (X, \partial X)$ determined by the embeddings $(A, \partial A) \hookrightarrow (P, P_0)$ and $(B, \partial B) \hookrightarrow (Q, Q_0)$.*

We introduce another construction of Poincaré embedding structure on $\iota : Y \to (X, \partial X)$ under a more general condition:

Construction 4.6. Assume there are maps: $H : P_0 \times (I, \{0, 1\}) \to Q_0 \times (I, \{0, 1\})$, and $F : \partial A \times (I, \{0, 1\}) \to P_0 \times (I, \{0, 1\})$, $G : \partial B \times (I, \{0, 1\}) \to Q_0 \times (I, \{0, 1\})$, which satisfy the following conditions:

i) H_0 is h-split along ∂B and $H_1 = \varphi$. Also $H(x, t) = (\varphi(x), t)$ for any $x \in \partial P_0$.

ii) F_0 is an embedding into Int $H_0^{-1} U_{\partial B} \subset P_0$, in which $U_{\partial B} \subset Q_0$ is an h-regular neighborhood of ∂B associated to the h-splitting of H_0. F_1 is the inclusion $\iota_{\partial A} : \partial A \hookrightarrow P_0$.

iii) $H_0 F_0 = \varphi \iota_{\partial A} : \partial A \to Q_0$.

iv) Both G_0 and G_1 are the inclusion $\partial B \hookrightarrow Q_0$.

v) The following diagram commutes up to a homotopy rel $\partial A \times \{0, 1\}$:

$$
\begin{array}{ccc}
\partial A \times I & \xrightarrow{F} & P_0 \times I \\
\downarrow{\scriptstyle \varphi' \times 1} & & \downarrow{\scriptstyle H} \\
\partial B \times I & \xrightarrow{G} & Q_0 \times I
\end{array} .
$$

vi) Finally, F, G are *concordances*.

Note. $H_s : P_0 \to Q_0, s = 0, 1$, denotes the map defined by $H(x, s) = (H_s(x), s)$. The maps $F_s, G_s, s = 0, 1$, should be understood likewise.

Under these conditions, we will define a Poincaré embedding structure on $\iota : Y \to (X, \partial X)$ which is uniquely determined by F, G and also by H. This is done as follows:

Choose collars of $\partial A \subset A$ and $\partial B \subset B$ respectively so that $A \cong \partial A \times I \cup_{A \times \{1\}} A'$ and $B \cong \partial B \times I \cup_{B \times \{1\}} B'$ for some submanifolds, $A' \subset A, B' \subset B$. Furthermore, we choose collars of $P_0 \subset P$ and $Q_0 \subset Q$ so that $P \cong P_0 \times I \cup_{P_0 \times \{1\}} P'$, $Q \cong Q_0 \times I \cup_{Q_0 \times \{1\}} Q'$ and $A \cap P' = A'$, $B \cap Q' = B'$ and the inclusions, $A \hookrightarrow P$, $B \hookrightarrow Q$ restrict to $\partial A \times I \hookrightarrow P_0 \times I$, $\partial B \times I \hookrightarrow Q_0 \times I$ respectively so that they are just $\iota_{\partial A} \times 1$, $\iota_{\partial B} \times 1$, etc.

Let $\tilde{F} : (A, \partial A) \to (P, P_0)$ be given by $\tilde{F}(y) = y$ if $y \in A'$ and $\tilde{F}(a, t) = F(a, t)$ if $(a, t) \in \partial A \times I$. Similarly, let $\tilde{G} : (B, \partial B) \to (Q, Q_0)$ be defined. Note that $\tilde{F}, \tilde{G}$ are well defined, and we may assume they are smooth embeddings.

Write $(\bar{X}, \partial \bar{X}) \equiv (P \cup_{H_0} Q, q(P_1 \sqcup Q_1))$. Note that $\partial \bar{X} = \partial X$. Then, the map $\iota_1 : Y \to (\bar{X}, \partial X)$ is well defined by the rule: $\iota_1[y] = [\tilde{F}(y)]$ for any $y \in A$, $\iota_1[y] = [\tilde{G}(y)]$ for any $y \in B$ (note that $\iota_{\partial B}\varphi' = H_0 F_0$ by iii) above).

By _4.5_, there is a unique Poincaré embedding structure on $\iota_1 : Y \to (\bar{X}, \partial \bar{X})$ determined by $\tilde{F}, \tilde{G}$.

The construction is complete except for the following:

Claim 4.A. _There is a homotopy equivalence rel $\partial \bar{X}$,_

$$\alpha : (\bar{X}, \partial \bar{X}) \to (X, \partial X),$$

so that $\alpha\iota_1$ is homotopic to ι.

The proof exploits the condition v) in the above, and it is provided in Section 5.

In fact, we can find maps H, F, G satisfying all the conditions of _4.6_, except for the requirement vi), simply by imposing a mild condition on the dimensions:

Lemma 4.7. _Assume_ $\dim P - \dim A \geq 3$ _and_ $2 \dim P \geq 3 \dim A + 2$. _Then, there are maps_ $H : P_0 \times (I, \{0, 1\}) \to Q_0 \times (I, \{0, 1\})$, _and_ $F : \partial A \times (I, \{0, 1\}) \to P_0 \times (I, \{0, 1\})$, $G : \partial B \times (I, \{0, 1\}) \to Q_0 \times (I, \{0, 1\})$, _which satisfy the following conditions:_

_i)_ H_0 _is h-split along_ ∂B _and_ $H_1 = \varphi$. _Also_ $H(x, t) = (\varphi(x), t)$ _for any_ $x \in \partial P_0$.

_ii)_ F_0 _is an embedding into_ $\mathrm{Int}\, H_0^{-1} U_{\partial B} \subset P_0$, _in which_ $U_{\partial B} \subset Q_0$ _is an h-regular neighborhood of_ ∂B _associated to the h-splitting of_ H_0. F_1 _is the inclusion_ $\iota_{\partial A} : \partial A \hookrightarrow P_0$.

iii) $H_0 F_0 = \varphi \iota_{\partial A} : \partial A \to Q_0$.

iv) Both G_0, G_1 _are the inclusion_ $\partial B \hookrightarrow Q_0$.

v) The following diagram commutes up to a homotopy rel $\partial A \times \{0, 1\}$:

$$
\begin{array}{ccc}
\partial A \times I & \xrightarrow{F} & P_0 \times I \\
\downarrow{\scriptstyle \varphi' \times 1} & & \downarrow{\scriptstyle H} \\
\partial B \times I & \xrightarrow{G} & Q_0 \times I
\end{array}
$$

The proof of _4.7_ is provided in the next section.

Note that F and G are homotopic to concordances rel $\partial A \times \{0, 1\}$ and rel $\partial B \times \{0, 1\}$, respectively, if we assume $\dim P \geq 2 \dim A + 1$ in addition to the dimensional conditions of _4.7_. Therefore, we have

Theorem 4.8. *Let* $\dim P \geq 2 \dim A + 1$ *and* $\dim P \geq 6$. *Then there is a Poincaré embedding structure on* $\iota : Y \to (X, \partial X)$.

Now, by *1.5*, we have

Proposition 4.9. *Assume* $\dim P = 2 \dim A$, $\dim P - \dim A \geq 4$. *Let* $T \subset Int\ Q_0$ *be a codimension* 1 *compact submanifold without boundary with a trivial normal bundle. Assume* φ *splits along* T. *Assume also* $\partial A \subset \varphi^{-1}T$ *and* $\partial B \subset T$, *and each path component of* T *contains at most one path component of* ∂B. *Furthermore, assume*

$$\text{Im}\ (\pi_1(\partial B) \to \pi_1(T)) < \pi_1(T)\ \textit{satisfies the square root closed condition}$$

at each path component. Then, F, G *of* 4.7 *can be chosen as concordances. In particular, there exists a Poincaré embedding structure on* $\iota : Y \to (X, \partial X)$.

Note. In the above, '*splitting*' is in the sense of Cappell ([2]). Say, let Y be a connected closed smooth manifold, $X \subset Y$ a connected closed submanifold of codimension 1 and Y_0, Y_1, submanifolds of Y such that $Y = Y_0 \cup Y_1$ and $Y_0 \cap Y_1 = X$. Then 'a homotopy equivalence $f : W \to Y$ from a closed manifold W splits along X' means '$f^{-1}X, f^{-1}Y_i, i = 0, 1$, are manifolds, and the restrictions of f to $f^{-1}Y_i \to Y_i, i = 0, 1$, and to $f^{-1}X \to X$ are all homotopy equivalences.'

Proof of 4.9. Note that $\varphi^{-1}T \subset P_0$ and $T \subset Q_0$ respectively have neighborhoods diffeomorphic to $\varphi^{-1}T \times [-1, 1]$ and $T \times [-1, 1]$.

Apply *Lemma 4.7* first to the inclusions

$$\partial A \hookrightarrow \varphi^{-1}T, \quad \partial B \hookrightarrow T$$

and the restriction of φ to $\varphi^{-1}T \to T$. Thus, we may arrange that

$$F : \partial A \times I \to P_0, \quad G : \partial B \times I \to Q_0$$

factor respectively through the inclusions

$$\varphi^{-1}T \hookrightarrow P_0, \quad T \hookrightarrow Q_0\ .$$

However, the maps

$$\bar{F} : \partial A \times I \to \varphi^{-1}T \times [-1, 1] \times I, \quad \bar{G} : \partial B \times I \to T \times [-1, 1] \times I$$

respectively defined by

$$\bar{F}(a, t) = (F(a, t), 0, t),\ \bar{G}(b, t) = (G(b, t), 0, t)$$

are homotopic to an embedding respectively rel $\partial A \times \{0, 1\}$ and rel $\partial B \times \{0, 1\}$ by applying *1.5*, which provides the concordances needed for us to apply *4.6*.

This completes the proof of *4.9*.

Next we prove the main theorem, *1.3*.

Proof of 1.3. For simplicity, we assume $\partial^- M_0 = \partial^+ M_1 = \emptyset$. Then, note that we have $\partial^+ M_0 = \partial M_0$ and $\partial^- M_1 = \partial M_1$, and also $X = M_0 \cup_f M_1$.

For each $i = 0, 1$, ∂M_i has a collar:

$$M_i \cong M_i' \cup_{\partial M_i \times \{1\}} (\partial M_i \times I)$$

for some compact smooth manifold $M_i' \subset M_i$. Therefore, presuming a fixed diffeomorphism, we shall identify M_i with $M_i' \cup (\partial M_i \times I)$.

$$\partial_1(M_0)^2 \qquad (\,M_0)^2 \qquad \begin{matrix} \partial_0(M_1)^2 \\ \partial_0(M_0)^2 \end{matrix} \qquad (\,M_1)^2 \qquad \partial_1(M_1)^2$$
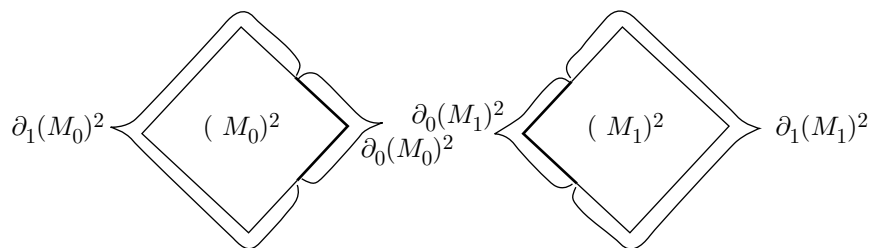
FIGURE 3

We shall denote $A \times A$ by A^2 for any space A. For example, $(M_i)^2 = M_i \times M_i$, $(\partial M_i)^2 = \partial M_i \times \partial M_i$, etc.. And let $\partial(M_i)^2$ denote the subspace

$$\partial M_i \times M_i \cup M_i \times \partial M_i$$

of $(M_i)^2$. We will also write

$$\partial(M_i)^2 = (\partial M_i \times \{0\}) \times M_i \cup M_i \times (\partial M_i \times \{0\}).$$

We define $\partial_0(M_i)^2 \subset \partial(M_i)^2$, $i = 0, 1$, by

$$\partial_0(M_i)^2 = (\partial M_i \times \{0\}) \times (\partial M_i \times I) \cup (\partial M_i \times I) \times (\partial M_i \times \{0\}).$$

Also we define $\partial_1(M_i)^2$ as the closure of $\partial(M_i)^2 - \partial_0(M_i)^2$ in $\partial(M_i)^2$. In other words, $\partial_1(M_i)^2$ is the space

$$M_i' \times (\partial M_i \times \{0\}) \cup (\partial M_i \times \{0\}) \times M_i'.$$

See Figure 3.

Now, consider the manifold triads $((M_i)^2; \partial_0(M_i)^2, \partial_1(M_i)^2; \partial\partial_1(M_i)^2)$, $i = 0, 1$, and the map

$$\hat{f} : (\partial_0(M_0)^2, \partial\partial_0(M_0)^2) \longrightarrow (\partial_0(M_1)^2, \partial\partial_0(M_1)^2)$$

defined by

$$\hat{f}(x, s, y, t) = (f(y), t, f(x), s)$$

for any $(x, s, y, t) \in (\partial M_0 \times I)^2$ such that $s = 0$ or $t = 0$.

We introduce the following identification:

$$\partial_0(M_i)^2 = (\partial M_i \times \{0\}) \times (\partial M_i \times I) \cup (\partial M_i \times I) \times (\partial M_i \times \{0\})$$
$$\cong (\partial M_i)^2 \times [-1, 1]$$

respectively given by the homeomorphisms

$$\begin{matrix} (x, s, y, t) \longrightarrow (x, y, s - t) & \text{if } i = 0, \\ (x, s, y, t) \longrightarrow (y, x, t - s) & \text{if } i = 1. \end{matrix}$$

Then, up to these identifications, $\hat{f}$ as defined above is simply $f \times f \times 1$. Let $(N, \partial N)$ denote the pair

$$((M_0)^2 \cup_{\hat{f}} (M_1)^2, \partial_1(M_0)^2 \cup_{\hat{f}'} \partial_1(M_1)^2),$$

where $\hat{f}'$ is the restriction of $\hat{f}$.

Note that there is a well-defined injection $\iota : X \to N$ given by the diagonal maps $(M_i, \partial M_i) \to ((M_i)^2, \partial_0(M_i)^2)$, $i = 0, 1$.

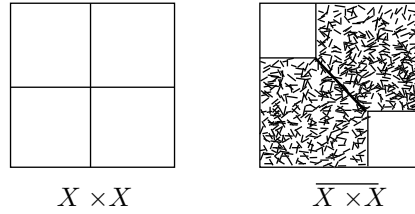$$X \times X \qquad\qquad \overline{X \times X}$$

FIGURE 4

Claim 4.B. *There is a space $\overline{X \times X}$ satisfying:*
   i) $\overline{X \times X}$ *admits a decomposition* $\overline{X \times X} = N \cup C$ *in which the equality, $N \cap C = \partial N$, holds and*
   ii) $(C, \partial N)$ *is a Poincaré pair. Furthermore,*
   iii) there is a homotopy equivalence $h : \overline{X \times X} \to X \times X$ so that the diagram,

$$
\begin{array}{ccc}
X & \overset{\Delta}{\to} & X \times X \\
\downarrow\iota & & \uparrow h \\
N & \hookrightarrow & \overline{X \times X}
\end{array}
$$

commutes, where Δ is the diagonal map (see Figure 4).

The proof of the claim is straightforward, and we provide only a sketch of it in the next section.

Then the assumptions of *4.9* is satisfied by $\iota : X = M_0 \cup_f M_1 \to (M_0)^2 \cup_{\hat{f}} (M_1)^2$ if we set $T = (\partial M_1)^2 \times \{0\}$. Thus, we may conclude there exists a Poincaré embedding structure on $\iota : X \to (N, \partial N)$, which is enough to prove *Theorem 1.3*.

Remark. The theorem is true in a more general setting. We could have replaced the manifold triads $(M_i; \partial^- M_i, \partial^+ M_i; \emptyset)$ by the more general

$$(M_i; \partial^- M_i, \partial^+ M_i; \partial\partial^- M_i)$$

and consider

$$(X, Y) \simeq (M_0 \cup_f M_1, q(\partial^- M_0 \sqcup \partial^+ M_1)),$$

in which $f : (\partial^+ M_0, \partial\partial^+ M_0) \to (\partial^- M_1, \partial\partial^- M_1)$ is a homotopy equivalence and $q : M_0 \sqcup M_1 \to M_0 \cup_f M_1$ is the quotient map.

5. PROOFS

Proof of Claim 4.A. We construct α by applying *2.5*. First of all, construct a homotopy equivalence,

$$\alpha' : P_0 \times I \cup_{H_0} Q_0 \to P_0 \times I \cup_{H_1} Q_0$$

rel $P_0 \times \{1\} \cup q(\partial P_0 \times I \sqcup Q_0)$ as in *2.5*, assuming $\partial P_0 \neq \emptyset$ (the simpler case $\partial P_0 = \emptyset$ can be dealt with similarly). To be more specific, P_0, Q_0 here respectively correspond to X, Y of *2.5*. Furthermore, we use a decomposition of P_0 in such a way that both A, B of *2.6* correspond to ∂P_0 and C, to $P_0 - \partial P_0 \times [0,1)$, where $\partial P_0 \times [0,1)$ is identified with an open collar of $\partial P_0 \subset P_0$.

Then we subsequently define $\alpha : (\bar{X}, \partial\bar{X}) \to (X, \partial X)$, so that α is the identity on $\bar{X} - (P_0 \times [0,1) \cup_{H_0} Q_0)$ and $\alpha = \alpha'$ on $P_0 \times I \cup_{H_0} Q_0$.

Now we prove that $\alpha\iota_1 \simeq \iota$.

Since F is an embedding and, by construction, $HF \simeq G(\varphi' \times 1) : \partial A \times I \to Q_0 \times I$ rel $\partial A \times \{0, 1\}$, we may apply a cofibration argument to replace the map $H : P_0 \times I \to Q_0 \times I$ with a new one, up to homotopy rel $\partial P_0 \times I \cup P_0 \times \{0, 1\}$, so that, still calling the new map H, we have

$$HF = G(\varphi' \times 1).$$

Thus we may assume α is constructed using such H. Then we have $\alpha \iota_1 \equiv \iota$ on $A' \sqcup B' \subset Y$.

By construction, α restricts to

$$\alpha' : P_0 \times I \cup_{H_0} Q_0 \times I \longrightarrow P_0 \times I \cup_{\varphi} Q_0 \times I.$$

Also ι_1 restricts to

$$\iota_1' : \partial A \times I \cup_{\varphi'} \partial B \times I \longrightarrow P_0 \times I \cup_{H_0} Q_0 \times I.$$

On the other hand, we may assume the restriction of ι to

$$\iota' : \partial A \times I \cup_{\varphi'} \partial B \times I \longrightarrow P_0 \times I \cup_{\varphi} Q_0 \times I$$

is given by $\iota'[y, t] = [y, t]$ for any $y \in \partial A \sqcup \partial B$. Now, to show $\alpha \iota_1 \simeq \iota$, it is enough to prove that

$$\alpha' \iota_1' \simeq \iota' \operatorname{rel}(\partial A \sqcup \partial B) \times \{1\}.$$

We may assume $pF(\partial A \times I) \subset P_0 - \partial P_0 \times [0, 1)$, where $p : P_0 \times I \to P_0$ is the projection. Then, by the definitions of α' and ι_1', it follows that

$$\alpha' \iota_1'[y, t] = \left\{ \begin{array}{ll} [y, 2t - 1] & \text{if } (y, t) \in \partial A \times [\frac{1}{2}, 1], \\ [HF(y, 2t), 0] & \text{if } (y, t) \in \partial A \times [0, \frac{1}{2}], \\ [G(y, t)] & \text{if } (y, t) \in \partial B \times I. \end{array} \right.$$

A homotopy

$$\mathbf{H} : (\partial A \times I \cup_{\varphi'} \partial B \times I) \times I \longrightarrow P_0 \times I \cup_{\varphi} Q_0 \times I$$

from $\alpha' \iota_1'$ to ι_1' rel $(\partial A \sqcup \partial B) \times \{1\})$ can be given, which 'cancels F with G' as follows:

$\mathbf{H}$ is defined as follows: if $y \in \partial A$ and $t, u \in I$, then

$$\mathbf{H}[y, t, u] = \left\{ \begin{array}{ll} [HF(y, 2t + u), 0)] & \text{if } 0 \le t \le \frac{1-u}{2}, \\ [y, \frac{2t-1+u}{1+u}] & \text{if } \frac{1-u}{2} \le t \le 1, \end{array} \right.$$

and, if $y \in \partial B$ and $t, u \in I$, then

$$\mathbf{H}[y, t, u] = \left\{ \begin{array}{ll} [G(y, t + u), 0] & \text{if } 0 \le t \le 1 - u, \\ [y, t] & \text{if } 1 - u \le t \le 1. \end{array} \right.$$

Note that $\mathbf{H}$ is well-defined, in particular, since we have: $HF = G(\varphi' \times 1)$.

We conclude that $\alpha \iota_1$ is homotopic to ι.

To provide a proof of *4.7*, we shall use the following theorem in addition to *3.1*.

Theorem 5.1 (S.E. Cappell)**.** *Let Y be a Poincaré complex of dimension $n + 1$, $n \ge 5$, and X a closed connected sub-Poincaré complex of dimension n of Y for which there are connected subcomplexes Y_0 and Y_1 satisfying $Y = Y_0 \cup Y_1$ and $X = Y_0 \cap Y_1$, and (Y_0, X), (Y_1, X) are Poincaré pairs. Assume one of the homomorphisms $\pi_1(X) \to \pi_1(Y_i)$ is an isomorphism. Then any homotopy equivalence $f : W \to Y$, where W is a smooth manifold, is splittable along X.*

In fact, *5.1* is a portion of what is stated in *Theorem 1* of [2] and the footnote below it (also see [13]). Note that the square-root closed condition is not needed here.

Proof of Lemma 4.7. First, we construct the map $H : P_0 \times I \to Q_0 \times I$ so that it satisfies i).

Let $(U_B, U_{\partial B}) \subset (Q, Q_0)$ be a regular neighborhood of $(B, \partial B)$ such that $U_B \cap Q_1 = \emptyset$. Then, since $\dim Q_0 - \dim \partial B \geq 3$ by assumption, $\partial U_{\partial B} \hookrightarrow U_{\partial B}$ induces an isomorphism between fundamental groups. Assume each path component of Q_0 contains at most one path component of B. Then, by applying a relative version of *5.1*, there is a homotopy $H' : P_0 \times I \to Q_0$ rel ∂P_0 such that H'_0 splits along $\partial U_{\partial B}$ and $H_1 = \varphi$. For the path components of Q_0 which contain more than one path component of ∂B, we apply the same theorem to the path components of ∂B, one by one, taking out the interior of the regular neighborhood from the target domain at each step. Then we define $H : P_0 \times I \to Q_0 \times I$ by the rule $H(x, t) = (H'(x, t), t)$.

As the second step, we construct $F : \partial A \times I \to P_0 \times I$ so that it satisfies ii).

Write $V_{\partial B} \equiv Q_0 - \text{Int } U_{\partial B}$ and set $U_{\partial A} \equiv H_1^{-1} U_{\partial B}$, $V_{\partial A} \equiv H_1^{-1} V_{\partial B}$. Let $\hat{H}_0$ be the inverse of $H_0 : (P_0, \partial P_0) \to (Q_0, \partial Q_0)$, which preserves the splitting of H_0; that is, $\hat{H}_0$ restricts to $U_{\partial B} \to U_{\partial A}$, $V_{\partial B} \to V_{\partial A}$, $\partial U_{\partial B} \to \partial U_{\partial A}$ etc.

Let $\bar{e}$ be the composite

$$\partial A \hookrightarrow (P_0, \partial P_0) \xrightarrow{H_0} (Q_0, \partial Q_0) \xrightarrow{\hat{H}_0} (P_0, \partial P_0).$$

Then it follows that $\bar{e}$ is homotopic to the inclusion $\iota_{\partial A} : \partial A \to P_0$, since $\hat{H}_0 \varphi$ is homotopic to the identity.

Note that

$$\bar{e}A \subset \text{Int } U_A.$$

The restriction e of $\bar{e}$ to $A \to \text{Int } U_A$ is a homotopy equivalence. Therefore, since $2 \dim P_0 \geq 3(\dim A + 1)$, e is homotopic to a smooth embedding by applying *3.1*.

Thus, we have shown there is a homotopy $F' : \partial A \times I \to P_0$ from $\bar{e}$ to $\iota_{\partial A}$. Define $F : \partial A \times I \to P_0 \times I$ by the rule $F(x, t) = (F'(x, t), t)$.

Now we modify $H : P_0 \times I \to Q_0 \times I$ so that iii) can be satisfied.

Let $\iota_{\partial B} : \partial B \hookrightarrow Q_0$, $\iota : \text{Int } U_{\partial B} \hookrightarrow Q_0$ denote the inclusions and $H_0'' : \text{Int } U_{\partial A} \to \text{Int } U_{\partial B}$, the restriction of H_0. Note that $e : \partial A \to \text{Int } U_{\partial A}$ is an embedding and $H_0'' e : \partial A \to \text{Int } U_{\partial B}$ is homotopic to $\iota_{\partial B} \varphi' : \partial A \to \text{Int } U_{\partial B}$. Therefore, by a cofibration argument, there is a homotopy from H_0 to a map, say $\tilde{H}_0$, so that $\tilde{H}_0 e \equiv \varphi'$ and $\tilde{H}_0$ preserves the splitting of H_0. We replace the homotopy H' by the new one, H', followed by the homotopy from H_0 to $\tilde{H}_0$. Keep calling it H'. Then, we define $H : P_0 \times I \to Q_0 \times I$ by the rule $H(x, t) = (H'(x, t), t)$.

Now we choose a $G : \partial B \times I \to Q_0$ so that iv) is satisfied.

Consider the base point preserving map,

$$(\varphi')^* : (\text{Map } (\partial B, Q_0), \iota_{\partial B}) \to (\text{Map } (\partial A, Q_0), \iota_{\partial B} \varphi').$$

$(\varphi')^*$ is a homotopy equivalence since φ' is a homotopy equivalence. Note, in particular, that $(\varphi')^*$ induces an isomorphism between the fundamental groups. Therefore, there is a loop,

$$g : (I, \{0, 1\}) \to (\text{Map } (\partial B, Q_0), \iota_{\partial B})$$

so that $(\varphi')^*g$ represents the loop $\mathrm{ad}(HF)$ in the pointed space

$$(\mathrm{Map}\,(\partial A, Q_0), \iota_{\partial B}\varphi'),$$

where $\mathrm{ad}(HF)$ is defined by the rule $\mathrm{ad}(HF)(t) = H_t F_t$. Let $G' : \partial B \times I \to Q_0$ be such that $G'_t = g(t)$. Define $G : \partial B \times I \to Q_0 \times I$ by the rule $G(x,t) = (G'(x,t), t)$. This completes the proof of *4.7*.

Sketch of the proof of Claim 4.B. Write:

$$K \equiv \partial M_0 \times I \cup_f \partial M_1 \times I.$$

Since K is a neighborhood of $\partial M_1 = \partial M_1 \times \{0\} \subset X$, $K \times K$ is a neighborhood of $(\partial M_1)^2$ in $X \times X$. We will write

$$\mathrm{Int} K \equiv \partial M_0 \times [0,1) \cup_f \partial M_1 \times [0,1), \quad \mathrm{Int}\,K^2 \equiv (\mathrm{Int}\,K)^2,$$

$$\partial K \equiv \partial M_0 \times \{1\} \sqcup \partial M_1 \times \{1\},$$

$$\partial(K \times K) \equiv \{([x,s],[y,t]) \in K \times K \mid s = 1 \text{ or } t = 1\}.$$

To construct $\overline{X \times X}$, we will replace $K \times K \subset X \times X$ by some other space, say $\overline{K \times K}$, which contains $\partial(K \times K)$ as a subspace and is homotopy equivalent to $K \times K$ rel $\partial(K \times K)$. In fact, we define $\overline{K \times K}$ as follows:

Note that $K \times K$ can be understood as a space formed by gluing the four spaces

$$(\partial M_0 \times I)^2, \quad (\partial M_0 \times I) \times (\partial M_1 \times I),$$

$$(\partial M_1 \times I) \times (\partial M_0 \times I), \quad (\partial M_1 \times I)^2.$$

We define $\overline{K \times K}$, say, *by rearranging the gluing* (see the figure below *Claim 4.B*).

First we glue $(\partial M_0 \times I) \times (\partial M_1 \times I) \sqcup (\partial M_1 \times I) \times (\partial M_0 \times I)$ to $(\partial M_1 \times I)^2$ using the gluing map:

$$h \equiv f \times (1 \times \mu) \sqcup (1 \times \mu) \times f :$$
$$(\partial M_0 \times \{0\}) \times (\partial M_1 \times I) \sqcup (\partial M_1 \times I) \times (\partial M_0 \times \{0\})$$
$$\longrightarrow (\partial M_1 \times \{0\}) \times (\partial M_1 \times I) \cup (\partial M_1 \times I) \times (\partial M_1 \times \{0\}),$$

where 1 represents the identity maps and $\mu : I \to I$ is defined by $\mu(t) = \frac{1}{2}(t+1)$. Then we write $\widehat{K \times K}$ for the resulting space.

Consider the subspace $\partial_0 \widehat{K \times K}$ of $\widehat{K \times K}$, defined as

$$((\partial M_1 \times \{0\}) \times (\partial M_0 \times I)) \cup_{h'} ((\partial M_0 \times [0, \tfrac{1}{2}]) \times (\partial M_1 \times \{0\}))$$

$$\cup ((\partial M_0 \times I) \times (\partial M_1 \times \{0\})) \cup_{h''} ((\partial M_1 \times \{0\}) \times (\partial M_0 \times [0, \tfrac{1}{2}])),$$

in which h' is the restriction of $(1 \times \mu) \times f$ to

$$(\partial M_1 \times \{0\}) \times (\partial M_0 \times \{0\}) \longrightarrow (\partial M_1 \times \{\tfrac{1}{2}\}) \times (\partial M_1 \times \{0\})$$

and h'' is the restriction of $f \times (1 \times \mu)$ to

$$(\partial M_0 \times \{0\}) \times (\partial M_1 \times \{0\}) \longrightarrow (\partial M_1 \times \{0\}) \times (\partial M_1 \times \{\tfrac{1}{2}\}) .$$

Also consider the subspace $\partial_0 (\partial M_0 \times I)^2$ of $(\partial M_0 \times I)^2$, which is defined as

$$(\partial M_0 \times \{0\}) \times (\partial M_0 \times I) \cup (\partial M_0 \times \{0\}) \times (\partial M_0 \times I)$$

Now we define the attaching map, $g : \partial_0(\partial M_0 \times I)^2 \longrightarrow \partial_0\widehat{K \times K}$, as follows:

$$g(x,s,y,t) = \begin{cases} [f(x),0,y,2t-1] & \text{if } s=0, \frac{1}{2} \le t \le 1, \\ [f(y),t,f(x),0] & \text{if } s=0, 0 \le t \le \frac{1}{2}, \\ [f(y),0,f(x),s] & \text{if } t=0, 0 \le s \le \frac{1}{2}, \\ [x,2s-1,f(y),0] & \text{if } t=0, \frac{1}{2} \le s \le 1. \end{cases}$$

To check the well-definedness of g is straightforward.

Finally, we set

$$\overline{K \times K} \equiv (\partial M_0 \times I)^2 \cup_g \widehat{K \times K}.$$

Note that there is an obvious identification between

$$\partial\overline{K \times K} \equiv \{[x,s,y,t] \in \overline{K \times K} | s=1 \ \text{ or } \ t=1\}$$

and

$$\partial(K \times K) = \{[x,s,y,t] \in K^2 | s=1 \ \text{ or } \ t=1\}.$$

Therefore, we may define $\overline{X \times X} \equiv (X \times X - \text{ Int } (K \times K)) \cup \overline{K \times K}$.

To show that $\overline{X \times X}$ satisfies $i), ii), iii)$ of the claim is straightforward.

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